DISTRIBUTION OF ELECTRONS IN THE IONOSPHERE

from vertical soundings

by

S. N. Sreenivasan

Physical Research Laboratory, Ahmedabad

A thesis submitted to

The Gujarat University, Ahmedabad

for the Doctorate in Philosophy

1958

C o n t e n t s

Preface

The fluid envelope of our earth presents many interesting and intriguing features. One of the most important of these is the distribution of electrons and ions in the ionosphere. During the three decades and over since the existence of ionized regions in the earth's atmosphere was proved by the convincing experiments of Sir Edward Appleton and others many of the broad features of the ionization have been elucidated. We have to-day a fairly understandable picture of how the ionization sets in and in what way the ionization changes with location, the time of the day, with the season and state of solar activity. Nevertheless, scietific inquiry into the different causes of the ionization and its varying distribution and the complex features of the dynamics of the ionosphere is far from complete.

A wealth of information is lying latent in the ionograms of various ionospheric stations round the world. More information has been collected during the IGY. Much of the existing literature in the field of ionospheric physics relates to a portrayal of the state of events that are taking place high up in the ionosphere and the forecasting of conditions which determine radio propagation. An important new development is the attempt to synthesize a coherent three-dimensional knowledge of the electron-distribution and its variations.

A beginning in this direction has been made recently by a number of workers and a systematic study of the distribution of electrons at a number of places has been started in the USA & UK. The following pages give an account of the effort made at Ahmedabad to analyse the distribution over tropical latitudes.

Certified that the thesis submitted by Mr S. A. Sreenivasan is an account of the research work carried out by him under my supervision and guidance at the Physical Research Laboratory, Ahmedabad.

K. R. Ramanathan

K. R. Ramanathan

Supervising Professor and Director, Physical Research Laboratory, AHMEDABAD

Dated: 26/1/59

SYNOPSIS

The thesis incorporates the results of an investigation into the distribution of electrons in the ionosphere. The study was carried out by the author under the supervision of Professor K.A.Ramanathan at the Physical Research Laboratory, Ahmedabad.

The investigation uses a rigorous method of evaluating the heights of reflection of radio-waves from the ionosphere by the application of Gaussian quadratures. The study is based on some of the ionograms obtained at the Radio Research Station of the laboratory during the years 1954 - 1956. The following studies are presented:

(a) Methods of determining the vertical distribution of electron-density in the ionosphere by evaluating the true geometric heights of reflection of radio waves from virtual height - frequency profiles.

(b) The quantitative derivation of the effect of the earth's magnetic field on the calculation of true heights of reflection of radio waves from the ionosphere over Ahmedabad and Delhi. The error in neglecting the effect of the geomagnetic field at Delhi serves as an upper limit to the error at all other Indian ionospheric stations.

(c) The distribution of electrons during some magnetically quiet days in the four seasons of the three years 1954-1956. The study includes the determination of the shape of the distribution, the quiet day type variation of parameters of

the distribution and the total electron content of the ionospheric regions in a column of unit cross section from the bottom of the region to the level of maximum electron density, finally the iso-electron density levels from the bottom of E-region to the level of maximum electron density of the F2 region.

(d) Ionospheric response to some magnetic storms during the year 1956. The perturbations induced in the variation of parameters, in the iso-electron density level patterns and the changes in the shape of the layer and in the total electron content in a column of unit cross-section are brought out.

(e) The distribution of electrons after correcting for the retardation produced by including the effect of the geomagnetic field is obtained to examine the nature of changes introduced. The procedure for obtaining the distribution corrected for the effect of the geomagnetic field is outlined.

The present investigation is a pilot study of the fine structure of the ionosphere of tropical latitudes by rigorous methods. It is believed that routine methods can now be applied on the lines outlined in the thesis to a larger mass of data and that it will lead to a better understanding of the physics of the ionosphere.

Acknowledgments to other workers have been made at appropriate places.

CHAPTER ONE

Methods of determining the electron-density distribution in the ionosphere

(ordinary ray)

CHAPTER ONE

Methods of determining the electron-density distribution in the ionosphere (ordinary ray)

1. Introduction

The height distribution of electron-density should necessarily include the determination of the heights of ionospheric regions, that is, the heights of reflection of radio waves. It is well known that there are three methods of doing this. Appleton (1928) has shown that all these methods measure essentially the same quantity, namely,

$$\int \frac{ds}{\mu} \qquad (1.1)$$

where μ is the phase refractive index at any point in the ionosphere and ds is an element of path-length of the atmospheric ray. We shall set out briefly the physical idea behind these methods and later describe the methods of studying the ionospheric electron-density distribution. The necessity for a rigorous treatment of the electron-distribution by more general and accurate methods of height determination will be explained. The derivation of the expression for the true height in terms of the virtual height and the operating frequency will be given. The accuracy and rigour imparted by Gaussian quadratures will be pointed out and this will be

followed by an outline of the actual procedure for getting the electron distribution. The limitations and criticisms of the method of quadratures will be recalled and answered. It should be noted that in this chapter and in chapter three the influence of the geomagnetic field has been neglected. The influence of the geomagnetic field has however been taken into account and the corrections to be applied are discussed separately in chapter two and the method of getting the corrected distribution will be outlined at the end of chapter three.

2. The three methods of height determination

One of the early methods to be used was the wave-length change method of Appleton and Barnett (1925). This method has been discussed by Appleton (1928) who used it with Barnett on transmissions from one of the B.B.C. stations at Bournemouth and the waves were received at Oxford. The method consists in measuring the variation in the number n of wave-lengths by which the atmospheric wave was behind the ground wave as revealed by the maxima and minima of the received signal current when the wave-length is varied over a small range. Hence, the name of the method.

The second method is again due to Appleton and Barnett (1925) who determined the angle of incidence of the downcoming radiowaves at Cambridge emitted by a B.B.C. transmitter at London. This method has also been used by Smith-rose and Barfield (1926). When the angle of incidence

is known the height of the ionospheric region could be obtained by triangulation. This method is therefore called 'angle of incidence method'.

A slight variant of this method was used by Mirick and Hentschel (1929) who observed the intensity of signals emitted by an aeroplane in flight and found that it underwent periodic fluctuations at a receiving point. Appleton (1929) has given a lucid explanation of the results of this experiment based on the Doppler shift in the frequency of the wave when the plane is moving with reference to an observing point and has shown that both these methods yield the angle of incidence of the wave at a point at the lower boundary of the region.

The third and now universally adopted method of sounding the ionosphere is due to Breit and Tuve.(1926) They sent up short pulses of mean frequency f and measured the time delay between the arrival of the waves coming via the atmosphere and of those coming over the ground. They found an expression for the apparent path of the atmospheric wave.

Appleton (1928) has discussed these experiments and has shown that all these experiments yield a measure of the same quantity

$$\int \frac{ds}{\mu} \qquad (1.1)$$

where ds is an element of path of the atmospheric ray and

μ is the phase refractive index at any point along the path. We shall point out the main physical idea behind all these experiments and state the assumptions made for deducing the expression for the height of reflection of the radio waves.

The experimental technique is to send up a radio pulse of frequency f and receive it at another point. There are two modes of propagation for the waves. One travels as a ground wave and the other goes up, gets refracted and arrives at the receiving point. This is illustrated in Fig. 1.

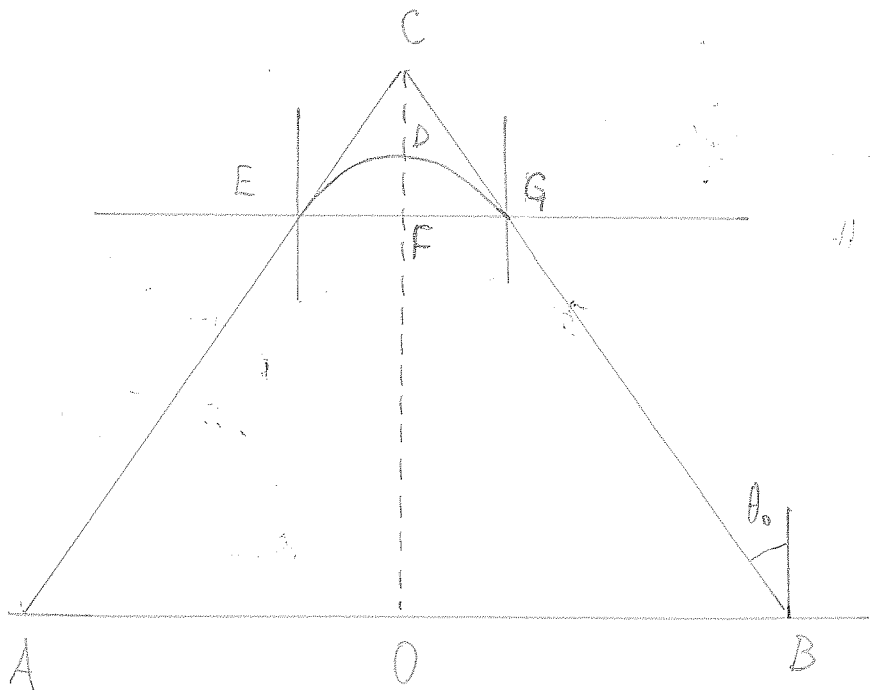


Figure 1

A wave is sent up from the transmitter at a point say A to the ionosphere. It gets bent and reaches a point B where it is received. Waves going along AOB and ADB interfere. The intensity at B will depend upon the path difference ADB - AOB. If it is equal to an integral number of wave-lengths the amplitude and intensity will be a maximum. But if it is equal to an integral multiple of $\lambda/2$ where λ is the wave-length, the intensity will be a minimum.

Assuming the ionosphere to be horizontally stratified, we can see that for a maximum of intensity path-difference P is given by the expression

$$P = n\lambda \qquad (1.2)$$

where λ is the wave-length in vacuum and n is an integer.

If μ is the refractive index at a point along the atmospheric path and ds is the element of path-length we have

$$P = \int_{AEDGB} \mu \, ds - AOB \qquad (1.3)$$

when the frequency (or the wave-length) changes over a small range it can be shown, following Appleton that the equivalent path-difference is equal to the product of the velocity in free space multiplied by the temporal retardation of a signal traversing the atmospheric path with the relevant group velocity as compared with a signal travelling along the ground.

Symbolically the path-difference is equal to

$$c \int \frac{ds}{u_g} - AOB \qquad (1.4)$$

When we neglect the effect of the magnetic field of the earth we know the well known expression for the group velocity is

$$u_g = c.\mu \qquad (1.5)$$

so that we have the equivalent path of the atmospheric wave as equal to

$$\int \frac{ds}{\mu} \qquad (1.1)$$

while the optical path would be $\int \mu \, ds$ (1.6)

Appleton has shown that one gets the same expression by the group retardation method. A point to be noted here is, as pointed out by Ratcliffe (Appleton 1928), that the component frequencies of the group do not all follow the same path. But Appleton has given a justification for the result on the analogy of the phenomenon of beats. A variational procedure thus applied gives the result that a maximum in the signal strength is produced again after a time

$$\int \frac{ds}{u_g} \qquad (1.7)$$

where u_g is the group velocity for the mean frequency. We can therefore conclude this section with the important result that all these three methods measure the same quantity

'the equivalent path of the atmospheric wave' =

$$\int \frac{ds}{\mu} \qquad (1.1)$$

We have now the optical path $P_o = \int \mu ds$ (1.6)

and the equivalent path $P_e = \int \frac{ds}{\mu}$ (1.1)

It can be shown that

$$P_e = P_o + f \frac{dP_o}{df} \qquad (1.8)$$

This follows from the expression for the relation between the phase velocity and the group velocity.

$$\frac{1}{u_g} = \partial/\partial f \left(\frac{f}{v_p} \right) \qquad (1.9)$$

Since $v_p = \frac{c}{\mu}$ we have on substitution,

$$\frac{1}{u_g} = \partial/\partial f \left(\mu f/c \right) = \frac{1}{c} \left(\mu + f \, \partial\mu/\partial f \right)$$

$$\therefore \quad c/u_g = \mu + f \, \partial\mu/\partial f$$

and $\int \frac{c}{u_g} ds = \int \mu ds + f \, \partial/\partial f \int \mu ds$

if $\int \frac{c}{u_g} ds$ is written as P_e and $\int \mu ds$ as P_o

we get $\quad P_e = P_o + f \frac{dP_o}{df} = d/df \, (P_o f)$

The optical path could be obtained on integration

$$P_o = \frac{1}{f} \int_0^f P_e \, df \qquad (1.10)$$

where $\underline{f}$ is the mean frequency of the waves. Since in an

ionized medium, μ is less than unity the equivalent path will be greater than the true path while the optical path will be less.

3. Vertical soundings and the ionogram

The method commonly used in the present day is the pulse-sounding method of Breit and Tuve. This consists in sending a pulse of mean frequency f vertically upwards and recording the time of its arrival at the ground after reflection. Assuming that the pulse travels vertically upward with a velocity c, the velocity of electromagnetic waves in free space, and returns after an interval Δt we have

$$P_e = c.\Delta t = 2h' \qquad (1.11)$$

where h' is the virtual height of the ionospheric region reflecting a pulse of mean frequency f. We shall denote this height by h'(f) in what follows. ~~The~~ By sending up in succession pulses of different frequency we get an ionogram giving the virtual height-frequency profile.

4. The refractive index of an ionospheric region

The refractive index of an ionospheric region which can reflect an electromagnetic pulse of mean frequency f could be derived on the lines suggested by Eccles (1912) and Larmor (1924) neglecting the influence of the geomagnetic field. This leads to the well known Sellmeyer dispersion formula

$$\mu^2 = 1 - \frac{4\pi N e^2}{m\omega^2} = 1 - \frac{f_N^2}{f^2} \qquad (1.12)$$

where N is the number of electrons per unit volume, e and m are the charge and mass of an electron in e.s.u., ω is the angular velocity ($2\pi f$) of the wave of frequency f which is reflected. We shall call f_N the plasma frequency of the region.

At the point of reflection we can set $\mu = 0$ so that we get an expression (when $f_N = f$)

$$N = (1.24 \times 10^4) . f^2 \qquad (1.13)$$

if we express N in electrons per c.c. and f in megacycles per second.

The formula for the refractive index is derived in text books on the Upper Atmosphere (for example the one by S.K.Mitra,1952) [illegible] However, We shall recall the assumptions made.

(i) The ionosphere could be regarded as a horizontally stratified medium. The electron density is a monotonic increasing function of height. The ray theory of geometric optics could be applied. (see Aydbeck 1940) This implies that the change of electron density within a wave-length is small in comparison with the dimensions of the inhomogeneities.

(ii) The effect of collisions of electrons could be neglected in most regions of the ionosphere. That is the collisional frequency is small compared with the angular frequency of the exploring wave. The only change in neglecting collisions would be that the calculated heights

may slightly be higher (see Rydbeck 1942).

(iii) The Lorentz polarisation term could be neglected. This is now a well accepted approximation and has been supported by the work of Darwin (1934, 1943), Ratcliffe (1939) and Beynon (1947) who have shown that the modification introduced by including the Lorentz polarisation term is not necessary in most considerations of wave propagation in the ionosphere.

(iv) The effect of geomagnetic field can be neglected to a first approximation. This is the most serious approximation and is not valid at stations which are not ~~very~~ close to the equator. The neglect of the field introduces a systematic error. This will be discussed ~~separately~~ in chapter two.

In our ~~an~~ initial work at Ahmedabad, we neglected the influence of the geomagnetic field. In ~~From~~ later calculations we determined ~~have found~~ the magnitude of the error. The error ~~This~~ is not serious in this latitude. ~~We have seen that the simple~~ The relation between the electron density N and the operating frequency f reduces the problem of finding the distribution of electrons to that of finding the heights at which pulses of different frequencies are reflected from a region.

5. Model methods

We shall now outline some of the earlier methods of determining the electron-distribution. The first attempt in

this direction was made by Appleton and Beynon (1940). This was immediately followed by the work of Booker and Seaton (1940) in America. In 1951, Ratcliffe gave a quick method of analysing ionospheric records, and in 1956, Beynon and Thomas outlined a slight extension of Appleton and Beynon's method. All these methods are what one may call "Model methods". ~~Since they~~ They presuppose some law of distribution of electrons with height ~~they are called model methods~~. Each is a slight variation of the Appleton-Beynon method and we shall treat them under one heading "Parabolic model methods".

It would be pertinent to inquire why the parabolic model was exploited. This ~~infact~~ came from Chapman's work. Chapman (1931) showed that if a monochromatic radiation were responsible for the ionisation of a plane iso-thermal atmosphere, then the rate of ionization of a constituent at an altitude h and at a solar zenith angle χ can be shown to be (Bates 1954)

$$I(x;h,\chi) = S\,A(x)\,n(x;h)\exp\left\{-A(x)\cdot H(x)\cdot n(x;h)\sec\chi\right\} \qquad (1.14)$$

where x is the constituent ionized, S is the incident photon flux in the spectral region concerned, A(x) is the ionization cross-section, n (x;h) is the particle concentration at the height h and H(x) is the scale height.

Chapman has shown that this rate is a maximum at a height

$$h_m(x;\chi) = H(x)\log\sec\chi + \text{a constant} \qquad (1.15)$$

where the particle concentration is $n_m(x;\chi) = \dfrac{\cos\chi}{A(x)\cdot H(x)}$ (1.16)

and this maximum value is given by

$$I_m(x;\chi) = \frac{S\cos\chi}{H(x)\cdot \exp 1} \qquad (1.17)$$

where exp1 is the Naperian base and is written in this way so that it may not ~~cause a confusion~~ be confused with the electronic charge.

One could make revised calculations by allowing for the finite radius of the earth. This has been done by Chapman and an auxiliary function called $Ch(\chi, x + h)$ ~~was~~ substituted for $\sec\chi$. This revision is ~~however an~~ important ~~except~~ when the sun is near or below the horizon. The theory could also be modified by taking the temperature gradient into consideration.

It is easier to study $n(e; h, \chi)$ the electron concentration than $I(x; h, \chi)$. This is related to the rate of ionization and the rate of disappearance of electrons which may be by recombination with positive ions or by attachment to neutral atoms. We have for the recombination type of loss the equation

$$\frac{dn(e;h,\chi)}{dt} = I(x;h,\chi) - \alpha n^2(e;h,\chi) \qquad (1.18)$$

where α = rate coefficient of recombination. As an approximate solution near noon when the electron concentration changes very slowly we have, if α is assumed constant,

$$n(e;h,\chi) = \left[\frac{I(x;h,\chi)}{\alpha}\right]^{\frac{1}{2}} \qquad (1.19)$$

and

$$n_m(e;\chi) = \left\{\frac{S}{\alpha\cdot H(x)\cdot \exp 1}\right\}^{\frac{1}{2}} \cdot \cos^{1/2}\chi \qquad (1.20)$$

Recent results (Ratcliffe and others 1956) have however shown that for the F2 region attachment is the more likely process and a similar result could be obtained for the attachment type of loss. In this case we have to use

$$dn/dt = I - \beta n \qquad (1.21)$$

Substituting (14) in (19) and expanding the exponentials in apower series, it can be shown after a slight simplification

$$n(\ell;h,\chi) \approx n_m(\ell;\chi)\left[1 - \frac{z^2}{4H^2(x)}\right] \qquad (1.22)$$

where $z = h - h_m(x;\chi)$

$\tau = 2H(x)$

where τ is the semithickness of an equivalent parabolic region.

The above equation to a parabola was the starting point for the model methods. (Appleton and Beynon 1940). It can be seen that except for the lowest end of the Chapman's distribution curve, the portion below the level of maximum electron density can be very well approximated to a parabola.

5a. Appleton and Beynon's work

For the sake of uniformity we shall define a parabolic distribution of electrons by the equation

$$(1 - N/N_m) = (1 - z/z_m)^2 \qquad (1.23)$$

where N is the number-density of electrons at a height Z measured from $z=0$: bottom of the region

above the ground measured from the base of the region, N_m is the maximum electron density of the parabolic region and Z_m is the height at which N has a maximum value. N stands for $(\pi m/e^2) \cdot f^2$ where m and e are the mass and charge on an electron in e.s.u. and $N_m = N_{max} = (\pi m/e^2) \cdot f_c^2$ where f_c is the critical frequency of the region or the highest frequency returnable from the ionized region.

Differentiating equation (1.23) we have,

$$\frac{dN}{N_m} = 2(1 - Z/Z_m) \cdot \frac{dZ}{Z_m}$$

i.e.

$$dZ = \frac{1}{2} \cdot \frac{Z_m}{(1 - N/N_m)^{\frac{1}{2}}} \cdot \frac{dN}{N_m} \qquad (1.24)$$

The refractive index in the field-free case is

$$\mu^2 = 1 - (Ne^2/\pi m f^2)$$

If we write $(Ne^2/\pi m f^2) = x = f_N^2/f^2$

where $\frac{Ne^2}{\pi m} = f_N^2$ or $f_N^2 = xf^2$

writing $f_c^2 = \frac{N_m e^2}{\pi m}$ where N_m = maximum electron density

we have $N/N_m = \frac{x f^2}{f_c^2}$

$$\therefore \frac{dN}{N_m} = \frac{f^2}{f_c^2} \cdot dx$$

We can rewrite equation (1.24) as

$$dz = \frac{1}{2} z_m \cdot (1 - x f^2/f_c^2)^{-1/2} \cdot (f^2/f_c^2) \cdot dx \qquad (1.25)$$

substituting this value of dz in

$$h' = h_0 + \int_{z=0}^{z_r} \frac{dz}{\mu}$$

or since $\mu \cdot \mu' = 1$ $\quad h' = h_0 + \int \mu' dz$

We define the group refractive index μ' as the free space velocity of light divided by the group velocity u_g of the wavepacket.

It is easy to see that μ' is related to μ, the phase refractive index by the general formula

$$\mu' = \partial/\partial f (\mu f) = \mu + f \, \partial\mu/\partial f$$

This relation follows from the formula for the group and phase velocities (1.9) u_g and v_p :

$$1/u_g = \partial/\partial f \cdot (f/v_p)$$

We therefore have for the virtual height the expression

$$h'(f) = h_0 + \int \mu' \left(\frac{\frac{1}{2} \cdot z_m \cdot (f^2/f_c^2)}{\sqrt{1 - (x f^2/f_c^2)}} \right) dx \qquad (1.26)$$

It can beseen that the integrand is not finite throughout the range of integration. It becomes infinite when $f_N = f_c$ In order to render the integrand finite throughout the range of integration one can make the transformation $x = 1 - t^2$ (Millington 1938).

We have therefore the equation (1.26) as

$$h'(f) = h_0 + z_m \cdot (f^2/f_c^2) \cdot \int_0^1 \frac{\mu' t}{(1 - x f^2/f_c^2)^{1/2}} \cdot dt \qquad (1.27)$$

Since $dx = -2t.dt$ and since we are dealing with the field-free case

$$\mu' = \frac{1}{\mu} = \frac{1}{(1-x)^{1/2}} = \frac{1}{(1 - f_N^2/f^2)^{\frac{1}{2}}} = \frac{1}{t} \qquad (1.28)$$

The expression reduces to

$$h'(f) = h_0 + z_m \cdot (f^2/f_c^2) \cdot \int_0^1 \frac{dt}{(1 - x f^2/f_c^2)^{1/2}} \qquad (1.29)$$

This expression could be integrated easily to give.

$$h'(f) = h_0 + \frac{1}{2} \cdot z_m \cdot (f/f_c) \log_e \left(\frac{f_c + f}{f_c - f} \right) \qquad (1.30)$$

This expression was derived by Appleton (1937) who showed that the parameter z_m of the parabolic distribution is twice the scale height of the atmosphere according to Chapman's theory and that one could express

$$N = N_m \left(1 - \frac{z^2}{4H^2} \right) \qquad (1.22)$$

where $H = kT/mg$ is the scale height of the atmosphere.

It seems proper at this stage to clarify the notation :

f = exploring frequency of the wave

f_N = any frequency which gets reflected from an ionospheric region whose density is N or the plasmafrequency of the region of density N

$\mu^2 = 1 - f_N^2/f^2$ = the square of the phase refractive index.

When $f = f_N$ the pulse is reflected ($\mu = 0$) and the electron

density in that region = $(\pi m/e^2) \cdot f^2$

f_c = the critical frequency or the frequency beyond which reflections do not occur from that region of the ionosphere. This is usually tabulated in all ionospheric data as f_o of that particular region meaning thereby that it is the ordinary wave critical frequency of the region specified by the first letter after f_o (for example : f_oE, f_oF1, f_oF2 etc.,).

We have from an ionogram - the virtual height - frequency profile - the following information :

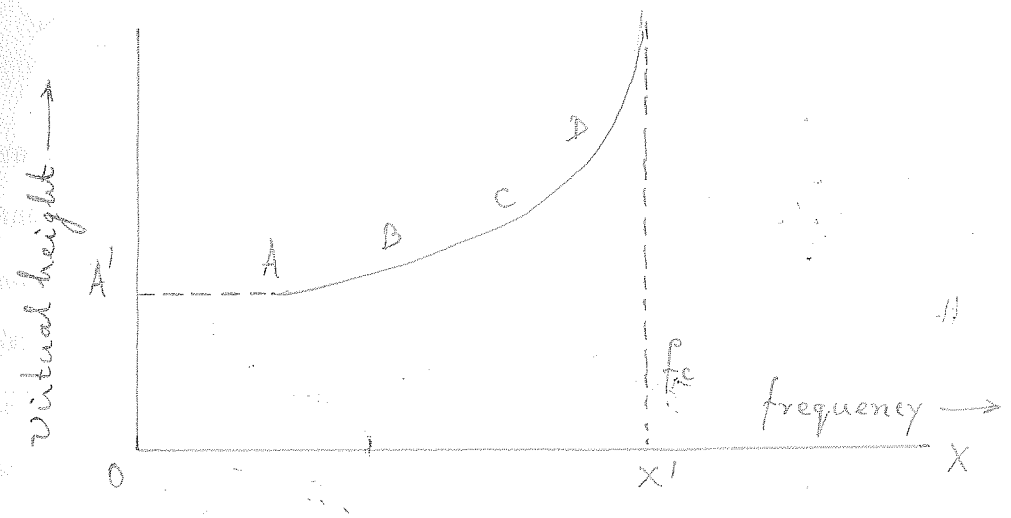


Fig. 2

(a) The minimum virtual height (OA' in figure above). The lowest height recorded in the ionosonde is called minimum virtual height - denoted by h'_{min} -

(b) The critical frequency of the region (OA' in above figure) f_c

The tangent to the h' - f profile at the right end goes off vertically that is the virtual height of reflection at the

critical frequency f_c is infinite (infinite retardation). The tangent is asymptotic to the h' - f profile. The wave penetratesthe layer and does not return downward. From the ionogram we can tabulate a set of values h'_1, f_1 ; h'_2, f_2 ; etc. for the points A, B, C, ... along the profile. The critical frequency f_c being known for that particular region, one could compute the values f_1/f_c, f_2/f_c, f_3/f_c,.....,f_i/f_c .. and also the function

$$\frac{1}{2}\cdot(f/f_c)\cdot \log_e\left(\frac{f_c+f}{f_c-f}\right)$$

If we put $f/f_c = \xi$ and

$$\psi(\xi) = \psi(f/f_c) = \frac{1}{2}\cdot(f/f_c)\cdot\log_e\left(\frac{f_c+f}{f_c-f}\right) = \frac{1}{2}\cdot(f/f_c)\log_e\left[\frac{1+f/f_c}{1-f/f_c}\right] \qquad (1.31)$$

we have a linear relation between h'(f) and $\psi(\xi)$ namely,

$$h'(f) = h_0 + \tau\,\psi(\xi)$$

and the h' - $\psi(\xi)$ plot would therefore be a straight line. This result was first given by Appleton and Beynon (1940) Let us examine the function $\psi(\xi)$ and the relation (1.30). The intercept on the height-axis made by the straight line

$$h' = h_0 + \tau\cdot\psi(\xi)$$

would give us the minimum height of reflection of the layer (theoretically at f = 0) or the <u>TRUE base of the layer</u>. This in the case of the lowest layer would be the length of the unionized column of air. The slope of the line would give us the semi-thickness of the layer z_m. This is the difference between the height of maximum electron density and the base of the layer. We shall denote, in what follows, the semi-thickness by $\tau = (h_m - h_0)$.

