RANDOM MATRIX THEORY
AND
THE STATISTICAL STUDY OF COMPLEX SPECTRA

BY
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DEDICATED

TO

MY PARENTS

AND

PROFESSOR N. ANANTHAKRISHNAN

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Prof. Jitendra Parikh,

May I beseech you to accept my profuse thanks for your all supreme supervision of my thesis work. I find myself privileged to have been in association with you the last half a decade the reminiscences of which would reflect back time and again to remind me of that ideal teacher.

Rhama Venkataraman

CERTIFICATE

I hereby declare that the work presented in this thesis is original and has not formed the basis for the award of any degree or diploma by any University or Institution.

Rhama Venkataraman

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Certified by:

Jitendra C Parikh

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JULY 21, 1981

CHAPTER I

INTRODUCTION

For studying the discrete energy levels of a quantal system, like a nucleus or an atom, powerful theoretical techniques like shell model and variational approaches are available. The successful application of these techniques is restricted to the ground state domain where the levels are found far apart. In contrast to this, levels in multitude at high excitation energies, which arise out of the excitation of many particles, are spread over a width which is a fraction of the mean around which the spread occurs. Such spectra are complex for in the extension of shell model techniques to such high excitation energies one encounters matrices of large dimensionalities. With such a large number of levels spread over a small width one inquires for the statistical properties of these levels wherein the average properties like the level density and the fluctuations about the averages are

studied. With the increasing need, with increasing excitation energy, for such a statistical study the techniques of the Random Matrix Theory (RMT) and the spectral distribution methods of French and coworkers gain increasing significance.

In the application of these two techniques one assumes that the statistical properties of spectra are insensitive to certain features of the Hamiltonian like for example the form of the interaction potential. The genesis of an ensemble of Hamiltonians finds its root in this assumption. Different members of the ensemble then correspond to different realistic interactions. In the event of the ensemble not being ergodic (when the fluctuations from member to member are large) the above assumption is violated.

In the Random Matrix Theory (RMT) ((1) & (2)) the Gaussian Orthogonal Ensemble (GOE) of real symmetric matrices gives a semi-circular level density when all the matrix elements are chosen to be statistically independent. But in physical situations one has a GOE only in the 'r' particle space corresponding to random r-body interaction which is then propagated to the many particles (say 'p' particle) space. Using the spectral distribution methods to evaluate the traces of different powers of the Hamiltonian (without actually diagonalising the matrix) Mon and French have shown (3) that the level density which is a semi-circle for p-body random interaction in 'p' particle space goes over to a Gaussian when the rank of interaction becomes much smaller than the particle number. The Gaussian distribution has also been observed in shell model calculations for levels

of fixed symmetry ((4) & (5)). In the second chapter, the effect of symmetry considerations on the rate of approach of the level density, with particle number, to normality is studied for two-body random interaction. The main part of this chapter is devoted to generating an ensemble of matrices, starting from purely mathematical considerations that would explain the other limit of Gaussian level density. In this attempt to pull the level density away from a semi-circle and towards a Gaussian, ensembles of matrices are generated with different characteristic level densities. The ergodicity of different ensembles is also discussed in detail.

With the sole objective of accounting for the increased departures found (6) in the partial level density of atoms, Chapter III investigates the effect of range on the partial level density besides its effect on the rate of approach of the level density to normality.

Embarking on the fluctuations, one finds that both the Gaussian Orthogonal Ensemble (GOE) of real symmetric matrices and the Two Body Random Ensembles (TBRE) for random two-body interaction exhibit identical features, in terms of spacing distributions for near neighbours. ((7) & (8)). In fact, Dyson suggests (9) that these fluctuation properties for a set of 'n' levels would not depend on the nature of the level density under the condition when the matrix dimensionality N ($N >> n$) goes to infinity. The fluctuation properties are sensitive only to mixing of levels of different symmetries and hence the only physical information that could be learnt till now from the fluctuations is about hidden symmetries.

Chapter IV investigates the fluctuation properties of the ensembles generated in Chapter II under conditions when $n << N$ *and also when* $n \simeq N$. *The fluctuations are studied through* Δ_3 *and* Q *(10) that measure the long and short range ordering in the level spectra. The variation of these fluctuation measures with range of interaction is also discussed. Chapter V gives a detailed account of the results.*

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CHAPTER II

ON LEVEL DENSITY DISTRIBUTIONS

1. Introduction:

With the advent of shell model and variational techniques, low-lying states of atomic and nuclear spectra were accounted for to an appreciable degree of accuracy. In the extension of these techniques to high excitation energies, the exponential rate of increase of the number of levels with excitation energy poses insurmountable technical difficulties. Choosing, therefore, to ignore the detailed structure of levels, one adopts statistical methods to inquire into the properties of such complex spectra. These methods project information on the global or average properties and the fluctuations about these averages as well.

First and foremost of the global properties is the level density that gives the number of levels per unit energy interval. The hypothesis that the level density and other statistical properties are insensitive to certain features of the Hamiltonian, like for example, the form of interaction potential, facilitates the use of an ensemble of Hamiltonians. Incomplete knowledge of the Hamiltonian often necessitates an ensemble approach. This concept of an ensemble is then founded on the randomisation of the unknown features of the Hamiltonian and stands past dispute as long as one is not interested in the detailed properties of level spectra.

A review of the level densities of physical models and the random matrix theory (RMT) in Section 2 shall be followed by the effect of symmetry considerations in Section 3. Section 4 gives a prescription for generating random matrix ensembles that give rise to different characteristic eigenvalue densities. It gives there to a comparison of the ergodicity of different ensembles. Section 5 summarises the results.

2. Fermionic Level Density and Random Matrix Theory:

Consider a system of 'p' particles distributed among 's' single particle states and interacting via r-body interaction. For an r-body random interaction, Mon and French ((1) & (2)) have shown that the ensemble averaged level density $\rho(E)$ *of the* ${}^{s}c_{r}$ *dimensional space, both for finite (finite 's' and 'p') and dilute (*$p \longrightarrow \infty$, $s \longrightarrow \infty$, $p/s \longrightarrow 0$*) systems is a semi-circle when all the particles interact*

simultaneously ($r = p$). As 'r' decreases there is a transition in the level density towards a Gaussian and in the limit when 'r' becomes much smaller compared to 'p' ($r \ll p$), it approaches a Gaussian.

The emergence of a Gaussian as a limiting distribution is easily seen in the case of a dilute system of non-interacting particles. In a dilute system, Pauli principle effects could be ignored and the many particle level density which is a convolution of the single particle level densities approaches a Gaussian in accordance with the Central Limit Theorem (CLT).

This analytical investigation of Mon and French was prompted by the numerical calculations (3) done in the shell model framework for levels of fixed symmetry. The level density called the partial level density in this case turned out to be a Gaussian for two body random interaction. This ensemble is called two body random ensemble (TBRE). From now on, the ensemble corresponding to r-body random interaction will be called RBRE for $r < p$ and PBRE for $r = p$.

In the theory of random matrices ((4), (5) & (6)), the Gaussian Orthogonal Ensemble (GOE) of real symmetric matrices gives a semi-circle for the eigenvalue density $\rho(E)$, in the limit of matrix dimensionality $N \longrightarrow \infty$. This is Wigner's law of semi-circle (7).

$$\rho(E) = 2/N\pi \sqrt{4N - E^2} \qquad (2.1)$$

where E is in units of $2\sigma N^{\frac{1}{2}}$ where σ is the width of a vast

majority of matrix elements (8) which are statistically independent.

In physical situations, an RBRE is characterised by a fixed set of independent matrix elements which are the r-body matrix elements (RBME). Essentially corresponding to r-body random interaction there is a GOE in 'r' particle space which is then propagated to the many particle space. The number of RBME increases with the rank of interaction and for r = p all the matrix elements are independent. Thus the random matrix theory (RMT) corresponds to the case when all the particles interact simultaneously. This is physically unrealistic for we know that realistic interactions are of low-particle rank in character. And it is this feature that gives a Gaussian level density (3). There are also examples of TBRE ((9) & (10)) in literature that suggest that symmetry considerations improve the rate of approach of the level density to normality. The following section elaborates on this subject.

3. *Symmetry Properties and Level Density:*

The effect of symmetry on level density is in a sense akin to that of a reduction in the rank of interaction. Just as a reduction in 'r' reduces the number of RBME bringing the level density more towards Gaussian, the conservation of quantum numbers corresponding to symmetry properties reduces the number of non-vanishing matrix elements in the 'r' particle space.

The level density under different conditions is usually studied through its shape parameters $S_p (p > 2)$ *which are given by*

$$S_p = K_p \,/\, (K_2)^{p/2} \qquad (2.2)$$

where K_p, the p^{th} cumulant is the coefficient of $(it)^p / p!$ in ln F(t), where F(t) is the Fourier transform of the level density

$$F(t) = \int_{-\infty}^{\infty} \exp(itE) \quad \rho(E) \; dE \qquad (2.3)$$

K_2 is the variance. The odd shape parameters give information on the asymmetry of the distribution. S_3 and S_4 are the standard measures of skewness and excess. All S_p's vanish for a Gaussian. In the event of higher shape parameters vanishing, a positive excess would correspond to a peaked distribution while a negative excess to a flattened one. S_4 for a semi-circle is -1.0.

Mon and French evaluated the shape parameters of the level density by using the elegant propagation technique. The Hamiltonian in 'r' particle space is given by

$$H = \sum_{Z_1, Z_2} W_{Z_1 Z_2} \quad Z_1(r) \; Z_2^{+}(r) \qquad (2.4)$$

The dimensionality of H in 'r' particle space is given by ${}^{s}c_r$ where 's' is the number of single particle states. In equation (2.4) Z_1 and Z_2^{+} are the 'r' particle creation and annihilation operators and W's are the r-body matrix elements. Since H is real symmetric, the number of W's is given by Y(Y + 1)/2, where $Y = {}^{s}c_r$ is the dimensionality of H. The average of different powers of H in 'p' particle space is then evaluated (11) using the relation

$$\langle O(r) \rangle^p = {}^{p}c_r \quad \langle O(r) \rangle^r \qquad (2.5)$$

where $O(r)$ is any r-body operator. Thus the traces of different powers of H are evaluated without actually diagonalising the matrix. This procedure is useful as long as one is interested only in low order shape parameters. Evaluation of traces of higher powers of H through straightforward becomes laborious.

The value of S_4 for TBRE for different particle number 'p' in (sd) shell calculated using this technique is given in Table 1. Since the number of single particle states 's' in (sd) shell is 24, the dimensionality of the two particle space is $^{24}C_2 = 276$ and the number of independent TBME is (276 x 277)/2 when no symmetry considerations are introduced.

TABLE 1

'p'	S_4 (without symmetry	S_4 (with symmetry)	S_4
4	-0.85	-0.58 (.14)	for Gaussian :
8	-0.59	-0.32 (.11)	0.0
12	-0.56	-0.22 (.12)	for Semi-Circle:
	(Ref:1)	(Ref:8)	-1.0

TABLE 1: Values of S_4 for different particle number 'p' in (sd) shell for the two cases without & with symmetry.

Potbhare has evaluated (9) S_4 for the same configurations by introducing conservation of J and T (total angular momentum and isospin quantum numbers) in 2 particle space. The number of TBME reduces to 63 and the 2 particle and 'p' particle matrices break down to block matrices along the diagonal. The values of S_4 for both these cases without and with symmetry are in Table 1. With symmetry the values of S_4 are smaller compared to those when no symmetry is introduced, and closer to that of a Gaussian.

For a complete knowledge of the level density distributions in both cases, one needs to know all the shape parameters, as S_3 and S_4 alone might not give a correct idea of the nature of distribution. This would become clear in the next section. Hence to ascertain the change in the level density distribution on introducing symmetry considerations, the following procedure is adopted.

Now that the propagation technique stands stripped of its simplicity in the evaluation of higher order moments of H, one has but one choice of matrix diagonalisation in the 'p' particle space. From the definition of r-body Hamiltonian in equation (2.4), the matrix elements and the Hamiltonian in 'p' particle space could be written as

$$H_{ij} = \sum_{Z_1,Z_2} W_{Z_1Z_2} \sum_{X} \langle p_i \mid Z_1(r) \mid (p-r)_x \rangle \times \langle (p-r)_x \mid Z_2^{+}(r) \mid p_j \rangle \qquad (2.6)$$

$$H = \sum_{Z_1 \leq Z_2} (A(Z_1) \overset{\frown}{A}(Z_2) + A(Z_2) \overset{\frown}{A}(Z_1))$$

where X's are (p-r) particle states and A(Z) the matrix that creates 'p' particle states from (p-r) particle states through 'Z'. Since the number of (p-r) and 'p' particle states is ${}^{s}c_{(p-r)}$ (X_1 say) and ${}^{s}c_{p}$ (X_2 say), A is an (X_2 x X_1) matrix; the rows and columns of A are labelled by 'p' and (p-r) particle states. Only those elements of A where the row and column indices have (p-r) states in common are non-vanishing. The fact that for fermions one uses determinantal basis states leads to the appropriate sign of the matrix elements in A for any permutation of the single particle state indices in many particle states. Now that the A(Z)'s are built in this fashion the Hamiltonian in 'p' particle space is known. Thus an ensemble of H's is generated for any rank of interaction.

