PRACTICAL METHODS OF HARMONIC ANALYSIS
FOR GEOPHYSICAL PROBLEMS

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I. INTRODUCTION

The use of Fourier series of the type
\[ a_0 + a_1 \cos \theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \ldots \ldots \]
\[ + b_1 \sin \theta + b_2 \sin 2\theta + b_3 \sin 3\theta + \ldots \ldots \]  \( (1) \)

is very common in solutions of a variety of problems in physics, meteorology and astronomy. An analysis of a function into a sum of trigonometric terms was first attempted in its simplest form in 1754-59 by Clairaut\(^1\) and Lagrange.\(^2\) They showed how to expand a function \( u(x) \) into a trigonometric series of finite number of terms such as
\[ u(x) = b_1 \sin x + b_2 \sin 2x + \ldots + b_{n-1} \sin (n-1)x \]  \( (2) \)

so as to give some known values \( u_1, u_2, \ldots, u_{n-1} \) of \( u \) for \((n-1)\) equally spaced values of the argument \( x \), e.g.,
\[ x = \pi/n, 2\pi/n, 3\pi/n, \ldots, (n-1)\pi/n. \]

Values of the coefficients \( b_1, b_2, \ldots, b_{n-1} \) could be evaluated from the general expression
\[ b_m = \frac{2}{n} \left\{ u_1 \sin \frac{m\pi}{n} + \ldots + u_{n-1} \sin \frac{(n-1)m\pi}{n} \right\} \]  \( (3) \)

The method gives trigonometrical expressions which agree very closely with the given function only over a certain restricted range of values of the argument \( x \). Outside this range the agreement may be far from satisfactory.

A more general formula involving both sine and cosine terms was suggested by Bessel\(^3\) in 1815. It enables one to obtain the best fitting series of terms when we have more data at hand than the minimum required.
Let the function

\[ u(x) = a_0 + a_1 \cos x + \ldots + a_r \cos rx \]
\[ + b_1 \sin x + \ldots + b_r \sin rx \]  \hspace{1cm} (4)

assume the values \( u_0, u_1, \ldots, u_{n-1} \) for values

\( 0, 2\pi/n, \ldots, 2(n-1)\pi/n \nespectively of the argument \( x \). We have thus \( n \) equations in all to determine the constants \( a_0, a_1, b_1, \ldots, a_r, b_r \) which are \( 2r + 1 \) in number. If \( n \) is greater than \( 2r \), the number of equations at hand is more than the minimum required and the conditions laid upon the values of the constants \( a_0, a_1, b_1, \) etc., for getting values \( u_0, u_1, \ldots, u_{n-1} \) for \( u(x) \) as accurately as possible, can be put down as follows:

\[
\begin{align*}
  u_0 &= a_0 + a_1 + \ldots + a_r \\
  u_1 &= a_0 + a_1 \cos \frac{2\pi}{n} + \ldots + a_r \cos \frac{r \cdot 2\pi}{n} \\
  &\quad + b_1 \sin \frac{2\pi}{n} + \ldots + b_r \sin \frac{r \cdot 2\pi}{n} \\
  \vdots \\
  u_r &= a_0 + a_1 \cos \frac{(n-1) \cdot 2\pi}{n} + \ldots + a_r \cos \frac{r \cdot (n-1) \cdot 2\pi}{n} \\
  &\quad + b_1 \sin \frac{(n-1) \cdot 2\pi}{n} + \ldots + b_r \sin \frac{r \cdot (n-1) \cdot 2\pi}{n}
\end{align*}
\]  \hspace{1cm} (5)

With the help of standard trigonometrical formulæ, one gets finally the set of values

\[
\begin{align*}
  a_0 &= \frac{1}{n} \sum_{k=0}^{n-1} u_k \\
  a_1 &= \frac{2}{n} \sum_{k=0}^{n-1} u_k \cos \frac{2k\pi}{n} \\
  \vdots \\
  a_r &= \frac{2}{n} \sum_{k=0}^{n-1} u_k \cos \frac{2kr\pi}{n}
\end{align*}
\]  \hspace{1cm} (6)
Practical Methods of Harmonic Analysis for Geophysical Problems

\[ b_1 = \frac{2}{n} \sum_{k=0}^{n-1} u_k \sin \frac{2k\pi}{n} \]

\[ b_r = \frac{2}{n} \sum_{k=0}^{n-1} u_k \sin \frac{2kr\pi}{n} \]

When \( r = n/2 \), the factor of \( a_r \) in front of the symbol \( \Sigma \) is \( 1/n \), not \( 2/n \).