5b. Booker and Seaton's work

Booker and Seaton (1940) have examined this suggestion in detail and given a table of values of the function

$$\frac{1}{2}\left(f/f_c\right)\cdot \log\left(\frac{f_c+f}{f_c-f}\right) - 1 \qquad (1.32)$$

They call it $\varphi(\xi)$ where $\xi = f/f_c$. Booker and Seaton use x instead of ξ but we have used x for fN^2/f^2 and ξ for f/f_c The above function is similar to our function $\psi(\xi)$ which is $1 + \varphi(\xi)$. Booker and Seaton have tabulated $\phi(\xi)$ when ξ goes from 0 to 2. It is interesting to see the values of $\phi(\xi)$ for some values of ξ. At a frequency 0.834 f_c, the virtual height is equal to the height of maximum electron density on the parabolic model. This is what is usually tabulated as $h_p = h'(0.834\ f_c)$ by ionospheric observatories. The following are some of the values of $\varphi(\xi)$ tabulated by Booker and Seaton (1940).

ξ	0	0.648	0.725	0.757	0.834	0.887	0.901	0.925	0.969
$\varphi(\xi)$	-1	$-1/2$	$-1/3$	$-1/4$	0	$1/4$	$1/3$	$1/2$	1

Table 1

The virtual height according to Booker and Seaton can be written as

$$h'(f) = h_m + T\cdot\varphi(\xi)$$

If we take $f = 0.648\ f_c$ and $f = 0.925\ f_c$ we have

$$h'(0.648\ f_c) = h_m - \tfrac{1}{2}T$$
$$h'(0.925\ f_c) = h_m + \tfrac{1}{2}T \qquad (1.33)$$

so that $\tau = h'(0.925\ f_c) - h'(0.648\ f_c)$.

Similarly $h'(0.834\ f_c) - h_o$ will again give τ

We have also $h_m = \frac{1}{2}(h' \text{ at } 0.925\ f_c + h' \text{ at } 0.648\ f_c)$.

Booker and Seaton have given an expression for reading h_m and τ from heights at more than two frequencies. These could be used for routine scaling of the records.

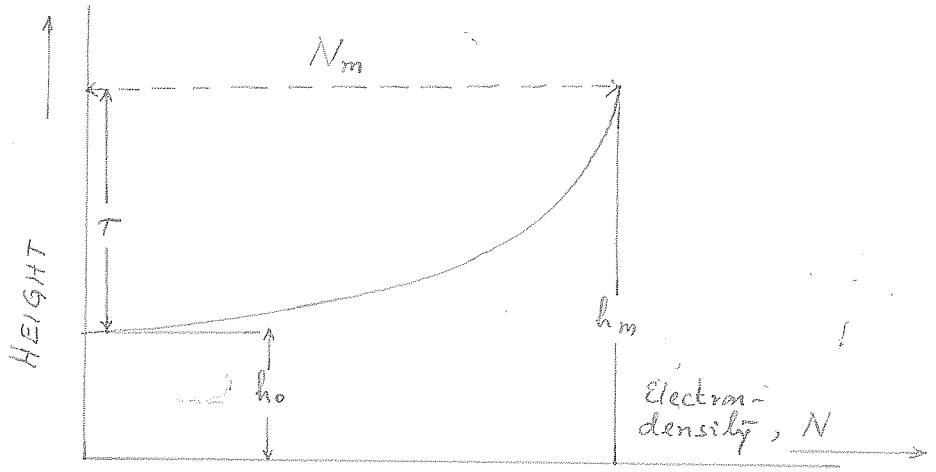


Fig. 3

We see that the Appleton-Beynon method together with the Booker-Seaton method provides us the basic parameters of the distribution h_o, h_m, τ, and N_{max}, on the assumption that the distribution with height is parabolic and that the effect of the geomagnetic field could be neglected.

There are some advantages as well as disadvantages in these methods.

(i) The Appleton-Beynon method while giving us the parameters when one fits a straight line to the observed points $(h', \varphi(\zeta) \text{ or } \psi(\zeta))$, also tells us whether there is any appreciable deviation from parabolicity of these regions. One observes that even at low latitudes where the correction

due to the effect of the geomagnetic field is quite small there are deviations from parabolicity. These were found to be few at Ahmedabad when one had allowed for the retardation in all the lower regions. The method of Booker-Seaton does not give us an opportunity to see the deviations from parabolicity. These are only minor differences and two methods are to be regarded as substantially similar. They provide a valuable means of studying the parameters of the distribution without much loss of generality at low latitudes. The drawback of the model methods apart from the lack of generality is that they are applicable only to single layers on quiet days during night hours. During the day, retardation due to the passage of the wave through lower regions is considerable and one will have to allow for this retardation and subtract the group retardation due to the lower regions before proceeding to further analysis.

5c. The quick method of Ratcliffe

Ratcliffe (1951) gave a few years ago a method for the routine scaling of ionograms. The analysis consists in constructing a series of h' - f curves on the assumption of parabolic distribution of electron density for a given set of semi-thicknesses. By matching these with the observed h' - f curves one can pick out the appropriate semi-thickness, minimum height h_0 or the height of maximum electron density. This is possible since we know the value of the function

$$\psi(\xi) = \xi/2 \cdot \log_e \left(\frac{1+\xi}{1-\xi}\right) \qquad (1.31)$$

for various values of ξ from Booker and Seaton's (1940) table and for different semi-thicknesses τ one can evaluate

$$h'(f) = h_0 + \tau \cdot \psi(\xi) \qquad (1.30)$$

for any defined value of h_0. Ratcliffe [illegible] used this method to analyse a number of ionograms at Huancayo. A useful result obtained by Ratcliffe (1951) was an expression for the retardation due to the passage through a lower region. We know that the formula

$$h'(f) = h_0 + \tau \cdot \tfrac{1}{2}(f/f_c) \cdot \log_e\left(\frac{f_c+f}{f_c-f}\right) \qquad (1.30)$$

is true for all values of f and f_c. It gives the virtual height of reflection for a wave of frequency f less than f_c. If f is larger than f_c we know that the wave will penetrate the region and get reflected from a higher region which lies beyond the height of maximum electron density of that region. The equivalent height of reflection in the latter case will therefore be much larger than when no layer is present below. The difference in the equivalent heights will therefore be the retardation in the half parabola below. If we denote the increase in virtual height by $\Delta h'$ then,

$$(1.34) \qquad \Delta h' = \tau \cdot \left[\tfrac{1}{2} \cdot (f/f_c) \cdot \log_e\left(\frac{f_c+f}{f_c-f}\right) - 1 \right]$$

or to use Booker-Seaton's notation $\Delta h' = \tau \cdot \varphi(f/f_c)$

$$\left\{ \text{Note that } \psi(\xi) = 1 + \varphi(f/f_c) \right\} \qquad (1.35)$$

This group retardation formula is true when the pulse has to

pass through a semi-parabolic distribution completely. One can also get an expression for the retardation when it passes through a part of a parabolic region (as is usually the case when the F1 just starts appearing in the morning or starts disappearing in the evening). This expression was also given by Ratcliffe (1951).

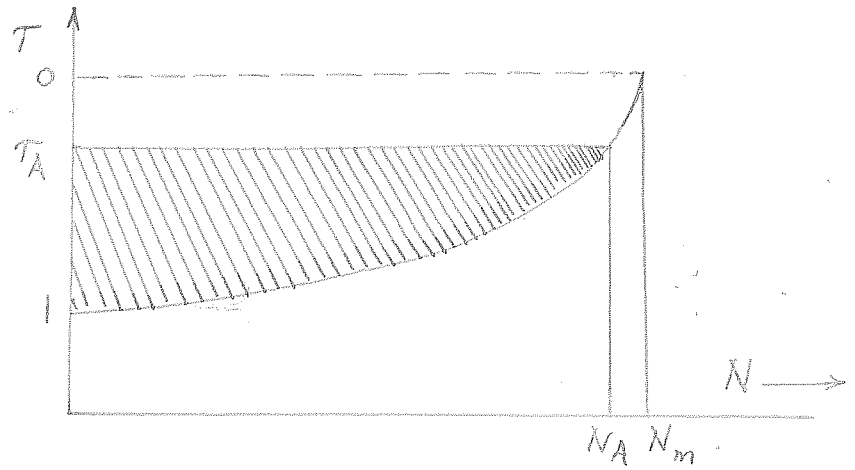


Figure 4

If the pulse has to pass through part of a parabola such as is given in the figure above the retardation could be obtained as follows :

The shaded portion will be given by a distribution function

$$N = N_m (1 - \tau^2)$$

within the limits $\tau = 1$ and $\tau = \tau_A$

The refractive index neglecting the effect of the geomagnetic field is given by

$$\mu^2 = 1 - \frac{Ne^2}{\pi m f^2}$$

where $N = N_m (1 - \tau^2)$

that is

$$\mu^2 = 1 - \frac{f_c^2}{f^2} (1 - \tau^2)$$

slightly modifying this equation we have,

$$\mu = (f_c/f)\cdot\sqrt{\left(\tau^2+(f/f_c)^2-1\right)}$$

The height traversed by a pulse of frequency f greater than

$$\sqrt{(e^2/\pi m)\cdot N_A}$$

(that is where it traverses the shaded portion of thickness $1-\tau_A$)

$$\Delta h' = \int_{\tau=1}^{\tau=\tau_A} \frac{d\tau}{\mu}$$

$$\therefore \quad \Delta h' = \int_{\tau=1}^{\tau=\tau_A} \left(\frac{f}{f_c}\right)\cdot\left\{\tau^2+(f/f_c)^2-1\right\}^{-1/2}\cdot d\tau$$

This could be integrated directly to yield

$$\Delta h' = (f/f_c)\cdot \log_e\left\{\frac{1+f/f_c}{\tau_A+\sqrt{\tau_A^2+(f/f_c)^2-1}}\right\}$$

putting f_A as the frequency which penetrates the shaded area; since $\mu = 0$ at the level τ_A where it gets reflected we have

$$0 = 1-(f_c/f_A)^2\cdot(1-\tau_A^2)$$

$$\text{or} \qquad \tau_A^2 = 1-f_A^2/f_c^2$$

so that

$$(1.36) \qquad \Delta h' = (f/f_c)\cdot\log_e\left\{\frac{(1+f/f_c)}{\sqrt{1-f_A^2/f_c^2}+\sqrt{(f/f_c)^2-(f_A/f_c)^2}}\right\}$$

Another idea suggested in Ratcliffe's analysis is the calculation of the total electron-content in a unit

column of the ionosphere. For a parabolic distribution the total electron-content would be

$$\frac{2}{3} \cdot T \cdot (\text{Maximum electron density}).$$

Since we can find T we can get

$$\frac{2}{3} \cdot T \cdot (1.24 \times 10^4) f_c^2 \qquad (1.37)$$

This is expressed as the number of electrons per column of unit cross section. Sheriff (1956) has analysed some Ahmedabad ionograms according to this method. Since we are dealing with only methods in this chapter we shall postpone the discussion of the results of Ratcliffe and Sheriff to a later chapter.

5d. Beynon and Thomas' method

A useful extension of the Appleton-Beynon method has recently been suggested by Beynon and Thomas (1956). This extends the Appleton-Beynon method to cases of reflections from contiguous parabolic sections and they have given an expression for the variation of virtual height with frequency in terms of the layer parameters. We shall illustrate this by a diagram below.

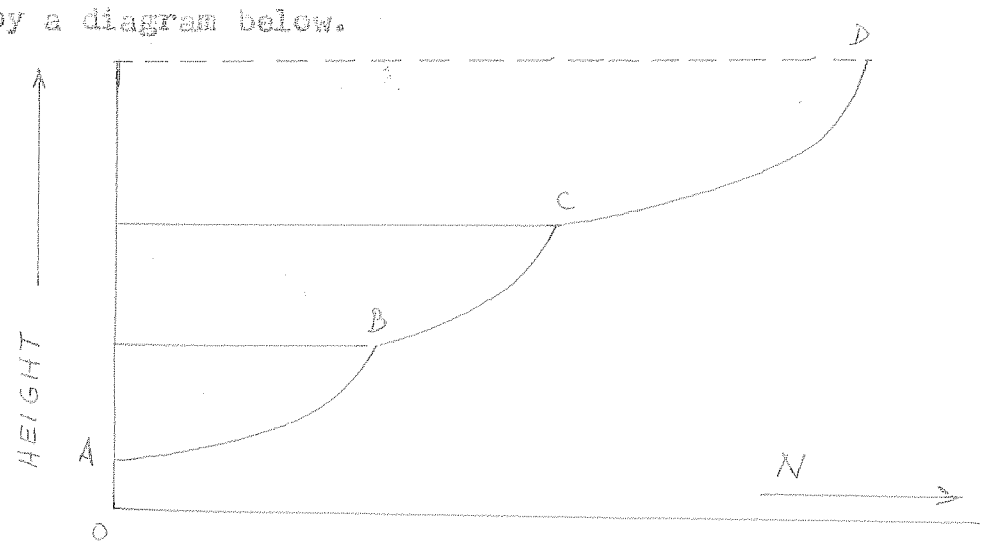


Figure 5

They consider the region consisting of three sections say E, F1 and F2. Region E is regarded as a parabola truncated on the upper side while F2 is treated as a parabola truncated below by F1. They obtained an expression for what they call a semi-equivalent path or the contribution to the virtual height of a signal passing through a parabolic region truncated on either side in the form

$$\Delta h' = \tau\xi \left\{ \text{arc sinh}\left(\frac{\tau'}{\tau\sqrt{\xi^2-1}}\right) + \text{arc sinh}\left(\frac{\tau''}{\tau\sqrt{\xi^2-1}}\right)\right\} \qquad (1.38)$$

where $\xi = f/f_c$ in our notation and τ = semi-thickness of the parabolic region and the ratios τ'/τ and τ''/τ are the degrees of truncation of the parabolic region on its lower and upper sides respectively.

It may be noted that the expression of Appleton and Beynon

$$h' = h_0 + \tau \cdot \tfrac{1}{2}\xi \log_e\left(\frac{1+\xi}{1-\xi}\right)$$

which we have derived earlier can be expressed as

$$\Delta h' = h' - h_0 = \tau\xi \cdot \text{arc tanh}\,\xi$$

since

$$\tfrac{1}{2}\log_e\left(\frac{1+\xi}{1-\xi}\right) = \text{arc tanh}\,\xi$$

This expression is a special case of the Beynon-Thomas expression for a single parabola truncated below and is got by putting $\tau' = \tau$. In the most general case when there are three regions E, F1 and F2 the expression is given by

$$\Delta h' = \tau_E \xi_E \left[\text{arc sinh}\left(\frac{1}{\sqrt{\xi_E^2 - 1}}\right) + \text{arc sinh}\left(\frac{\tau''_E}{\tau_E \sqrt{\xi_E^2 - 1}}\right)\right]$$

$$+ \tau_{F_1} \xi_{F_1} \left[\text{arc sinh}\left(\frac{\tau'_{F_1}}{\tau_{F_1} \sqrt{\xi_{F_1}^2 - 1}}\right) - \text{arc sinh}\left(\frac{\tau''_{F_1}}{\tau_{F_1} \sqrt{\xi_{F_1}^2 - 1}}\right)\right]$$

$$+ \tau_{F_2} \xi_{F_2} \left[\text{arc cosh}\left(\frac{\tau'_{F_2}}{\tau_{F_2} \sqrt{1 - \xi_{F_2}^2}}\right)\right] \qquad (1.39)$$

This expression holds when the situation is as shown in figure 5. The expression for the group retardation obtained by Ratcliffe (1951) can be derived from the expression of Beynon and Thomas if we put $\tau' = \tau = 1$ and $\tau'' = -\tau_b$ so that we have

$$\Delta h' = \xi \left\{ \text{arc sinh}\left(\frac{1}{\sqrt{\xi_E^2 - 1}}\right) - \text{arc sinh}\left(\frac{\tau_b}{\sqrt{\xi_E^2 - 1}}\right)\right\} \qquad (1.40)$$

The identity of this expression and Ratcliffe's

$$\Delta h' = (f/f_c) \cdot \log_e \left\{ \frac{1 + f/f_c}{\tau_b + \sqrt{\tau_b^2 + (f/f_c)^2 - 1}} \right\}$$

follows by recalling that $f/f_c = \xi$ and arc sinh $\xi = \log_e (\xi - \sqrt{\xi^2 + 1})$.

This method of Beynon and Thomas is an interesting extension of the original Appleton-Beynon method and could be used in places where the correction to the geomagnetic effect is very small. But this consumes as much time as the no-field method of Kelso and hence Kelso's method seems preferable since it does not presuppose parabolicity of regions.

Recapitulating the results;

(a) All the experiments point to the measurement of the same quantity $\int ds/\mu$ - the virtual height of the

ionospheric region which returns a pulse of frequency f. μ is a function of the operating frequency f and the plasma frequency f_N when the effects of electron collision, Lorentz polarization and the effect of geomagnetic field are neglected and the ray treatment of geometrical optics is applied.

(b) The ionospheric parameters could be studied after the method of Appleton-Beynon developed further by Booker and Seaton and by Ratcliffe. All these methods rest on the same assumption that the distribution of electron density is a parabolic function of height. These methods allow for the group retardation of a pulse which penetrates a lower region to get reflected from a higher one. Besides the usual parameters h_o, h_m and τ one could get an idea of the total electron content in a unit column of the ionosphere hereafter designated n_T and the nature of variation of these quantities can be studied quickly.

6. Integral methods

The urge for finding integral methods emanates from two serious drawbacks of model methods. They are

(i) The <u>apriori</u> assumption of a parabolic distribution of electron density.

(ii) The neglect of the influence of the earth's magnetic field on the distribution and heights of ionized regions.

We shall first discuss (i); the method of overcoming the second difficulty will be dealt with under a separate head.

We have seen that the virtual height is given by the expression

$$h' = \int \mu' \, ds \qquad (1.41)$$

where μ' is the group refractive index. We shall first develop the integral method neglecting the effect of the earth's magnetic field. μ' is then a function of the exploring frequency f and the plasma frequency f_N. We can write the equation as

$$h'(f) = \int_{z=0}^{z_r} \mu'(f, f_N)\, dz = \int \mu'(f, f_N) \cdot \frac{dz}{df_N} \cdot df_N \qquad (1.42)$$

where z_r is the height of reflection and $\mu' = \dfrac{1}{\sqrt{1 - f_N^2/f^2}}$

We have from the ionospheric records the virtual height at a number of frequencies f_i. The refractive index, since it involves only f and f_N, is also known. What we do not know is the function (dz/df_N). The unknown quantity (dz/df_N) occurs inside the integral sign. In model methods a law for the function (dz/df_N) is assumed. But the problem can be treated without any assumption as to the nature of (dz/df_N) except for the condition that it should be a monotonic increasing function of height. This assumption has to be allowed on both physical and mathematical grounds. Physically one can see that it is impossible to measure electron densities beyond the height of the first maximum of electron density by the aid of radio reflections. An exploring frequency greater than the highest plasma frequency of that region (the first maximum of density) will not be returned since it will

penetrate the layer indicating that the region immediately above the height of first maximum has lesser density. The only means of knowing electron densities beyond the first maximum of the region is by the aid of rockets and artificial earth satellites or from extra terrestrial sources of radiation. This is beyond the scope of the present thesis and hence will not be discussed here. Mathematically, the inversion of the integral is valid only if dz/df_N is assumed single valued and monotonically increasing. Under this restricting assumption it is fairly straightforward to solve the integral equation. Following Abel who treated a dynamical problem of a similar nature (1881) and de Groot (1930) who pointed out that this was essentially the same problem one can give the solution. Abel found out the curve in which the time of descent of a particle to the lowest point. is a given function of the vertical height of descent. The solution was first given, in the ionosperic case by de Groot (1930) and Appleton (1930). The same problem has been treated by Aydbeck (1942) Pekeris (1940) and Manning (1947). The solution could be arrived in two slightly different ways. We shall briefly indicate the solution. We have

$$h' = \int_0^{z_r} \mu' \, dz \qquad z_r = \text{height of reflection}$$

and $$\mu' = \frac{1}{\sqrt{1 - f_N^2/f^2}}$$

h' could be written as $$= \int_0^{f} \mu' \cdot (dz/df_N) \cdot df_N$$

$$= \int_0^{z_r} \frac{dz}{(1 - f_N^2/f^2)^{\frac{1}{2}}} = \int_0^{f} \frac{1}{(1 - f_N^2/f^2)^{\frac{1}{2}}} \cdot \frac{dz}{df_N} \cdot df_N$$

$$= \int \frac{f}{(f^2 - f_N^2)^{\frac{1}{2}}} \cdot \frac{dz}{df_N} \cdot df_N$$

If h_0 is the height of the unionized column

$$P' = 2h' = 2h_0 + 2\int \frac{f}{(f^2 - f_N^2)^{\frac{1}{2}}} \cdot \frac{dz}{df_N} \cdot df_N$$

Let us rewrite $(P' - 2h_0)/f$ as $\Phi(f)$ and dz/df_N as $G(f_N)$ we then have

$$\Phi(f) = 2\int_0^{f} \frac{G(f_N) \cdot df_N}{\sqrt{f^2 - f_N^2}} \qquad (1.43)$$

This is an Abelian integral equation and could be inverted (Abel 1881) to give

$$G(f_N) = dz/df_N = \frac{d}{df_N} \frac{2}{\pi} \int_0^{f_N} \frac{\Phi(f)\, df}{\sqrt{f_N^2 - f^2}}$$

Integrating both sides with respect to f_N we have

$$z(f_N) = \frac{2}{\pi} \int_0^{f_N} \frac{\Phi(f) \cdot df}{(f_N^2 - f^2)^{\frac{1}{2}}} \qquad (1.44)$$

Realising that $\Phi(f)$ is the virtual height function we have the expression for the true height of reflection where the

plasma frequency is f_N as

$$\frac{2}{\pi}\int_0^{f_N} \frac{\text{Virtual height}}{\sqrt{f_N^2 - f^2}} \cdot df$$

One can make the transformation $\sin\theta = (f/f_N)$ since f is less than f_N and get by simple change of variables the expression

$$\text{True height at } f_N = h_0 + \frac{2}{\pi}\int_0^{\pi/2} h'(f_N \sin\theta) \cdot d\theta \qquad (1.45)$$

6b. The method of Laplace-transforms

To derive the result

$$Z(f_N) = h_0 + \frac{2}{\pi}\int_0^{f_N} \frac{h'(f)\, df}{(f_N^2 - f^2)^{\frac{1}{2}}} \qquad (1.44)$$

we shall start with the three fundamental relations (1) expressing the refractive index in terms of the electron density and operating frequency, (2) the relation that the distance traversed is velocity times the time interval for propagation and lastly (3) the definition of group refractive index. (Manning (1947).

$$\mu_g = \mu' = \frac{c}{u_g} \qquad (1.46)$$

$$\mu' = \frac{1}{\left(1 - \frac{kN}{f^2}\right)^{\frac{1}{2}}} \qquad (1.47)$$

where $k = \left(\frac{e^2}{\pi m}\right)$.

$$\text{velocity} = \frac{dz}{dt} \qquad (1.48)$$

The time taken by a wave of velocity say v(z) to reach a height z is

$$t = \int_0^z \frac{dz}{\text{velocity}} \qquad (1.49)$$

Substituting from (46) & (47) in (49) we have

$$t = \frac{f}{c} \int_0^z \frac{dz}{(f^2 - kN)^{1/2}}$$

defining virtual height as c.t we have

$$h' = f \int_0^{z_m} \frac{dz}{(f^2 - M)^{\frac{1}{2}}} \qquad (1.50)$$

where M = kN

under the assumptions we have made in the previous case that height z is a single valued or monotonic increasing function of N (or kN = M). It is to be noted here that M is a function of z. Since a wave of frequency f goes to a height z, z is an inverse function of f or f^2 :- $z = M^{-1}(f^2)$. Equation (50) is based on the assumption that to each height z there corresponds a single value of electron-density. The solution contends that to each value of electron-density there corresponds a single height so that we have z(N) to represent the electron distribution. The main shortcoming of this procedure is that although N(z) is necessarily single valued, z(N) is not necessarily so. We have as our solid rock the fact that according to physical considerations it is impossible to determine exactly the electron distribution beyond the first maximum. It is therefore valid to assume that the function z(N) upto the first maximum is monotonic

Since at the point of reflection $M = f^2$ the single valued function obtained by taking the least value of $z(M)$ will therefore be the true height function. We shall make certain tranformations of variables for the sake of convenience.

Putting $g = f^2$ and $P(g) = h'(f)/f$ and changing the variable of integration to M, we write $z = z(M)$ and $dz/dM = Z'(M)$. When z is the height of reflection $M = g = f^2$ and when $z = 0$ $M = 0$. The equation (1.50) is transformed to

$$P(g) = \int_0^g \frac{Z'(M)}{(g-M)^{\frac{1}{2}}}\, dM \qquad (1.51)$$

This is a special case of Abel's integral equation.

Putting $P_1(g) = g^{-\frac{1}{2}}$ and $Z'(M) = P_2(M)$ the integral becomes

$$P(g) = \int_0^g P_1(g-M) \cdot P_2(M) \cdot dM \qquad (1.52)$$

This is what is called a 'Faltung integral' in the language of operational calculus. We define the Laplace transform of a function $P(g)$ as $p(s)$ where

$$p(s) = L[P(g)] = \int_0^\infty P(g) \cdot e^{-sg} \cdot dg.$$

Let

$$p(s) = L[P(g)]$$

$$p_1(s) = L[g^{-1/2}]$$

$$p_2(s) = L[Z'(g)]$$

$$p_1(s) = \int_0^\infty g^{-1/2} \cdot e^{-sg}\, dg$$

on making the substitution $x = sg$ $\quad dx = s.dg$

$$\rho_1(s) = \int_0^\infty (x/s)^{-1/2} \cdot e^{-x} \cdot s^{-1} \cdot dx$$

$$= s^{-1/2} \int_0^\infty x^{-1/2} \cdot e^{-x}\, dx$$

$$= s^{-1/2} \int_0^\infty x^{1/2-1} \cdot e^{-x}\, dx$$

Since $\int_0^\infty e^{-x} \cdot x^{1/2-1} \cdot dx$ is a Γ-function $= \Gamma(\frac{1}{2}) = \sqrt{\pi}$

we have $\quad \rho_1(s) = (\pi/s)^{1/2}$

Let us take $\quad \rho_2(s) = \int_0^\infty Z'(g) \cdot e^{-sg}\, dg$

integrating by parts we have

$$\rho_2(s) = \left[Z(g) \cdot e^{-sg} \right]_0^\infty - \int_0^\infty Z(g) \cdot (-s)\, e^{-sg}\, dg$$

$$= s \int_0^\infty Z(g) \cdot e^{-sg} \cdot dg + \left[0 - Z(0) \right]$$

But $\int_0^\infty Z(g) \cdot e^{-sg} \cdot dg$ = Laplace transform of $Z(g)$

$= \zeta(s)$, say

$$\therefore \quad \rho(s) = \rho_1(s) \cdot \rho_2(s)$$

$$\rho(s) = (\pi/s)^{1/2} \left[s\zeta(s) - Z(0) \right]$$

or solving for $\zeta(s)$ we have,

$$\zeta(s) = (\pi/s)^{1/2} \cdot \frac{1}{\pi} \cdot \rho(s) + \frac{1}{s} \cdot Z(0)$$

Reverting to the procedure from which we took the Laplace transforms, we have

$$Z(M) = \frac{1}{\pi}\int_0^M \frac{P(g)}{(M-g)^{\frac{1}{2}}}\, dg + Z(0) \qquad (1.53)$$

We shall go back to the old notation - recalling $P(g) = h'(f)/f$ and $g = f^2$, $dg = 2f.df$ and when $g = f^2 = M$, $f = \sqrt{M}$

$$Z(M) = \frac{2}{\pi}\int_0^{\sqrt{M}} \frac{h'(f)}{(M-f^2)^{\frac{1}{2}}}\, df + Z(0) \qquad (1.54)$$

The only assumption we have made in deriving this result is that $M(z)$ is a single valued function. This means that the virtual height frequency curve should be bounded. This could be relied on because, as we shall show later, we do not need the virtual height at f = critical frequency to get the true height at the critical frequency. We need only the virtual height at $0.9878\, f_c$. This will be shown when we come to the practical procedure for getting the true height. Nevertheless, it is true that at $f = f_c$, $h' = \infty$.