To study the effect of symmetry considerations on level density using the above scheme the case of p = 4, s = 8 and r = 2 is chosen. The dimensionalities of the 4 particle and 2 particle spaces are 70 and 28 respectively. Without symmetry considerations one has (28 x 29)/2 TBME and an ensemble of H's generated and diagnoalised. This is the first case without symmetry. In the second case, each of the single particle states is assigned a quantum number 'm' corresponding to any additive symmetry (say J_z). With the conservation of total 'm' in 2 particle space, one has only 50 non-vanishing TBME. Again an ensemble of H's is generated and diagonalised. The shape parameters S_3 to S_8 for the two cases along with those for a Gaussian and semi-circle are shown in Table 2. It also contains the fluctuations or the RMS deviations of the shape parameters. The shape parameters go down in magnitude when symmetry considerations are

TABLE 2

Ensemble Dimensionality	S_3 (S_3RMS)	S_4 (S_4RMS)	S_5 (S_5RMS)	S_6 (S_6RMS)	S_7 (S_7RMS)	S_8 (S_8RMS)
47 without symmetry	0.01 (0.06)	-0.84 (0.06)	-0.02 (0.28)	3.64 (0.49)	0.10 (2.82)	-35.79 (7.00)
50 with symmetry	-0.08 (0.18)	-0.57 (0.25)	0.29 (0.67)	1.78 (1.68)	-2.14 (6.12)	-13.90 (18.56)
Semicircle	0.0	-1.0	0.0	5.0	0.0	-56.00
Gaussian	0.0	0.0	0.0	0.0	0.0	0.0

TABLE 2: Shape parameters & RMS deviation of level density for 70 x 70 matrices ($p = 4$; $s = 8$; $r = 2$) for the indicated ensemble dimensionalities.

introduced thus taking the 'p' particle level density closer to Gaussian. What the level density looks like in the second case would become clear in the next section where there are distributions with such values of S_p's. It should be noted at this stage that for the case with symmetry the RMS deviations of the shape parameters are larger than those for the first case. The discussion on these fluctuation properties is deferred for the present and shall be taken up in Section 4a.

The conclusion then that symmetry considerations accelerate the rate of approach of the level density to Gaussian is also aided by numerical examples of shell model studies found in literature (10). One finds for example that for 4 particles in (sd) shell, S_4 is -0.85 (1) for the level density in $^{24}C_4$ dimensional space, while its submatrix of dimension 56 for $J = 2$, $T = 0$ gives a Gaussian with $S_4 = -0.54$ (10).

4. In Search of Gaussian Level Density:

In the field of real symmetric matrices, a rigorous derivation for the eigenvalue density has been given only for GOE. In general an ensemble that is invariant under orthogonal transformation is defined through

$$P(H)dH = f(\text{tr } H, \text{tr } H^2 \ldots\ldots \text{tr } H^N)\ dH \qquad (2.7)$$

where $P(H)$ is the probability of finding H in the volume element dH. f is positive definite and is such that

$$\int P(H)dH = 1$$

Every such function 'f' would define an ensemble with a characteristic level density. The GOE is a special case defined by

$$P(H)dH = \exp\left(-(a+b\ \text{tr } H + c\ \text{tr } H^2)\right) \prod_{i \leq j} dH_{ij} \qquad (2.8)$$

and owes its simplicity to the physically unrealistic assumption of statistical independence of matrix elements. Mehta and Gaudin (12) have derived the eigenvalue density of this ensemble for finite matrix dimensionality (N). The eigenvalue densities for other forms of 'f' in

equation (2.7) have not been derived rigorously because of the mathematical intractability.

Wigner dealt with another kind of ensemble (7) called the Random Sign Ensemble (RSE) in which the diagonal matrix elements are zero and the off-diagonal matrix elements are the same in magnitude excepting for a difference in phase. These carry a positive or negative phase with 50% probability. This ensemble also giving a semi-circle in the limit $N \longrightarrow \infty$, he suggested that a large variety of ensembles would yield a semi-circle as long as the matrix elements are statistically independent.

To rephrase Wigner, the eigenvalue density does not depend explicitly on the nature of distribution of the matrix elements. Truely enough, the results of RBRE are also independent of the nature of distribution of the RBME.

The contrast in the nature of level density of GOE and physical models prompted the need for generating other kinds of random matrix ensembles with different level densities. Balian (13) came up with a prescription based on information theory for generating an ensemble with a given level density. Dubener (14) defined another ensemble of matrices that gives an ensemble averaged Gaussian level density. Both these ensembles are not invariant under orthogonal transformation. The undesirable features of Dubener's and the difficulty one encounters in following Balian's way shall be pin-pointed in Section 4a on ergodicity of ensembles.

Hence the aim now is to generate an ensemble of matrices starting from purely mathematical considerations. One begins by building these ensembles on those features that are characteristic of RBRE. An RBRE is characterised by a set of independent variables that are the RBME and the matrix elements in many particle space occur as linear combinations of these RBME.

$$\text{i.e.} \quad H_{ij} = \sum_k C_{ij}^k X_k \qquad (2.9)$$

where X_k's are the RBME and C_{ij}^k's are coefficients that define the geometry of the space. Unless one follows the physical models the explicit nature of the correlation between the different matrix elements arising out of C_{ij}^k's is irretrievable.

With the above factors in mind, the search for the ensemble which gives a Gaussian level density is begun. The dimensionality of the matrix is chosen to be 50 for all the following ensembles. That of the ensemble is also 50 in each of the cases.

In the first case, 20 independent matrix elements (X_k's) are chosen whose positions are fixed at random. The rest of the matrix elements are given by

$$H_{ij} = \sum_k C_{ij}^k X_k \qquad (2.10)$$

where the correlation coefficients C_{ij}^k's are chosen at random and fixed over the ensemble. The X_k's in this and the following cases are chosen to be uniform deviates around zero mean. This correlated ensemble (case I) yields a semicircular level density despite the fact that the number of independent matrix elements is highly restricted. The shape

parameters S_p ($p = 3$ to 8) of the level density are shown in Table 3 along with those for RSE. In RSE even the diagonal matrix elements have been taken to be ± 1. The RMS deviations of S_p's for case I are small and close to those of RSE. These ensembles are ergodic; the deviation, on the average, of the level densities of the individual members from the ensemble average is small. The merit of RSE or GOE lies in its ergodicity which is a consequence of its invariance under orthogonal transformation. The correlated ensemble (case I) emerges out to be equally ergodic, though in this case the invariance property is not so easily seen.

The equivalence of GOE and RSE in respect of the statistical independence of matrix elements, level density and the associated ergodicity suggests the next natural prescription for producing an ensemble with a Gaussian level density. In the second case, 20 independent X_k's are chosen while the rest of the matrix elements are given by

$$H_{ij} = \pm \left| \frac{\sum_k X_k}{20} \right| \qquad (2.11)$$

which is the mean of X_k's except for a difference in phase. This ensemble shall be referred to as Random Phase Ensemble I (RPE I, case II). The ensemble averaged shape parameters (Table 3) upto S_8 are close to zero. A glimpse of the RMS deviations, also given in Table 3, tells one that the ensemble is not ergodic. In fact an analysis of the members of the ensemble shows that the average has sprung out of Gaussians, semi-circles and other peaked distributions. This ensemble consists of a vast

majority of inadmissible Hamiltonians. It is found that different level densities arise from different values of the ratio

$$R = \sum_k X_k \Big/ \sqrt{\sum_k X_k^2}$$

Hence other ensembles are generated following the same scheme choosing different values for R.

Case III (RPE II) and case IV (RPE III) correspond to R ranging from 0.494 to 0.506 for two different sets of positions of X_k's. Figures (1) and (2) show that the level densities (Table 3) are close to Gaussian. For R ranging from 0.85 to 0.95 the level density (Figure 3) is a semi-circle (case IIIa; RPE IV) and for R ranging from 0.32 to 0.38 (case IIIb; RPE V), the level density is a triple humped distribution (Figure 4).

Keeping the mean value of R at 0.5 when the number of X_k's is increased to 100 the level density is seen to slide towards a semi-circle (case V; RPE VI and case VI; RPE VII). These two cases V and VI (Table 3; Figures 5 and 6) correspond again to two different sets of positions of X_k's and the results are identical. For the same ratio when the number of X_k's is increased to 500, the level density turns out to be a semi-circle. This case is not included in Table 3.

Table 3 shows the effect of S_6 on the distribution. Looking at S_4 for cases III and IIIb one would be tempted to conclude that IIIb is a better Gaussian. Comparison of Figures 1 and 4 reveals that the former is a better Gaussian though it has a higher value of S_4. Similarly, the effect of S_6 (greater than 1.0) can be seen from Figures

Figs. 1 & 2 : The exact level density (histogram) and the Gaussian level density (continuous curve) for RPE II and RPE III described in the text.

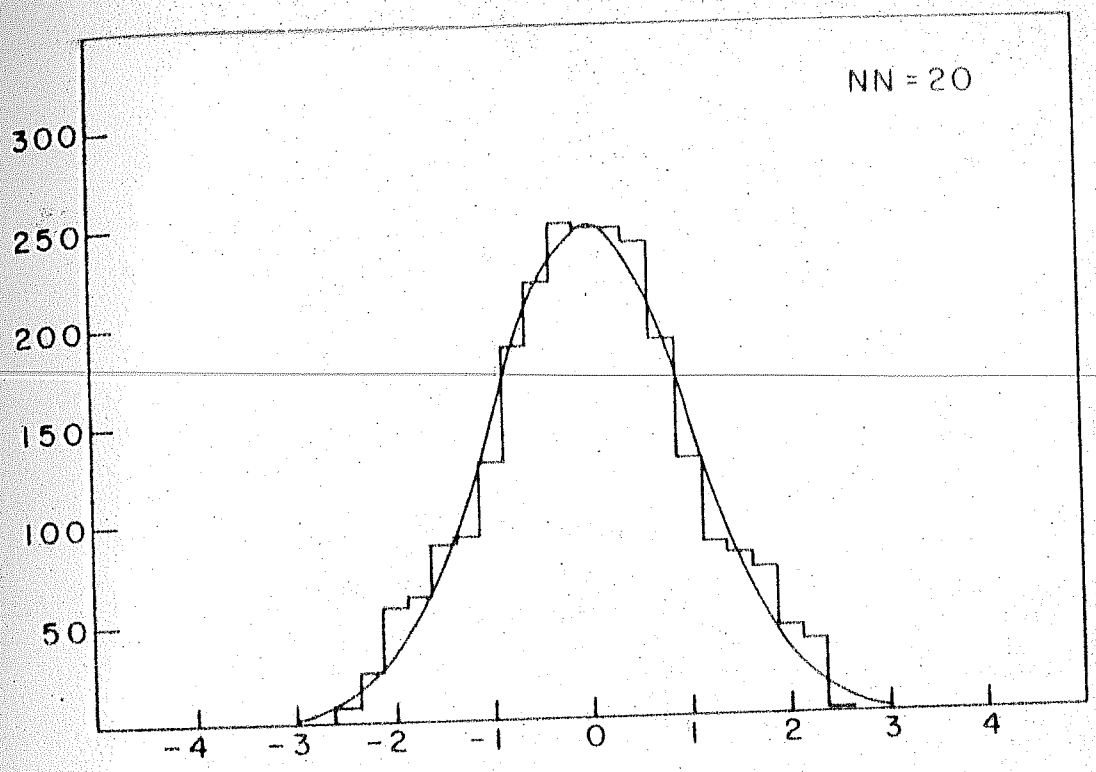
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Fig. 1