II. APPLICATION OF HARMONIC ANALYSIS TO GEOPHYSICAL PROBLEMS

The method of harmonic analysis is very useful for studying solar and sidereal diurnal variations of geophysical elements. For a period of one day, the variation can be broken up into harmonic components of periods 24 hours, 12 hours, etc. This can be done either from 24 values spaced at hourly intervals or from 12 values spaced at bihourly intervals. For a 24 value scheme one has for analysis 24 values \( u_0, u_1, \ldots, u_{23} \) which when substituted in equation (6), yield the coefficients \( a_0, a_1, \ldots, a_{11}, b_1, \ldots, b_{11} \). For the 12 value scheme, \( u \) assumes the 12 bihourly values \( u_0, u_1, \ldots, u_{11} \) giving the coefficients \( a_0, a_1, \ldots, a_6, b_1, \ldots, b_6 \). For harmonics of low order (first, second, etc.) the values obtained for the coefficients by both the schemes are almost identical. Since, in the majority of cases encountered in geophysics, only the first two harmonics are of significance, the adoption of the bihourly scheme saves a lot of computational labour.

Analysis of a daily variation into its harmonic components enables one to study the physical phenomena underlying them in a simple and precise way. The results can be presented with advantage on a harmonic dial which is essentially a vector diagram of a particular harmonic component, the length and direction of the vector representing the magnitude and hour of maximum of the particular harmonic component of the daily variation.

For a set of 12 bihourly values \( u_0, u_1, \ldots, u_{11} \) the coefficients \( a_0, a_1, b_1, \) etc., in the equation

\[ u(x) = a_0 + a_1 \cos x + \ldots + a_5 \cos 5x + a_6 \cos 6x \]
\[ + b_1 \sin x + \ldots + b_5 \sin 5x \]
are given by

\[12 \alpha_0 = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8 + u_9 + u_{10} + u_{11}\]

\[6 \alpha_1 = u_0 + u_1 \sqrt{3}/2 + u_2 \cdot 1/2 - u_4 \cdot 1/2 - u_5 \cdot \sqrt{3}/2 - u_6 - u_7 \cdot \sqrt{3}/2 - u_8 \cdot 1/2 + u_{10} \cdot 1/2 + u_{11} \cdot \sqrt{3}/2\]

\[6 \alpha_2 = u_0 + u_1 \cdot 1/2 - u_2 \cdot 1/2 - u_3 - u_4 \cdot 1/2 + u_5 \cdot 1/2 + u_6 + u_7 \cdot 1/2 - u_8 \cdot 1/2 - u_0 - u_{10} \cdot 1/2 + u_{11} \cdot 1/2\]

\[6 \alpha_3 = u_0 - u_2 + u_4 - u_6 + u_8 - u_{10}\]

\[6 \alpha_4 = u_0 - u_1 \cdot 1/2 - u_2 \cdot 1/2 + u_3 - u_4 \cdot 1/2 - u_5 \cdot 1/2 + u_6 - u_7 \cdot 1/2 - u_8 \cdot 1/2 + u_0 - u_{10} \cdot 1/2 - u_{11} \cdot 1/2\]

\[6 \alpha_5 = u_0 - u_1 \cdot \sqrt{3}/2 + u_2 \cdot 1/2 - u_4 \cdot 1/2 + u_5 \cdot \sqrt{3}/2 - u_6 + u_7 \cdot \sqrt{3}/2 - u_8 \cdot 1/2 + u_{10} \cdot 1/2 - u_{11} \cdot \sqrt{3}/2\]

\[12 \alpha_6 = u_0 - u_1 + u_2 - u_3 + u_4 - u_5 + u_6 - u_7 + u_8 - u_9 + u_{10} - u_{11}\]

\[6 b_1 = u_1 \cdot 1/2 + u_2 \cdot \sqrt{3}/2 + u_3 + u_4 \cdot \sqrt{3}/2 + u_5 \cdot 1/2 - u_7 \cdot 1/2 - u_8 \cdot \sqrt{3}/2 - u_0 \cdot \sqrt{3}/2 - u_{11} \cdot 1/2\]

\[6 b_2 = u_1 \cdot \sqrt{3}/2 + u_2 \cdot \sqrt{3}/2 - u_4 \cdot \sqrt{3}/2 - u_5 \cdot \sqrt{3}/2 + u_7 \cdot \sqrt{3}/2 + u_8 \cdot \sqrt{3}/2 - u_{10} \cdot \sqrt{3}/2 - u_{11} \cdot \sqrt{3}/2\]

