A Deconvolution method for Astronomical Photometric and Spectroscopic Observations.

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Abstract:

A numerical method for deconvoluting one-dimensional optical imaging is described along with its algorithm. This method can   very successfully be used for achieving image enhancement and restoration with minimised noise, spurious as well as systematic, in the case of spectroscopic and photometric data in optical or infrared astronomy. An example has been given to illustrate the importance of the smoothing and restoration parameters in the method.

1. Introduction:

In most of the astronomical photometric and spectroscopic work, the observed image of a celestial object under study is inevitably modified by what is known as instrumental function which is essentially defined by (i) aperture function of the instrument or the point-spread function, (ii) filter response, (iii) the response time of the detector used and many other limitations in the experimental and observational process. In all these cases, it becomes necessary to retrieve the actual signal due to the object without the modifications introduced by the extraneous agencies from the noisy image observed, given a priori the nearest equivalent of the instrumental profile.

Let us suppose that we observed an object with a certain instrument under certain conditions pertaining to various extraneous factors which influence the data. Then, in general, the measured function $s(t)$ does not truly represent the object function $p(t)$; but, on the other hand, it represents the convolution of the object function with a modulation or transfer functions $a(t)$. This can be written down as,

$$s(t) = p(t) * a(t) \qquad \qquad \ldots\ldots(1)$$

where, the symbol * represents the convolution operation.
We can write the above equation also in the following way,

$$s(t) = \int_{-\infty}^{\infty} p(t') \, a \, (t - t') \, dt' \qquad \ldots\ldots(2)$$

where the integral on the right-side represents the convo-
lution integral. Knowing $s(t)$ and $a(t)$, it is indeed
possible to retrieve the object $p(t)$ provided the observa-
tions are noise-free, by direct methods.

For instance, one can take the Fourier transform on
both sides of the equation (1) and then a simple division
followed by an inverse Fourier transformation would give
the desired object function $p(t)$, in terms of the observed
image $s(t)$ and the instrumental function $a(t)$ which is
also called as the point-spread function (p.s.f). However,
the p.s.f in many applications has dominant values in the
central part of the data and falls off steeply to near-zero
values on either side of central maximum. This may lead to
practical difficulties in obtaining the discrete Fourier
transform.

Alternatively, we can write the integral equation (2)
as a set of linear equations and can, in principle, solve

for the unknown function p(t) by matrix inversion. As stated already, the matrix elements formed by the function a(t) are usually very small and this fact renders the matrix ill-conditioned. Therefore, eventhough, the errors (or noise) are negligibly small in s(t), they will contribute to large errors in p(t). Thus, it is desirable to approach the problem as if s(t) has errors (however small they may be) and minimise these errors in obtaining p(t).

In this note, we describe a method and its algorithm to find a solution to the problem addressed here using the above mentioned approach.

2. Mathematics of the Method.

Writing down the convolution integral and adding the noise term n(t), the eqn. 2 becomes,

$$s(t_m) = \int_{-\infty}^{\infty} p(t) \, a \, (t_m - t) \, dt + n(t_m) \qquad \ldots\ldots(3)$$

where m = 1,2,3,....N, N being the number of data points. Approximating the integral as a finite sum and assuming that there are closely spaced data-points, eqn.(3) can be written as,

$$s(t_m) = \sum_{i=1}^{N} p(t_i) \, a(t_m - t_i) \, w_i + n_m \qquad \ldots\ldots(4)$$

$$m = 1, 2, \ldots\ldots N.$$

where w_i are the weightage factors for the numerical method used for integration, $w_i = k_i \, \triangle t$ with k_i some constant and $\triangle t$, the step - size in t. The summation approximation in eqn (4) is valid when a (| t |) is a fast decreasing function of t. This property is satisfied in most of the applications.

From the eqn. (4) the variance in noise is written as,

$$\sigma^2 = \frac{1}{N} \sum_{m=1}^{N} (n_m^2) = \frac{1}{N} \sum_{m=1}^{N} \left[s(t_m) - \sum_{i=1}^{N} p(t_i) a(t_m - t_i) w_i \right]^2$$

$$\ldots \; (5)$$

Now the condition for minimisation of σ^2 is,

$$\frac{\partial \sigma^2}{\partial p_i} = 0 \quad \text{for } i = 1, 2, \ldots\ldots N \qquad \ldots\ldots(6)$$

Substituting eqn. (5) in eqn (6), we obtain the following

set of linear equations,

$$\sum_{j=1}^{N} A_{ij} P_j = B_i \quad , \quad i = 1,2,\ldots\ldots N \quad \ldots\ldots(7)$$

where, $A_{ij} = \dfrac{1}{N} \sum_{m=1}^{N} w_i \, a(t_m - t_i) \, w_j \, a(t_m - t_j)$

and $B_i = \dfrac{1}{N} \sum_{m=1}^{N} w_i \, a(t_m - t_i) \, s_m$ $\ldots\ldots(8)$

the object function P_j can now be obtained by solving the eqn.(7).