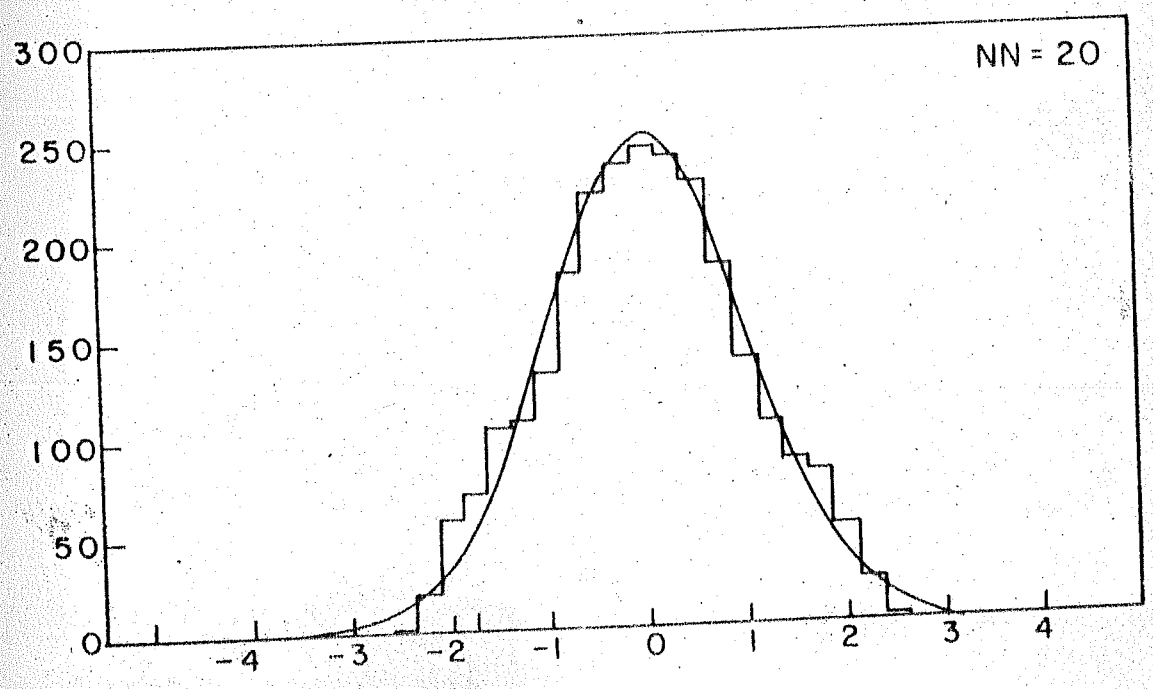
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Fig. 2

Figs. 3 & 4 : *The exact level density (histogram) and the semicircular/Gaussian level density for RPE IV/RPE V described in the text.*

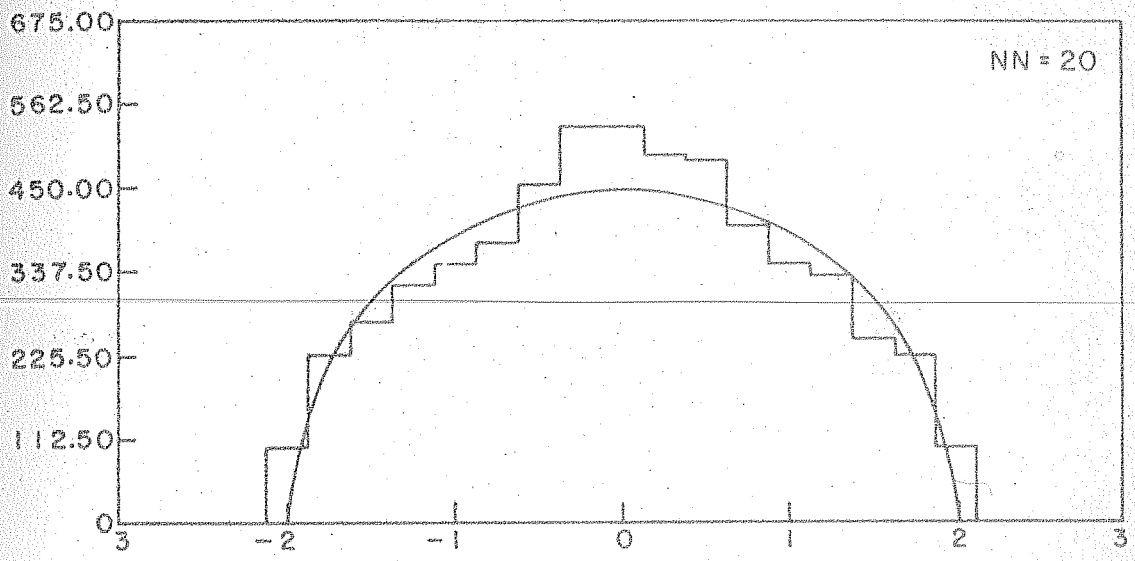
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Fig. 3

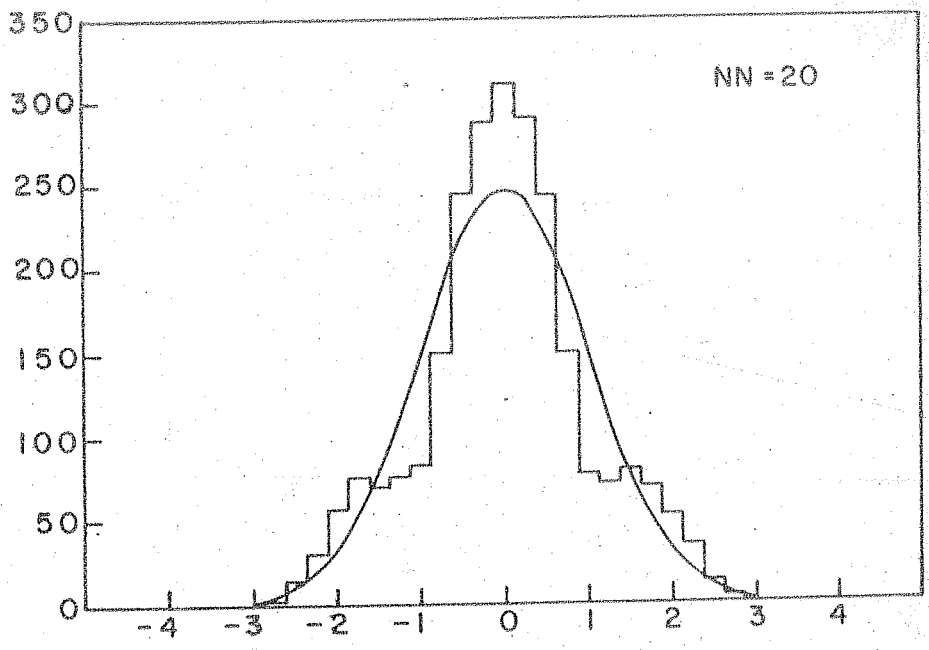
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Fig. 4

Figs. 5 & 6 : The exact level density (histogram) and the Gaussian level density (continuous curve) for RPE VI and RPE VII discussed in the text.

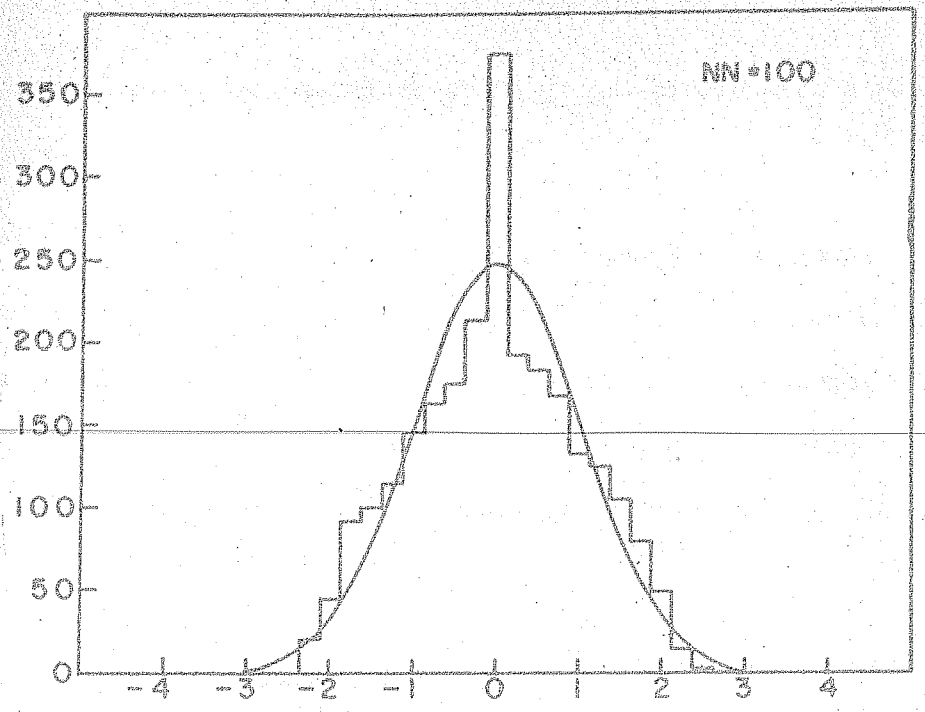
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Fig. 5

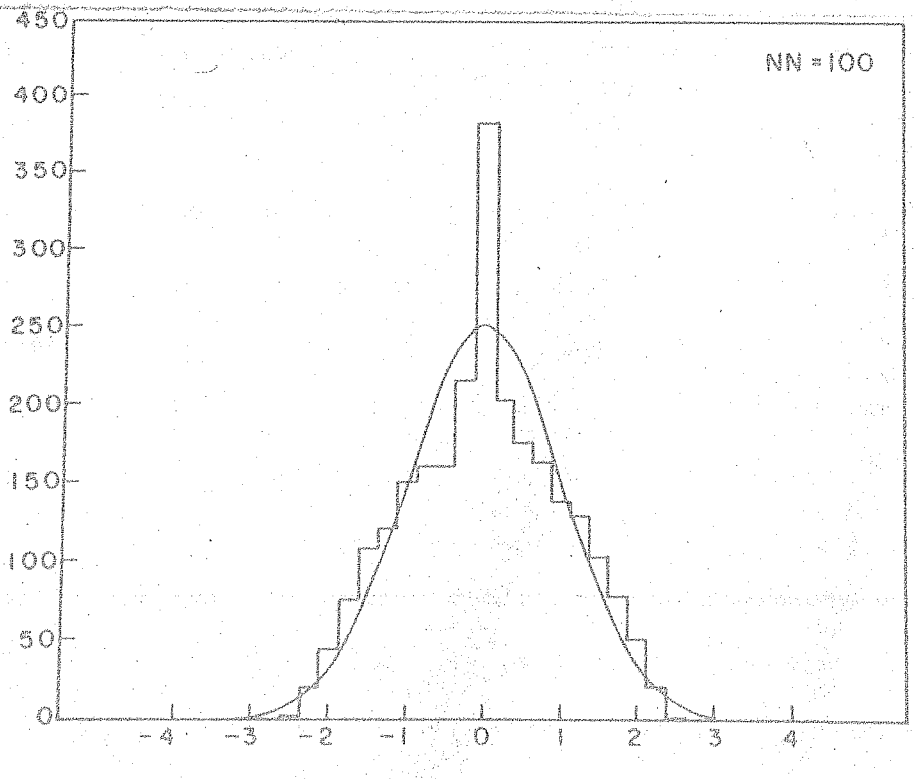
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Fig. 6

5 and 6 where the distribution is a semi-circle with a peak at the centre.

To conclude then on the level density of Random Phase Ensembles it is found that each value of the ratio R gives rise to a characteristic level density that goes from a semi-circle to other distributions when R is decreased from 1.0. Again for the same value of R, when the number of independent matrix elements is increased the level density goes over to a semi-circle.

4a. *Ergodicity of Ensembles:*

Any average property ought to be accompanied by ergodicity in a proper ensemble. It is common knowledge that GOE & RSE are ergodic. The correlated ensemble (case I) is also equally ergodic while RPE I is not ergodic at all.

To examine the ergodicity of the other RPE, the RMS deviations of S_p's are given in Table 3. These are small in cases III and IV and comparable to those found in TBRE. But these ensembles are still not as ergodic as GOE; the RMS deviation of S_4 is small while that of S_6 suggests that distributions of the type corresponding to case V (RPE VI) are also found in the ensemble. A further decrease in the range of the ratio might probably lead to better ergodic conditions. In cases III b and IIIc ergodic conditions are quite good. The inference is that a semi-circle always turns out to be more ergodic than any other distributions.

Case No.	NN	R	S_3 (RMS)	S_4 (RMS)	S_5 (RMS)	S_6 (RMS)	S_7 (RMS)	S_8 (RMS)
		RSE	0.00 (0.04)	-1.03 (0.06)	0.02 (0.22)	5.23 (0.54)	-0.21 (2.63)	---
I	20	---	0.00 (0.06)	-0.98 (0.05)	0.17 (0.31)	4.83 (0.38)	0.16 (3.53)	---
II	20	---	-0.01 (0.05)	0.25 (0.89)	0.04 (0.26)	-0.33 (5.83)	-0.37 (2.67)	-18.34 (9.20)
III	20	0.494 -0.506	0.00 (0.05)	-0.34 (0.25)	0.00 (0.19)	-0.28 (1.52)	0.10 (1.58)	8.49 (14.89)
IV	20	0.494 -0.506	0.00 (0.05)	-0.50 (0.18)	0.02 (0.21)	0.80 (1.22)	-0.19 (1.76)	-0.81 (9.85)
IIIa	20	0.850 -0.950	-0.01 (0.05)	-0.82 (0.07)	-0.05 (0.25)	3.30 (0.62)	-0.58 (2.54)	-30.22 (8.73)
IIIb	20	0.32 -0.38	0.01 (0.04)	0.08 (0.32)	-0.02 (0.16)	-3.41 (1.74)	-0.13 (1.34)	31.27 (5.34)
V	100	0.498 -0.500	0.00 (0.06)	-0.58 (0.16)	-0.00 (0.25)	1.28 (1.33)	0.02 (2.18)	-4.72 (1.35)
VI	100	0.490 -0.505	0.00 (0.04)	-0.57 (0.15)	0.00 (0.19)	1.28 (1.07)	-0.03 (1.64)	-4.92 (12.31)

TABLE 3 : Shape parameters and RMS deviations for the different classes of ensembles as indicated in the text.