\[6 b_3 = u_1 - u_3 + u_5 - u_7 + u_9 - u_{11}\]

\[6 b_4 = u_1 \cdot \sqrt{3}/2 - u_2 \cdot \sqrt{3}/2 + u_3 \cdot \sqrt{3}/2 + u_4 \cdot \sqrt{3}/2 + u_6 \cdot \sqrt{3}/2 - u_7 \cdot \sqrt{3}/2 + u_8 \cdot \sqrt{3}/2 - u_{10} \cdot \sqrt{3}/2 - u_{11} \cdot \sqrt{3}/2\]

\[6 b_5 = u_1 \cdot 1/2 - u_2 \cdot \sqrt{3}/2 + u_3 + u_4 \cdot \sqrt{3}/2 + u_5 \cdot 1/2 - u_7 \cdot 1/2 + u_8 \cdot \sqrt{3}/2 - u_9 + u_{10} \cdot \sqrt{3}/2 - u_{11} \cdot 1/2\]

For a set of known values \(u_0, u_1, \ldots, u_{11}\), one can thus calculate all the coefficients \(\alpha_0, \alpha_1, b_1, \ldots, \alpha_r, b_r\) and get an expression for the function \(u(x)\) in terms of a sum of trigonometric functions.

III. Evaluation of Harmonic Coefficients

It is easily seen that the task of evaluating the various coefficients from the abovementioned formulae is a difficult one from the point of view of practical computation. It is necessary that the set of values \(u_0, u_1, \ldots, u_{11}\)
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should be handled in a convenient form to enable the evaluation of the various coefficients with the expenditure of the least amount of labour and time. Computation forms designed for this purpose with the help of the suggestions derived from many writers (C. Runge\(^4\) in particular) are given by Whittaker and Robinson\(^5\) for the 12 value and the 24 value schemes.

Another method of presentation of the same data in a different form is evolved by Joseph Lipka\(^6\) and Terebesi\(^7\). Computation chart for their 12 ordinate scheme is given by Henney\(^8\). The method is essentially similar to the one given by Whittaker and Robinson. But the tabulation arrangement is terminated at an earlier stage and resort is taken to picking up terms suitably for combining together to give the various coefficients. It is rather difficult to judge which one of the two methods is easier and more practical. With an added mental strain, the latter seems to be quicker than the former one.

IV. THOMPSON’S METHOD

In some physical problems, it is generally known that the third and the higher harmonics are negligibly small compared to the first and the second ones. It is then only necessary to evaluate accurately the latter two harmonics. Common examples are the daily variations of atmospheric pressure and temperature at any station. It was pointed out by Thompson\(^9\) for a case of the type

\[
u(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x\]

that the various coefficients could be evaluated by a simple averaging process as follows:

\[
a_0 = \frac{1}{12} (u_0 + u_1 + \ldots + u_{11})
\]
\[
a_1 = \frac{1}{2} (u_0 - u_3) - u_3
\]
\[
a_2 = \frac{1}{4} (u_0 - u_3 + u_6 - u_9)
\]
\[
a_3 = \frac{1}{6} (u_0 - u_3 + u_4 - u_6 + u_8 - u_{10})
\]
\[
b_1 = \frac{1}{2} (u_3 - u_6) + b_3
\]
\[
b_3 = \frac{1}{6} (u_1 - u_3 + u_5 - u_7 + u_9 - u_{11})
\]

For evaluating \(b_2\), the complete graph of the function \(u(x)\) is to be plotted and the values \(u_1, u_3, u_5\) and \(u_7\) for \(x = 45^\circ, 135^\circ, 225^\circ\) and \(315^\circ\) respectively are to be read off. The value of \(b_2\) is then given by

\[
b_2 = \frac{1}{4} (u_1 - u_3 + u_5 - u_7)
\]
Table I (a). Method of evaluating the first and second harmonics of a daily variation by the 12 value scheme