We now get another interesting similarity between the forms of the expression for virtual height and true height. While we have

$$h'(f) = \int_0^{f^2} \frac{Z'(M)}{(1 - M/f^2)^{1/2}}\, dM \qquad (1.55)$$

we get
$$Z(f_N) = \frac{2}{\pi}\int_0^{f_N} \frac{h'(f)}{(f_N^2 - f^2)^{\frac{1}{2}}} \cdot df + Z_0 \qquad (1.56)$$

These two form a function-transform pair between the virtual height function and the distribution function. It would be appropriate to use the term 'Manning-transform' for this operation.

6c. Procedure for the calculation of true heights from the above results

Several procedures have been suggested for the application of the formula

$$\text{True height for a frequency } f_N = \frac{2}{\pi}\int_0^{f_N} \frac{h'(f)}{\sqrt{f_N^2 - f^2}} \cdot df \qquad (1.44)$$

Manning and Pekeris suggested planimetric integration. Kelso taking his inspiration from Professors Kraus and Friedman of New York University's Institute of Mathematical Sciences (1950), first made a systematic approach to integration by numerical methods. The method consists in applying Gaussian quadrature to the integral as it cannot be solved by analytical methods exactly. Kelso uses Gauss-Christoffel quadratures to evaluate the integral by numerical analysis (1952). But Mehler (1864) - see also Kopal (1955) - had given the solution to this particular class of functions long ago.

We shall briefly discuss the technique of Gaussian quadratures to bring out the range of validity of such

methods. This has been questioned by some workers notably Manning himself (1955). Shinn (1955) and Kraus (1955) have given an explanation which seems adequate enough but we shall go over the ground here in order to emphasize the basic soundness of Gaussian quadrature and to point out the rigour and exactness which an analytical method, if it were available would have given.

The purpose of Gaussian quadratures is to approximate the definite integral of a function by a certain number of finite summations. In other words to replace

$$\int_a^b f(x)\,dx \quad \text{by a sum} \quad \sum_{j=1}^{n} H_j \cdot f(a_j)$$

This method has its foundations in a theorem due to Weierstrass (1903) that any function which is continuous can be approximated within an interval (a,b) by a polynomial of sufficiently high degree to any accuracy. This can be done without loss of generality at least as long as (a,b) is finite. A polynomial of degree m is specified by m + 1 constants uniquely. The right hand side of equation

$$\int_a^b f(x)\,dx = \sum_{j=1}^{n} H_j \cdot f(a_j) \qquad (1.57)$$

contains 2n arbitrary constants - the n a_j's and the n H_j's - which is adequate for the complete specification of the definite integral of a polynomial of degree 2n - 1. In other words the degree of precision of the quadrature formula is 2n - 1.

Hence if the above equation is to be exact, the degree m of the polynomial approximating f(x) should not be

greater than 2n - 1. If m = 2n - 1 none of the a_j's and H_j's will be arbitrary. The quadrature becomes approximate only if m is greater than 2n - 1. Geometrically it means the following :

The integral on the left hand side represents an area, being the definite integral included between the ordinates at x = a and x = b, the x - axis and the curve f(x) of say $(2n - 1)^{th}$ degree. On the right hand side the summation contains n discrete values of $f(a_j)$ whose arguments are not so far specified. Whatever be the arguments it is always possible to pass through these n points a power polynomial of degree (n - 1). Let us call it $F_{n-1}(x)$ (see figure below).

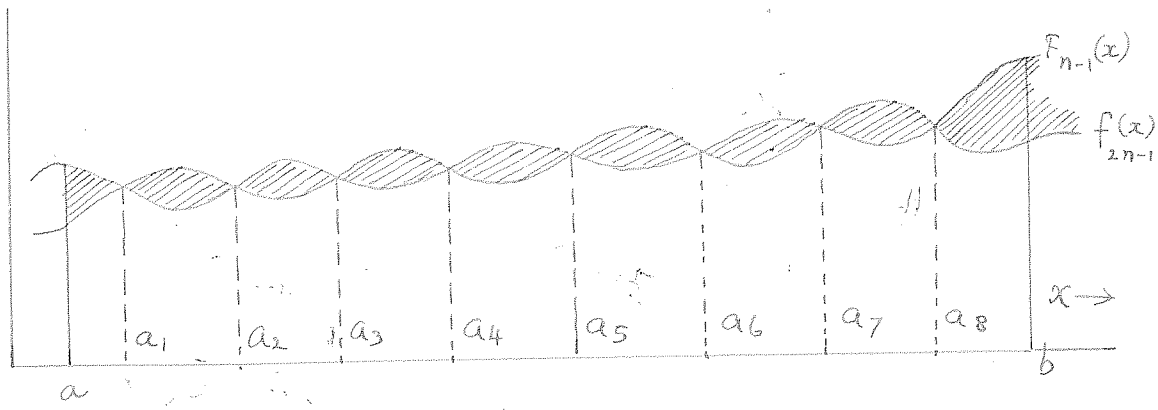


Figure 6

This area which is delimited as shown in the figure could be expressed always as a certain weighted mean of n discrete values of $f(a_j)$'s. The two functions F_{n-1} and f_{2n-1} coincide at n discrete values of x but will differ between these points It is indeed possible to adjust the values of the a_j's so that the areas subtended by both the curves between x = a and

$x = b$ are identical provided the a_j's are so chosen that the shaded area above the curve $f_{2n-1}(x)$ cancels that below it in the interval. This can be shown as follows :

If $F_{n-1}(x)$ is an arbitrary polynomial of x of $(n - 1)$th degree then an arbitrary polynomial of x of degree $2n - 1$ coinciding with $F_{n-1}(x)$ at n points $a_1, a_2, \ldots, a_j, \ldots, a_n$ is

$$f_{2n-1}(x) = F_{n-1}(x) + G_{n-1}(x) \left\{ (x - a_1).(x - a_2)\ldots(x - a_n) \right\}$$

where $G_{n-1}(x)$ is any power of polynomial in x of degree $n - 1$. Integrating both sides with respect to x between (a, b) we find that the area

$$\int_a^b f_{2n-1}(x)\, dx = \int_a^b F_{n-1}(x)\, dx$$

if

$$\int_a^b (x - a_1).(x - a_2) \ldots (x - a_j)\ldots(x - a_n).G_{n-1}(x).dx = 0 \qquad (1.58)$$

If this is true for any arbitrary polynomial of degree $n - 1$ $G_{n-1}(x)$, then it must be true for each one of its terms. i.e.

$$\int_a^b (x - a_1).(x - a_2)\ldots(x - a_j)\ldots(x - a_n).(x)^k.dx = 0$$

for $k = 0, 1, 2, \ldots, n - 1$.

The above condition represents n indepndent conditions for the determination of n unknown constants a_j which specify the location of the desired ordinates in the figure 6. The values of a_j are the roots of a polynomial

$$p_n(x) = (x - a_1).(x - a_2)\ldots(x - a_j)\ldots(x - a_n) \qquad (1.58')$$

which according to (1.58)

$$\int_a^b p_n(x).\, G_{n-1}(x) = 0 \qquad (1.59)$$

is orthogonal to any polynomial or power in x of degree less than n. If we multiply out the equation

$$p_n(x) = (x - a_1).(x - a_2)\ldots(x - a_j)\ldots(x - a_n)$$

$p_n(x)$ will be

$$x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \ldots + c_0 \quad \text{, identically.} \qquad (1.58'')$$

substituting in

$$\int_a^b p_n(x).G_{n-1}(x) = 0 \qquad (1.59)$$

we get by abbreviating

$$\int_a^b x^k\,dx = \alpha_k = \frac{b^{k+1} - a^{k+1}}{k+1} \qquad (1.60)$$

the n separate conditions

$$\alpha_{i+n} + \sum_{k=0}^{n-1} C_k\,\alpha_{i+k} \qquad (i = 0, 1, 2, \ldots, n-1) \qquad (1.61)$$

The orthogonality of the function guarantees that all the roots are real and distinct and lie in the interval (a, b). The system of these equations is non-homogeneous and the determinant of its coefficients is not zero. Hence the solution is unique and nonvanishing. Once the roots of the equation, that is, the a_j's are determined then by inserting them in

$$\alpha_k = \sum_{j=1}^{n} H_j\,a_j^k \qquad (k = 0, 1, 2, \ldots, 2n-1) \qquad (1.62)$$

We get a system of n linear equations for n corresponding coefficients H_j. In this way we determine the a_j's and H_j's If we apply Gaussian quadrature to polynomials of degree

equal to or greater than 2n, then the equation

$$\int_a^b f(x)\,dx = \sum_{j=1}^{n} H_j \cdot f(a_j) \qquad (1.57)$$

<u>is no longer exact</u>. One will have to add an error term E_n. The error term can be expected to be of the form

$$E_n = K \cdot f^{(2n)}(\xi) \qquad (1.63)$$

where $a < \xi < b$ and K is some constant. If $f(x) = x^{2n}$ say, then

$$f^{(2n)}(x^{2n}) = (2n)! \quad \text{and} \quad \int_a^b x^{2n}\,dx = \alpha_{2n}$$

K can be solved from

$$\alpha_{2n} = \sum_{j=1}^{n} H_j\, a_j^{2n} + (2n)!\,K$$

$$\therefore \quad E_n = \frac{f^{(2n)}(\xi)}{(2n)!}\left[\alpha_{2n} - \sum_{j=1}^{n} H_j \cdot a_j^{2n}\right] \qquad (1.64)$$

The above proof can be given a purely analytical approach also by having recourse to Hermite's interpolation formula. (Z. Kopal 1955).

Generalising the above one can seek a quadrature formula of the form

$$\int_a^b w(x) \cdot f(x)\,dx = \sum_{j=1}^{n} H_j \cdot f(a_j) \qquad (1.65)$$

This formula is <u>exact</u> for all polynomials of degree less than or equal to 2n - 1 regardless of the character of w(x) -

except that it does not vanish throughout the interval of integration and is integrable -

The proof is essentially similar except that

$$\alpha_k \text{ will now be } \int_a^b \omega(x) \cdot x^k \, dx \quad \text{and} \tag{1.66}$$

$$\sum_{j=1}^{n} H_j = \int_a^b \omega(x) \cdot dx \tag{1.67}$$

If all the 'moments' α_k exist for k less than 2n, we can determine the c_k's, a_j's and finally the H_j's.

Mehler's quadrature formulae of the Gauss type or the Gauss-Mehler formulae are associated with a function of the type

$$\omega(x) = (1-x)^{\alpha} (1+x)^{\beta} \tag{1.68}$$

where α and β are numbers greater than -1. This is required to render the integral

$$\int_{-1}^{+1} (1-x)^{\alpha} (1+x)^{\beta} f(x) \, dx = \sum_{j=1}^{n} H_j \cdot f(a_j) \tag{1.69}$$

convergent. The solutions a_j's and H_j's are given by Mehler (see Kopal 1955). The a_j's are the roots of the polynomial

$$Q_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n \cdot n! \, (1-x)^{\alpha}(1+x)^{\beta}} \cdot \frac{d^n}{dx^n} \left[(1-x)^{\alpha+n} \cdot (1+x)^{\beta+n} \right] \tag{1.70}$$

These abscissae a_j are roots of the Jacobi polynomials

$${}_2F_1 \left\{ -n, n+\alpha+\beta+1, \alpha+1, \tfrac{1}{2}(1-x) \right\} = G_n \left\{ \alpha+\beta+1, \alpha+1, \tfrac{1}{2}(1-x) \right\} \tag{1.71}$$

of the nth degree and H_j's are explicitly

$$H_j = \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{\Gamma(n+1)\cdot\Gamma(n+\alpha+\beta+1)} \cdot \frac{2^{\alpha+\beta+1}}{(1-a_j^2)\left\{Q_n^{I}(a_j)\right\}^2} \qquad (1.72)$$

(see Szegö 1939). p. 67

We are interested in the case when $\alpha = \beta = -\frac{1}{2}$ when we have

$$\int_{-1}^{1} \frac{f(x)}{(1-x^2)^{\frac{1}{2}}} \cdot dx = \sum_{j=1}^{n} H_j \cdot f(a_j) \qquad (1.73)$$

we have $H_j = \pi/n$; $a_j = \cos \frac{2j-1}{2n} \cdot \pi$ (1.74)

The limits of integration can always be modified with the formula

$$\int_a^b f(x)\,dx = \frac{b-a}{\beta-\alpha} \int_\alpha^\beta g(y)\,dy \qquad (1.75)$$

by putting $(\beta-\alpha)x = (b-a)y + (a\beta - b\alpha)$ (1.75')

with particular reference to our example in the ionosphere:

$$\frac{2}{\pi}\int_0^1 \frac{h'(f/f_N)}{\left[1-(f/f_N)^2\right]^{\frac{1}{2}}} \cdot d(f/f_N) = \frac{2}{\pi}\int_0^1 \frac{h'(\xi)}{(1-\xi^2)^{\frac{1}{2}}} \cdot d\xi$$

$$H_j = \frac{\pi}{2n} \qquad a_j = \cos\left(\frac{2j-1}{2n}\right)\cdot\pi$$

by putting $\xi = \cos\theta$ or $\sin\theta$ one gets $\int_{-1}^{1} \frac{f(\xi)}{(1-\xi^2)^{\frac{1}{2}}} \cdot d\xi$

as $$\int_0^{\pi} f(\cos\theta)\, d\theta = \pi/n \sum_{j=1}^{n} f\left[\cos\left(\frac{2j-1}{2n}\right)\pi\right]$$

or $$\int_0^{\pi/2} f(\cos\theta)\, d\theta = \pi/2n \sum_{j=1}^{n} f\left[\cos\left(\frac{2j-1}{2n}\right)\pi\right]$$

therefore

$$\frac{2}{\pi}\int_0^{\pi/2} f(\cos\theta)\, d\theta = \frac{1}{n}\sum_{j=1}^{n} f\left[\cos\left(\frac{2j-1}{2n}\right)\pi\right] \qquad (1.76)$$

It now follows that one has to take the mean of n values on the right hand side for the value of the integral on the left.

Thus we see that the application of Gauss-Mehler quadrature to the no-field case of true height determination reduces to a mere summing up as many values as we need and dividing it simply by the number of values to give the true height. In concluding it should be pointed out that this method is exact and not approximate. Occasionally we do observe that the electron density distribution does not vary in so simple a way as the parabolic law (a quadratic function in z) but it should be possible at any stage to represent it with ample accuracy by a polynomial with terms upto even the 9th power in z when we take a summation to only five terms. Even in this extreme case, the Gaussian quadrature is exact and mathematically sound. This analysis is sufficient to justify the quadrature method.

6d. The procedure for analysing the record

The procedure is not too taxing as the experience of the author has shown in analysing typical and good records in all the four seasons of the three years 1954 to 1956. The procedure was carried out with a desk calculator (Facit - electrically driven) by the author. It is outlined below. It should not be difficult with a set of three or four workers to analyse a good number of records. On an average it takes for a single man around 45 minutes for one plus [illegible] record including plotting etc. The procedure is given in the form of tables (given as Appendixes) and certain standard forms were prepared by the author in the laboratory.

(1) Note down from the ionogram a series of values of h' vs frequency, and the critical frequency of the region for each of the regions present. One can have as many pairs of values as possible. Usually it is possible to read accurately in a record heights correct to about ¼ (one-fourth) of each division - in our record it was 12.5 km - with suitable transparent scales with divisions marked on it.

(2) Compute the values f/f_c where f is any frequency for which you read a virtual height and f_c is the critical frequency for the region, for all frequencies and the value of the function $\psi\,(f/f_c)$. The function

$$\psi(f/f_c) = f/2f_c \cdot \log_e\left(\frac{1+f/f_c}{1-f/f_c}\right) = \frac{\xi}{2}\cdot\log_e\left(\frac{1+\xi}{1-\xi}\right)$$

(where $\xi = f/f_c$)

is tabulated and is given in the Appendix at the end. This

This function is equal to $1+\varphi(\xi)$ of Booker & Seaton (1940) and the table of values of ξ from $\xi = 0$ to $\xi = 1.0$ in steps of 0.01 are prepared from their table given in 1940.

(3) Plot the value of h' vs $\psi(\xi)$. The plot is a straight line. This follows from the relation

$$h' = h_0 + \tau \cdot \psi(\xi)$$

The intercept on the height axis gives the value of h_0 and an ordinate at $\psi(\xi) = 1$ intersects the straight line

$$h'(f) = h_0 + \tau \cdot \psi(\xi)$$

at a point whose ordinate gives the height of maximum electron density. The difference of h_m and h_0 is τ, the semi-thickness. All these are true values according to the parabolic model. If the points $[h', \psi(\xi)]$ fit well to a straight line, the inference is that the distribution is parabolic. As is the experience of the author, the distribution is very satisfactorily parabolic in most cases. The deviations from parabolicity were noticed in very few cases during the analysis of records on quiet days in 1954-56.

(4) The h' - f curve is now transferred on to a graph paper and the lowe end is extrapolated down to h_0 given by (3).

(5) The frequencies $f_N \sin\theta$ are tabulated for each integral frequency from 1 to 25 Mc/s and is given in the Appendix. Virtual heights at these frequencies are noted down in the standard form prepared for this purpose. The mean value of these virtual heights gives the true height for that particular frequency.

This procedure is repeated for all frequencies from 1 to the critical frequency in step of 1 Mc/s. We now have the true heights at every integral frequency and the height of maximum electron density is also obtained from the mean of five virtual heights in a similar way. This gives the distribution of electron density with height.

Case (ii) - <u>When there are more regions than one</u>.

(1) - Repeat the procedure as in the case of single region for the lowest region.

(2) The function

$$\varphi(\xi) = \tfrac{1}{2}\cdot\xi\cdot\log_e\left(\frac{1+\xi}{1-\xi}\right) - 1 \qquad \text{where } \xi = f/f_c,\ f > f_c$$

is tabulated and given in the Appendix. These are the same as those given by Booker and Seaton (1940) in the range

$$f_c < f < 2f_c$$

(3) Compute the value $\tau\cdot\varphi(f/f_c)$ for each of the frequencies, for which the pulse gets reflected from a higher region. In this case τ and f_c refer to the lower region. This is the group retardation for passage through lower region of the pulse. If we do not have a complete region below, we have to apply the formula (Ratcliffe 1951)

$$\Delta h' = \frac{f}{f_c}\cdot\log_e\left\{\frac{1+f/f_c}{(1-f\lambda^2/f_c^2)^{\frac{1}{2}} + \left[(f/f_c)^2 - (f\lambda/f_c)^2\right]^{\frac{1}{2}}}\right\}$$

(4) Subtract from h' the corresponding value $\Delta h'$ for the group retardation and find the differences $h' - \Delta h'$. These are the virtual heights to be used for later calculations.

(5) The procedure is now repeated according to (3)-case for a single layer - and onwards.

(6) In the case of F2 region calculations, allowance has to be made for retardation in E, F1 and F1.5 regions before deriving the true heights.

CHAPTER TWO

The effect of the Geomagnetic field on the calculation of true geometric heights of reflection

CHAPTER TWO

The effect of the Geomagnetic field on the calculation of true geometric heights of reflection.

1. Introduction

We have already seen that the earlier work on electron density distribution in the ionosphere was done on the assumption that the effect of the geomagnetic field could be neglected. That this was a serious limitation was known to workers in ionospheric physics. The main difficulty was the complexity of the integral equation relating the virtual height function to the true height function. The early workers had therefore recourse to 'model methods'. The papers by Shinn and Whale (1952), by Kelso and by Budden in 1954 and 1955 respectively opened the way to calculate the true height distribution of electrons from vertical ionospheric soundings without assuming any special form for the distribution and with due allowance for the geomagnetic field. The paper by Jackson (1956) gave a similar method without having to make use of electronic computation. We shall discuss this paper together with the previous ones and point out how the effect of the magnetic field can be calculated with the help of an ordy electrically driven desk computing machine. We shall later report the results for Ahmedabad and Delhi and establish that the effect of the field is not serious for stations near the equator. For the sake of comparison we shall give the results for England to demonstrate the difference at higher latitudes.

A few instances of the distribution of electrons over Ahmedabad will be given to lend factual evidence to this.

2. Group velocity and group refractive index

The virtual height of reflection of a pulse of mean frequency f is given by

$$h'(f) = h_0 + \int_0^{z_r} \mu' \, dz \qquad (2.1)$$

where h_0 is the true base of the region (height of reflection at f = 0, z_r is the height of reflection of a pulse of frequency f and μ' is the group refractive index defined as the group velocity dividing the free-space velocity of electromagnetic radiation — $\mu' = c/v_g$ (2.2)

Besides, μ' obeys a relation

$$\mu' = \mu + f \, \partial\mu/\partial f = \partial/\partial f \, (\mu f) \qquad (2.3)$$

where μ is the phase refractive index $= c/v_p$ (2.4)

and f is the frequency of the wave.

We have so far used in our calculation of true heights the formula given by Eccles (1912) named after Sellmeyer

$$\mu^2 = 1 - \frac{Ne^2}{\pi m f^2} \qquad (2.5)$$

and the equation $\mu' = \frac{1}{\mu}$; that $\mu\mu' = 1$ can be shown by simple differentiation of the Selleyer dispersion relation in juxtaposition with the relation

$$\mu' = \mu + f \cdot d\mu/df \qquad (2.3)$$

We shall include the effect of the earth's magnetic field in

our calculations. In this case we shall have to use the general refractive index formula given by the magneto-ionic theory of Appleton and Hartree in 1932. We could write the formula for the phase refractive index as

$$\mu^2 = 1 - \frac{2x(1-x)}{D} \qquad (2.6)$$

where $x = f_N^2/f^2 = Ne^2/\pi m f^2$; f_N = plasma frequency

N = the number density of electrons

e = the charge on the electron in e.s.u.

m = mass of the electron in grams

f = exploring frequency of the wave-packet

$D = 2(1 - x) + S - y^2 \sin^2\theta$

$y = f_H/f$

$f_H = He/2\pi mc$

H = total field intensity of the earth's magnetic field of the place in Oersted.

θ = propagation angle - the angle between the direction of the earth's field and the wave-normal = the complement of the angle of dip at that place $= \frac{\pi}{2} - \phi$

$S = \pm\sqrt{y^4 \sin^4\theta + 4y^2(1-x)^2\cos^2\theta}$

c = velocity of electromagnetic waves in free-space in cms/sec

The above formula for the refractive index was given by Appleton (1932). It neglects the effect of electron collisions and the Lorentz polarisation term. The positive sign for the radical in the expression for S is to be taken

for the ordinary component and the negative sign for the extraordinary component of the wave. The expression can also be used with the angle of dip φ instead of the propagation angle θ. In this case $\sin\theta$ will have to be replaced by $\cos\varphi$ and $\cos\theta$ by $\sin\varphi$.

Whale and Stanley (1951) followed by Shinn and Whale (1952) have studied the behaviour of group velocities and group heights of reflection of radio waves. We shall briefly recapitulate the results. It is well known that the group velocity and therefore the group height of reflection depends on the strength and direction of the magnetic field at a particular place. The dependence is rather involved. Let us see the equations

$$\mu^2 = 1 - \frac{2x(1-x)}{D} \qquad (2.6)$$

This is the phase refractive index, and the group index has to be obtained from the relation

$$\mu' = \mu + f\,\partial\mu/\partial f$$

Let us examine the nature of the function μ. This depends on the exploring frequency f and the plasma frequency f_N corresponding to an electron density $N\left(=\sqrt{\frac{Ne^2}{\pi m}}\right)$; the strength of the field at the place or f_H, the Larmor frequency of the electron in a field of strength H $(= He/2\pi mc)$ and the propagation angle θ or the angle of dip φ. We can write μ briefly as

$$\mu = \mu\left(f, f_N, f_H, \theta\right) \qquad (2.7)$$

at a given place. We have neglected the effect of electron

collisions. It can be seen that μ is in fact a function of the two ratios f_N^2/f^2 which we shall call x in what follows and f_H/f which we call y and θ. That is

$$\mu = \mu(x, y, \theta) \qquad (2.8)$$

differentiating $\quad (\mu^2 - 1) = -2x(1-x)/D$

we get $\quad 2\mu \cdot \partial\mu/\partial f = -\partial/\partial f \left(\dfrac{2x(1-x)}{D} \right)$

we have then,

$$\mu \cdot \partial\mu/\partial f = -\partial/\partial f \left(\frac{x(1-x)}{D} \right)$$

It is easier to differentiate (2.6) from the relation

$$(\mu^2 - 1) D f^2 = 2x(x-1) \cdot f^2 \qquad (2.9)$$

The differentiation is a little lengthy although straightforward (Shinn and Whale 1952) and yields after a little simplification

$$(\mu\mu' - \mu^2) D = 2\left[1 - \mu^2 - x^2 + \tfrac{1}{5} \cdot (1-\mu^2)(1-x^2) y^2 \cos^2\theta\right] \qquad (2.10)$$

from which one can get $\mu\mu'$ and finally μ'. The above formula is suitable for tabulation of μ' values which we would need in deriving heights including the effect of the geomagnetic field (that is when y is not zero). This formula is more convenient and is accurate for the computation of μ' than when done in the way suggested by Jackson (1956). When we are evaluating μ' at any one place θ will be constant and the expression $y \cos\theta$, $y \sin\theta$ and the different functions of $y \cos\theta$ and $y \sin\theta$ which we shall abbreviate as y_L and y_T (Longitudinal and Transverse components of y) can be tabulated

for different values of y. This would give the accurate values of $\mu\mu'$ and μ'. There is however a shortcut suggested by Jackson (1956). Since for a given θ, μ is only a function of x and y only (i.e., x and y only vary) we can differentiate (2.8) and get —

$$\mu^2 = F(x, y) \qquad (2.8')$$

$$2\mu \cdot \partial\mu/\partial f = \partial F/\partial x \cdot \partial x/\partial f + \partial F/\partial y \cdot \partial y/\partial f$$

since $x = f_N^2/f^2$; $dx/df = -2x/f$

and $y = f_H/f$; $dy/df = -y/f$

$$2\mu \, \partial\mu/\partial f = -2x/f \cdot (\partial F/\partial x) - y/f \cdot (\partial F/\partial y)$$

since $\mu' = \mu + f \, \partial\mu/\partial f$

$$\mu\mu' = \mu^2 + \mu f \cdot \partial\mu/\partial f$$

The above equation becomes

$$\mu\mu' = \mu^2 - x(\partial F/\partial x) - \tfrac{1}{2} \cdot y \, (\partial F/\partial y) \qquad (2.11)$$

Since we calculate μ' for each value of y, we have in a given operation x varying and y is fixed so that $\partial F/\partial y$ is zero in that operation. and

$$\mu\mu' = \mu^2 - x \, \partial F/\partial x \qquad (2.12)$$

$\partial F/\partial x$ can be determined by numerical methods , by differentiation

$$\partial F/\partial x = \frac{1}{2} \cdot \left\{ \frac{F(x+\Delta x) - F(x-\Delta x)}{\Delta x} \right\} \qquad (2.13)$$

where y=constant

It is possible to get $\partial F/\partial y$ from the family of curves $\mu^2(x, y)$. Unfortunately this formula is not exact and so cannot be used for inclusion in true height calculations. Nevertheless, it gives an idea of the variation of $\mu\mu'$. It is therefore preferable to use

$$(\mu\mu' - \mu^2) = \frac{2}{D}\left(1-\mu^2-x^2 + \frac{1}{3}(1-\mu^2)(1-x^2)y^2\cos^2\theta\right) \quad (2.10)$$

This formula is especially suited for computing μ' for different values of 'f'. This will be needed at a later stage. It is interesting to examine the nature of variation of μ, $\mu\mu'$, μ', given by the general formula from the magneto-ionic theory with the values given by the Sellmeyer formula

Table 2

When field is neglected ($f_H = 0$)	When field is taken into account
1. $\mu^2 = 1 - f_N^2/f^2 = 1 - x$	1. $\mu^2 = 1 - \frac{1}{D}\left[2x(1-x)\right]$ $= 1 - x\left\{\frac{2(1-x)}{D}\right\}$
2. $\mu\mu' = 1$	2. $\mu\mu' = \mu^2 + \frac{2}{D}\left[1-\mu^2-x^2 + \frac{1}{3}(1-\mu^2).(1-x^2)y_L^2\right]$
3. $\mu' = \frac{1}{\mu} = (1-x)^{-\frac{1}{2}}$	3. $\mu' = \mu + \frac{2}{\mu D}\left[1-\mu^2-x^2 + \frac{1-\mu^2.1-x^2.y_L^2}{3}\right]$

One can appreciate the differences better if one draws these functions on a graph paper. We give below the curves for Ahmedabad, Delhi and England. The quantities shown are μ^2 for Ahmedabad and $\mu'.t$ for Ahmedabad, Delhi and England. The quantity $(\mu'.t)$ is particularly interesting since as we shall show later, is the ratio of μ' with the field-effect included to μ' without the field.