These results are not surprising; even in physical models a Gaussian is never as good a Gaussian as a semi-circle is a semi-circle. The RMS deviations become larger as one moves away from a semi-circle. To recall, a similar trend in the fluctuation properties is found in TBRE calculations discussed in Section 2. The large fluctuations are due to the fact that TBRE (RBRE) is invariant under orthogonal transformation only in 2(r) particle space and not in 'p' particle space. Instead of a GOE characterised by N (N+1)/2 (say X) independent matrix elements in 'r' particle space, if one starts with an orthogonal ensemble (the traces of all powers of H are fixed) characterised by (X-N) independent variables, one could probably ensure better ergodic conditions in the many particle level density.

It is at this juncture concerning ergodicity that Balian's approach needs a mention. His prescription (13) for defining an ensemble of matrices with a given level density does not accompany the invariance property of the ensemble. To include the two together in his scheme, there is no clue yet available.

Dubener's ensemble for real symmetric case is defined through

$$H = X_o + \sum_{k=1}^{M} X_k \, (S^k + S^{k+}) \qquad (2.12)$$

where the matrix dimensionality $N = 2M + 1$. X_k's are independent random variables identically distributed. S is an elementary cyclic matrix of order N. The Fourier transform R(w) of the ensemble averaged level density $\rho(E)$ is given by

$$R(w) = V_0(w) \prod_{k=1}^{M} V_k(4w \cos 2\pi k/N) \qquad (2.13)$$

where V_k's are Fourier transforms of X_k's, with $w = 4w \cos 2\pi k/N$. Equation (2.13) shows that $\rho(E)$ depends explicitly on the nature of distribution of X_k's, turning out to be the Gaussian.

$$\frac{1}{\sqrt{2\pi\,\sigma_A^2}} \exp(-E^2/2\,\sigma_A^2) \qquad (2.14)$$

when X_k's are Gaussian with a width σ.

$$\sigma_A^2 = \sigma^2 \left(1 + \sum_{k=1}^{M} 4 \cos^2 2\pi k/N\right) \qquad (2.15)$$

This dependence of $\rho(E)$ on the nature of distribution of X_k's is an undesirable feature. Moreover, this ensemble is accompanied by large fluctuations as in RPE I. In this case, the violation of invariance property can be seen easily as any power of a member H in the ensemble, say H^p is given by

$$H^p = X_o' E + \sum_{k=1}^{M} X_k' (S^k + S^{k+}) \qquad (2.16)$$

occurs in the ensemble with a non-zero probability since no condition is imposed on X_k's. Thus Dubener's ensemble violates conditions of ergodicity.

5. Conclusion:

It is found that the introduction of symmetry considerations accelerates the rate of approach of the level density to normality.

Semi-circle emerges also from correlated ensembles under certain conditions. The trend in the variation of the level density in Random Phase Ensembles calls for an analytical treatment of the problem. Since the level density distributions of Random Phase Ensembles, other than semi-circle, are not as ergodic as GOE, the analytical treatment of RPE would in all probability shed more light on the invariance property and the ergodicity thus paving the way for generating a class of ensembles of different characteristic level densities.

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CHAPTER III

LEVEL DENSITY WITH RANGE OF INTERACTION

1. Introduction:

For realistic interactions the partial level density in nuclei could be well approximated by a Gaussian distribution ((1) & (2)). A survey of the atomic energy levels (3) indicates relatively larger departures from Gaussian in the partial level density of atoms. Clearly, a major difference between the two systems lies in the range of interaction. It is common knowledge that nuclear interactions are characterised by a short range while the Coulombic interaction between electrons is of infinite range in character. This chapter is therefore devoted to the study of effect of range on level density distributions and the aim is to see if the long range of interaction (potential) accounts for the increased departures from Gaussian, observed in the partial level density of atoms.

Since asymptotically (in the limit when the rank of interaction 'r' becomes much smaller than the particle number 'p') the level density approaches normality irrespective of any other parameters of the interaction, it is to be emphasised that whatever effect one looks for by varying the range could be observed only in finite spaces. In addition to the nature of level density distribution, the rate of approach to normality could also vary with the range of interaction (potential). With this end in view, the following calculations are done.

In section 2, the rate of approach of the level density to normality is studied for a system of non-interacting particles where the single particle spectrum is one generated by an infinite range potential. The effect of range of interaction on the level density of an interacting system is studied in section 3 through shell model calculations and scalar averages. The two body interaction is chosen to be Yukawa interaction. Section 4 deals with ensemble evaluation of scalar averages in (sd) shell. This ensemble is a modified TBRE with range built in as a parameter. In section 5 the effect of range of interaction on the distribution of matrix elements in the defining space is studied. Section 6 gives a summary of the results and conclusions.

2. Range of Potential and Level Density in Non-Interacting System:

In the case of a dilute system where the number of single particle states 's' is much higher than the particle number 'p', Pauli principle effects could be ignored and the 'p' particle level density $\rho_p(E)$ is given by the convolution of the single particle level

densities $\rho_1(E)$.

$$\rho_p(E) = \int_{-\infty}^{\infty} dE_1 \ldots\ldots\ldots \int_{-\infty}^{\infty} dE_{(p-1)} \times \rho_1(E)\ \rho_1(E_2)\ldots\ldots\ \rho_1(E-(E_1+E_2+E_{(p-1)})) \qquad (3.1)$$

the q^{th} cumulant of which is given by

$$K_q(p) = p\ K_q(1) \qquad (3.2)$$

Since the shape parameters

$$S_q(p) = \frac{p\ K_q(1)}{(p\ K_2(1))^{q/2}} \qquad (3.3)$$

vary inversely as 'p', in the limit when $p \longrightarrow \infty$, all S_q's $\longrightarrow 0$ yielding a Gaussian.

French (6) has shown that in a uniform spectrum of six equally spaced levels each with a degeneracy two, the six particle level density is a Gaussian. This is the case with a uniform spectrum. What happens when the single particle spectrum arises from a long range potential?

To answer this question the spectrum of hydrogen atom is chosen. This spectrum arises out of an infinite range potential. Its n^{th} eigenvalue is given by

$$E_n = - Rhc/n^2 \qquad (3.4)$$

and the degeneracy g_n for the n^{th} level including the degeneracy due to spin 1/2 for the particle is given by

$$g_n = 2\,n^2 \qquad (3.5)$$

Because of the infinite range of the potential, there is an infinite number of bound states. The spacing between the successive levels decreases monotonically thus making the single particle spectrum non-uniform. From equations (3.4) & (3.5) the single particle level density could be deduced. For large 'n',

$$\rho_1(E) = 1/(1-E)^{5/2} \qquad (3.6)$$

where E is the excitation energy above the ground state ($n = 1$) in units of Rhc. $E = 0$ corresponds to the ground state and $E = 1$ to the zeroth energy state. $\rho_1(E)$ diverges at $E = 1$ indicating that there is an infinite number of bound states.

From this single particle spectrum, different stretches of levels running from $n = N_1$ to $n = N_2$ are chosen and the rate of the approach of the level density to normality studied through skewness (S_3) and excess (S_4). When $|S_3|$ and $|S_4|$ are less than or equal to 0.30, the level density is considered to have approached normality. Strictly speaking, one has to evaluate other shape parameters as well.

TABLE 1

N_1	N_2	p	S_3	S_4
10	20	23	-0.30	0.06
10	40	135	-0.30	0.11
1	100	200	-21.86	666.9
201	400	19	-0.30	0.05
200	800	119	-0.29	0.11

TABLE 1: The quantum numbers N_1 and N_2 of the levels chosen as limits of integration together with the skewness S_3 and the excess S_4 for p number of convolutions.

Table 1 shows the values of S_3 and S_4 for different particle number 'p' for the indicated stretches of levels. The particle number also indicates the number of convolutions. When the number of levels is increased, the rate of approach of the level density to normality goes down and the higher up one goes the faster becomes the rate. When the lower region of the spectrum is chosen, even after 200 convolutions skewness retains a high value. Thus the non-uniformity in the single particle spectrum reduces the rate of approach of the level density to normality in comparison with that found in a uniform spectrum (6).

3. Effect of Range on an Interacting System:

For studying the effect of range on interacting particles, numerical calculations were carried out in 2s-1d shell region of nuclei. The single particle states consist of three levels $1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$. The Hamiltonian is given by

$$H = H(1) + H(2) \tag{3.7}$$

where the one-body Hamiltonian $H(1)$ is given by $E(1d_{5/2})$, $E(2s_{1/2})$, $E(1d_{3/2})$ which give the energies of the three single particle states.

$$H(2) = \sum_{i<j} V_{ij} \tag{3.8}$$

where V_{ij} is a central, attractive potential with Yukawa radial dependence.

$$V_{ij} = V_o \exp(-r_{ij/a}) / (r_{ij/a}) \tag{3.9}$$

where $r_{ij} = |r_i - r_j|$. $V_o > 0$ is the strength of the potential. The choice of Yukawa interaction is made as in the limit when the range of interaction 'a' goes to infinity, it reduces down to the Coulomb form. The single particle wave functions are harmonic oscillator wave functions and hence 'a' in equation (3.9) is in units of the oscillator size parameter 'b'.

$$b = \sqrt{\hbar/m\omega} \tag{3.10}$$

In equation (3.10) 'm' is the particle mass and $\hbar\omega$ is the energy of one oscillator quantum. The range is deemed short or long with respect to 'b', the size of the system. Calculations are done for six different values of 'a' from 0.04 to 2.0. a = 0.6 corresponds to the nuclear range and the corresponding strength is chosen to be 40 MeV. The strength for the other ranges is calculated from the relation

$$V_o a^3 = \text{a constant} = 40 \times (0.6)^3 \tag{3.11}$$

which arises from the normalisation of the potential. The calculations are done for two different sets of single particle energies:

$$\text{(i) } E(1d_{5/2}) = E(2s_{1/2}) = E(1d_{3/2}) = 0.0 \qquad (3.12)$$

$$\text{(ii) } E(1d_{5/2}) = -4.15 \text{ MeV}$$

$$E(2s_{1/2}) = -3.28 \text{ MeV}$$

$$E(1d_{3/2}) = 0.93 \text{ MeV} \qquad (3.13)$$

The first set called without SPE takes all the single particle states to be degenerate in energy. The second set labelled With SPE corresponds to the observed levels in ^{17}O nucleus. It is for this second set the strengths for different ranges of the potential need to be renormalized according to equation (3.11). For the first case, the strength of the interaction is an inessential parameter.

3a. Matrix Calculations and Scalar Averages:

Shell model matrices corresponding to the case $p = 8$, total angular momentum $J = 0$ and total isospin $J = 2$ are generated and diagonalised for different ranges. The dimensionality of the space being 287 the results obtained could be taken to be statistically meaningful. The eigenvalue spectrum provides the partial level density in each case and from the exact spectrum S_3 and S_4 are evaluated.

Figure 1 shows the exact distribution function (histogram)

$$F(E) = \int_{-\infty}^{E} \rho(E')\, dE' \qquad (3.14)$$

Fig. 1 : The exact cumulative level density (histogram) and the Gaussian distribution function (of corresponding mean and variance) depicted by the continuous curve for the cases Without SPE and with SPE for the indicated ranges.