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>S</th>
<th>A'</th>
<th>B'</th>
<th>C'</th>
<th>D'</th>
<th>A''</th>
<th>B''</th>
<th>C''</th>
<th>D''</th>
<th>( A_2 )</th>
<th>( B_2 )</th>
<th>( C_2 )</th>
<th>( D_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_0 )</td>
<td>( \mu_0 - \mu_5 )</td>
<td>( \mu_2 - \mu_0 )</td>
<td>( \mu_0 + \mu_6 )</td>
<td>( \mu_3 + \mu_0 )</td>
<td>( \mu_1 + \mu_{11} )</td>
<td>( \mu_2 + \mu_7 )</td>
<td>( \mu_4 + \mu_4 )</td>
<td>( \mu_6 + \mu_{10} )</td>
<td>( \mu_2 + \mu_{10} )</td>
<td>( \mu_4 + \mu_3 )</td>
<td>( \mu_1 + \mu_5 )</td>
<td>( \mu_7 + \mu_{11} )</td>
<td>( \mu_4 + \mu_5 )</td>
<td>( \mu_4 + \mu_{10} + \mu_{11} )</td>
<td>( \mu_4 + \mu_5 )</td>
<td>( \mu_4 + \mu_{10} + \mu_{11} )</td>
</tr>
</tbody>
</table>

Checks:—
1. \( 0.500 \times (A_1 + B_2 + C_1 + D_1) = 0.866 \times (A_2 + B_2 + C_2 + D_2) = 0.866 \times (X + Y) \)
2. \( M' + N' + R + S = Z \)
3. \( P + Q + R + S = 2 (\mu_0 + \mu_3) \)

12 \( a_0 = Z \)
- \( a_1 = A_1 - B_4 + A_2 - B_2 + P \)
- \( b_1 = C_1 - D_1 + C_2 - D_2 + Q \)
- \( a_2 = R - S + Y - X \)
- \( b_2 = M - N \)
Table I (b). Supplementary chart for evaluating the third and higher harmonics of the daily variation

\[ \times 0.866 \]

| \( u_1 + u_4 \) | \( L_1 \) |
| \( u_7 + u_{10} \) |   |
| \( u_2 + u_5 \) | \( L_2 \) |
| \( u_8 + u_{11} \) |   |

CHECKS:
1. \( u_0 = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \)
2. \( u_1 - u_{11} = b_1 + 2b_3 + b_5 + 1.732 (b_2 + b_4) \)

\[ 12a_0 = R - S + A' + B'' - C'' - D'' \]
\[ b_6 = \text{Zero.} \]

The method, though apparently simple looking, is still difficult to handle and does not save much time especially because of the time required for evaluating \( b_2 \) which is an important term and hence cannot be neglected.

V. A SIMPLER METHOD OF EVALUATING THE HARMONIC COEFFICIENTS IN A 12 VALUE SCHEME

Because of the difficulties of Thompson’s method, it was felt necessary to evolve a different type of presentation which would involve the handling of the terms \( u_0, \ldots, u_{11} \) in the most convenient way and without much mental strain. Moreover, the additional advantage of the two rigorous methods referred to earlier, in that the coefficients obtained therein are required to satisfy a few checks, cannot be overlooked. At some stage, not too far away from the end, it is desirable that the correctness of calculations can be checked. With this view, a new presentation is evolved which satisfies most of these requirements. The presentation given in Table I (a) is useful for finding only the first and the second harmonics. The supplementary chart in Table I (b) enables one to evaluate all the harmonics without much extra labour and also provides a check for the correctness of all of them.

The multiplication of some of the terms by a factor of \( 0.866 \) can be done orally by a number of approximate methods.

(i) The method mentioned by Whittaker and Robinson is to multiply by a factor \( (1 - 1/10 - 1/30) \) which is a very close approximation to \( 0.866 \). Effectively, it reduces to multiplying the original number by \( 2 \), dividing the product by \( 15 \) and subtracting the quotient from the original number. For example,

\[ 107 \times 0.866 = 107 - (107 \times 2)/15 \]
\[ = 107 - 214/15 \]
\[ = 107 - 14.3 \]
\[ = 92.7 \]
**Table II. Method of evaluating the first two harmonics from 24 hourly values of a daily variation**