3. Smoothing:

The solution obtained by solving the eqn.(7), is known to be contaminated, in general, by spurious ripples (Phillips, 1962). This is because of the ill-conditioning of the matrix $\begin{bmatrix} A \end{bmatrix}$ involving the p.s.f, $a(t)$. To avoid these spurious ripples, in addition to the minimisation of the variance of the noise, we seek to impose smoothness constraints on the solution in which we choose

an operator C together with a parameter γ and minimise

$$\| ap - s \|^2 + \gamma \| cp \|^2 .$$ In this method,

known as regularization (Phillips, 1962), the operator

C can be I, $\dfrac{d}{dt}$ or $\dfrac{d^2}{dt^2}$ and so on. A proper selection of γ

removes the ill-posed nature of the problem.

Here in our present discussion, we use C as $\dfrac{d^2}{dt^2}$

and use the numerically equivalent second differences for

minimisation.

Let us now define

$$\epsilon^2 = \frac{\gamma}{(N-1)} \sum_{i=2}^{N} \left[p(t_{i+1}) - 2\, p(t_i) + p(t_{i-1}) \right]^2$$

$$........(9)$$

Then the condition for minima in eqn. (6) becomes,

$$\frac{\partial (\sigma^{-2} + \epsilon^2)}{\partial p_i} = 0 \qquad ....(10)$$

This leads to the following equation,

$$\sum_{j=1}^{N} \left(A_{ij} + C_{ij} \right) P_i = B_i \ , \ i=1,2,\ldots.N \quad \ldots.(11)$$

which is a modified version of eqn.(7) with A_{ij} and B_i having the same definitions as before, and with,

$$C_{ij} = \frac{1}{(N-1)} \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & \cdots \\ -2 & 5 & -4 & 1 & 0 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & 0 & \cdots \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

The parameter γ is to be so chosen as to give importance to both the observed data and the regularization or smoothing required. In other words, there will be a particular value of γ, say γ_c, above which the smoothing process dominates and we start sacrificing the information or the self-consistency, while below the value of γ_c, the ripples will dominate the solution. In general the value of γ_c has to be chosen in accordance with the signal - to - noise ratio and after a few trials we can fix the value of γ_c for the desired and meaningful solution p(t).

4. Algorithm:

The algorithm of the method described above is as follows:

(i) We have the observed data set $s(t)$ at $t = t_1, t_2, \ldots t_N$

(ii) We assume the point-spread-function $a(t)$ at $t = t_1, t_2, \ldots t_N$

(iii) Then we form the constant matrix C

(iv) We choose the weightage factors w_i $(i = 1, 2, \ldots N)$ depending upon the numerical method of integration. And we fix the step-size $\triangle t$.

(v) The column matrix elements

$$B_i = \frac{1}{N} \sum_{m=1}^{N} w_i \, a(t_m - t_i) s_m \; ; \; i=1, 2, \ldots N$$

are computed and the matrix B is formed.

(vi) Then the matrix elements

$$A_{ij} = \frac{1}{N} \sum_{m=1}^{N} w_i \, a(t_m - t_i) w_j \, a(t_m - t_j)$$

for $i=1, 2, 3, \ldots N$ and $j = 1, 2, \ldots N$ are computed and the matrix A is formed.

(vii) We assign a certain value for γ

(viii) Then the solution P_i ($i=1,2,\ldots.N$) is initialized

by assuming, for instance,

$$P_i^0 = 10^{-10} \quad \text{for all i}$$

(ix) The matrix equation

$$(A + \gamma C) \quad P = B$$

is solved for P

(x) The error, then is computed by

$$E = - \gamma B P$$

and check whether $\eta = \max_i \left| E_i \right| \leq \eta_0$

a constant value representing the variance of the noise
originally present in the data-set. If this inequality
is satisfied then the deconvolution procedure ends. If
not then we start from the step (vii) all over again by
changing the value of γ. Sometimes, one may use the
value of η for γ and go to step (vii) - in each iteration
substituting the new value of η for γ.