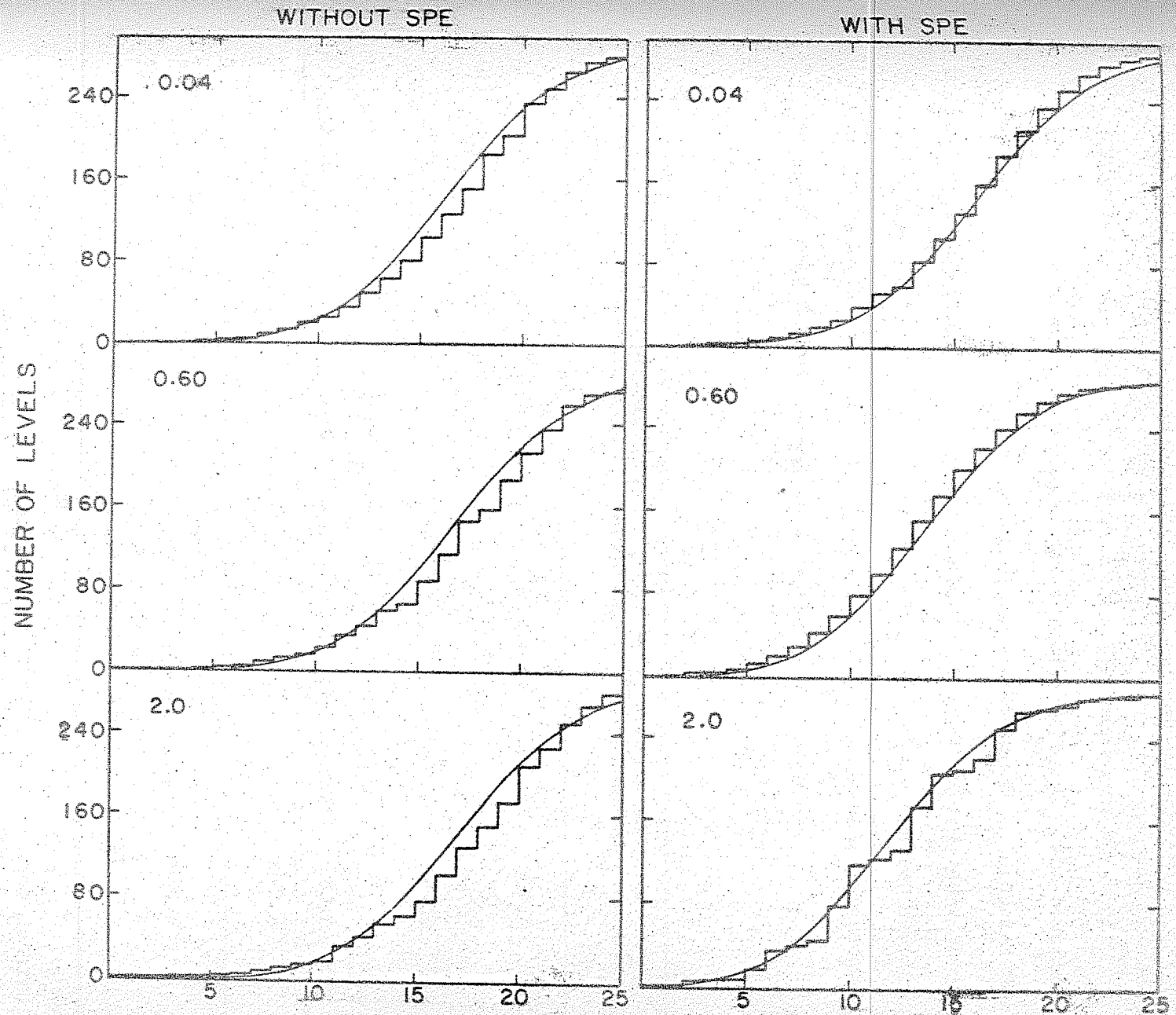
WITHOUT SPE
WITH SPE
0.04
0.04
0.60
0.60
2.0
2.0
NUMBER OF LEVELS
0
80
160
240
5
10
15
20
25
MeV

Fig.1

resulting from the exact eigenvalue spectrum for the three different ranges 0.04, 0.6 and 2.0 and for the two different sets of single particle energies given in equations (3.12) and (3.13). The continuous curve gives the distribution function corresponding to a Gaussian frequency defined by the mean energy and the variance of the exact frequency function. In the case without SPE, there is a greater deviation of the histogram from the normal distribution with increase in range. With SPE, the fit with Gaussian is best for the range 0.6.

TABLE 2

Range	Without SPE			With SPE		
a	S_3	S_4	χ^2	S_3	S_4	χ^2
0.04	-0.57	0.25	12.4	-0.33	-0.21	5.8
0.20	-0.61	0.28	13.4	-0.51	0.17	5.2
0.60	-0.73	0.45	21.1	-0.18	-0.07	1.0
1.00	-0.76	0.51	24.9	0.03	-0.13	12.6
1.50	-0.78	0.53	23.8	0.08	-0.11	63.8
2.00	-0.78	0.54	24.1	0.09	-0.11	101.0

χ^2 *(critical) = 14.067*

TABLE 2. The skewness S_3 *and the excess* S_4 *for different ranges 'a' for* $p = 8$, $J = 0$, $T = 2$ *with dimensionality* $d = 287$.

Table 2 gives the values of S_3 *and* S_4 *of the level density along with* χ^2 *which gives a measure of the goodness of fit of the exact distribution function with a Gaussian distribution function. For the case without SPE the values of* S_3 *and* S_4 *increase and saturate with*

range. The values of χ^2 also show a similar trend. For the case With SPE both $|S_3|$ and $|S_4|$ are ≤ 0.3 for most of the ranges. From the values of χ^2 it is seen that the best fit is for the range 0.6. For longer ranges χ^2 values are greater than χ^2 critical. This shows the effect of higher cumulants on the level density distribution.

These results obtained for one specific case could be generalized to other equally large spaces. To improve upon the reliability of these results one should in principle study a large number of such cases. Since matrix diagonalisation is very time consuming the alternative choice of scalar average evaluation is resorted to, for a more exhaustive study of the effect of range.

Using the propagation technique ((7) and (8)) the low order shape parameters S_3 and S_4 of the level density are evaluated for different particle number in the 2s-1d shell for different ranges. To recapitulate the principle underlying this technique, the average of an r-body operator in 'p' particle space is given in terms of its average in r-particle space by the relation

$$\langle O(r) \rangle^{p} = {}^{p}C_{r} \langle O(r) \rangle^{r} \qquad (3.15)$$

Tables 3 and 4 give the values of S_3 and S_4 for the two cases Without SPE and With SPE. For the case Without SPE, S_3 and S_4 increase with range for a given particle number. Consistent with the theories ((4) and (5)) S_3 and S_4 are lowest around mid-shell where the dimensionality of the space is the largest. Since at mid-shell the values of S_3 and S_4 increase with range, it is concluded that the rate

TABLE 3

p \ a	0.04	0.20	0.60	1.00	1.50	2.00
			S_3			
9	-0.531	-0.597	-0.751	-0.793	-0.810	-0.816
10	-0.510	-0.580	-0.741	-0.784	-0.801	-0.807
11	-0.499	-0.570	-0.735	-0.779	-0.795	-0.801
12	-0.496	-0.568	-0.733	-0.776	-0.791	-0.797
13	-0.500	-0.571	-0.735	-0.775	-0.789	-0.794
14	-0.513	-0.582	-0.739	-0.777	-0.789	-0.793
15	-0.534	-0.599	-0.748	-0.781	-0.790	-0.793
			S_4			
9	0.214	0.308	0.570	0.639	0.664	0.674
10	0.132	0.240	0.530	0.605	0.632	0.642
11	0.087	0.202	0.507	0.584	0.612	0.622
12	0.074	0.191	0.499	0.576	0.602	0.611
13	0.091	0.205	0.505	0.578	0.602	0.610
14	0.140	0.245	0.527	0.592	0.611	0.617
15	0.225	0.316	0.566	0.619	0.631	0.633

TABLE 3. The skewness S_3 and excess S_4 for different ranges 'a' for different particle numbers 'p' for the case without SPE.

of approach of the level density to normality decreases with range. It should be noted that in no case $|S_3|$ and $|S_4|$ are less than 0.30. It means that with this two body interaction, the asymptotic limit has not been reached. A similar trend is observed in the shell model results (Table 2).

TABLE 4

p \ a	0.04	0.20	0.60	1.00	1.50	2.00
			S_3			
9	-0.490	-0.474	-0.181	0.023	0.067	0.074
10	-0.473	-0.484	-0.197	0.001	0.043	0.048
11	-0.464	-0.479	-0.211	-0.021	0.019	0.024
12	-0.461	-0.478	-0.221	-0.040	-0.004	0.000
13	-0.464	-0.480	-0.230	-0.060	-0.027	0.023
14	-0.474	-0.487	-0.237	-0.079	-0.051	0.048
15	-0.492	-0.498	-0.244	-0.099	-0.075	-0.073
			S_4			
9	0.158	0.164	-0.121	-0.180	-0.157	-0.158
10	0.087	0.117	-0.116	-0.172	-0.147	-0.138
11	0.047	0.090	-0.113	-0.177	-0.142	-0.132
12	0.034	0.081	-0.111	-0.164	-0.139	-0.130
13	0.048	0.091	-0.110	-0.164	-0.140	-0.131
14	0.089	0.118	-0.110	-0.164	-0.145	-0.137
15	0.160	0.166	-0.114	-0.173	-0.153	-0.146

TABLE 4. The skewness S_3 and excess S_4 for different ranges 'a' for different particle numbers 'p' for the case With SPE.

Table 4 shows that $|S_3|$ and $|S_4|$ are $\leq$ 0.30 for all ranges starting from 0.60. This is in contrast with the first case revealing the dominance of the single particle energies over two body interaction. The distributions for all the ranges greater than 0.2 cannot be taken for a Gaussian since neither higher shape parameters nor χ^2 values are made available in this procedure.

It is inferred from the above calculations that the level density departs more and more from Gaussian with range. Its rate of approach to normality also declines with range.

5. Fixed Range Two Body Random Ensemble (FRTRE):

To investigate the effect of range for two body random interaction, the range of interaction is introduced as a parameter in the conventional TBRE. The two body matrix elements of this modified ensemble are generated as follows.

TBME are known to occur as linear combinations of radial integrals which again are linear combinations of Talmi integrals for a given interaction. Given the space, the number of Talmi integrals (N) that one encounters in the evaluation of TBME is fixed. And given the radial dependence of the potential in any two body interaction that includes central and non-central parts, the range dependence of TBME could be explicitly built in through the appropriate Talmi integrals. What remain to be randomised then are the strengths corresponding to different parts of the interaction.

Thus the TBME of FRTRE are given by

$$X_i = \sum_{k=0}^{N-1} a_k T_k \qquad (3.16)$$

where T_k's are the appropriate Talmi integrals and a_k's are random numbers chosen to be uniform deviates with zero mean. With this FRTRE, scalar averages are evaluated in 2s-1d shell for the three different ranges 0.04, 0.060 and 2.00. The five Talmi integrals required in (sd) shell calculations are evaluated for Yukawa radial dependence of two body random interaction. The calculations are done only for the case Without SPE. To eliminate the effect of induced single particle energies, the diagonal two body matrix elements are chosen to be zero. This gives rise to exact particle-hole symmetry. The results of 50 dimensional ensemble calculations are shown in Table 5 for the three different ranges 0.04, 0.06 and 2.00.

For most of the cases S_3 and S_4 are quite close to those of a Gaussian for all the ranges. and there is no significant change in the values with range. The trend in the fluctuations of S_3 and S_4 with 'p' is in accord with the analytical expressions given by Mon & French (4) for dilute systems. In any ensemble calculation, one cannot look for non-vanishing values of S_q's for any odd 'q', since the matrix elements in the defining space are symmetrically distributed. So from FRTRE calculations, it is concluded that there is no change in S_4 with range. S_3 changes with range as observed in the shell model and scalar moment calculations in Section 3. Though at the risk of repetition, it is to be mentioned that one cannot conclude from the results of FRTRE alone about the change in level density with range. An indubitable conclusion can be drawn only from shell model calculations of FRTRE.

TABLE 5

a	0.04				0.60				2.00			
p	S_3	S_3 RMS	S_4	S_4 RMS	S_3	S_3 RMS	S_4	S_4 RMS	S_3	S_3 RMS	S_4	S_4 RMS
4	.009	.112	-.566	.289	.033	.120	.561	.124	.041	.105	-.562	.126
5	.003	.097	-.477	.239	.025	.109	.477	.090	.035	.097	-.479	.088
6	-.001	.091	-.406	.206	.021	.105	.409	.080	.031	.094	-.409	.079
7	-.003	.089	-.352	.183	.018	.103	.357	.078	.028	.094	-.357	.080
8	-.005	.089	-.313	.169	.016	.103	.319	.080	.027	.095	-.318	.085
9	-.006	.089	-.285	.160	.014	.103	.292	.083	.026	.096	-.290	.089
10	-.007	.089	-.266	.154	.013	.103	.273	.085	.025	.096	-.271	.093
11	-.007	.089	-.256	.151	.013	.103	.263	.086	.024	.097	-.261	.095
12	-.007	.089	-.252	.150	.013	.103	.259	.086	.024	.097	-.257	.095

TABLE 5. Skewness S_3 and excess S_4 along with RMS deviations for different particle number 'p' for FRTRE.

5. Distribution of Matrix Elements:

Porter (9) has shown that for realistic nuclear interactions, the many particle matrices in general belong to GOE. The diagonal and the off-diagonal matrix elements fall on a Gaussian around zero mean with the variance of the diagonal matrix element distribution twice that of the off-diagonal one. In this section, the effect of range of interaction on the distribution of matrix elements in the defining space is studied.