<table>
<thead>
<tr>
<th>$u_0$</th>
<th>$u_0 - u_{12}$</th>
<th>$P$</th>
<th>Sum</th>
<th>$L = S_1 = x \cdot 0.961$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$</td>
<td>$u_6 - u_{13}$</td>
<td>$Q$</td>
<td>$u_1 + u_{23}$</td>
<td>$A_1$</td>
</tr>
<tr>
<td>$u_2$</td>
<td>$u_0 + u_{13}$</td>
<td>$R$</td>
<td>$u_{11} + u_{13}$</td>
<td>$B_1$</td>
</tr>
<tr>
<td>$u_3$</td>
<td>$u_5 - u_{19}$</td>
<td>$S$</td>
<td>$u_5 - u_{19}$</td>
<td>$C_2$</td>
</tr>
<tr>
<td>$u_4$</td>
<td>$u_1 - u_{17}$</td>
<td>$T$</td>
<td>$u_11 - u_{17}$</td>
<td>$D_2$</td>
</tr>
<tr>
<td>$u_5$</td>
<td>$u_{23}$</td>
<td>$M$</td>
<td>$u_{10} + u_{22}$</td>
<td>$A'$</td>
</tr>
<tr>
<td>$u_6$</td>
<td>$u_0 + u_{21}$</td>
<td>$N$</td>
<td>$u_{10} + u_{22}$</td>
<td>$B'$</td>
</tr>
<tr>
<td>$u_7$</td>
<td>$u_{13}$</td>
<td>$K'_1$</td>
<td>$u_7 + u_{17}$</td>
<td>$C'$</td>
</tr>
<tr>
<td>$u_8$</td>
<td>$u_5 + u_{10}$</td>
<td>$K'_2$</td>
<td>$u_1 - u_{23}$</td>
<td>$D'$</td>
</tr>
<tr>
<td>$u_9$</td>
<td>$u_{11} + u_{13}$</td>
<td>$K'_3$</td>
<td>$u_{12}$</td>
<td>$E$</td>
</tr>
<tr>
<td>$u_{10}$</td>
<td>$u_2 + u_{14}$</td>
<td>$K''_1$</td>
<td>$u_{12}$</td>
<td>$F$</td>
</tr>
<tr>
<td>$u_{11}$</td>
<td>$u_8 + u_{20}$</td>
<td>$K''_2$</td>
<td>$u_{10} + u_{16}$</td>
<td>$G$</td>
</tr>
<tr>
<td>$u_{12}$</td>
<td>$u_4 + u_{18}$</td>
<td>$K''_3$</td>
<td>$u_{10} + u_{16}$</td>
<td>$H$</td>
</tr>
<tr>
<td>$u_{13}$</td>
<td>$u_{10} + u_{22}$</td>
<td>$K''_4$</td>
<td>$u_{10} + u_{16}$</td>
<td>$I$</td>
</tr>
<tr>
<td>$u_{14}$</td>
<td>$u_{23}$</td>
<td>$I'$</td>
<td>$u_{10} + u_{16}$</td>
<td>$J$</td>
</tr>
<tr>
<td>$u_{15}$</td>
<td>$u_{23}$</td>
<td>$I''$</td>
<td>$u_{10} + u_{16}$</td>
<td>$K$</td>
</tr>
<tr>
<td>$u_{16}$</td>
<td>$u_{23}$</td>
<td>$I'''$</td>
<td>$u_{10} + u_{16}$</td>
<td>$L$</td>
</tr>
<tr>
<td>$u_{17}$</td>
<td>$u_{23}$</td>
<td>$I''''$</td>
<td>$u_{10} + u_{16}$</td>
<td>$M$</td>
</tr>
<tr>
<td>$u_{18}$</td>
<td>$u_{23}$</td>
<td>$I'''''$</td>
<td>$u_{10} + u_{16}$</td>
<td>$N$</td>
</tr>
<tr>
<td>$u_{19}$</td>
<td>$u_{23}$</td>
<td>$I''''''$</td>
<td>$u_{10} + u_{16}$</td>
<td>$O$</td>
</tr>
<tr>
<td>$u_{20}$</td>
<td>$u_{23}$</td>
<td>$I''''''''$</td>
<td>$u_{10} + u_{16}$</td>
<td>$P$</td>
</tr>
<tr>
<td>$u_{21}$</td>
<td>$u_{23}$</td>
<td>$I'''''''''$</td>
<td>$u_{10} + u_{16}$</td>
<td>$Q$</td>
</tr>
<tr>
<td>$u_{22}$</td>
<td>$u_{23}$</td>
<td>$I''''''''''$</td>
<td>$u_{10} + u_{16}$</td>
<td>$R$</td>
</tr>
<tr>
<td>$u_{23}$</td>
<td>$u_{23}$</td>
<td>$I'''''''''''$</td>
<td>$u_{10} + u_{16}$</td>
<td>$S$</td>
</tr>
</tbody>
</table>

**CHECKS:**

1. $(S_1/0.961) + (S_2/0.257) = 2(u_1 + u_5 + u_7 + u_{11})$
2. $(S'/0.866) + (S''/0.500) = 2(u_2 + u_4 + u_8 + u_{10})$
3. $L = 2(u_0 + u_6)$
4. $M + N = A_3 + B_3$
5. $S_2 = 2(u_3 + u_9) \times 0.707$
6. $(L' \times 0.500) = (L'' \times 0.866)$