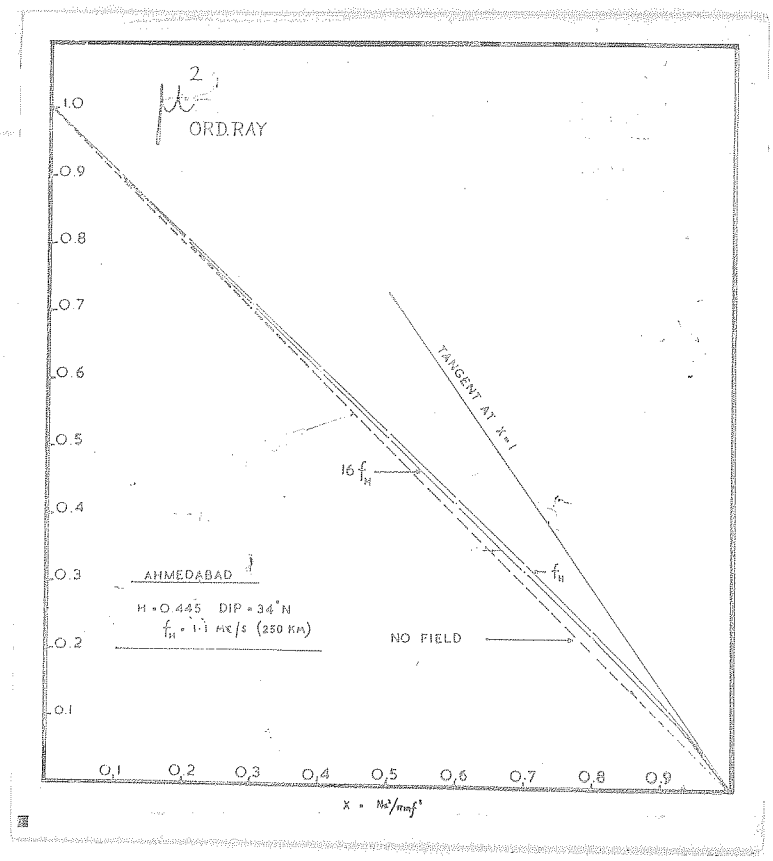


Figure 7

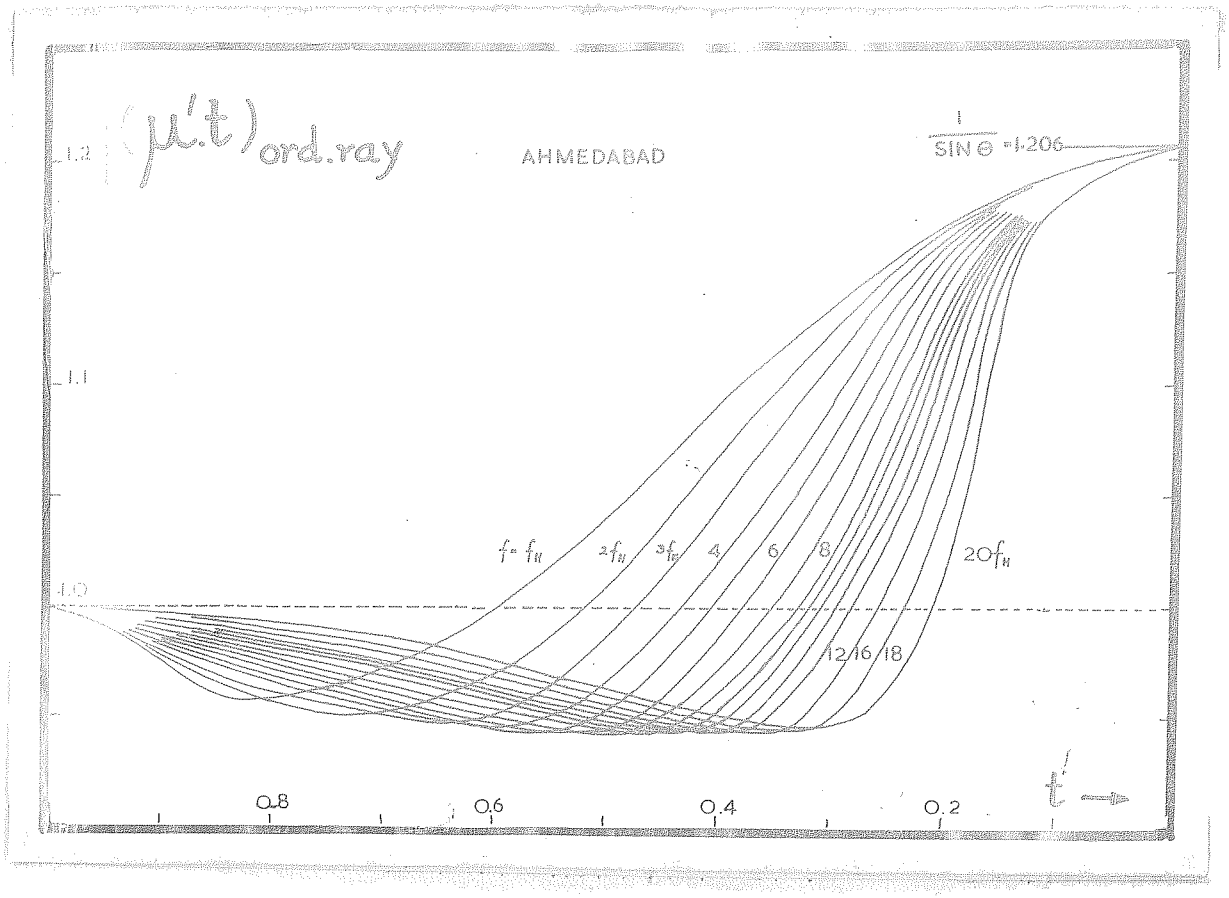


Figure 8

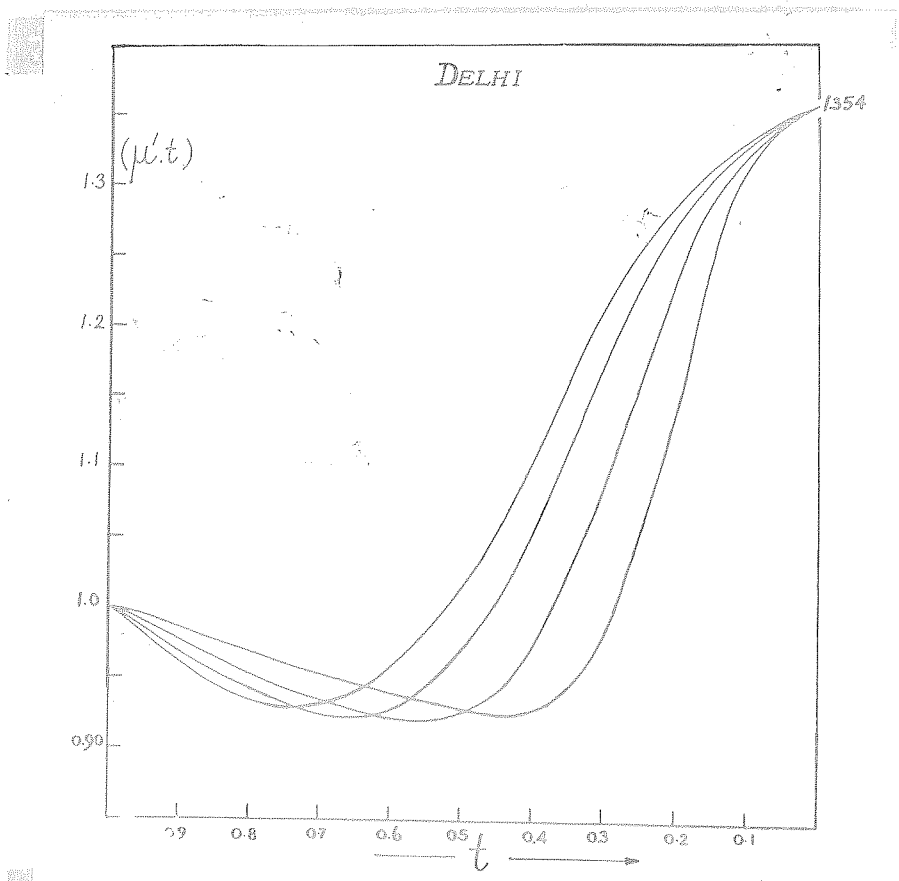


Figure 9

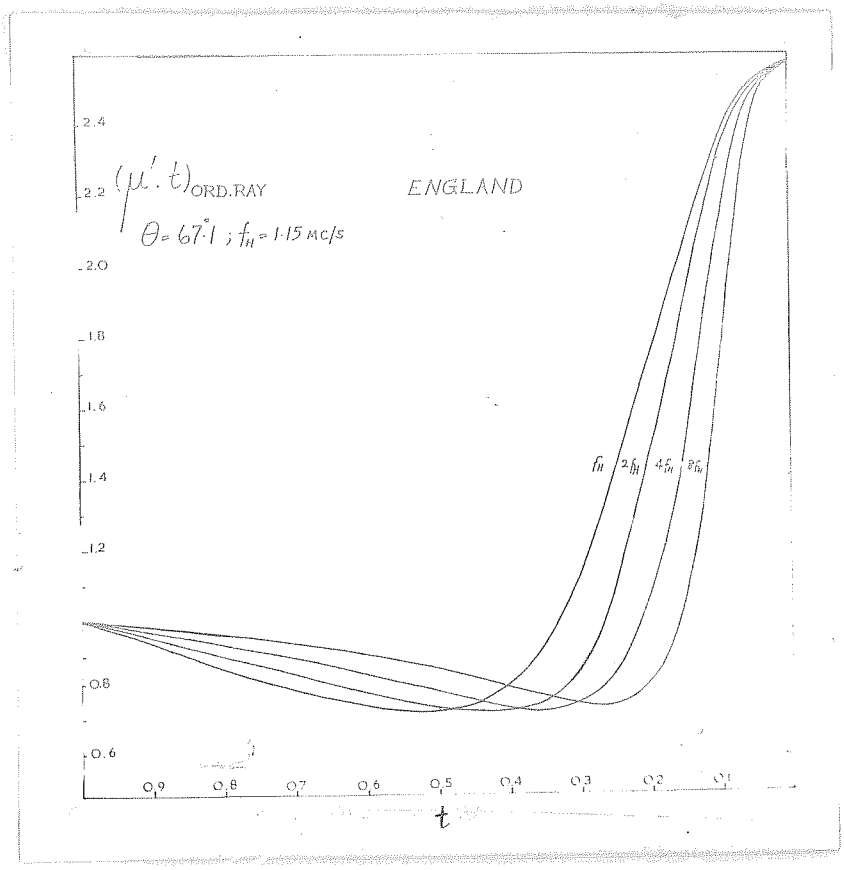


Figure 10

The first attempt to reveal the effect of the magnetic field of the earth on the electron distribution was by Shinn and Whale (1952). They first considered a parabolic distribution of electron density, of the form

$$(1 - N/N_{max}) = (1 - z/z_{max})^2 \qquad (2.14)$$

Differentiating this with respect to z and substituting in

$$h'(f) = h_0 + \int_0^{z_r} \mu' \, dz \qquad (2.1)$$

and making the transformation $t^2 = 1 - x$ we get the expression

$$h'(f) = h_0 + \tau_0 \cdot \frac{f^2}{f_c^2} \cdot \int_0^1 \frac{\mu' . t}{(1 - x f^2/f_c^2)^{\frac{1}{2}}} \, dt \qquad (2.15)$$

Inserting the value of μ' given by the magneto-ionic theory one can compute the value of the integral

$$I(f/f_c) = (f/f_c)^2 \cdot \int_0^1 \frac{\mu' t}{(1-x f^2/f_c^2)^{\frac{1}{2}}} \cdot dt \qquad (2.16)$$

Let us write this equation as

$$h'(f) = h_0 + \tau \cdot I(\xi) \qquad (2.17)$$

where $\xi = f/f_c$. If, therefore, we plot h'(f) vs I(ξ) we will get a straight line whose slope will be the semi-thickness of the parabolic region and the intercept on the height axis will be the base of the layer. It is to be noted that this will include the effect of the magnetic field. The integral I(ξ) can be evaluated by Simpson's rule or by any other suitable numerical integration formula. We will have to repeat that although this includes the effect of the field the presumption of parabolic distribution will have to be justified and we are therefore obliged to look for other methods to lend generality to the conclusions reached.

This has been done by Budden (1955) and it has been applied by Schmerling (1957) and Thomas et al (1958) to some records. Budden's method is based on the method of solving an integral equation by treating it as the limit of a matrix equation. This is a well known standard method of numerical analysis (Jeffreys and Jeffreys 1950). It is fortunate that this could be applied successfully to the ionospheric problem and has been done by Budden.

We start from the well known relation

$$h'(f) = \int_0^{z_r} \mu'(f, f_N)\, dz \qquad (2.18)$$

The problem is to determine $z(f_N)$. Treating f_N as the variable, we have, since $z = z_r$ when $f = f_N$

$$h'(f) = \int_{f_N=0}^{f_N=f} \mu'(f, f_N) \cdot \frac{dz(f_N)}{df_N} \cdot df_N \qquad (2.18')$$

The basic assumption here is of course, as has been stated earlier, that $z(f_N)$ is a monotonic function of f_N in $0 \leq f_N \leq f$ The left hand side of the equation is known. Let us say we have a tabulated set of h' values against f. That is we have

$h'(f_1)$

$h'(f_2)$

$h'(f_3)$

$h'(f_4)$

$\vdots$

$h'(f_n)$

Let us suppose that these f's are such that $f_1 = \Delta f$; $f_2 = 2\Delta f$ $f_3 = 3\Delta f$; etc... $f_n = n\Delta f$ that is we have divided the frequencies in integral intervals of Δf. Let us call $h'(\Delta f)$ as h'_1, $h'(2\Delta f)$ as h'_2 etc....$h'(n\Delta f)$ as h'_n. We can regard these h's as forming a column matrix

$$\underline{h}' = \begin{pmatrix} h'_1 \\ h'_2 \\ h'_3 \\ \vdots \\ h'_n \end{pmatrix} \qquad (2.19)$$

Now the range of integration (0, f) is divided into n discrete intervals. Let the mth interval be such that $(m - 1)\Delta f$ is

less than f_N is less than $m\Delta f$; Where m is less than or equal to n. Assume that the intervals Δf are small enough so that

$$\frac{dz}{df_N} \approx \frac{\Delta z}{\Delta f_N} \qquad (2.20)$$

can be regarded as fairly constant and that it could be written

$$\frac{dz}{df_N} \approx \frac{\Delta z}{\Delta f_N} = \frac{z_m - z_{m-1}}{\Delta f_N} \qquad (2.20')$$

We shall abbreviate $z(m.\Delta f)$ as z_m. The assumption that dz/df is constant over fairly small intervals of df, although rigorously speaking questionable, is fundamental and that for the accuracy available in ionospheric work one can expect it as not unreasonable. We are now able to replace the integral equation

$$h'(f) = \int_0^{z_r} \mu' \, dz \qquad (2.18)$$

by a summation over a finite number of discrete values as

$$h'_n \approx \sum_{m=1}^{n} \int_{(m-1)\Delta f}^{m.\Delta f} \mu'(n\Delta f, f_N) \cdot \frac{z_m - z_{m-1}}{\Delta f} \cdot df_N$$

Since we have assumed dz/df constant in this small interval it can be taken outside the sign of integration so that we have now

$$h'(n\Delta f) = \sum_{m=1}^{n} \left(\frac{z_m - z_{m-1}}{\Delta f}\right) \int_{(m-1).\Delta f}^{m.\Delta f} \mu'(n\Delta f, f_N) \, df_N \qquad (2.21)$$

Let us write $\frac{1}{\Delta f} \int_{(m-1).\Delta f}^{m.\Delta f} \mu'(n\Delta f, f_N) \cdot df_N$ as M_{nm}

$m \leq n$ (2.22)

when $m \leq n$ and M_{nm} will be zero when m is greater than n. It can be seen that M_{nm} will form a lower triangular matrix. We can rewrite equation (2.18) as a matrix equation :

$$\begin{pmatrix} h_1' \\ h_2' \\ h_3' \\ \vdots \\ h_j' \\ \vdots \\ h_{n-1}' \\ h_n' \end{pmatrix} = \begin{pmatrix} M_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ M_{21} & M_{22} & 0 & 0 & \cdots & & & & & \\ M_{31} & M_{32} & M_{33} & 0 & \cdots & & & & & \\ \vdots & & & & 0 & \cdots & & & & \\ \vdots & & & & & & & & & \\ M_{j1} & M_{j2} & M_{j3} & \cdots & & & & & & \\ \vdots & & & & & & & & & \\ M_{n-1,1} & M_{n-1,2} & M_{n-1,3} & \cdots & & & M_{n-1,n-1} & & 0 & \\ M_{n1} & M_{n2} & M_{n3} & \cdots & & & M_{n-1,n} & & M_{nn} & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots \\ -1 & 1 & 0 & \cdots \\ 0 & -1 & 1 & 0 \cdots \\ 0 & 0 & -1 & 1 \\ \cdots & & & \\ \cdots & 1 & & \\ \cdots & & & \\ \cdots & & & \\ 0 & 0 & 0 \cdots & -1, 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ \vdots \\ z_j \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix}$$

(2.23)

Symbolically we can write $\underline{h}' = \underline{M}\ \underline{D}\ \underline{z}$ (2.23')

Multiplying out $\underline{M}$ and $\underline{D}$ one can easily see that $\underline{MD}$ is also lower triangular and non-singular

$$\underline{M}\ \underline{D} = \begin{pmatrix} M_{11} & 0 & 0 & 0 & \cdots \\ M_{21}-M_{22} & M_{22} & 0 & 0 & \cdots \\ M_{31}-M_{32} & M_{32}-M_{33} & M_{33} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{n1}-M_{n2} & M_{n2}-M_{n3} & M_{n3}-M_{n4} & \cdots & M_{nn} \end{pmatrix} \tag{2.24}$$

We now have an equation $\underline{h}' = \underline{M}\ \underline{D}\ \underline{z}$ where we know $\underline{h}'$ from the ionogram, $\underline{M}\ \underline{D}$ can be evaluated after computing the elements M_{nm} by numerical integration. The unknown is $\underline{z}$. It can be solved by inverting $\underline{M}\ \underline{D}$. This can be done by a method

due to Cholesky (see Hartree 1952) and get a lower triangular matrix $\underline{L}$ where $\underline{L} = (\underline{A}\,\underline{D})^{-1}$. The solution is therefore

$$\underline{x} = \underline{L}\,\underline{h}' \qquad (2.25)$$

The integral

$$\frac{1}{\Delta f}\int_{(m-1)\cdot\Delta f}^{m\cdot\Delta f} \mu'(n\Delta f, f_N)\, df_N$$

can be computed either by Simpson's rule or by some other suitable numerical integration formula and the operation will be a little easier if one puts $f_N = f \sin\vartheta$ and the integral changes to

$$\int_{\arcsin(m-1)\cdot\Delta f}^{\arcsin m\cdot\Delta f} \left[\mu'(f, f\sin\vartheta)\, f \cos\vartheta\right] d\vartheta \qquad (2.26)$$

The integrand remains finite as f_N tends to f or as $\vartheta \longrightarrow \pi/2$ tends to

Budden's method is comparatively straightforward. One has to give for a given place the values of the matrix $\underline{L}$. (i.e. only f_N at a desired level - say some mean level and the angle of dip at that place are needed for this calculation) Budden choses a step of 0.2mc/s and uses at best a five-ordinate integration method. The method used seems best suited for programming and feeding to an electronic computer. The job is not heavy at places where he has used it - SE England - and where the critical frequency does not shoot up to very high values. It has now been reported by R.L.Smith-Rose (1958) that routine analysis, on the lines suggested by Budden (1955) are being made on a large scale, of records at some stations operated by the DSIR England on selected days during the IGY.

These results when they become available should prove useful to understand the ionosphere of those latitudes. The accuracy of Budden's method can be increased by having the frequency tabulation at steps of 0.1 Mc/s and by taking more ordinates in the numerical integration. One can also use the appropriate values of f_H for the three regions E, F1 and F2 instead of the one value he uses for f_H at 300 km. One has to guard against rounding off errors while carrying out matrix inversion by Cholesky's method. These could be considerable. When one has insured against these errors, a set of values of the matrix $\underline{L}$ once calculated for a station is available for use with any amount of data. It is to be emphasized that it is worth all the trouble and the results would be useful. The trouble of inversion could be saved if one proceeds stepwise in solving the equation for $\underline{z}$, when $\underline{h}'$ is known. One has if one writes $\underline{A}$ for $\underline{M}\,\underline{D}$

$$
\begin{aligned}
h'_1 &= A_{11}\, h_1 \\
h'_2 &= A_{21}\, h_1 + A_{22}\, h_2 \\
h'_3 &= A_{31}\, h_1 + A_{32}\, h_2 + A_{33}\, h_3 \\
&\ldots \quad \ldots \quad \ldots \quad \ldots
\end{aligned}
\tag{2.27}
$$

we have solving for h_1

$$
\begin{aligned}
h_1 &= h_1'/A_{11} \\
h_2 &= -(A_{21}/A_{22})h_1 + h_2'/A_{22} \\
h_3 &= -(A_{31}/A_{33})h_1 - (A_{32}/A_{33})h_2 + h'_3/A_{33} \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots
\end{aligned}
\tag{2.28}
$$

This has also been pointed out by Caminer to Thomas et al (see Thomas et al 1958) who have applied Budden's Tables to analysis of data at Slough, England. Schmerling (1957) also has carried

out certain test reductions for Washington and State College (Pennsylvania), on the same procedure.

We conclude the section with the remark that Budden's method is a really good and reliable one for the calculation of group heights including the earth's magnetic field effect. It is especially suited where one has access to electronic computing devices. For workers in laboratories where there are only facilities for desk computations this method is certainly laborious and will take considerable time for the preparation of the $\underline{L}$ matrix elements. This suffers from another drawback that the group heights of reflection from regions higher than one-E for example - do not get corrected for the group retardation in the lower regions. This ultimately will take very considerable time.

Jackson (1956) has reported an essentially similar method which is more suitable for places which do not possess modern computing devices. Another effort to include the effect of the magnetic field is due to Whale (1951). When one neglects collision effects the group height of reflection of a signal of frequency f incident on the region and reflected from a height z_r ($z = 0$ reckoned as the bottom of the layer) is given by

$$h'(f) = \int_0^{z_r} \mu'(z) \cdot dz \qquad (2.18)$$

where μ' is the group refractive index. When we neglect the magnetic field we can write

$$h'(f) = \int_0^{z_r} \frac{dz}{(1 - f_N^2/f^2)^{\frac{1}{2}}} \qquad f_N^2 = \frac{Ne^2}{\pi m}$$

When one includes the effect of the field μ' is to be got from

$$\mu^2 = 1 - \frac{2x(1-x)}{D}$$

and $$\mu' = \mu + f \; d\mu/df$$

in the usual notation. Whale has tried to include the effect of the field by choosing one value of $y = f_N/f$ and treating the group refractive index as given for a small range of frequencies centred about the one given by y. He chose to change any of the other $\mu' - x$ curves into a curve of this shape by a suitable transformation of the variable. Let us replace f by some function of f say ζ (f). Then μ' will be

$$\mu' = \frac{1}{\left\{1 - fN^2/\zeta^2(f)\right\}^{1/2}} \qquad (2.29)$$

Whale assumed that ζ would be of the same form over a spread (although small) of frequencies say from $y = \alpha$ to $y = \beta$. It is then possible to invert the integral since

$$\int \mu' \, dz$$

would still be an Abelian integral. What has probably been overlooked by Whale is that μ' is a function of f_N and ζ both and therefore the function considered by Whale, ζ, is not a function of f alone. It is also a function of f_N. Therefore the inversion of the integral is no longer valid. Although this method is faulty it does give some reasonable results. But one cannot accept therefore the findings due to

the inherent flaw in the argument. Before we return to the most suitable method from our point of view and therefore the one adopted here we shall briefly review another method due to Kelso (1954).

This uses the Gaussian quadrature - this time a repeated quadrature to include the effect of the field. μ', the group refractive index is a known function of f_N and f. Precisely it is function of f_N^2 and f^2. Let us define it as a function $G(f_N^2, f^2)$ such that

$$G(f_N^2, f^2) = (1 - f_N^2/f^2)^{1/2} \cdot \mu'(f_N^2; f^2)$$

The virtual height is given by

$$h'(f) = \int_0^{z_r} \frac{G(f_N^2, f^2)\, dz(f_N)}{(1 - f_N^2/f^2)^{1/2}} \qquad (2.30)$$

Since the variable is f_N^2 let us make a transformation $f_N^2 = \xi$ therefore $dz/d\xi$ will be another function say $u(\xi)$. Let us put $f^2 = F$ so that we have when $z = z_r$, $F = f_N^2 = f^2$.

$$h'(f) = \int_0^F f \frac{G(\xi, F)\, u(\xi)\, d\xi}{(F - \xi)^{1/2}} \qquad (2.31)$$

$$\text{i.e.} \quad H'(F) = \frac{h'(f)}{f} = \int_0^F \frac{G(\xi, F)\, u(\xi)\, d\xi}{(F - \xi)^{1/2}} \qquad (2.31')$$

This is a singular kernel integral equation of the first kind called Volterra's type equation. The solution to this was given by Picard (1904). It consists in splitting the above equation into two types

$$H'(F) = \int_0^F \frac{u(\xi)\, d\xi}{(F - \xi)^{1/2}} + \lambda \int_0^F \frac{[G(\xi, F) - 1]}{(F - \xi)^{1/2}} \cdot u(\xi)\, d\xi \qquad (2.32)$$

this is done by putting

$$G(\xi, F) \cdot u(\xi) = u + \lambda (G-1) \cdot u \qquad (2.33)$$

when $\lambda \to 1$ L.H.S $\leftrightarrow$ R.H.S.

Let us expand $u(\xi)$ as $u_0 + \lambda u_1 + \lambda^2 u_2 + \cdots + \lambda^j u_j + \cdots$ (2.34)

and insert the value in the above equation. Identifying the coefficients of like powers of λ on either side, we have the identities

$$H'(F) = \int_0^F \frac{u_0(\xi)}{(F-\xi)^{\frac{1}{2}}} \, d\xi \qquad (2.35)$$

$$\int_0^F \frac{u_k}{(F-\xi)^{1/2}} \cdot d\xi = -\int_0^F \frac{[G(\xi,F)-1]}{(F-\xi)^{\frac{1}{2}}} \cdot u_{k-1}(\xi) \, d\xi \qquad (2.36)$$

for $k = 1, 2, 3, \ldots$

But $H'(F) = \displaystyle\int_0^F \frac{u_0(\xi)}{(F-\xi)^{\frac{1}{2}}} \cdot d\xi$

is an Abelian integral equation and $u_0(\xi) = \dfrac{dz_0}{d\xi}$ (2.37)

Hence solving for $u_0'(\xi)$ and then integrating with respect to ξ we get

$$z_0(f_N) = \frac{2}{\pi} \int_0^{f_N} \frac{h'(f) \, df}{(f_N^2 - f^2)^{\frac{1}{2}}} \qquad (2.38)$$

This is exactly the equation we had in the field-free case. Let us take the next equation;

$$\int_0^F \frac{u_k(\xi) \, d\xi}{(F-\xi)^{\frac{1}{2}}} = -\int_0^F \frac{G(\xi,F)-1}{(F-\xi)^{1/2}} \cdot u_{k-1} \cdot d\xi$$

multiplying this by $\dfrac{dF}{\sqrt{\zeta - F}}$ and integrating (2.36)

from 0 to ζ

$$\int_0^{\zeta} \frac{dF}{(\zeta - F)^{\frac{1}{2}}} \int_0^{F} \frac{u_k(\xi)\, d\xi}{(F-\xi)^{\frac{1}{2}}} = -\int_0^{\zeta} \frac{dF}{(\zeta - F)^{\frac{1}{2}}} \int_0^{F} \frac{(G-1)\cdot u_{k-1}}{(F-\xi)^{1/2}} \cdot d\xi \qquad (2.39)$$

Changing the order of integration

$$\int_0^{\zeta} u_k(\xi)\, d\xi \int_0^{F} \frac{dF}{(\zeta - F)^{1/2}(F-\xi)^{1/2}} = -\int_0^{\zeta} \frac{dF}{(\zeta - F)^{1/2}} \int_0^{F} \frac{(G-1)\, u_{k-1} \cdot d\xi}{(F-\xi)^{1/2}} \qquad (2.39')$$

put $$\int_0^{F} \frac{dF}{(\zeta - F)^{1/2}(F-\xi)^{1/2}} = \int_0^{\xi} + \int_{\xi}^{\zeta} + \int_{\zeta}^{F}$$

$$= \int_{\xi}^{\zeta} \frac{dF}{(\zeta - F)^{1/2} \cdot (F-\xi)^{1/2}} \qquad (2.40)$$

(by Dirichlet's integral formula).