In order to have reasonable statistics, the TBME of Yukawa interaction in equation (3.9) are evaluated in the model space consisting of three ($N = 0, 1, 2$) major harmonic oscillator shells for the three different ranges 0.04, 0.6 and 2.0. In this model space, there are 332 matrix elements - 92 diagonal and 240 off-diagonal. These fall under different matrices characterised by different J, T and π. Excluding the elements corresponding to one dimensional matrices and also those for which the variances are different by orders of magnitude from those of the rest, the distributions of the matrix elements are presented in Figs. 2, 3, 4 and Figs. 5, 6 and 7.

The distributions of the diagonal matrix elements are highly skewed and show no significant change with range. The off-diagonal ones are close to Gaussian and there is an increasing asymmetry with range.

Figs. 2, 3 and 4 : *The exact frequency function (histogram) for 68 diagonal matrix elements for the indicated ranges and the Gaussian distribution (depicted by the continuous curve).*

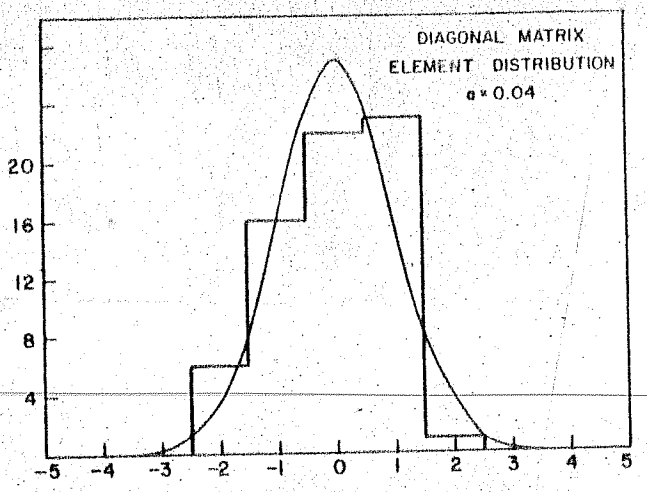

Fig. 2

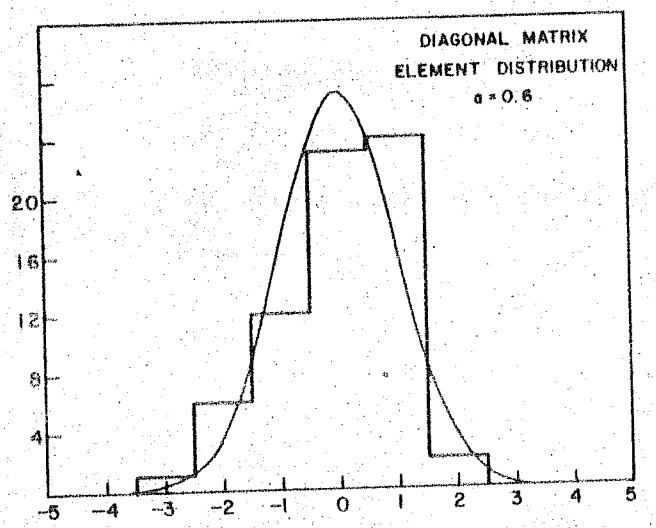

Fig.3

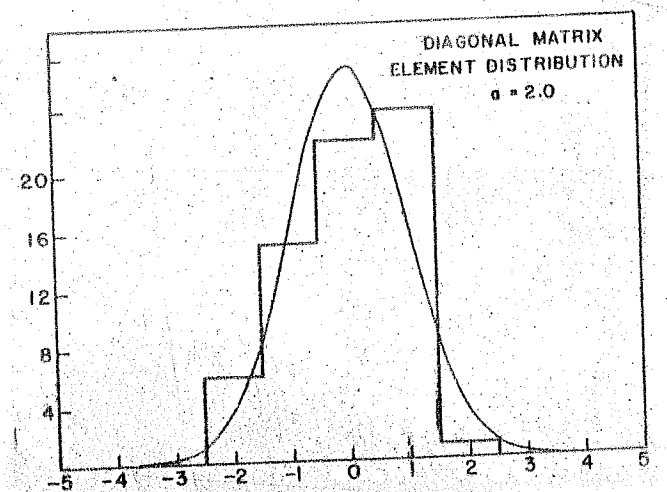

Fig.4

Figs. 5, 6 & 7 : The exact frequency function (histogram) for 208 off-diagonal matrix elements for the indicated ranges and the Gaussian distribution (depicted by the continuous curve).

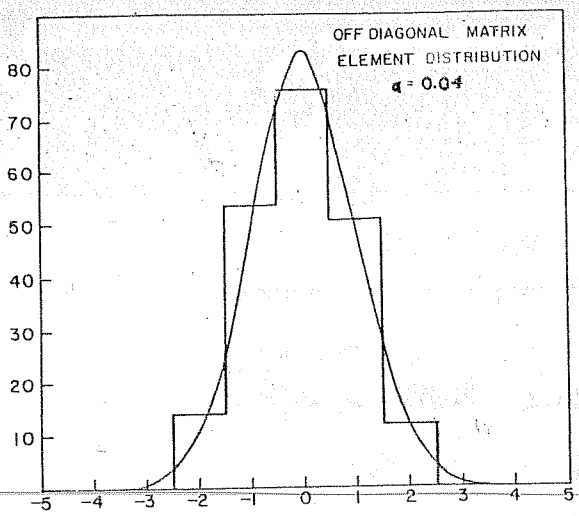

Fig 5

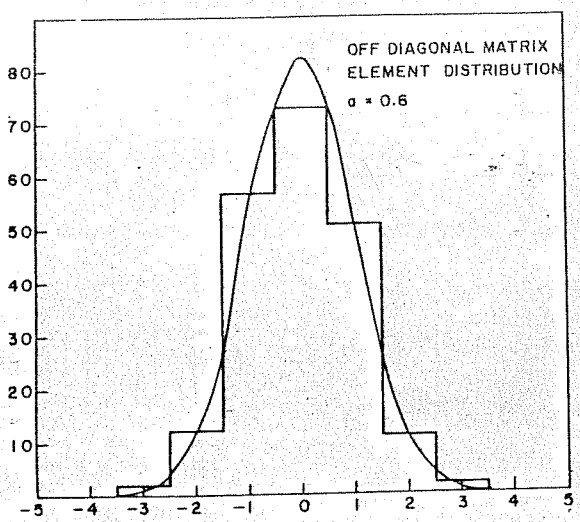

Fig. 6

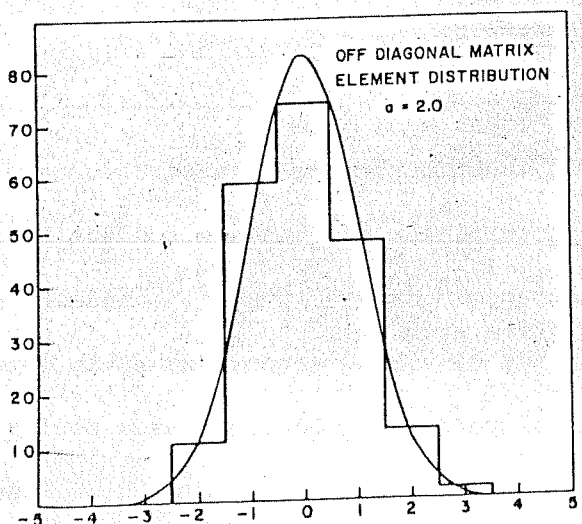

Fig 7

TABLE 6

d	2T	2J	π	0.04	0.6	2.0
3	0	0	-	1.25	1.10	0.90
7	0	2	-	5.43	1.62	0.51
6	0	4	-	1.22	0.61	0.49
3	0	6	-	1.28	0.48	0.30
3	2	0	-	10.21	169.14	150.0
7	2	2	-	0.91	0.54	0.28
6	2	4	-	0.78	0.62	0.35
3	2	6	-	21.17	1.55	0.39
11	0	2	+	1.71	1.20	0.79
6	0	4	+	2.94	3.33	2.49
6	0	6	+	3.13	1.96	1.58
7	2	0	+	6.16	2.27	1.23
5	2	2	+	0.99	1.00	0.47
9	2	4	+	3.59	2.14	1.50
3	2	6	+	0.18	0.98	0.78
Average				2.45	1.39	0.86

TABLE 6. The ratio of dispersion of off-diagonal to diagonal matrix element distributions for the matrices characterised by quantum numbers J, T and π with dimensionality 'd' for the three different ranges 0.04, 0.6 and 2.0. The underlined values were excluded for the evaluation of the average.

Table 6 gives the dimensionalities of the different J T π matrices together with the ratio of dispersions of off-diagonal to diagonal matrix element distributions (0.5 for GOE) for the three different ranges. The ratio of dispersions fluctuates strongly but by and large this ratio goes down with range. So does the average value. The underlined cases in Table 6 are not used in evaluating the averages.

6. Conclusions:

From the shell model and scalar moment calculations, it is concluded that with increase in range there is increasing departure from Gaussian in the partial level density. This urges one to believe that the infinite range of the Coulombic potential is one of the factors contributing to increased departures from Gaussian in the partial level density of atoms. As regards the matrix element distributions, only the off-diagonal ones change with range.

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CHAPTER IV

FLUCTUATION MEASURES AND RANGE OF INTERACTION

1. Introduction:

Fluctuations shed light on the local features of level spectra as against the average properties that depict the global features. Fluctuations about eigenvalue density distributions are related to the local correlation functions of 'n' levels in an N ($N > n$) dimensional matrix. These are investigated through near neighbour spacing distributions and also through statistics like Δ_3 and Q(1) that give measures of long and short range ordering in level spectra. Analysis ((2) and (3)) of TBRE and GOE suggest that these fluctuation properties are identical for both these ensembles. In fact Dyson (4) suggests that as long as the number of levels 'n' involved in the analysis is much smaller than N, the matrix dimensionality, all ensembles of real symmetric matrices will yield identical fluctuation properties.

irrespective of the nature of level density. One such example is the Wigner surmise, for the nearest neighbour spacing distribution which is valid in the limit of large N. All these fluctuations properties are found to be sensitive only to mixing of levels of different symmetries. For example the level repulsion in Wigner surmise is modified when the level sequence is not pure. Hence the only physical information that could till now be learnt from the fluctuations is about hidden symmetries.

In this chapter the fluctuation properties of Random Phase Ensembles (RPE) discussed in Chapter II are investigated. Since the level density is found to depart more and more from Gaussian with range of interaction (Chapter III) the trend exhibited by the fluctuation measures with range is also analysed.

Section 2 discusses the nature of different fluctuation properties as derived in random matrix theory and Section 3 those of Random Phase Ensembles. The effect of range on fluctuations is contained in Section 4 which is followed by a discussion of the outcomes in Section 5.

2. Fluctuations:

For the nearest neighbour spacing distribution Wigner surmised the following, based on the theory of Wigner-Von Neumann level repulsion (5).

$$P(x^0) = \pi x^0 / 2D^2 \exp(-\pi x^{0^2} / 4D^2) \qquad (4.1)$$

where the superscript '0' of 'x' indicates the probability of finding '0' levels in an interval 'x'. This expression was shown to be very close to the exact distribution for the central region of the infinite GOE. Analytic expressions $P(x^k)$ are available for more distant neighbours but the spheroidal functions that occur in these expressions are tabulated only upto k=4. Hence for more distant neighbours comparison with the analytical form of GOE is not feasible.

In addition to spacing distributions there are statistics defined by Dyson and Mehta (1) that quantify the fluctuations. "A statistic is a quantity which can be computed from an observed sequence of levels alone without further information and for which the average value and the variance are known from the theoretical model. A suitable statistic is one which is sensitive to the property to be compared and insensitive to other details".

The C^k statistic measures the average correlation coefficient between $(k+1)^{th}$ near neighbours. For a run of (n+1) levels C^0 is defined by

$$C^0 = \left(\sum_{i=1}^{n} (s_i^0 - D)(s_{i+1}^0 - D) n \right) / \left(\sum_{i=1}^{n+1} (s_i^0 - D)^2 / (n+1) \right) \qquad (4.2)$$

From the Coulomb gas model (6), Dyson has given a value of -0.266 for C^0. The negative sign of C^0 indicates that a larger than average spacing always tends to be followed by a smaller than average spacing.