$24 a_0 = Z$
$12 a_1 = A_1 - B_1 + A_2 - B_2 + A_3 - B_3 + A' - B' + A'' - B'' + P$
$12 a_2 = C_1 - D_1 + C_2 - D_2 + C_3 - D_3 + C' - D' + C'' - D'' + Q$
$12 a_3 = X'_1 - Y'_1 + X''_1 - Y''_1 + R - S$
$12 b_2 = X'_2 + Y'_2 + X''_2 + Y''_2 + M - N$
(ii) Another method is to multiply by \( \cdot 9 \) (which should be possible orally) and then subtract 4 for every 100 (i.e., 4\%) from the product. Thus

\[
107 \times \cdot 9 = 96.3 \quad (9 \frac{1}{2} \times 10)
\]

Subtracting \((\cdot 4 \times 9\frac{1}{2}) = 3.8 \) from 96.3, we get

\[
107 \times \cdot 866 = 92.5
\]

(iii) A different method is to evaluate the fractions 1/10, 3/100 and 5/1000 of the original number and subtract their sum from the original number. Thus,

\[
107 \times \cdot 866 = 107 - (10.7 + 3.2 + 0.5) = 107 - 14.4 = 92.6
\]

The method of harmonic analysis presented in Table I (a) and (b) has been found, after trial, to be easier and shorter than any of the methods referred to earlier. The only defect of the method given in Table I (a) is that there is no check to find out whether the coefficients \( a_0, a_1, b_1, \ldots \), \( a_{23}, b_{23} \) have been correctly computed from the factors A, B, C, D, etc. Such a defect will, however, be inherently present in any method which gives only the first two or three harmonics while the other higher harmonics are not absent. The difficulty is removed when all the harmonics are evaluated by use of the supplementary chart in Table I (b) which provides a check for all the coefficients.

VI. EVALUATION OF THE HARMONIC COEFFICIENTS IN A 24 VALUE SCHEME

A similar presentation can be evolved for the harmonic analysis of 24 hourly values. An elaborate method for such an analysis has already been given by Whittaker and Robinson. It gives all the coefficients \( a_0, a_1, b_1, \ldots, a_{23}, b_{23} \) with methods for checking their correctness. The specimen chart given by these authors is similar to the one used by them for the analysis of 12 values, and is more elaborate and lengthy because of the additional terms. Table II gives a chart which can give the first two harmonics in an easier way.

The advantages and drawbacks of this method are similar to those for the method used for the harmonic analysis of 12 values. In addition, there is the problem of multiplication by factors like \( \cdot 707, \cdot 257 \) and \( \cdot 961 \). In their method for harmonic analysis of 24 values, Whittaker and Robinson have managed to do away with the latter two factors in an interesting manner.
These two factors arise from the terms $\sin 15^\circ$ and $\cos 15^\circ$ which they have expanded as

$$
\sin 15^\circ = \sin (45 - 30)^\circ = (1/\sqrt{2}) (\sqrt{3}/2 - 1/2)
$$

$$
\cos 15^\circ = \cos (45 - 30)^\circ = (1/\sqrt{2}) (\sqrt{3}/2 + 1/2)
$$

The factors involved are thus only ($\sqrt{3}/2$) and ($1/\sqrt{2}$) which are $\cdot 866$ and $\cdot 707$ respectively. The only additional factor (as compared to the method adopted for 12 values) is therefore $\cdot 707$ instead of the 3 factors required in the method given in Table II. From this point of view, Whittaker and Robinson's method seems to have a definite advantage over the present one. Whether this can compensate for its lengthiness will have to be judged by workers who will use both these methods and study their comparative advantages and disadvantages.

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**SUMMARY**

The use of a Fourier series in expanding a function into a sum of trigonometric terms, and its application to the problem of resolving a daily variation into its harmonic components are discussed. Various practical methods adopted for the harmonic analysis of a daily variation are reviewed. A new form of analysis of a 12 ordinate scheme is suggested. This is both quicker and easier to handle than the earlier methods. It enables one to evaluate in about 5 minutes the first and the second harmonics from 12 bi-hourly values of a daily variation. Also, with the help of a supplementary chart, the higher harmonics can be evaluated without much extra labour.

A similar method of harmonic analysis for a 24 ordinate scheme is suggested.

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