This is an Euler B-integral and is equal to π .

To get back to the previous step, we have to differentiate (2.39') with respect to ζ . We get on the left hand side

$$u_k(\zeta) = -\frac{1}{\pi} \cdot \frac{d}{d\zeta} \int_0^{\zeta} \frac{dF}{(\zeta - F)^{1/2}} \cdot \int_0^{F} \frac{(G-1)\, u_{k-1}(\xi)\, d\xi}{(F-\xi)^{1/2}}$$

Since

$$u_n(\zeta) = \frac{dz_n(\zeta)}{d\zeta} = z_n'(\zeta) \text{ , say}$$

$$\frac{dz_k(\zeta)}{d\zeta} = -\frac{1}{\pi} \frac{d}{d\zeta} \left[\int_0^{\zeta} \frac{dF}{(\zeta - F)^{1/2}} \int_0^{F} \frac{(G-1)\, u_{k-1}\, d\xi}{(F-\xi)^{1/2}} \right]$$

therefore

$$Z_k(\zeta) = -\frac{1}{\pi}\int_0^{\zeta}\frac{dF}{(\zeta - F)^{1/2}}\int_0^{F}\frac{(G-1)\,u_{k-1}}{(F-\xi)^{1/2}}\cdot d\xi \qquad (2.41)$$

Now

$$\int_0^{F}\frac{[G(\xi,F)-1]}{(F-\xi)^{1/2}}\cdot Z'_{k-1}(\xi)\cdot d\xi$$

can be seen in comparison with $\displaystyle\int_0^{F}\frac{G(\xi,F)\,u(\xi)\,d\xi}{(F-\xi)^{1/2}}$

to be analogous to a group height as a function of frequency.

Since

$$\left.\begin{aligned} x &= f_N^2/f^2, \quad = \xi/F \\ \text{Let } \alpha &= F/\zeta \quad \text{where } \zeta = f_v^2 \end{aligned}\right\} \qquad (2.42)$$

Then we write (2.42) as follows

$$\zeta\alpha = F \qquad dF = \zeta\cdot d\alpha$$

$$xF = \xi \qquad d\xi = F\cdot dx$$

Let us write $\displaystyle\int_0^{1}\frac{Z'_{k-1}(xF)\cdot[G(x)-1]\,dx}{(1-x)^{1/2}}$ as $\phi_{k-1}(F)$ (2.43)

so that

$$Z_k = -\frac{2}{\pi}\int_0^{1}\frac{\phi_{k-1}(\alpha f_v)\cdot\alpha^2\,d\alpha}{(1-\alpha^2)^{1/2}} \qquad (2.44)$$

We can evaluate these two integrals considering $(1 - x)^{-\frac{1}{2}}$ as

a weight function over the interval (0, 1). The result is given by Kelso (1954)

$$\begin{aligned}\phi_{k-1}(f^2) = \; & 0.1338\, Z'_{k-1}(x_1 f^2)\,[\,G(x_1)-1\,] \\ & + 0.2992\, Z'_{k-1}(x_2 f^2)\,[\,G(x_2)-1\,] \\ & + 0.4382\, Z'_{k-1}(x_3 f^2)\,[\,G(x_3)-1\,] \\ & + 0.5382\, Z'_{k-1}(x_4 f^2)\,[\,G(x_4)-1\,] \\ & + 0.5905\, Z'_{k-1}(x_5 f^2)\,[\,G(x_5)-1\,]\end{aligned} \qquad (2.45)$$

here $x_1 = 0.0517$; $x_2 = 0.2523$; $x_3 = 0.5390$; $x_4 = 0.8125$
$x_5 = 0.9779$ (2.46)

Now one has to introduce the numerical values of $G(x_i)$. One makes a choice of a value of y (constant one) and for the propagation angle at this place. Since μ' is known (to be calculated) $G(x_i)$ will be easy to calculate, so that ϕ_{k-1} can be evaluated. Then one has to integrate the formula (2.46) for $Z_k(f_v)$ using $\alpha^2/(1-\alpha^2)^{\frac{1}{2}}$ as the weight function over (0, 1) and finally one gets $Z_k(f_v)$.

(2.47) ∴ The true height of reflection is therefore $\sum_k Z_k + Z_0$. The process is convergent in general according to Picard. Z_0 is the field-free value and the Z_k's are the corrections. Usually one can go as far as one's patience permits. Kelso has evaluated actually certain heights and he is of the opinion that a single h'-f curve needs twelve hours of labour.

After one has gone through the mathematical frame work one need not hesitate in remarking about the practicability of this method. More than that it does not appear reasonable to

choose only a fixed value for y and evaluate μ' and $G(x)$. This could well amount to be erroneous. At any rate this is certainly not suited for reduction of a large number of records.

Lastly, we shall outline the procedure followed at the Physical Research Laboratory in Ahmedabad. The method has been outlined by Jackson (1950). This method has been modified and has been used conveniently even with a desk calculator. It can be shown that this method can give the correction to the earth's magnetic field effects, that is, to the integral

$$\int \mu' \, dx$$

where $\int \mu' \, dz = \int \mu' \, dz/dx \cdot dx = \frac{dz}{dx} \int \mu' \, dx$ if taken in small steps

We know that

$$h_0 + \int_0^{z_r} \mu' \, dz = h'(f)$$

. Let us consider the integral $\int_0^{z_r} \mu' \cdot dz$

Now, $\mu' = \mu'(f^2, f_N^2)$ is a function of x i.e. f_N^2/f^2

$$\int_0^{z_r} \mu' \, dz$$

on changing the variable from z to x we have $\int_0^1 \mu' \cdot \frac{dz}{dx} \cdot dx$

$$\therefore \quad h'(f) = h_0 + \int_0^1 \mu' \cdot \left(\frac{dz}{dx}\right) \cdot dx$$

Let $x = x_0 = 0$ and $x = x_n = 1$ and let the interval $(0, 1)$ be divided into n parts we have then

$$h'(f) = h_0 + \int_{x_0}^{x_n} \mu' \cdot \left(\frac{dz}{dx}\right) dx$$

$$= h_0 + \int_{x_0}^{x_1} \mu' \cdot \left(\frac{dz}{dx}\right) \cdot dx + \int_{x_1}^{x_2} \mu' \cdot \left(\frac{dz}{dx}\right) \cdot dx + \cdots + \int_{x_{j-1}}^{x_j} \mu' \cdot \left(\frac{dz}{dx}\right) \cdot dx +$$

$$\cdots + \int_{x_{n-1}}^{x_n} \mu' \cdot (dz/dx) \cdot dx$$

If the intervals $(x_0, x_1), (x_1, x_2), \ldots, (x_{j-1}, x_j), \ldots, (x_{n-1}, x_n)$ are small enough then $\left| dz/dx \right|_{x_{j-1}}^{x_j}$ can be assumed constant. Therefore

$$\int_{x_{j-1}}^{x_j} \mu' \cdot \left(\frac{dz}{dx}\right) \cdot dx = \left(\frac{dz}{dx}\right) \cdot \int_{x_{j-1}}^{x_j} \mu' \cdot dx$$

putting $(x_j - x_{j-1}) = dx_j$ and let $dz_j = z_j - z_{j-1}$ then

$$\int_{x_{j-1}}^{x_j} \mu' \cdot \frac{dz}{dx} \cdot dx = \frac{(z_j - z_{j-1})}{(x_j - x_{j-1})} \cdot \int_{x_{j-1}}^{x_j} \mu' \, dx$$

since

$$\int_{x_{j-1}}^{x_j} \mu' \, dx = \int_0^{x_j} \mu' \, dx - \int_0^{x_{j-1}} \mu' \, dx \quad \text{and}$$

Letting

$$\int_0^{x_j} \mu' \, dx = G(x_j) \quad \text{we can write}$$

then,

$$h'(f) = h_0 + \sum_{j=1}^{n} \left(\frac{z_j - z_{j-1}}{x_j - x_{j-1}}\right) \cdot \left\{G(x_j) - G(x_{j-1})\right\} \tag{2.48}$$

μ' can be computed from

$$\mu^2 = 1 - \frac{2x(1-x)}{D}$$

and $$\mu' = \mu + f\,\partial\mu/\partial f$$

h_0 can be found out by the standard Appleton-Beynon method.

Then in the equation

$$h'(f) = h_0 + \sum_{j=1}^{n}\left[\frac{G(x_j) - G(x_{j-1})}{(x_j - x_{j-1})}\right](z_j - z_{j-1}) \qquad (2.48')$$

The only unknown is $(z_j - z_{j-1})$. This could be solved step-wise. That is

when $n = 1$ we have $h'(f) = h_0 + (z_1 - z_0).\left[\frac{G(x_1)-G(x_0)}{(x_1-x_0)}\right]$

i.e. $$[h'(f)-h_0] = \left[\frac{G(x_1)-G(x_0)}{(x_1-x_0)}\right].(z_1 - z_0)$$

We can find out $(z_1 - z_0)$: let us call this dz_1.

when $n = 2$ $$h'(f) = h_0 + \left\{\frac{G(x_1)-G(x_0)}{x_1 - x_0}\right\}(z_1 - z_0) + \left\{\frac{G(x_2)-G(x_1)}{x_2 - x_1}\right\}(z_2 - z_1)$$

Let us call $h'(f)$ when $n = 1$ as $h'_1(f)$

then $$h'_1(f) = h_0 + \left\{\frac{G(x_1) - G(x_0)}{x_1 - x_0}\right\}(z_1 - z_0)$$

Now, $$h'_2(f) = h'_1(f) + \left\{\frac{G(x_2) - G(x_1)}{x_2 - x_1}\right\}(z_2 - z_1)$$

i.e. $$h'_2(f) - h'_1(f) = \left\{G(x_2) - G(x_1)/(x_2 - x_1)\right\}.(z_2 - z_1)$$

so $(z_2 - z_1)$ is easily solved for.

In general we have from

$$h'_j(f) - h'_{j-1}(f) = \frac{G(x_j) - G(x_{j-1})}{(x_j - x_{j-1})}.(z_j - z_{j-1}) \qquad (2.49)$$

the solution of $(z_j - z_{j-1})$. Let us call $(z_i - z_{i-1})$ as dz_i

then the <u>true height including the effect of the field</u> is

$$z = \sum_{i=1}^{n} dz_i \qquad (2.50)$$

The problem now reduces to the evaluation of the integral

$$G(x_j) = \int_0^{x_j} \mu' \, dx \qquad (2.51)$$

putting $t^2 = 1 - x$ we have $2t.dt = -dx$ when $x = 0$ $t = 1$ and when $x = 1$ $t = 0$. Therefore

$$G(x_j) = -\int_1^{\sqrt{1-x_j}} \mu'.\, 2t\, dt = -2\int_1^{\sqrt{1-x_j}} (\mu'.t)\, dt \qquad (2.52)$$

that is $G(x_j) = -2 \times$ (area under the curve $\mu'.t - t$ lying between the t-axis and the ordinates at $t = 1$ and $t = \sqrt{1-x_j}$).

Thus $G(x_j)$ could be easily evaluated by numerical integration for different frequencies or values of y after evaluating for different frequencies. We have given the curves of for Ahmedabad, Delhi and England (see figures 11, 12 and 13).

There is another interesting point. When the effect of the field is ignored, we know that

$$\mu' = \frac{1}{\mu} = (1-x)^{-\frac{1}{2}} \qquad (2.53)$$

so that in a given distribution (when dz/dx is ~~known~~ the same) we have

$$h'(f) = h_0 + \int \mu' \cdot \left(\frac{dz}{dx}\right) \cdot dx = h_0 + \left(\frac{dz}{dx}\right) \cdot \int \mu' \, dx$$

$$= h_0 + \left(\frac{dz}{dx}\right) \cdot \int \frac{dx}{(1-x)^{1/2}} \qquad (2.54)$$

when the field effect is included

$$h'(f) = h_0 + \left(dz/dx\right) \int \mu' \, dx \qquad (2.55)$$

therefore the effect of the field on the height can be seen from the difference :

$$\int_0^x \frac{dx}{(1-x)^{\frac{1}{2}}} - \int_0^x \mu' \, dx \qquad (2.56)$$

for the same dz/dx.

$$\int_0^x \frac{dx}{(1-x)^{\frac{1}{2}}}$$

can be integrated easily by setting $(1 - x) = \theta$ and $dx = -d\theta$ the value of the integral is

$$\left[2 - 2(1-x)^{1/2} \right] \qquad (2.57)$$

therefore the difference is

$$\left[\left\{ 2 - 2(1-x)^{\frac{1}{2}} \right\} - \int_0^x \mu' . dx \right. \qquad (2.58)$$

Since $\int_0^{x_j} \mu' . dx = G(x_j)$ can be calculated, it is easy to see the effect of the field. Graphs of the G-functions $G(x_j)$ for different frequencies at Ahmedabad and Delhi have been prepared. A graph of $2 - 2(1-x)^{1/2}$ which we call $G_0(x)$ and $G_H(x) = \int_0^x \mu' . dx$ is given for $f = f_H$ at Ahmedabad, Delhi and England. (see figures 11, 12 and 13). A table of values of

$$\left\{ 2 - 2(1-x)^{1/2} \right\} - \int_0^x \mu' . dx$$

is prepared and the differences are shown as percentages. (see table) in Appendix.

The procedure for determining the true heights including the effect of the geomagnetic field.

(i) Determine the true height neglecting the field. The procedure for this has been given in chapter one. $= Z_0(f)$

(ii) Determine the true height including the effect of the field. The procedure is contained in the body of chapter two. i.e. determine dz_j from

$$h'(f) = h_0 + \sum_{j=1}^{n} \frac{dz_j}{(x_j - x_{j-1})} \cdot [G(x_j) - G(x_{j-1})]$$

then the true height $= Z = \sum_{j=1}^{n} dz_j$.

The value of x_i can be obtained from f_N^2/f_i^2 for different values of f_i. The value of $G(x_i)$ can also be obtained and tabulated. The value of h_0 is obtained from Appleton-Beynon's method (the justification for this is discussed in chapter three).

(iii) It will be seen that the true height obtained from (ii) above will always be larger than the value obtained from (i) above. This is so because the nett effect of the geomagnetic field is to retard the pulse (this has been discussed in chapter three) and therefore to increase the apparent height. This means that ~~the height of reflection in the absence of the geomagnetic field or the true geometric height~~ is less than the apparent height affected by the magnetic field. Calling the height obtained in (i) above as $Z_0(f)$ and the height obtained by including the field in (ii) above as $Z(f)$, we see that a wave which has traverse a height $Z(f)$ has effectively suffered a retardation $Z(f) - Z_0(f)$. But the true height $Z_0(f)$ has been derived in case (i) above from a virtual height which has already suffered retardation due to the presence of the geomagnetic field although we have neglected its effect in the calculation of the group refractive index. Therefore the true geometric height corrected for the retardation due to the geomagnetic field is therefore

$$Z_0(f) - \left\{ \frac{Z(f) - Z_0(f)}{Z(f)} \right\} . \; Z_0(f)$$

or $$Z_0(f) \left\{ 1 - \frac{Z(f) - Z_0(f)}{Z(f)} \right\} \qquad (2.59)$$

This we shall call the true geometric height $Z_g(f)$ of a pulse of mean frequency f. This is less than the height derived by neglecting the effect of the field.

We shall give in chapter three a few instances of the height distribution of electron density corrected for the effect of the geomagnetic field for January 1957 at Ahmedabad. It can be seen from the table showing the error in neglecting the effect of the field at Ahmedabad and Delhi that the correction at Delhi being the greatest, for all Indian ionospheric stations will be the maximum limit of the error for all Indian ionospheric stations. The effect at Ahmedabad is about 5.2 % (shown in graph) and is only of the order of experimental uncertainties in the ionosonde height determination. This is discussed in chapter three. The error for South-East England is about 19.4% and is very considerable as compared to that for Delhi which is about 9.0%.

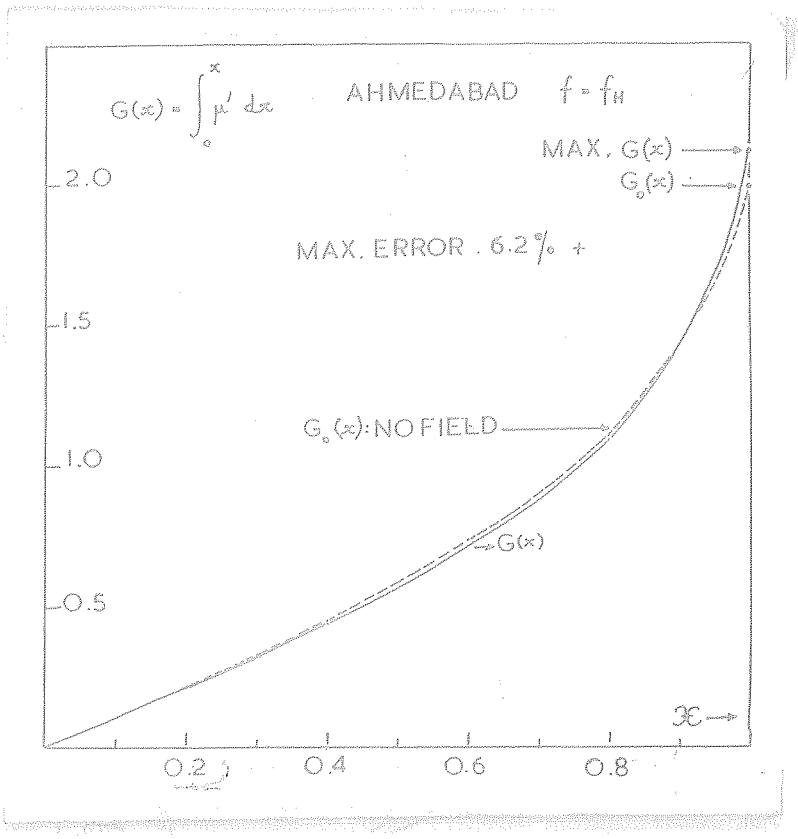

$G(x) = \int_0^x \mu' \, dx$
AHMEDABAD $f = f_H$
MAX. $G(x)$
$G_0(x)$
MAX. ERROR . 6.2% +
$G_0(x)$: NO FIELD
$G(x)$
2.0
1.5
1.0
0.5
x
0.2
0.4
0.6
0.8

Fig 11

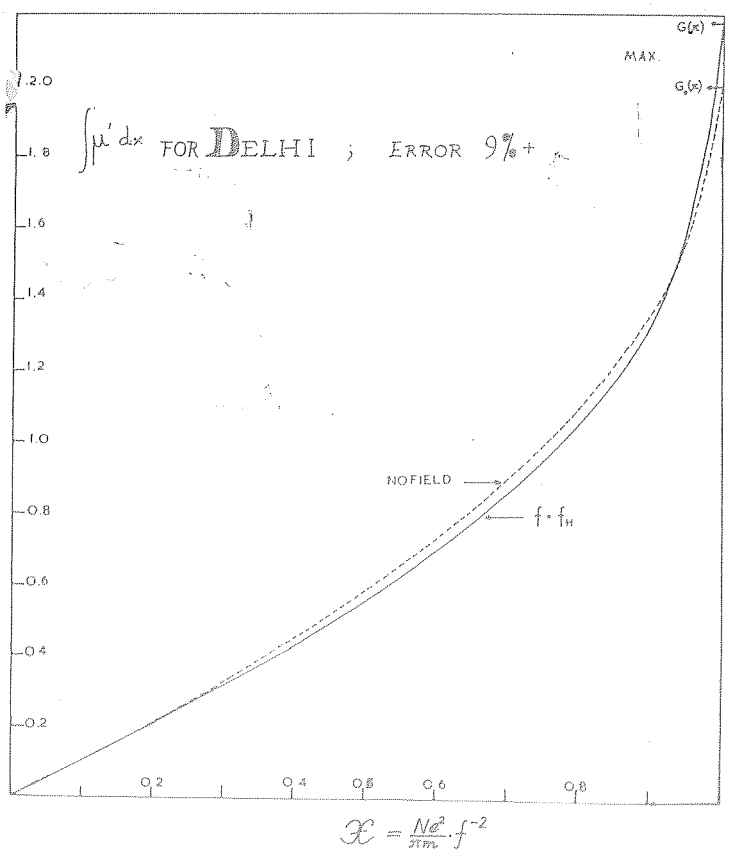

$G(x)$
MAX.
$G_0(x)$
$\int \mu' \, dx$ FOR DELHI ; ERROR 9% +
NO FIELD
$f = f_H$
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
$x = \frac{Ne^2}{\pi m} \cdot f^{-2}$

Fig 12

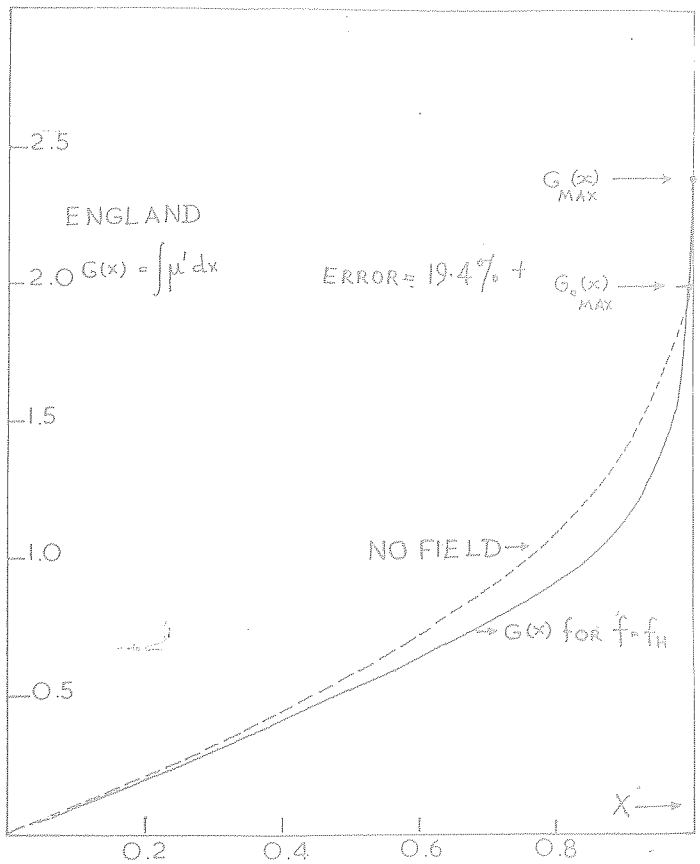

2.5
ENGLAND
2.0 G(x) = ∫μ' dx
ERROR = 19.4% +
G (x) MAX
G₀ (x) MAX
1.5
1.0
NO FIELD
G(x) FOR f = f_H
0.5
X
0.2
0.4
0.6
0.8

Figure 13

CHAPTER THREE

The distribution of electrons in the ionosphere over Ahmedabad on some selected magnetically quiet days during 1954 - 1956

CHAPTER THREE

The distribution of electrons in the ionosphere over Ahmedabad on some selected magnetically quiet days during 1954 - 1956

1. Introduction

We have examined the methods of derivation of the distribution of electrons in the ionosphere. We shall now present the results of the calculation of true heights on some selected days over Ahmedabad. We first outline briefly how the ionograms are analysed and later describe the major features of the distribution, the parameters of the distribution and the total electron content in a unit column of the F region of the ionosphere. The Results are summarised at the end and a comparative study of the features is presented ~~at the end~~.

The Physical Research Laboratory at Ahmedabad has an ionospheric station attached to it and systematic studies are in progress to understand several aspects of the physics of the ionosphere. The Indian Council of Scientific and Industrial Research is financing the project and the station is in charge of Professor A.M.Ramanathan.

The ionospheric sounding equipment was installed in 1953 at the laboratory's research station (23°02'N latitude, 72°38'E longitude) and is of the British NPL type. It has been described in detail in earlier studies published by

and the Doctoral dissertations presented by Messrs Rastogi (1956) and Kotadia (1958). The author's problem was to study the height distribution of electron density in the ionosphere based on the analysis of some selected ionograms. He has had the opportunity of working with the radio group and would like to express his thankfulness for all their help. A study of the total electron content in a unit column of the F region over Ahmedabad based on some selected records taken in 1953 (January to December), has already been reported by Sheriff (1956). He used a method outlined by Ratcliffe (1951). We shall return to his results a little later.

The need for a systematic investigation into the fine structure of the ionosphere based on an accurate and rigorous method of height derivation had been felt by the radio group. So the present study was commenced by the author in April, 1956. In the first phase of the investigation, the distribution of electrons was calculated on magnetically quiet days without taking into account the effect of the earth's magnetic field. At a later stage the effect of the magnetic field was calculated. The laboratory does not possess modern electronic computing facilities. All the computations were made by the author with the help of an electrically driven desk (Facit) calculator. The author had to restrict himself to a few typical days in each of the four seasons - Autumn, Spring, Summer and Winter - in each of the years 1954 to 1956 because of the large time involved for the computations. The following months were selected as typical - January, April, June/July and October. A set of five

and ten internationally quiet days is published every month by the Central Radio Propagation Laboratory of the National Bureau of Standards, Boulder, Colorado, United States of America. The quiet days were selected on the following criteria :

(a) The record should be clear and good throughout the 24 hour-period, and

(b) The day must be magnetically quiet. It was easy to select the days since Mr Kotadia scales and tabulates ionospheric data for each day regularly and the 'quietness' of the ionosphere could be ascertained from the variation of the critical frequencies.

2. The procedure

The ionograms were read region by region every hour on each day analysed and a set of as many pairs of values of h' and f as could be read was written down. Two tabular forms were prepared by the author setting down the procedure (forms given in Appendix).