Besides C^0, there is yet another statistic Q that gives a measure of the short range order in level spectra. For a stretch of 'n' levels running from -L to +L in the limit of matrix dimensionality N going to infinity, Q is given by

$$Q = - \sum_{i<j} f(E_i,E_j) \ln |(E_i-E_j)/R|$$

$$+ \sum_{ij} F(E_i)\, U(E_j) - 1/2\, U_o \sum_{ij} F(E_i)\, F(E_j)$$

$$+ 1/2 \left(\sum_i F(E_i) \right) \ln\left(2\pi R \sum_j F(E_j) \Big/ \int F(E)\,dE\right) \qquad (4.3)$$

where $R = mD = m(2L)/n$ ($m = 2, 3$ or 4)

$f(x,y) = 1$ for $|x-y| < R$, $|x| < L$, $|y| < L$

$= 0$ otherwise

$F(x) = 1$ for $|x| < L$

$= 0$ otherwise

$U_0 = (-R/L) + 1/8\,(R/L)^2$

$U(x) = -R/L$ for $|x| < L-R$

$= -1/2L\,(R+(L-|x|)\,(1-\ln((L-|x|/R)))$

for $L-R < |x| < L$

The fact that $f(x,y)$ vanishes for $|x-y| > mD$ shows that Q measures the short range ordering in level spectra. The average value of Q and its variance (1) are given by

$$\langle Q \rangle = n(U+1/\pi^2 M) \tag{4.4}$$

where $U = 0.365$.

$$V_Q \simeq n(C^0 + 1/\pi^2 M) \tag{4.5}$$

To measure the long range order Dyson and Mehta have defined the Δ_3 statistic which measures the deviation of the cumulative level density (staircase graph) from a straight line.

$$\Delta_3 = \underset{A,B}{Min}\ (1/2L \int_{-L}^{L} (N(E)-AE-B)^2 dE) \tag{4.6}$$

where $N(E)$ is the cumulative level density. In terms of the eigenvalues Δ_3 can be reduced to the following (3):

$$\Delta_3 = n^2/16 - 1/4L^2 \left(\sum_{i=1}^{n} E_i \right)^2 + 3n/8L^2 \left(\sum_{i=1}^{n} E_i^2 \right) - 3/16L^4 \left(\sum_{i=1}^{n} E_i^2 \right)^2 + 1/2L \left(\sum_{i=1}^{n} (n-2i+1)E_i \right) \tag{4.7}$$

The average value of Δ_3 and its variance are given by

$$\Delta_3 = 1/\pi^2 (\ln n - 0.0687) \tag{4.8}$$

$$V_{\Delta_3} = 1/\pi^4\ 1.1690 = 0.12 \tag{4.9}$$

Equation (4.8) shows that for Δ_3 to reach a value of 1.0 one should have about 20,000 levels. Thus the logarithmic dependence of Δ_3 on 'n' shows that the deviation of the staircase graph from a straight line

is on the average less than one spacing unit for a smaller number of levels. Dyson calls it appropriately the crystalline structure of the eigenvalue lattice. "Even at large separations, two eigenvalues feel the natural periodicity of the lattice and have a slight preference for separations which are an integer multiple of the mean spacing".

In the following sections these statistics are evaluated for finite matrix dimensionalities (N) under conditions when n $\gtrsim$ N. The speculation is that while C, Q and near neighbour spacing distributions would be identical for ensembles of different eigenvalue densities, $\triangle_3$ would discriminate between these ensembles.

3. Fluctuations in Random Phase Ensembles:

The ensembles that give a Gaussian, intermediate (semi-circle with a peak at the centre) and semicircular eigenvalue densities are chosen for the purpose of analysis of fluctuations. These correspond to RPE II (Fig.1), RPE VI (Fig. 5) and RPE IV (Fig. 3). In this chapter these will be denoted by Case I, II and III respectively. In the 50 dimensional matrices 46 levels are chosen leaving two on either side to eliminate end effects. To remove the effects of secular variation in the level density arising out of the finite size of the matrix the spectrum is unfolded - each spacing is expressed in units of the local mean spacing. In the evaluation of Q local mean spacing is found by taking a couple of spacings on either side of each spacing, Q is evaluated using equation (4.5) and the parameter R has been taken to be 3D in all the cases.

TABLE 1

Case No.	Q (RMS)
Case I	12.46 (7.46)
Case II	12.45 (5.60)
Case III	11.90 (5.92)
GOE (Theoretical)	15.24 (3.72)

TABLE 1. Values of Q and RMS deviations for the three Random Phase Ensembles.

Table 1 gives the values of Q for all the three cases of RPE together with the RMS deviations. The corresponding theoretical values as calculated from equations (4.3) and (4.4) are also given. There is not much of a variation in the average value of Q. The fluctuation in case I is larger than that of the second and conforms with the relatively poorer ergodic conditions in the level density of case I (Chapter II).

In the evaluation of Δ_3 two different ways of unfolding have been adopted. While unfolding the spectrum case should be taken to see that along with the secular variations, fluctuations are also

not washed off. If the fluctuations are washed off too, one would always end up with long range order ($\Delta_3 < 1$). An instance of this will be seen to crop up later. Hence the unfolding has to be done using an optimum number of spacings to include the largest possible number of levels. There is no unique prescription for this optimum unfolding. For every spectrum one has to find that out by trial and error methods.

TABLE 2

k	NE	Case I	Case II	Case III	GOE (Theoretical)
5	42	0.22 (0.03)	0.22 (0.03)	0.21 (0.02)	0.37 (0.12)
11	36	0.29 (0.04)	0.40 (0.07)	0.28 (0.05)	0.36 (0.12)
17	30	0.33 (0.07)	0.66 (0.20)	0.33 (0.07)	0.34 (0.12)
23	24	0.31 (0.06)	0.79 (0.28)	0.34 (0.08)	0.32 (0.12)
29	18	0.28 (0.07)	0.65 (0.25)	0.30 (0.07)	0.29 (0.12)
35	12	0.22 (0.05)	0.37 (0.17)	0.23 (0.07)	0.25 (0.12)
41	6	0.15 (0.06)	0.17 (0.05)	0.15 (0.05)	0.17 (0.12)

TABLE 2. Values of Δ_3 and RMS deviations for NE levels for the RPE corresponding to the first way of unfolding mentioned in the text.

Table 2 gives the values of Δ_3 corresponding to one way of unfolding. In this case local mean spacing is found by taking (k-1)/2 spacings on either side of each spacing. The values of 'k' and the number of levels used for analysis along with Δ_3 and its RMS deviation are given in Table 2 for all the three cases of RPE. For k = 5, the ensemble averaged Δ_3 even for a set of 247 random levels is less than 1 (Table 6). Table 6 gives the values of Δ_3 for higher 'k' as well for this set of random sequences. k = 5 is hence inadmissible for the evaluation of Δ_3. To decide the optimum stage of unfolding, firstly one looks for a minimum difference in the values of Δ_3 for two consecutive 'k' values. If this difference increases on either side then any value of 'k' in between would be optimum provided again, corresponding to a random sequence long range order is violated. For the three cases of RPE, the upper limit of 'k' has been underlined in Table 2. The upper limit of 'k' turns out to be 23 for all the three cases. But the Δ_3 value for the second case is different from that of the first or the third. The violation of long range order ($\Delta_3 > 1$) could probably be seen in higher dimensional matrices. What is important is the difference in the values of Δ_3 between case II and case I (case III), corresponding to different level densities.

This way of unfolding has two demerits. The first is associated with the fact that the number of levels made available for analysis decreases with increasing 'k' while one actually wants to use the highest 'k' for unfolding. This difficulty could be circumvented if the matrix dimensionality is high. For example while studying the effect of range on Δ_3, this procedure is adopted since the matrix dimensionality is as high as 247 (Section 4).

The second demerit concerns the RMS deviation in Δ_3. To recall, the RMS deviation in Q was largest in case I in accordance with the ergodicity observed in the eigenvalue density (Table 3, Chapter II). But this procedure gives the largest RMS deviation for Case II. Hence the second method for unfolding the spectrum.

In this scheme the whole spectrum under investigation is split into different regions and for each of these regions mean spacing is calculated. This is tantamount to fitting the cumulative level density by more than one straight line. The spacings in each region are expressed in units of the mean spacing corresponding to that region. However, high be the 'k' value in each region, the total number of levels is always kept in tact for analysis with this method.

TABLE 3

k	Case I	Case II	Case III	GOE (Theoretical)
5	0.20 (0.05)	0.31 (0.04)	0.29 (0.05)	0.38 (0.12)
10	0.46 (0.11)	0.61 (0.18)	0.39 (0.08)	0.38 (0.12)
15	1.29 (0.39)	0.74 (0.18)	0.54 (0.14)	0.38 (0.12)
20	1.96 (0.45)	2.54 (0.85)	0.97 (0.26)	0.38 (0.12)

TABLE 3. Values of Δ_3 and RMS deviations for 46 levels of Random Phase Ensembles.

Table 3 shows the values of Δ_3 and RMS deviations for the RPE for the second way of unfolding. The optimum 'k' is decided using the same criterion of minimum difference in Δ_3 values for two consecutive 'k's and the upper limit for 'k' underlined in each case. k = 5 has been eliminated since Case III gives a much smaller value compared to the theoretical estimate. In fact an analysis of the members of Case III shows that for semi-circle k = 15 gives the theoretical estimates. It is found that Δ_3 value increases from semi-circle to Gaussian. In the case of a Gaussian (Case I) long range order is violated. In accord with the RMS deviations in level density and Q values, RMS deviation in Δ_3 also increases from semi-circle to Gaussian. It is concluded from these results that while short range correlation property Q remains invariant Δ_3 does indicate the difference in the nature of level density distributions when the number 'n' of levels used is comparable to the dimensionality of the matrix. The RMS deviations in the statistics indicate as to what extent the ensembles are ergodic in the level density distribution.

4. Fluctuations with Range:

The shell model matrices for the three different ranges 0.04, 0.60 and 2.00 discussed in Section 3 of Chapter III are analysed for their fluctuation properties. From the matrices of dimensionality 287, 247 levels are chosen from the centre. The spectrum of hydrogen atom (discussed in Section 2 of Chapter III), a uniform spectrum and an ensemble of random levels are also analysed for their fluctuation properties.

The first three near neighbour spacing distributions for the shell model matrices of different ranges are shown in Figures 1, 2, 3, 4, 5, 6, 7, 8 and 9 respectively. The spacing is expressed in units of the n^{th} order (n = 1, 2, 3) local mean spacing. In Figures 1, 2 and 3, the nearest neighbour distribution is compared to Wigner surmise (equation 4.1) depicted by the continuous curve. Deviation from Wigner surmise even for the nuclear range is attributed to inadequate statistics. With range there is not much of a change in the near neighbour spacing distributions.

TABLE 4

	C	Q
Shell Model Matrix a = 0.04	-0.28	123.90
Shell Model Matrix a = 0.60	-0.37	100.31
Shell Model Matrix a = 2.00	-0.29	120.40
Hydrogen Atom	0.71	1.54
Uniform Spectrum	0.00	5.80
Random Numbers	-	195.44 (20.00)
GOE (Theoretical)	-0.27	82.50 (8.4)

TABLE 4. Values of C and Q for the shell model matrices and other spectra.

Figs. 1, 2 & 3 : The nearest neighbour (k = 0) spacing distribution (histogram) along with the Wigner surmise for the indicated ranges.

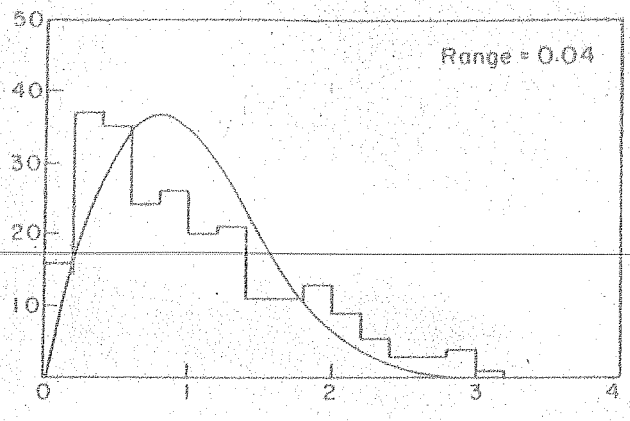

Fig. 1

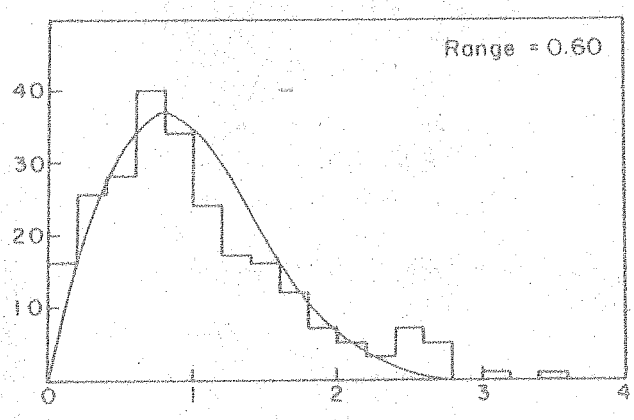

Fig. 2

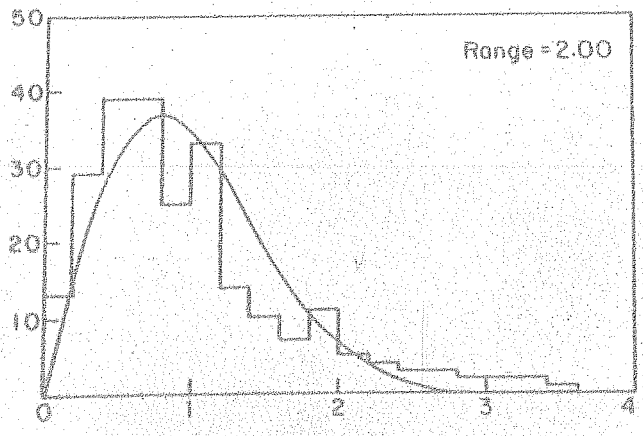

Fig. 3

Figs. 4, 5 & 6 : The second near neighbour (k = 1) spacing distribution (histogram) for the indicated ranges.