We have used Gauss-Seidel iterative method for
solving the matrix equation above, and found that a
reasonably fast convergence to a solution is achieved.

5. Example:

We have used the above discussed deconvolution method to far-infrared photometric data successfully. The data were obtained by a far-infrared photometer at the Cassegrain focus of a 32.5 cm telescope on board a Caravelle aircraft. The specific data used were on the planet Venus in 100 micron photometric band. The planet was observed in a scanning mode with a field aperture of 1.4 arc minutes. Since the planet's angular-size is about 8 arc seconds one can consider this as a point-source and use it to obtain the instrumental profile or the point-spread function. In order to check this, we have used the actual observations on Venus and deconvolved these with a point-spread function of Gaussian shape with a full-width at half-maximum intensity of 1.4 arc minutes. The result is shown in the figure 1. One can see that there is almost a perfect agreement between the actual data and the deconvolved profile. This proves that a point-source like a planet can be used to obtain the point-spread function even in case of far-infrared wavelengths (Van der Wal et al, 1985).

6. Conclusions:

The following conclusions are in order regarding the application of the method:

(i)   While approximating the convolution integral

as a finite sum, in many cases, the Simpson's rule is adequate; if the data points are sufficiently closely spaced, even the trapezoidal rule may be used.

(ii) The operator C can, in general, be taken as a second-order difference; however, if necessary, in some cases one may have to go to higher order differences.

(iii) Choosing of the critical parameter γ_c is dependent on the observational errors and usually it can be determined by trial basis.

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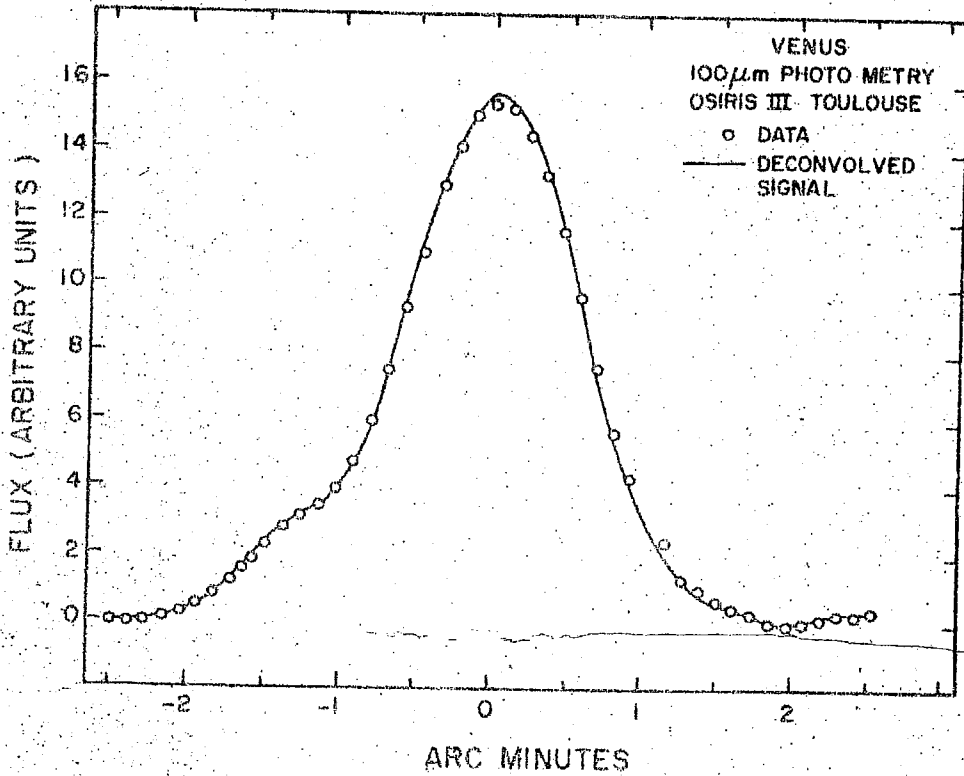
Figure 1 : Far-infrared (100 microns) photometry of
Venus; the actual data are deconvoluted
using a Gaussian of FWHM of 1.4 arc min
(after Van der Wal, 1985).