The Ahmedabad ionosonde records the h' - f profiles at every integral hour (75° E.M.T.). The height markers are set 50 km apart and the frequency scale is marked for every 0.1 Mc/s. The frequency band goes from 1.5 Mc/s to 25 Mc/s. Frequencies could be read to an accuracy of 0.1 Mc/s. The pulse width is about 150 microseconds. (Approx. 45 km). The lower end of the profile trace is taken for reading heights. Heights could be read correct to about 12.5 km when there are

no multiple reflections. In the latter case of course the accuracy in reading heights could be increased. The ionogram is transferred on to a linear graph paper. The frequency axis is set to a scale 1 Mc/s = 2 cm. The critical frequency is read off. We now have say,

frequencies f_1 f_2 f_3 ....f_i ....f_{n-1}, f_c

virtual hts. h'_1 h'_2 h'_3...h'_i.... ... h'_{n-1}, ∞

One then calculates the ratios

$$f_1/f_c \;,\; f_2/f_c \;,\; \ldots \;,\; \ldots\ldots\ldots,\; f_{n-1}/f_c$$

and $\psi(f/f_c)$: ψ_1 , ψ_2 , ... , ..., ..., ψ_{n-1}

from the values given in the appendix. The table of values of this function as has been said earlier were prepared from Booker and Seaton's (1940) table. We use ξ for f/f_c and x for f_N^2/f^2. This notation will be uniform in our treatment. The function $\psi(\xi)$ as well as ξ will be given correct to two places of decimals. The plot of h' vs $\psi(\xi)$ will therefore yield us the basic parameters h_o, h_m and τ the base, height of maximum ionization and the semi-thickness on a parabolic distribution model. We have

$$h'(f) = h_o + \tau\left[\tfrac{1}{2}\cdot(f/f_c)\cdot \log_e\{(f_c+f)/(f_c-f)\}\right]$$

or $$h'(f) = h_o + \tau\cdot\psi(\xi) \text{ , briefly ,}$$

The plot of h' - $\psi(\xi)$ is a straight line. The intercept of this line on the height axis gives h_o. The ordinate at $\psi(\xi) = 1$ intersects the line $h' = h_o + \tau\cdot\psi$ at a point whose height-coordinate = h_m and $\tau = h_m - h_o$ i.e. $h' = h_m$ (when $\psi = 1$) $= h_o + \tau$

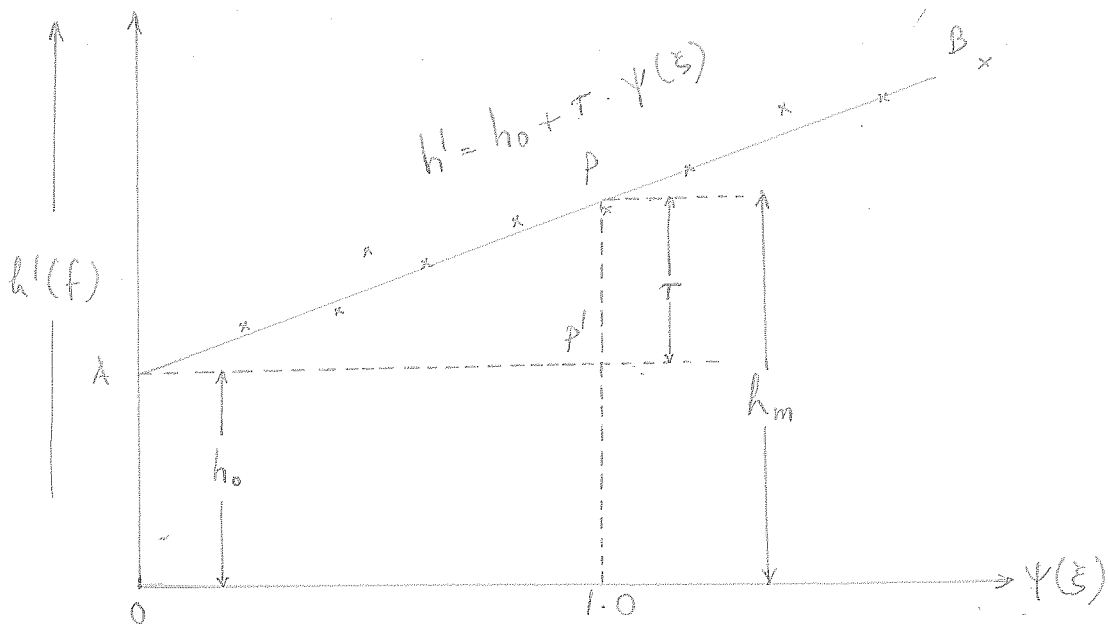


Figure 14

slope = $PP'/AP' \Rightarrow PP' = h_m - h_o = \tau$ (since $AP' = 1$)

The purpose of doing this is two-fold.

(a) To test whether the region is parabolic, which will be the case if all the points fit well on a straight line ; the scatter from linearity will reveal the deviation from parabolicity.

(b) To get an idea of the base of the region. The analysis for deriving the formula

$$h(f_N) = \frac{2}{\pi}\int_0^{f_N} \frac{h'(f)}{(f_N^2 - f^2)^{1/2}} \cdot df$$

rests on the condition that the virtual height is given from $f = 0$ to $f = f_N$. This is one of the differences between our method and that of Schmerling and Thomas (1956). Schmerling and Thomas have extrapolated beyond the minimum height of reflection to $f = 0$ at the same value. This will tend to increase the true height especially when one has not allowed for the retardation in lower regions as they have done. They have explained this

by saying that since the major contribution to the real height comes from the region of the h'f profile nearer the critical frequency, the extrapolation at the same value of h'_{min} will not invalidate their results. But the left end of h'-f profile could be out by a considerable height when one has not allowed for retardation in the lower regions. This especially ~~so~~ happens when there is a thick F1 region and we are considering reflections from F2.

and -(c) To examine and appreciate the differences between h_pF2 as tabulated by routine method of scaling reported by ionospheric observatories. The value obtained by the Appleton-Beynon method (by the straight line fit), the value given by true height calculation and the value tabulated as h_pF2 do not always agree. It has been observed that h_pF2 which is equal to the virtual height at 0.834 times the critical frequency of F2 or for that matter the h_p of any region is mostly greater than h_m obtained by the Appleton-Beynon method $\geqslant$ true h_m i.e. the true height at $f = f_c$ (note that while the virtual height at $f = f_c$ is infinite the true height is not). The reason for the discrepancy is that whereas h_p is scaled at 0.834 f_c, h_m takes into consideration whether the distribution is parabolic and if it is so gives the value obtained by the points that fit the straight line (that answer parabolicity). It is natural to expect, experimentally that not all points will sit on a straight line. In the former case (i.e. in the case of h_pF2) one does not know whether the distribution is at all parabolic.

When once the h_o, h_m and τ have been determined the later analysis is mechanical. In form 8 is given a table which gives

f_N	f_1 f_2 f_3 f_4 f_5 h'_1 h'_2 h'_3 h'_4 h'_5	$\sum_{i=1}^{5} h'(f_i)$ sum	$1/5 \sum_{i=1}^{5} h'(f_i)$ mean height	N in 10^5/cc

f_N is the plasma frequency for which the height to which a pulse whose frequency is equal to the plasma frequency ascends. f_1 to f_5 are the five frequencies given by the quadrature method (the value of the a_j's) at which we read off the virtual heights. The mean of these heights is the true height of reflection of a pulse whose frequency $f = f_N$. The table gives, in the Appendix, the f_i's for f_N going from 1 to 25 Mc/s in steps of 1 Mc/s. $N = 1.24 \times 10^4 \times f_N^2$ is also given in units of 10^5 electrons per cc. at the extreme right. We therefore have a set of values of h and N. This is the height distribution of electron density.

Case (2) When there are more regions than one.

The true height derivation in this case is a little more time-consuming. One has to repeat the above procedure for the lowest region. One has then to subtract from each virtual height of the next higher region the group retardation corresponding to the pulse frequency. If a pulse of frequency f is traversing a region whose critical frequency is f_c to get reflected from a region above it, it suffers a retardation as long as $f_c < f < 2f_c$. In such a case one calculates the value $\Delta h' = \tau \left[f/2f_c \cdot \log\left(\frac{f_c+f}{f_c-f}\right) - 1 \right]$ and subtracts it from the virtual height and then proceeds with the analysis. When we are

considering height of reflection from say the F2 region one has to allow for retardation in all the lower regions. i.e.

$$h'(f) - \left\{ (\Delta h')_E + (\Delta h')_{F1} + (\Delta h')_{F1.5} \right\}$$

The second formula of Ratcliffe (1951) namely

$$\Delta h' = \xi \log_e \left\{ \frac{1+\xi}{(1 - f_A^2/f_c^2)^{\frac{1}{2}} + (\xi^2 - f_A^2/f_c^2)^{\frac{1}{2}}} \right\}$$

is to be used when pulses are not traversing complete parabolic regions. The retardation in the E region for a pulse which is returned from the F2 may not be very significant but it does contribute for answering the test for parabolicity. This is the second point of difference between our approach and that of Schmerling and Thomas (1956) and that of Thomas, Haselgrove and Robbins (1958). We believe that our results are not vitiated by the retardation in the lower regions. When this has been done we have the complete distribution of ionization in all the regions of the ionosphere.

One of the most gratifying conclusions is that the true distribution of electrons is to a satisfactorily high degree parabolic. This is particularly interesting when taken in conjunction with the result of Duncan (1956) that a Chapman region retains its shape under gravity, diffusion and vertical drift.

The results of the distribution of electrons are best presented diagrammatically. We therefore give the iso-electron density levels for the four seasons typified by January, April, July and October in the three years of growing solar activity

1954-1956 sequentially. From form 8 it can be seen that we have readily heights at densities $k.1^2$, $k.2^2$, ... etc. where $k = 1.24 \times 10^4$. The numerals squared are the frequencies corresponding. If we plot these heights we get what we call iso-electron density levels. These are the levels which reflect pulses of frequency 1, 2, 3, ...Mc/s. The circles marked at about 6 and 18 hours in the diagrams showing iso-dense levels on quiet days are the times of sun rise and sun set at those levels. This enables one to see readily the sun rise and sun set effects. The diagram presents sequentially the fine structure of the ionosphere. (King (1956) has given one such diagram). This diagram from night to night reveals an expansion and contraction in the ionosphere. If one restricts himself to any particular level it also shows the fluctuation in the electron density at that constant level. The nature of variation of the height of maximum electron density is to be seen from a upper most curve at the top beyond which the ionosonde beeps down nothing. It appears as though the three fibres representing the density levels $k.1^2$ $k.2^2$ and $k.3^2$ swing down after sun rise and bump up after sun set. The ionosphere is more shrunk during the dark hours revealing a higher gradient, while it relaxes and expands during sunlit hours. The formation of E, F1 and /or F1.5 appears as though a few chains snap out and form a ring and there appears a discontinuity in the gradient of electron density and we see different region profiles in the ionograms. These can be seen from the figure. Normally the F1 region is one of low density gradient while F2 and E have a higher

density gradient. We shall illustrate a few more points with reference to the following diagram.

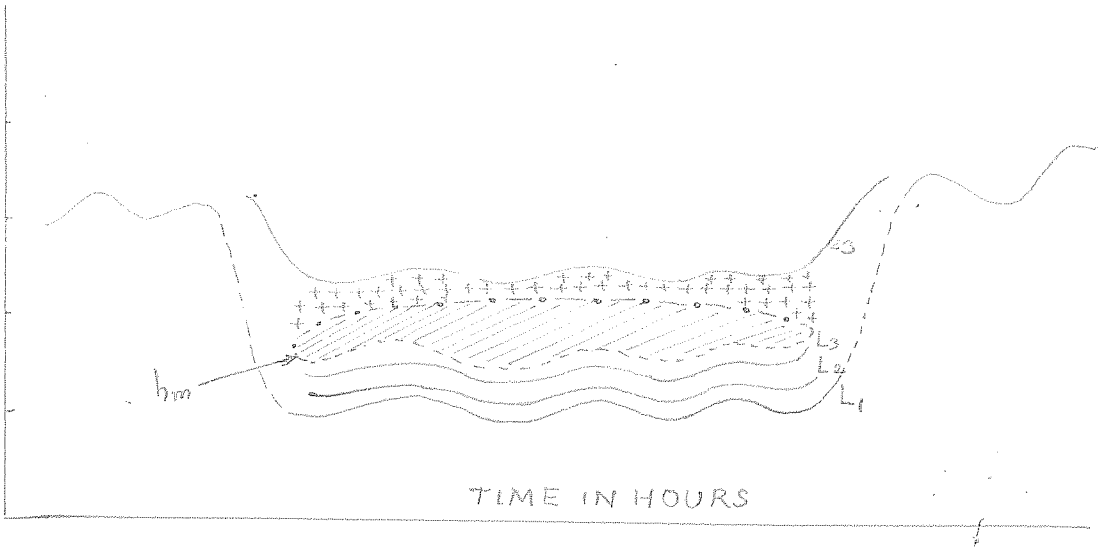


Figure 15

We have the three levels say, L_1, L_2 and L_3 where the densities are proportional to 1^2, 2^2 and 3^2 in the region E. We have then the h_m- line. If we complete mentally the curve by -.-.-.-. meaning thereby that at some height above it such a density level could exist, then the oval shaded area represents a region of density equal to or higher than the density $k\ f_c^2$. If it were higher then the level of maximum electron density would have gone a little higher. We have another 3-line in F1 also. This is not meaningless because we have allowed for the retardation in E for pulses getting reflected from F1 so that F1 could be considered as an independent region for purposes of true height calculation. It is not unlikely that the electron density falls off beyond the height of maximum density of any region below the value $k.f_c^2$. Otherwise there would have been no h_m level at all.

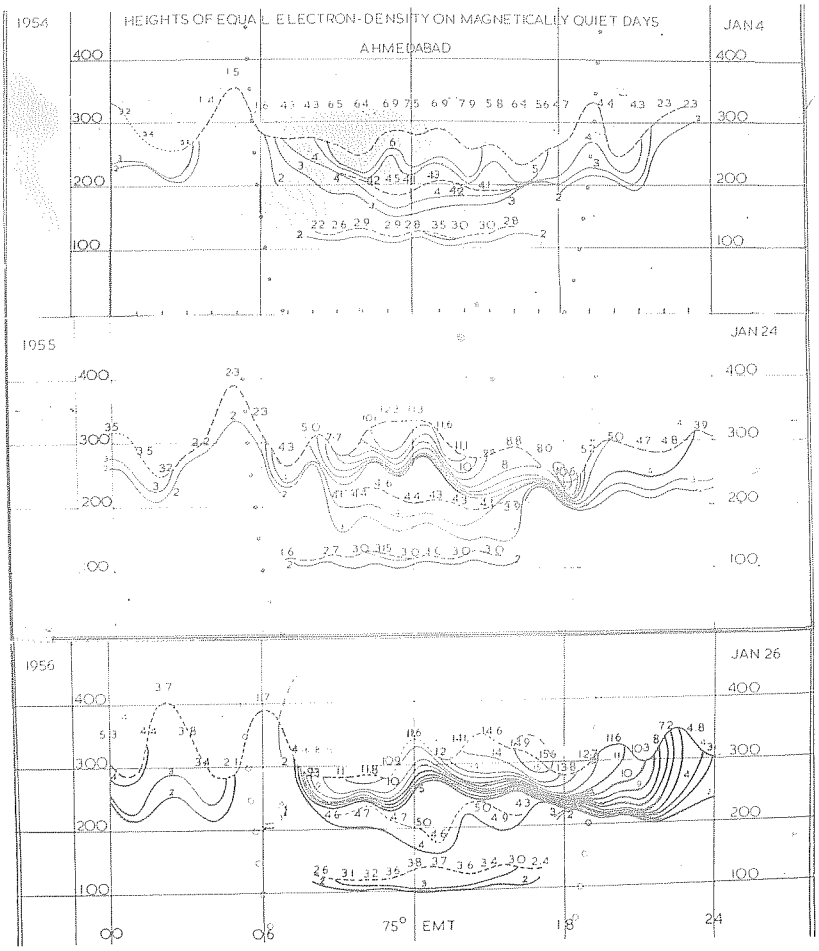


FIGURE 16

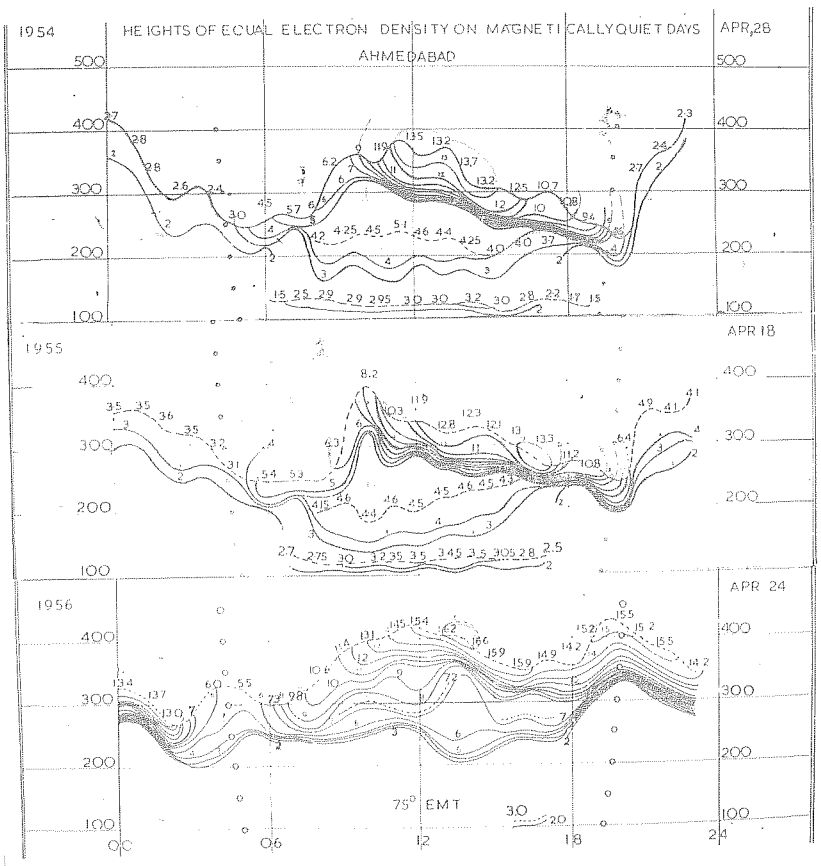


FIGURE 17

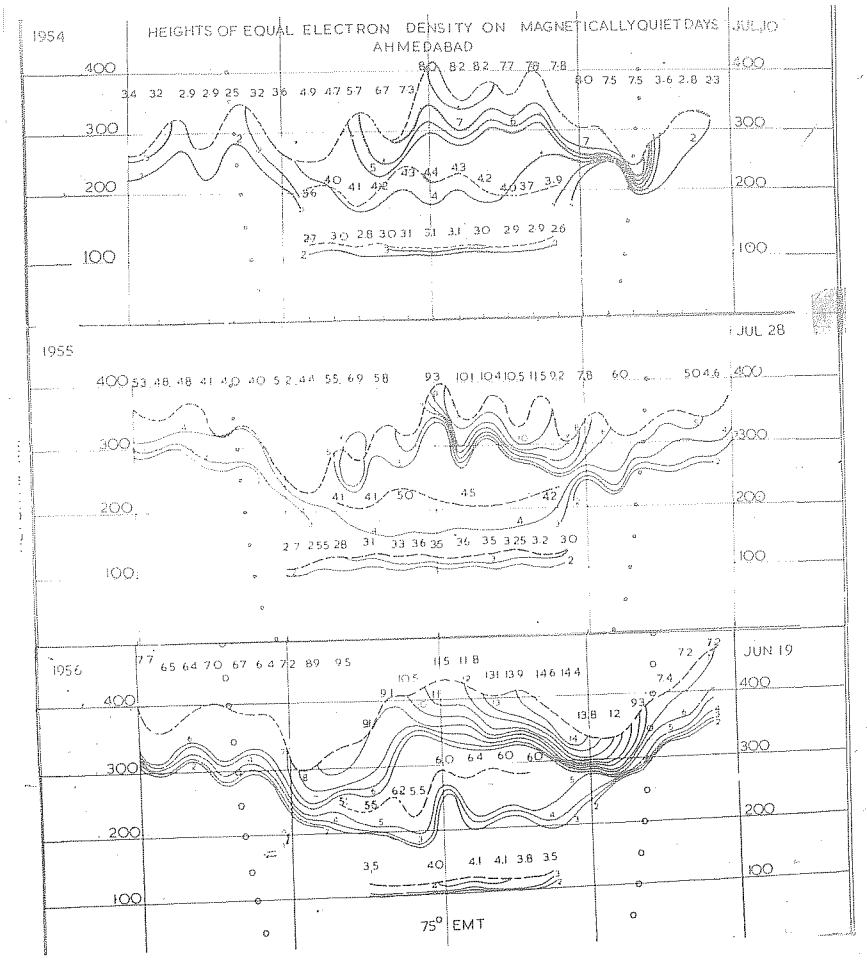


FIGURE 18

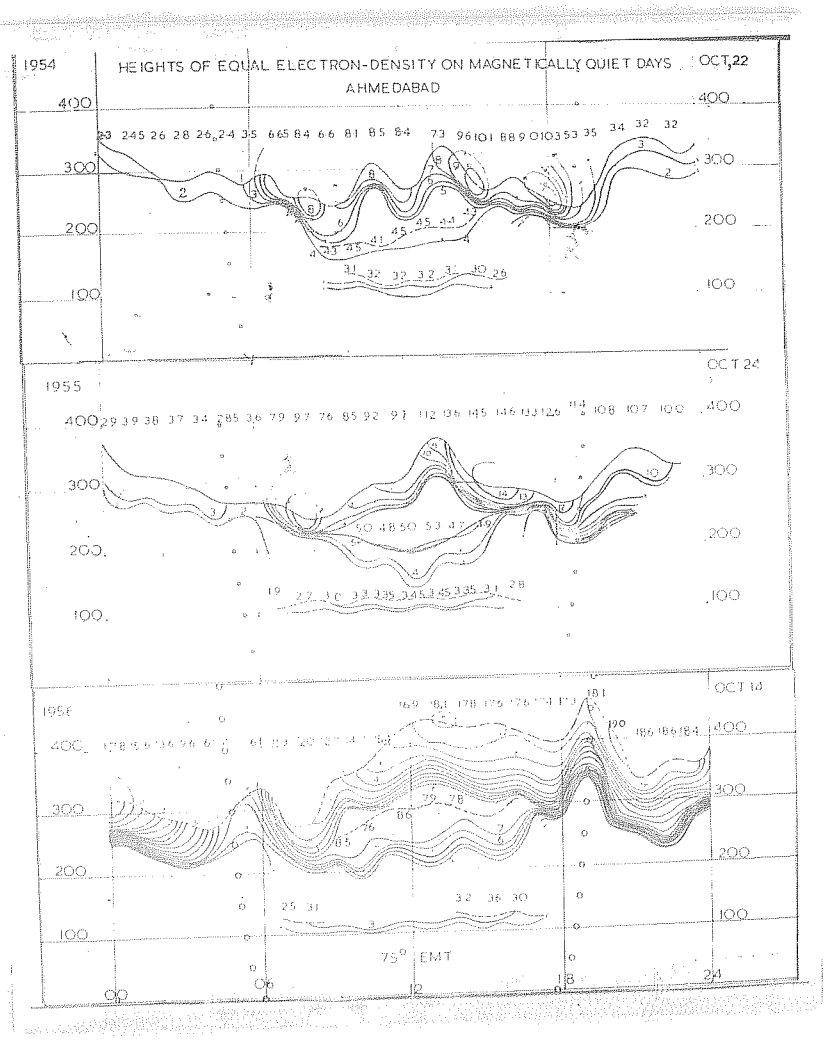


FIGURE 19

Rocket results bear evidence to this observation. But by ionospheric methods we are unable to isolate regions of decreasing electron density or valleys as they are called. But it is possible to estimate the extent in height of these valleys as follows. Let us take a look at a typical electron distribution curve.

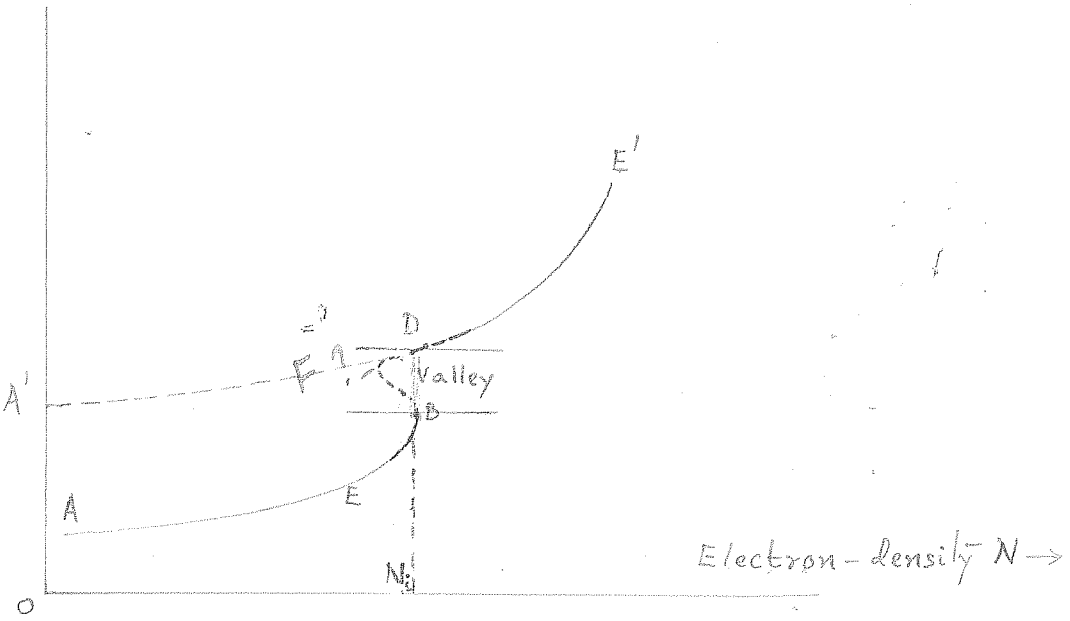


Figure 20

The curve AEB is the h(N) curve for the E region, say. What we know from Gaussian methods is the distribution AB. We know the distribution A'D', the h(N) curve for region F1. Since we have allowed for the retardation in E, for pulses getting reflected from F1, the distribution F1 is independent of the E distribution. The part of the F1 distribution AD is what it would have been had there been no E distribution or the whole thing would have come down. If the distribution above B has to fall in density and again rise to a value higher than E-density values, since the atmosphere is a

continuous medium, the curve has to pass though a point of inflexion. Hence there ought to be two heights with the same density (e.g. in the above figure that given by ON_1). Therefore if we allow for retardation in a lower region it is not meaningless to speak of another height with a density equal to that observed a little below. However it should be remarked that we allow for retardation at best in a half parabolic region that is for retardation below the level of maximum-electron density. Since it is not possible to determine with precision the actual shape of the distribution curve above the first maximum we cannot ~~therefore~~ make any legitimate allowance for retardation above the maximum. This in our opinion should tell us therefore the width of the valley ~~are~~ or gap in the iso-dense level pattern. Although it is not in principle possible to isolate the region of inflecting electron density (region marked +++++) it is possible at least to get a fair idea of the vertical extent of the gap. It is not very meaningful to speak of that part of the F1 distribution represented by AD since we have no practical evidence for it on the ionogram. The distribution AEB may well go off above as AEBFD or it may be different. Similar arguments could be advanced for the F1 and F1.5 ovals. Such diagrammatic representation of density levels is illuminating since they tell us about the dynamics of the ionosphere. This is particularly so when we consider the ionospheric response to a magnetic storm. This will be discussed in greater detail under chapter four.

A'B

Another use of true height curves is in studying the

density fluctuations in the atmosphere at various heights. This should throw light on the state of events happening at certain crucial heights like 100-110 km; 180-220 km; 250 km etc., which mark in a way the boundaries of regions. Secondly it could tell us about the penetration of solar radiation at different levels.

Lastly we shall study the variation of the parameters h_o, h_m, τ and n_T. They are plotted on a graph and are shown below. The total electron content in a unit column of the ionosphere below h_m is

$$n_T = 2/3 \; . \tau . \; N_{max} \quad \text{expressed in } 10^{12} \text{ electrons.}$$

This is a very useful quantity as it gives an idea of the total ionization. Ratcliffe reported that the variation in n_T is more faithful to cos χ than the critical frequency variation (Ratcliffe 1951). Sheriff of our laboratory found out n_T by finding τ from Ratcliffe method. By n_T we shall mean the total electron content in region F1 + region F2. The nature of the variation of n_TF2 is not altered by including the contribution from the F1 region contrary to Chatterjee's results. This was observed by Sheriff (1956) also. (See Chatterjee 1953). By averaging over a large number of days the variation of n_T would be more faithful to the variation of cos χ .

3. Effect of the earth's magnetic field.

The distribution studied earlier neglected the effect of the geomagnetic field. The effect of the magnetic field was

worked out later. It is satisfactory to find that the initial neglect of the field did not seriously vitiate the results of our study. It has been shown in chapter two that the percentage error

$$\left(\int \mu' dx\right)_{no\text{-}field} - \left(\int \mu' dx\right)_{with\text{-}field}$$

is not greater than about 6.2% for Ahmedabad. Assuming the pulse width to be about 150 microseconds, it can be seen that the reading errors due to the profile width alone are of the order of 22.5 km. Heights could be read to an accuracy of 12.5 km since the height markers are 50 km apart, normally, so that while reading true geometric heights of say 350 km we would be correct with an error margin of ± 10%. Secondly this work of ours is only a pilot study in Ahmedabad and it was not possible to have the analysis made with more data. It was therefore thought that the results could be best appreciated from graphs and we have given full illustrations by means of diagrams.

The first series of diagrams illustrate the shape of the distribution. It would be seen that on most occasions the distribution is very satisfactorily parabolic. For each day there are at least six diagrams representing :

(a) The midnight distribution - around 00 hour.

(b) The pre-sunrise or near sunrise distribution round about 0600 hour.

(c) The midday distribution.

(d) A post-sunrise distribution round about 0900/1000.