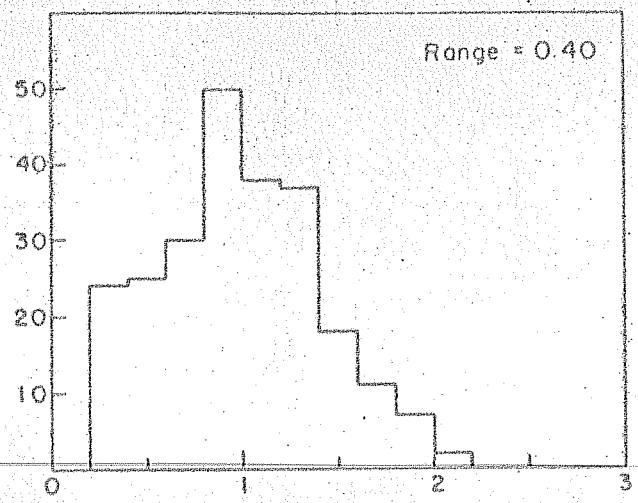

Fig. 4

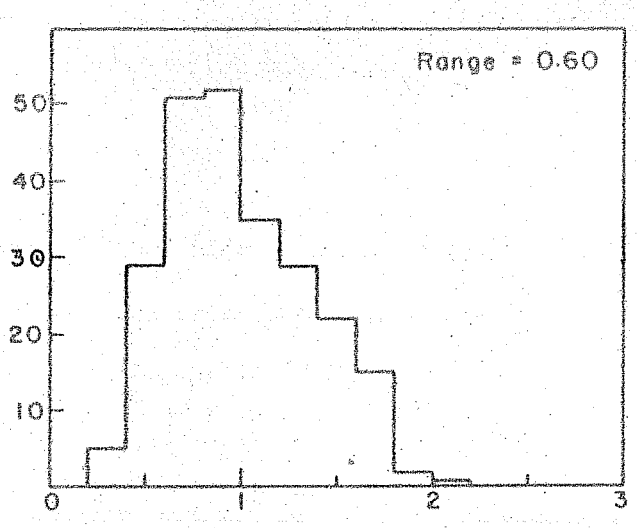

Fig. 5

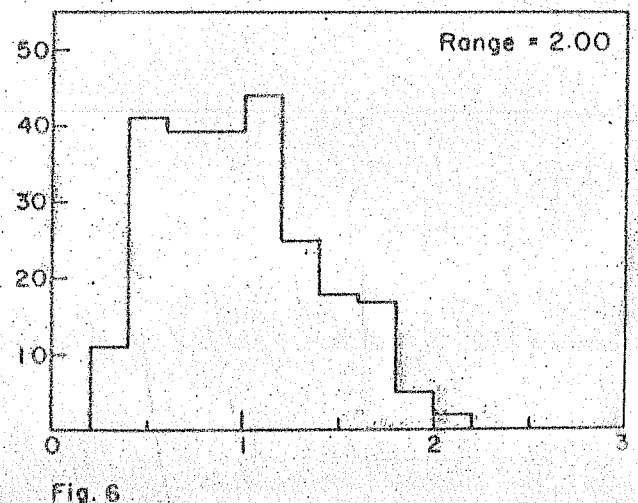

Fig. 6

Figs. 7, 8 & 9 : The third near neighbour (k = 2) spacing distribution (histogram) for the indicated ranges.

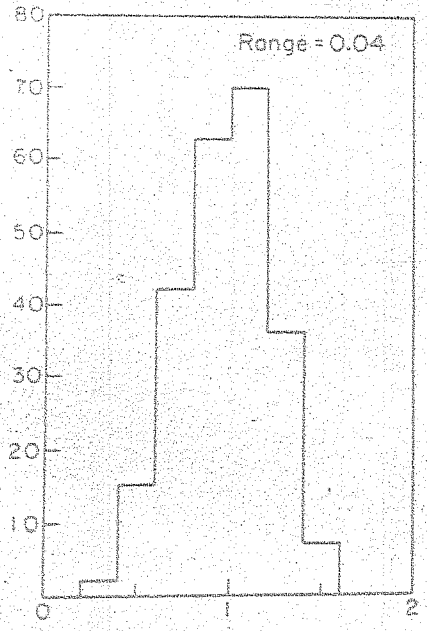

Fig. 7

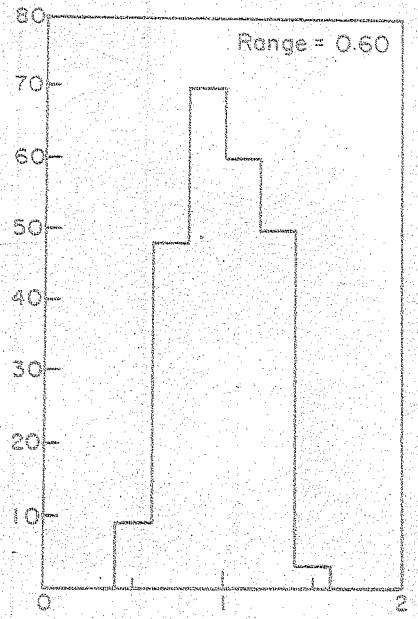

Fig. 8

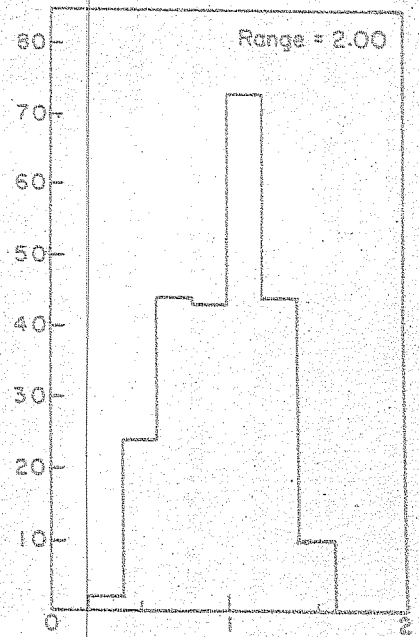

Fig. 9

The values of C^0 and Q for all the cases are given in Table 4 for 247 levels after unfolding in the same way as was used for the evaluation of Q in RPE. For the uniform spectrum, Q is much smaller than the theoretical estimate (GOE) whereas that of random sequence is much higher. Q is smallest for the nuclear range 0.6 and increases on either side suggesting increased short range disorder. The values of C^0 are negative in all the cases excepting for hydrogen atom.

For evaluating Δ_3 unfolding has been done using the first procedure discussed in the earlier section. Table 5

TABLE 5

k	NE	a = 0.04	a = 0.60	a = 2.00
5	50	0.33 (2.11)	0.31 (0.93)	0.44 (2.92)
11	44	0.92	0.52	0.69
17	38	1.25	0.53	1.22
23	32	1.85	0.55	1.60

TABLE 5. Values of Δ_3 for NE levels for the shell model matrices of different ranges. The theoretical estimate for 50 levels is 0.39 ± 0.12.

gives the values of Δ_3 for 50 levels from the centre of the shell model matrices. Values of Δ_3 without unfolding are given in brackets. For the nuclear range, even without unfolding one finds long range order. The optimum 'k' is underlined in each case. Comparison of Δ_3 for k = 17, shows that as one goes away from the nuclear range on either side, long range order decreases.

Table 6 gives the values of Δ_3 for 247 levels for unfolding with respect to 'k' spacings using the first procedure. For 212 levels even for the nuclear range the long range order is absent. Again comparing the values of Δ_3 for a fixed 'k' shows increased long range disorder away from the nuclear range. In the spectrum of hydrogen atom long range order is violated. It is inferred from these calculations that both short range and long range disorder increase away from the nuclear range on either side. Even for the nuclear range since Δ_3 is greater than 1, the extension of the long range order or the crystalline structure to ground state region is not valid. The violation of long range order for Case I in RPE also confirms this. This fact is in contradiction with the results of French et al (2). In their analysis unfolding has been done using a four moment function when the level density itself is described exactly by a four moment function (7). Thus the fluctuations are also washed away in the unfolding.

TABLE 6

k	NE	0.04	0.60	2.00	Hydrogen Atom	Random Sequence
5	242	0.76	0.37	0.43	0.14	0.94 (0.27)
11	236	1.10	0.62	0.74	0.43	1.33 (0.34)
17	230	1.60	0.94	1.32	1.12	1.76 (0.34)
23	224	2.30	1.27	1.78	2.04	2.20 (0.46)
29	218	3.01	1.64	2.57	3.15	2.70 (0.55)
35	212	4.02	1.82	3.53	4.42	3.17 (0.73)
41	206	5.11	2.01	4.20	5.77	3.70 (0.89)
47	200	5.95	2.33	4.89	7.15	4.17 (1.04)
53	194	6.68	2.92	5.31	8.52	4.62 (1.19)

TABLE 6. Values of Δ_3 for NE levels for the shell model matrices and Hydrogen Atom and random spectrum. The theoretical estimate for 242 levels is 0.55 ± 0.12.

5. *Conclusion:*

The decoupling between the level density and fluctuation properties is valid only in the limit when the number of levels 'n' is much smaller than the dimensionality of the matrix. This is evident in the variation depicted by Δ_3 *in RPE with level density while the short range correlation Q remains invariant. This point is also confirmed by the fact that long range order decreases away from the nuclear range on either side. A similar change in* Δ_3 *values should be observed in those of RBRE for different ranks of interaction when the level density changes from semi-circle to Gaussian. Since both long range and short range order decrease away from the nuclear range on either side, it is concluded that in addition to hidden symmetries one could learn about the range of interaction from the fluctuation measures. For the nuclear range the extension of the long range order or the crystalline structure to ground state region is not valid.*

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CHAPTER V

SUMMARY

In the hunt for a random matrix ensemble with a Gaussian level density, more ensembles with semicircular eigenvalue density gathered. The semicircle which emerges under the assumption of statistical independence of matrix elements in the GOE is also found to emerge when the matrix elements are taken to be linear combinations of a fixed set of independent matrix elements with random weightage coefficients.

The Random Phase Ensembles characterised by a small set of independent matrix elements give different characteristic level densities under different conditions. There emerge semi-circle, Gaussian and other distributions for eigenvalue density. When one starts with the conditions that correspond to a Gaussian eigenvalue density and

increases the number of independent matrix elements, then the level density goes quickly over to a semi-circle as in RBRE. These Random Phase Ensembles are as ergodic as RBRE but still not as ergodic as GOE. While all these Random Phase Ensembles are identical in short range correlation measure Q, Δ_3 does discriminate between the different eigenvalue densities, when a large fraction of the levels is analysed.

The partial level density of TBRE has been shown to approach Gaussian at a faster rate on the introduction of a general additive symmetry. It also departs more and more from Gaussian with increasing range of interaction. This suggests that the infinite range of the Coulombic interaction is one of the factors contributing to increased departures from Gaussian in the partial level density of atoms.

With respect to fluctuations, both short and long range ordering are found to decrease away from the nuclear range on either side. The change in the short range ordering is not very well understood as one would expect that short range correlation would be identical for all matrices. The shell model calculations of FRTRE would lead to indubitable conclusions on the effect of range. The violation of long range order even for the nuclear range and for the RPE with Gaussian level density show that the extension of the long range order or the crystalline structure to the ground state region of nuclei is not valid.

The variation in Δ_3 *with level density in RPE under conditions when the number of levels 'n' used is comparable to the matrix dimensionality couples this fluctuation property to the level density. The decoupling between level density and fluctuations is found only under conditions when* $n \ll N$. *The RBRE for different values of the rank of interaction should also exhibit the same trend in the values of* Δ_3 *as observed in RPE.*

Finally, for an extension of the RPE from the real to the complex domain, an analytical treatment of these ensembles is called for. Such an analytical approach would provide a clue to making these ensembles invariant under the appropriate transformations thus improving upon their ergodicity.