(e) The pre-sunset and post-sunset distribution. It was thought desirable to illustrate by as many as six figures because the striking evidence for parabolicity simplified many physical considerations and lends full meaning to the total electron content in a unit column of the ionospheric region from its base to its first maximum level of electron density. It is easier and at the same time more meaningful to speak in terms of the n_p rather than in terms of N_m and h_m. On inspection one can see that in the figures marked 'Ch' the distribution resembles a Chapman's distribution or as we shall term it, it is 'Chapman-like'. It has been shown earlier that the distribution which is a first satisfactory approximation to the Chapman distribution is the parabolic distribution. We have observed that where the distribution has not been parabolic, it has been Chapman-like. It is significant that the shape of the distribution is not other than either the parabolic or the P-type or the Chapman type - the Ch-type. While the thesis was being written there appeared a paper by Duncan (1958) which reports that the distribution during night is Chapman-like over Brisbane. There is a point of difference here.

One knows that on an ionogram the h'-f trace does not start from f = 0. One also knows that for obtaining the true height-frequency profile one should have a continuous curve of h'-f. This follows from the consideration that N is a monotonically increasing function of height and that it is given from N = 0 to N = N_{max} for a region. One has therefore

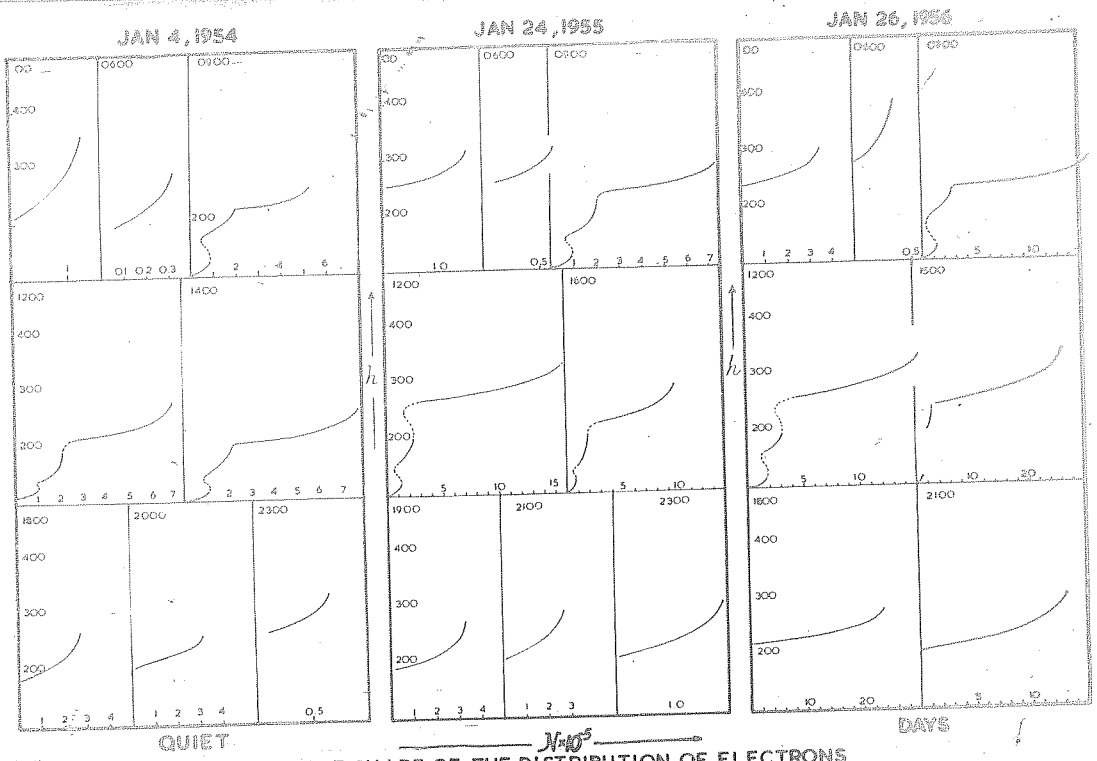


Figure 21

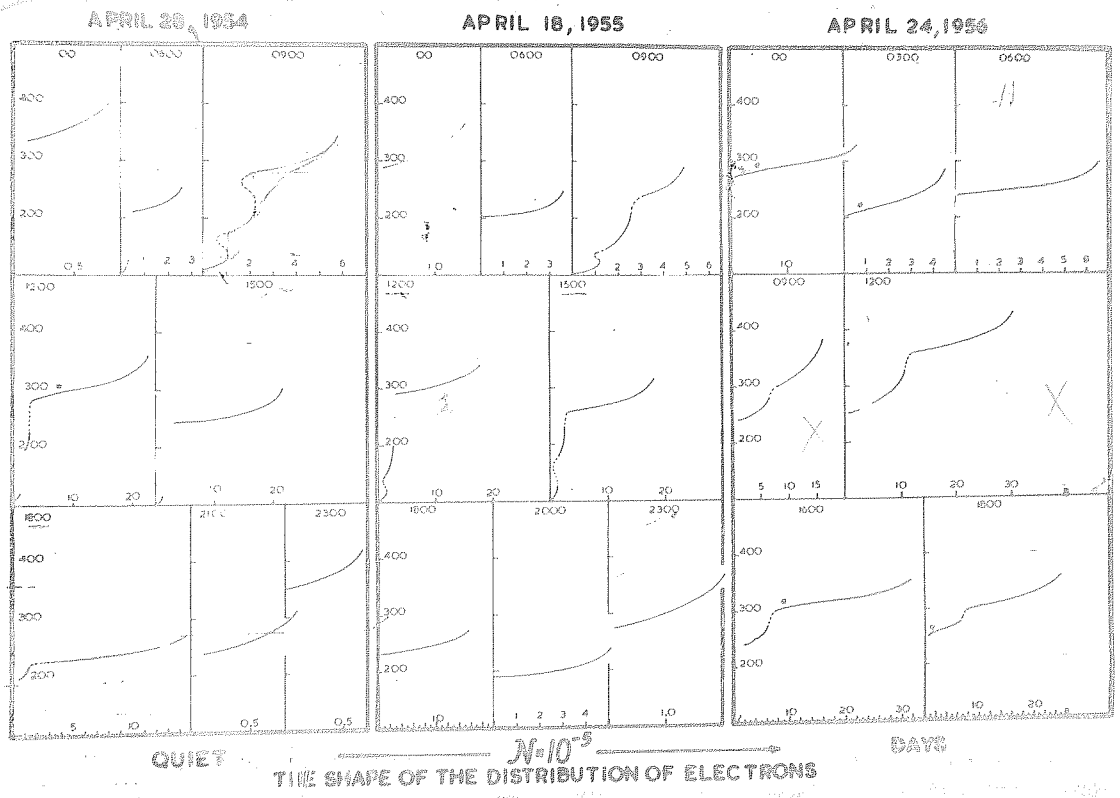
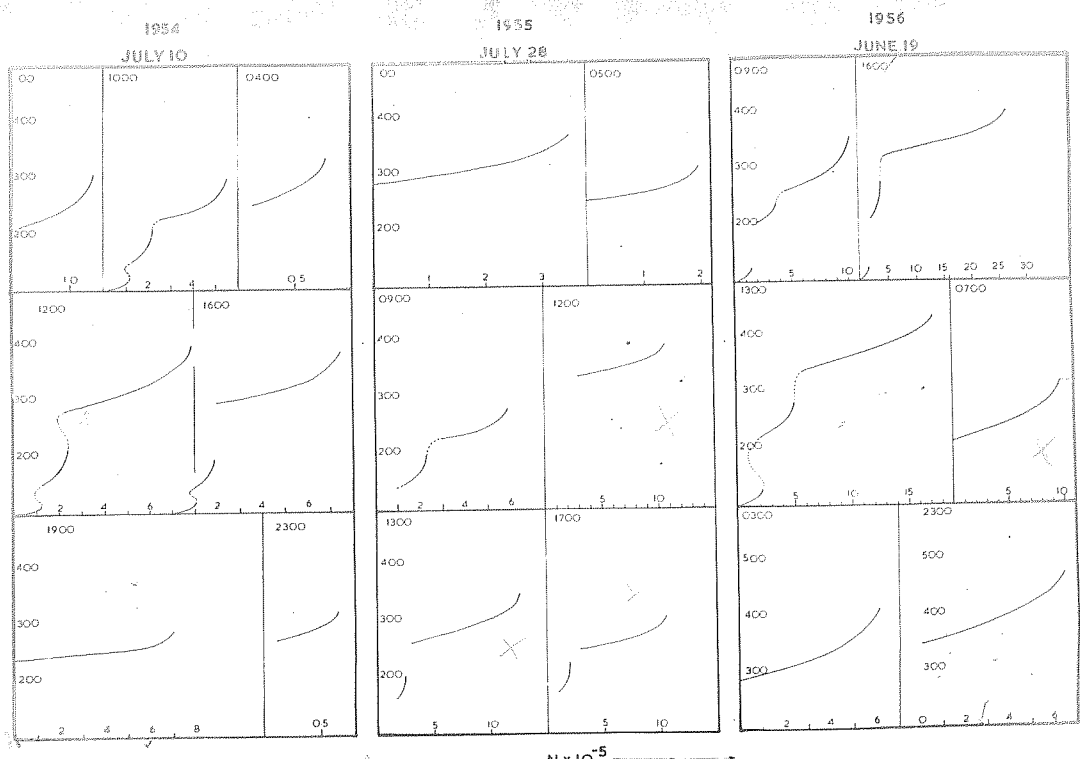
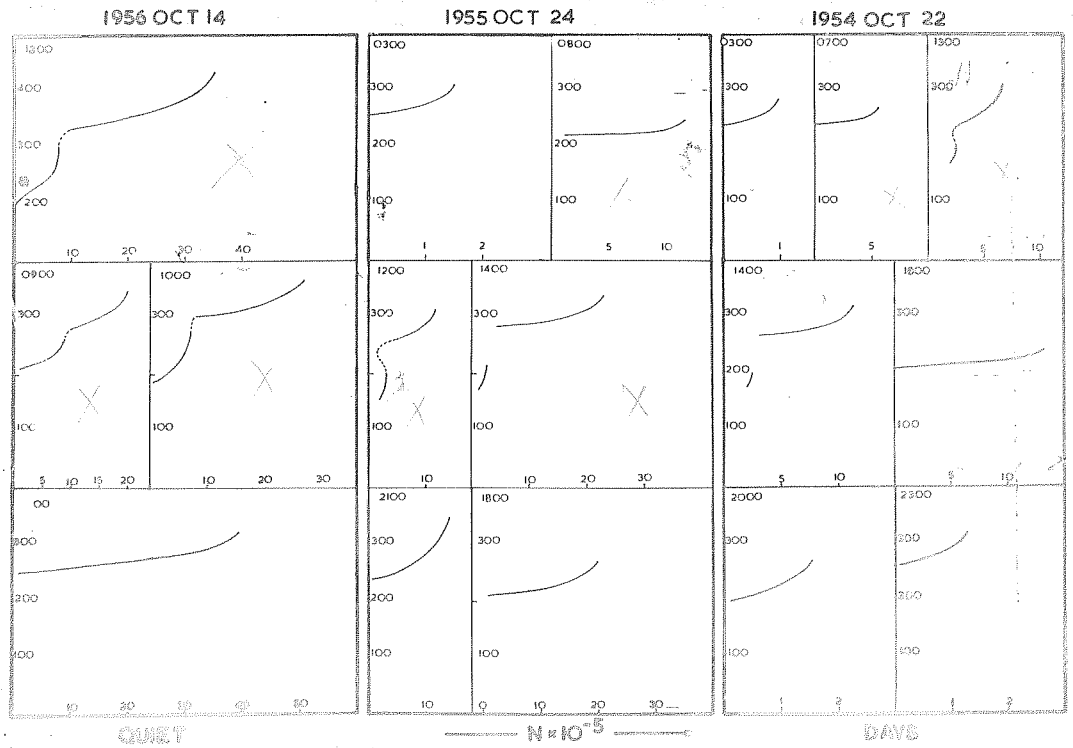


figure 22



THE SHAPE OF THE DISTRIBUTION OF ELECTRONS (QUIET DAYS)

Figure 23



THE SHAPE OF THE DISTRIBUTION OF ELECTRONS

Figure 24

to extrapolate the h'-f curve beyond a certain frequency say f_{min} for which we have h'_{min} to zero frequency. Schmerling and Thomas (1956) have extrapolated h'_{min} at the same value to f = 0. Duncan (1958) and Jackson (1956) have extrapolated on the assumption that the density falls to zero linearly beyond h'_{min} or f_{min}. Duncan has given a formula for obtaining the extrapolated value of height at f = 0 while Jackson has given an essentially similar procedure graphically. If the h'-f trace in the neighbourhood of f_{min} were almost parallel to the f axis, Schmerling and Thomas's method would not be inaccurate. If the h'-f trace is not parallel to the f axis, as is often the case, this would tend to increase the height of reflection. Besides this point there is the serious objection that the virtual height traces are not corrected for retardation in lower regions when they are present in the work of Schmerling and Thomas and Thomas et al. We do not know if the retardation has been corrected for in the later work of Thomas and others (this is to appear as we are told in their paper of 1958). Likewise, Duncan and Jackson's extrapolation will tend to lower the height of reflection by extrapolating the virtual height to f = 0 linearly and in addition will bias the lower part of the true height profile in favour of a Chapman-like distribution. This could be seen as follows.

We have exaggerated the extrapolation somewhat to illustrate what could happen to the shape of the distribution in the figure given overleaf. The method of our extrapolation

seems to avoid ~~to~~ much error on either side. The Appleton-Beynon test gives a very satisfactory answer in favour of parabolicity and the virtual height profile has been extrapolated accordingly.

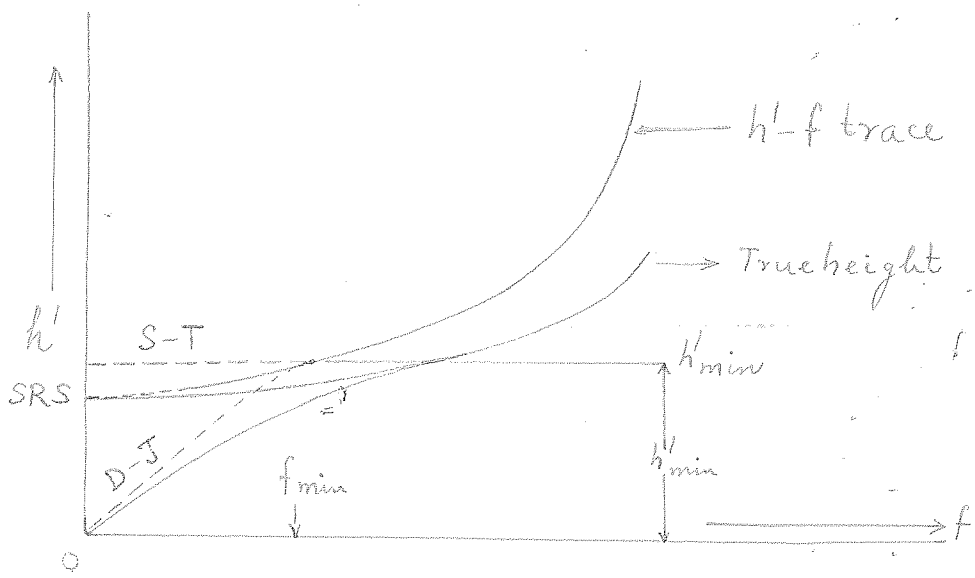


Figure 25

The significant point that emerges out is that the distribution is sometimes Chapman-like even during sunlit hours besides being so sometimes during night. This could be noticed from the N-h diagrams. It is on this account that it was thought worthwhile to illustrate this point in detail. The percentage of occasions of the departure from parabolicity to Chapman-like of the distribution is very small. We can therefore conclude the section with the remark that the distribution over Ahmedabad is parabolic on most occasions and Chapman-like on some.

It may be recalled that Martyn has shown that any

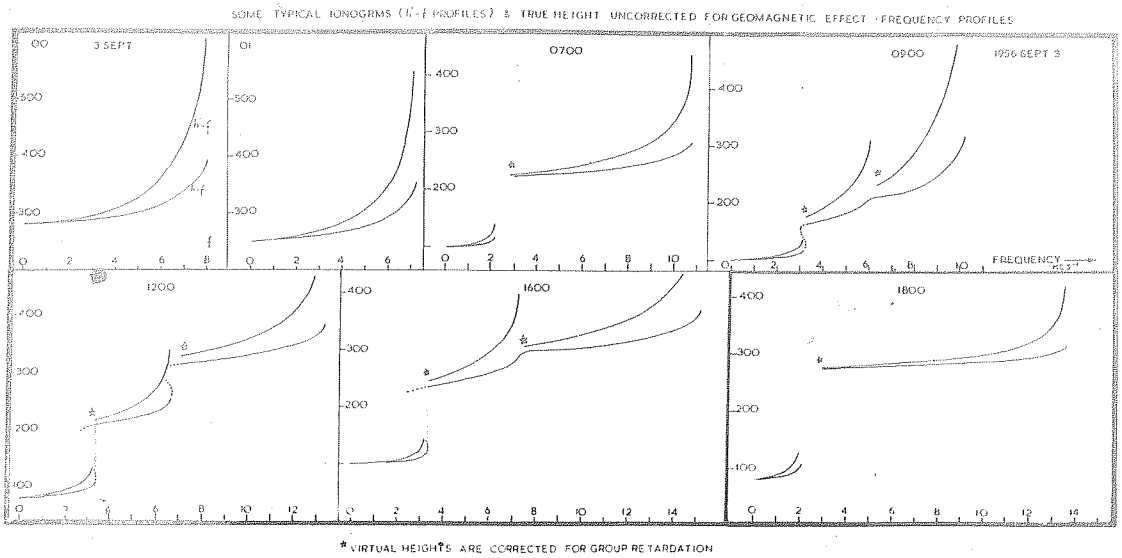
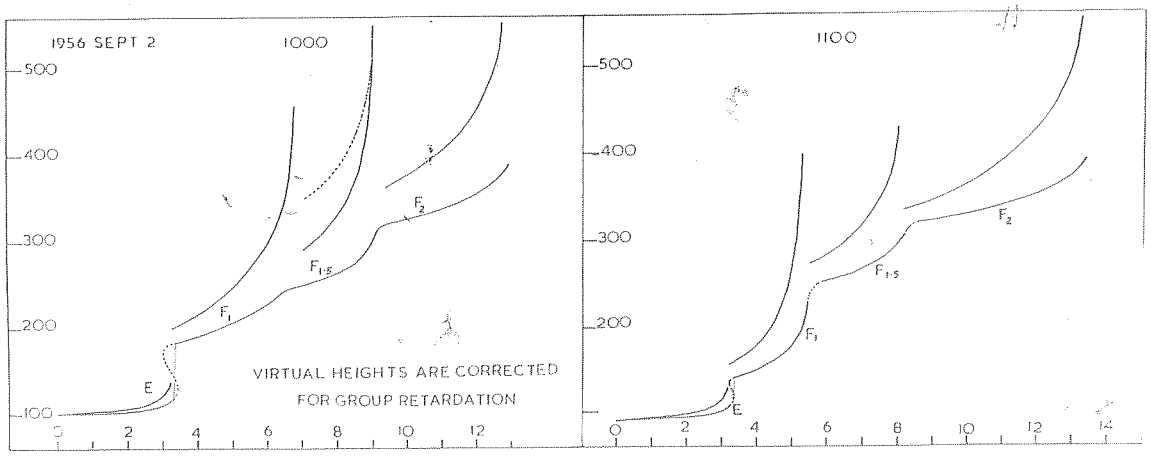


Figure 26



TRUE HEIGHT PROFILES (NO FIELD)
WHEN REGION $F_{1.5}$ WAS PRESENT IN 1956

Figure 27

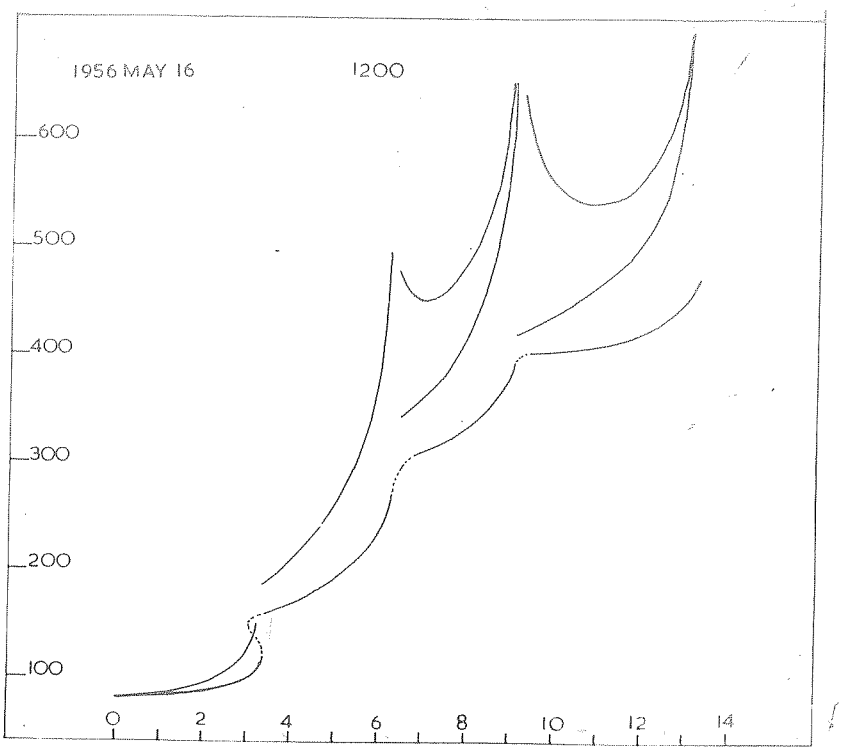


Figure 28

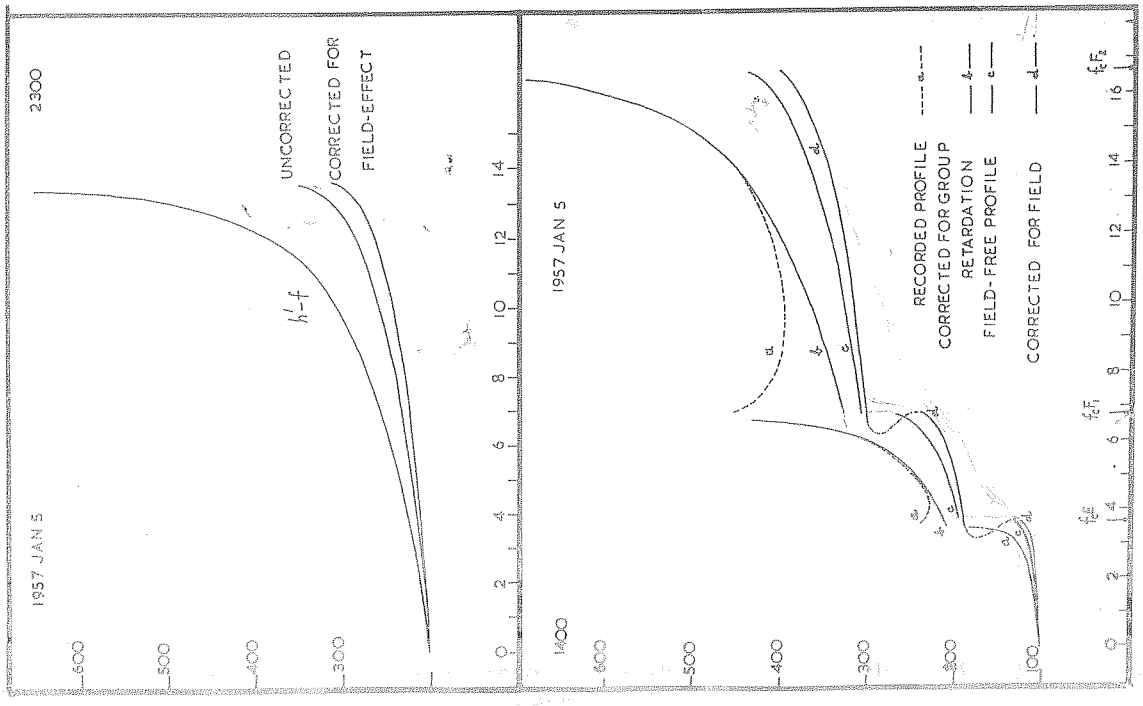


Figure 29

arbitrary distribution will tend to be Chapman-like due to diffusion in the presence of gravity and that a Chapman region preserves its shape in the presence of vertical drift diffusion and gravity (Duncan 1956). It should however be still shown that any distribution of electron density necessarily tends to be Chapman-like under the most general conditions.

Another important result which gains strength from the evidence for parabolicity and Chapman distribution is the expression for retardation that could be found for a pulse traversing an ionized region below. This is very significant because our knowledge of the F2 distribution which is what is needed badly at the moment could be vitiated if we did not correct rightly for the retardation in the lower regions.

4. The variation of the parameters of the distribution

We are reasonably satisfied that the distribution is parabolic over Ahmedabad. The parameters of the distribution are :

(a) The minimum height or the base of the region (this is denoted by h_o)

(b) The height of maximum electron density of the region (denoted by h_m)

(c) The semi-thickness of the region which is just the difference $h_m - h_o$ this we shall call

(d) The maximum electron density of the region, we shall call this N_m

(e) The total number of electrons in a column of unit cross section from the bottom of the region to the maximum density-level (called n_p).

The nature of the variation of f_c, the critical frequency of the region, the minimum virtual height h'_{min}, the virtual height at (0.834 x critical frequency) which is called h_p have already been studied by Mr R.M.Kotadia (Ph.D. thesis 1958 - unpublished) for the F region and by Dr A.C. Kastogi for the E region (Ph.D. thesis 1956 - unpublished).

We may note one point here regarding the quantity we call as h_p. This is based on assumption that the distribution is parabolic. While the assumption of parabolicity is certainly valid, unfortunately it cannot be applied to a profile which is not corrected for retardation in the lower regions. It has been noticed in our investigation that the Appleton-Beynon test is satisfactorily answered only by the profile corrected for retardation. That is, each region is by itself parabolic or as we have seen in certain cases is Chapman-like. The test is not answered when we apply it to the profiles uncorrected for retardation.

The total electron content for the F region for the year 1953 was studied by Mr R.M.Sheriff of this laboratory who made use of the method suggested by Ratcliffe in 1951 (Sheriff 1956).

We shall now study the parameters for the years 1954-56. The parameters are plotted in the accompanying diagrams. h_m and

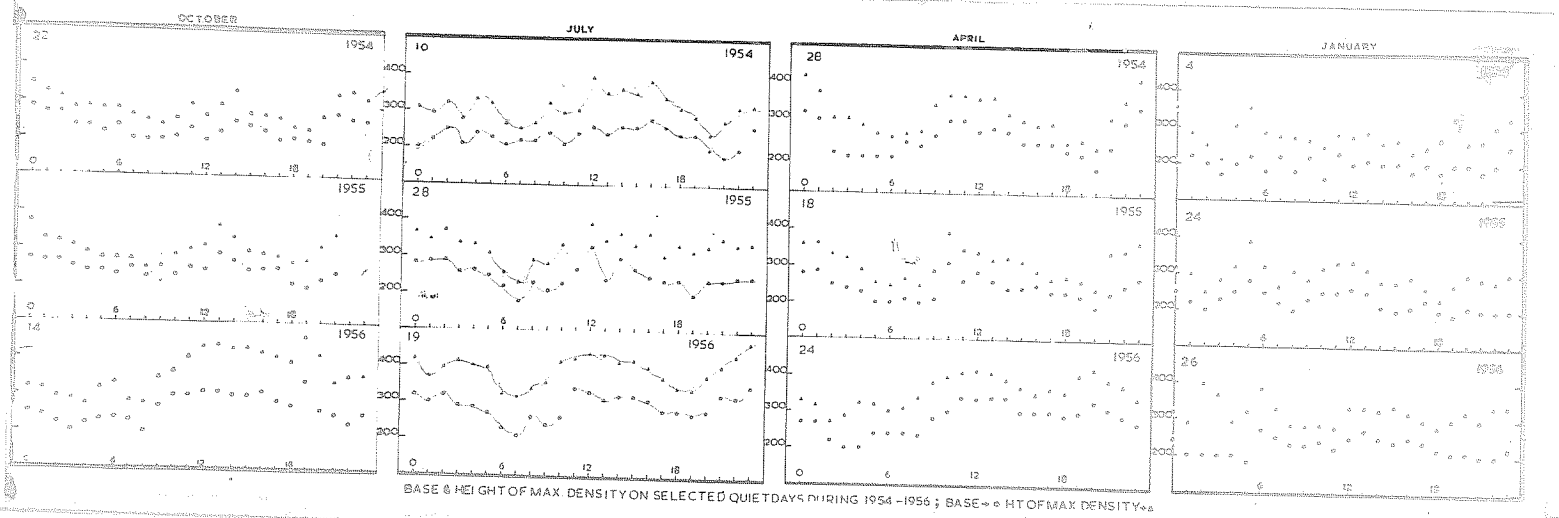
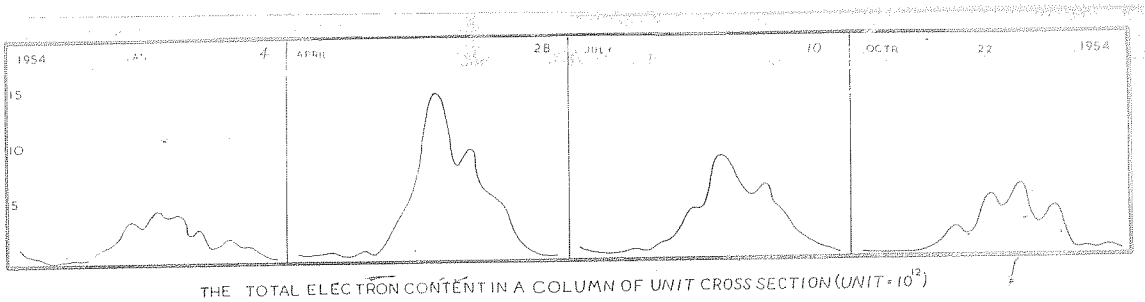
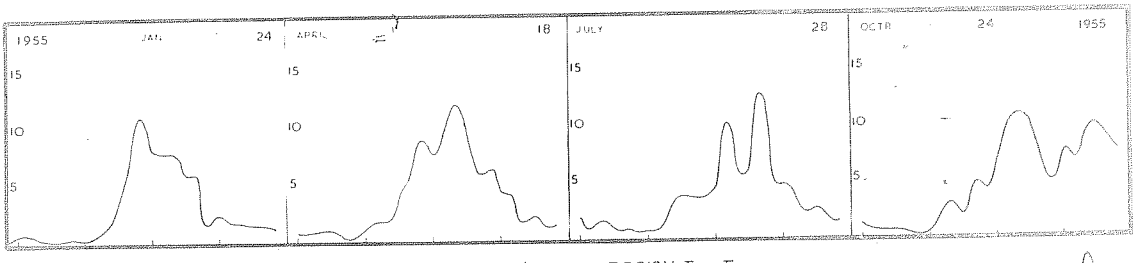

OCTOBER
JULY
APRIL
JANUARY
1954
1955
1956
BASE & HEIGHT OF MAX. DENSITY ON SELECTED QUIET DAYS DURING 1954-1956 ; BASE-•-• HT OF MAX DENSITY-•-•

Figure 30



THE TOTAL ELECTRON CONTENT IN A COLUMN OF UNIT CROSS SECTION (UNIT = 10^{12})



QUIET DAYS 1954 - 1956 REGION F1 + F2

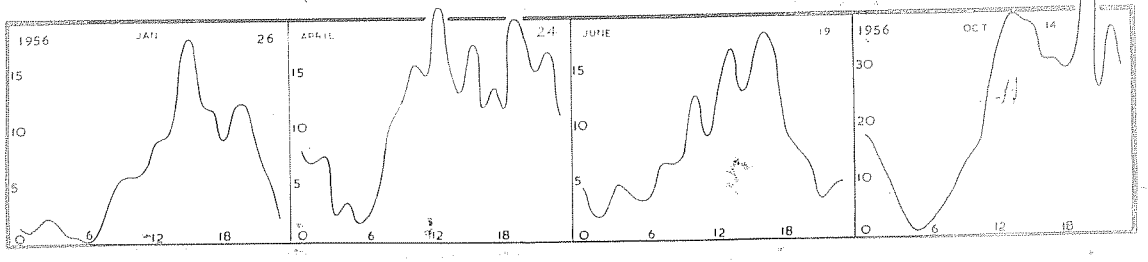


Figure 31

h_o are plotted in one diagram for 1954-56. The difference $h_m - h_o$ in the same diagram shows the variation of τ. We shall not repeat the variation of f_c since it has been treated by Mr Kotadia in his Ph.D. thesis. We have given the variation of n_f for quiet days in 1954-56 in a separate diagram. Since these do not represent the average behaviour we shall not discuss them at length. A comparative discussion of the quiet and disturbed features of these quantities is presented in chapter four.

5. Distribution corrected for the effect of the earth's magnetic field

The detailed analysis and procedure for the evaluation of the effect of the magnetic field of the earth has been described earlier. It may be worthwhile to emphasize one point here. Let us take a look at the figures (7-9) showing the variation of $\mu'.t$ vs t or x. We know that the group index of refraction is given by

$$\mu' = c/u_g$$

where u_g is the group velocity of the wave-packet. We know that if we neglected the influence of the magnetic field of the earth

$$\mu\mu' = 1 \quad \text{or} \quad \mu' = \frac{1}{\mu} = \frac{1}{(1-x)^{1/2}} = \frac{1}{t} \cdots (\text{no field})$$

$$\text{so that} \quad \mu'.t = \frac{\mu'}{1/t} = \frac{\mu'(\text{with field})}{\mu'(\text{when field is neglected})}$$

Hence the variation of $\mu't$ with x will give us the variation of the ratio of the group refractive indices.

When we neglect the effect of the field, the curve $\mu't$ vs t is always equal to 1 and is therefore a straight line parallel to the t axis. The curve $\mu't$ vs t when we include the field effect is not a straight line but is like what is shown in the following figure.

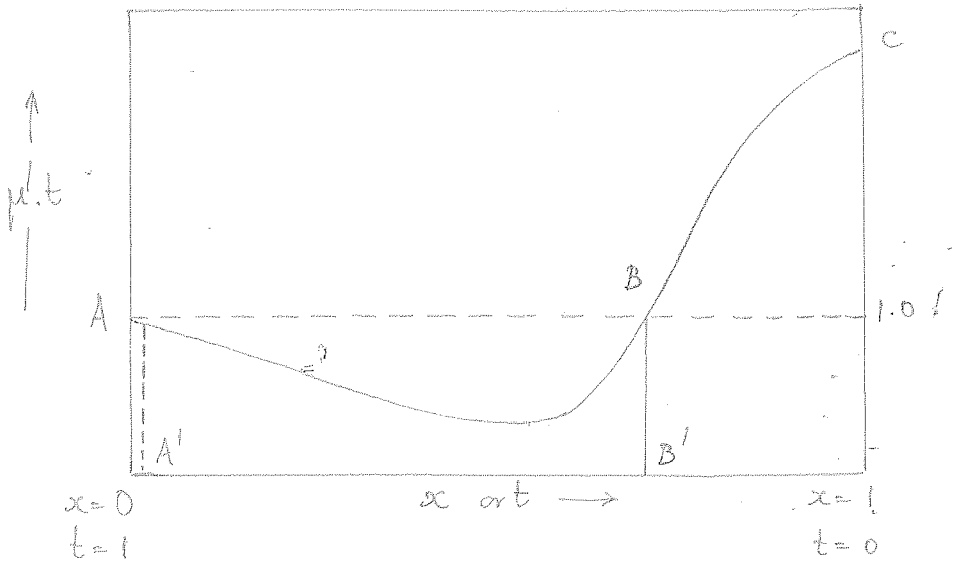


Figure 32

μ' when field is included is equal to μ' when field is neglected for x = 0 (at A) and for the abscissa of the point B. The point B shifts as y varies. But in the range $A' < x < B'$, $\mu't < 1$. i.e. μ' when the field effect is included is less than μ' with field ignored. In other words from the formula given above $\mu' = c/u_g$, the group velocity in the range $A' < x < B'$ of the pulse is larger than the group velocity of the wave when the field effect is ignored. But in the range $B' < x < C'$ the situation is the reverse. What is the same, in the range $A' < x < B'$ the pulse travels faster when the field is included than when it is neglected, while it travels slower in the range $B' < x < C'$. But the value of the integral

$$\int_0^1 \mu' \, dx \quad \text{is greater than} \quad 2\int_{t=(1-x)^{1/2}}^{1} \mu' \cdot t \, dt$$

this means therefore that the nett result of including the effect of the magnetic field of the earth in our consideration is to retard the pulse. This is in accordance with the findings of Goubau (1934), Poverlein (1951) and Shinn and Whale (1952).

This means that when we include the effect of the magnetic field we should expect the heights to be higher than when it is neglected. However, the true geometric height <u>corrected for</u> the retardation due to the effect of the magnetic field of the earth will be <u>lower</u> than when the field effect is ignored.

We have illustrated this point for two quiet days in 1957 and have given diagrams for a couple of hours. It is expected to analyse more records with the application of the geomagnetic correction to the heights of reflection on the method suggested in this thesis and will be reported by other workers of this laboratory in due course. However, it is well to remember that the correction is quite small for Ahmedabad.

CHAPTER FOUR

Ionospheric response to magnetic storms

CHAPTER FOUR

Ionospheric response to magnetic storms

1. Introduction

It has been known for many years that a magnetic storm is associated with perturbations in the ionosphere. One of the changes about which there is no doubt is a reduction of the critical frequencies of the regions during some part of the period of storm activity. It was felt that a study of the behaviour of electron density distribution over Ahmedabad during a few magnetic storms would prove interesting.

To study the effect of the storm it is necessary to follow the changes from a little prior to the commencement of the storm to the time when the ionosphere is restored to normalcy. Our study is based on the evaluation of the heights of reflection of radio waves by the technique of Gaussian quadratures. The procedure is set out in detail in chapter two. We confined our attention to storms in a year of high solar activity, 1956. There were about 33 storms during the year. We have analysed the effects during six storms. The effect of the geomagnetic field on the calculation of the actual heights was not taken into account.

2. The effect of magnetic storms.

The storms were of two types, those with gradual

commencement and those with sudden commencement. They are again classified as severe, moderately severe and moderate as regards their intensity. This is according to the maximum changes in the horizontal component of the earth's magnetic field as recorded by the magnetogram and expressed in γ's ($1\ \gamma = 10^{-5}$ gauss). The storms prevail from over twelve hours to a few days. We have normally started our analysis well before the storm-commencing time and followed it till after it has ended and the ionosphere has attained normalcy. The following table gives some details of the storms whose effects have been studied. The data presented below in the table were taken from the Indian Journal of Meteorology and Geophysics, Vols. 7 and 8 (1956 and 1957).

Table 3

There was one storm which was severe in activity (Feb. 25) but the ionospheric record was not very good for that day. We have analysed three storms each, for two types and of two different strengths. The effect of the storm on the ionosphere could be well appreciated with reference to the iso-dense levels of the following days which are given below :

Quiet type	Storm type
A. January 26, 1956	i. January 27-30, 1956
	ii. February 22, 1956
B. April 24, 1956	May 16-17, 1956
C. June 19, 1956	June 24-25, 1956
D. October 14, 1956	i. September 2-3, 1956
	ii. October 26-28, 1956

The study of the storm type changes are classified as :

(1) Changes in the ionosperic parameters, namely the base, height of maximum ionization, semi-thickness and the total electron content of the region.

(2) Changes in the shape of the distribution.

(3) Changes in the iso-dense level pattern.

We shall treat the changes in the iso-dense level patterns first, after examining the above diagrams. We have explained the quiet type patterns earlier. We shall explain the disturbed type pattern. The X-axis represents time in hours (75° E.A.T.). The Y-axis represents the true height uncorrected for field effect in km. The dotted vertical lines

show the commencement and ending times of the storm. The level marked 1, 2, 3, etc. refer to heights of equal electron density $k.1^2$, $k.2^2$, $k.3^2$ etc. The dotted line above the level 3 is the h_m of E region. Similarly the other two are the levels of maximum electron density in the F region. When there is a third dotted line, the lines correspond to h_mF1, $h_mF1.5$ and h_mF2 in that order. The numbers given above the dotted lines indicate the critical frequency of the region at those hours. All values of f_c are not marked since it would mean overcrowding with numbers. Usually they are given for alternate hours.

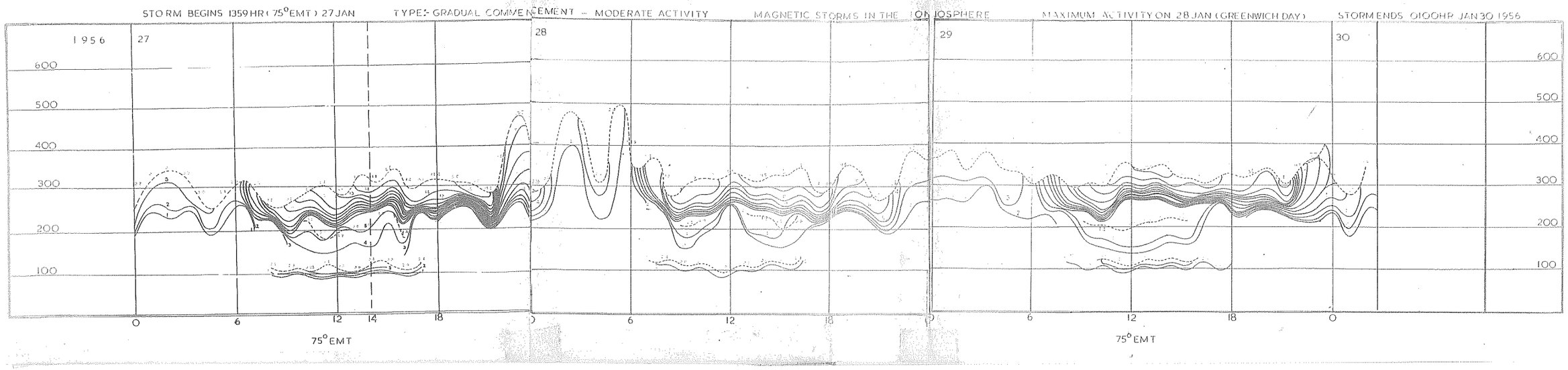
(i) The first clear effect is that any significant ionospheric disturbance does not set in simultaneously with the onset of the magnetic storm. The ionosphere generally responds after 6-8 hours.

(ii) The second clear effect is that the distance between two consecutive iso-dense levels increases, particularly during night.

(iii) The pattern of the quiet day is greatly disturbed.

(iv) The valley between consecutive ionospheric regions increases, the upper layer going upward and the lower layer going downward. The level of maximum electron density is brought down during day and pulled up during night.

In the case of a storm lasting for more than a day the effect on the second day is usually the most severe. This observation from the iso-dense pattern confirms the magnetic


STORM BEGINS 1359HR (75°EMT) 27JAN
TYPE:- GRADUAL COMMENCEMENT - MODERATE ACTIVITY
MAGNETIC STORMS IN THE IONOSPHERE
MAXIMUM ACTIVITY ON 28JAN (GREENWICH DAY)
STORM ENDS 0100HR JAN30 1956
1956
27
28
29
30
600
500
400
300
200
100
0
6
12
14
18
0
6
12
18
0
6
12
18
0
75°EMT
75°EMT

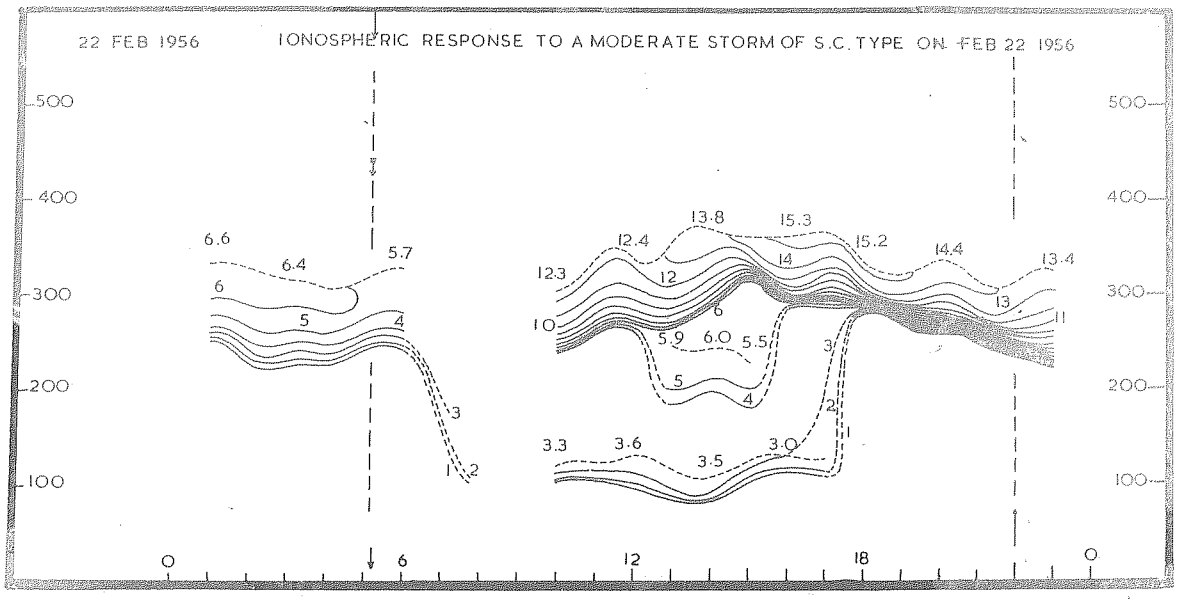


Figure 34

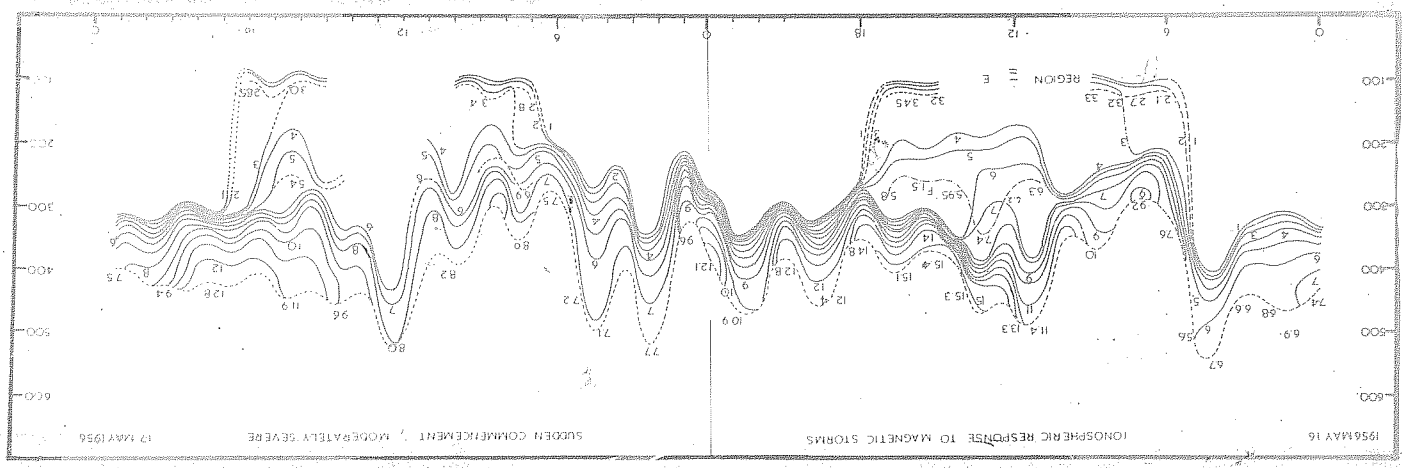


Figure 35

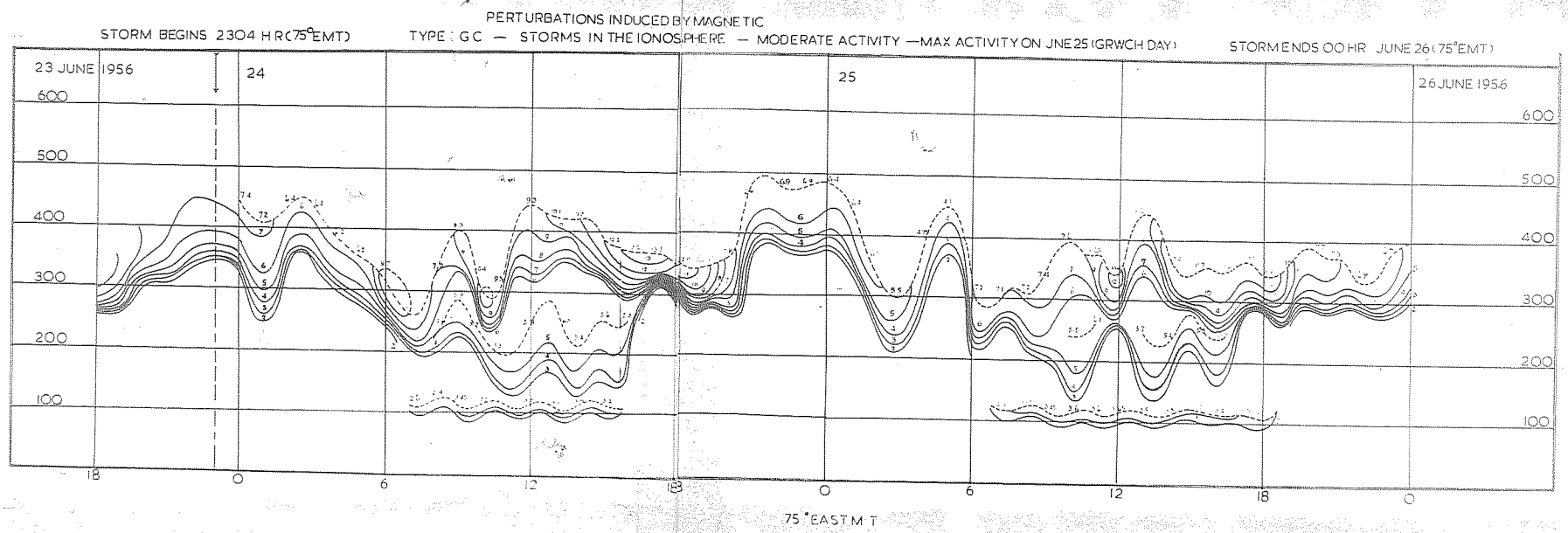

PERTURBATIONS INDUCED BY MAGNETIC
STORM BEGINS 2304 HR (75° EMT)
TYPE : GC — STORMS IN THE IONOSPHERE — MODERATE ACTIVITY — MAX ACTIVITY ON JNE 25 (GRWCH DAY)
STORM ENDS 00 HR JUNE 26 (75° EMT)
23 JUNE 1956
24
25
26 JUNE 1956
600
500
400
300
200
100
18
0
6
12
18
0
6
12
18
0
75° EAST M T

Figure 36

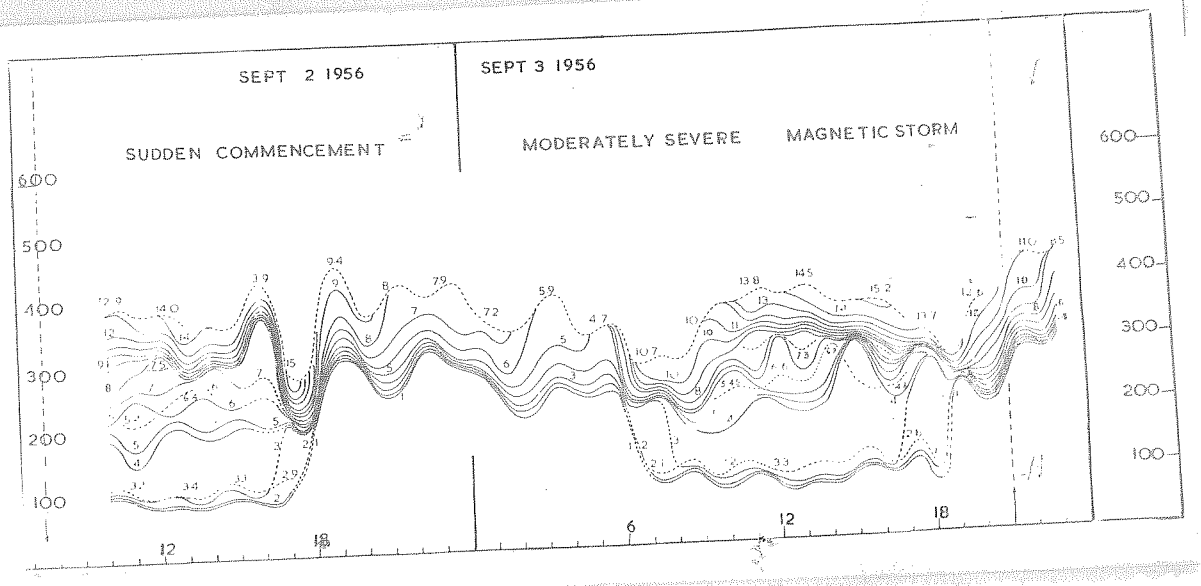

SEPT 2 1956
SUDDEN COMMENCEMENT
SEPT 3 1956
MODERATELY SEVERE MAGNETIC STORM
600
500
400
300
200
100
12
18
6
12
18

Figure 37

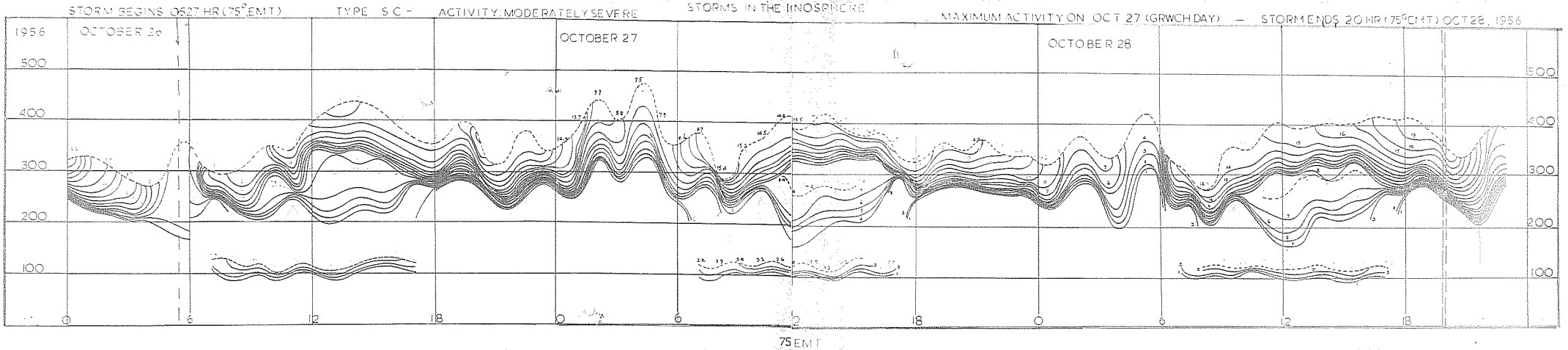

PERTURBATIONS INUCD BY MAGNETIC
STORMS IN THE IONOSPHERE
STORM BEGINS 0527 HR (75° EMT)
TYPE S C -
ACTIVITY: MODERATELY SEVERE
MAXIMUM ACTIVITY ON OCT 27 (GRWCH DAY)
STORM ENDS 20 HR (75° EMT) OCT 28, 1956
1956
OCTOBER 26
OCTOBER 27
OCTOBER 28
500
400
300
200
100
0
6
12
18
75 EMT

Figure 38.

observations.

3. The effect of the storm on the shape of the region

The only effect that we observe is that there are more deviations from parabolicity on stormy days than on quiet days. But the deviation is towards a Chapman-like distribution. This can be seen from the diagrams given at the end of the chapter.

4. The effect on the parameters of the distribution

In this section we shall discuss the type of changes a storm induces on h_0', h_m, τ and n_T and shall compare our results with those obtained by Ratcliffe (1951), Sheriff (1956) Schmerling and Thomas (1956) and Thomas, Haselgrove and Robbins (1958).

Ratcliffe has reported from his analysis of data from Watheroo, Huancayo and College, Alaska that F1 and F2 were approximately parabolic during day and that near sunset and there-after during the night assumed an accurately parabolic form. The significant importance of parabolicity is now clear from our results. If the distribution were corrected for retardation in the lower regions Ratcliffe also would have probably observed better agreement with parabolicity even during day. Sheriff assumed the parabolicity of the regions and carried out his analysis. But his results can be substantiated now that we have a satisfactory answer to the test of the shape of the distribution. Ratcliffe analysed

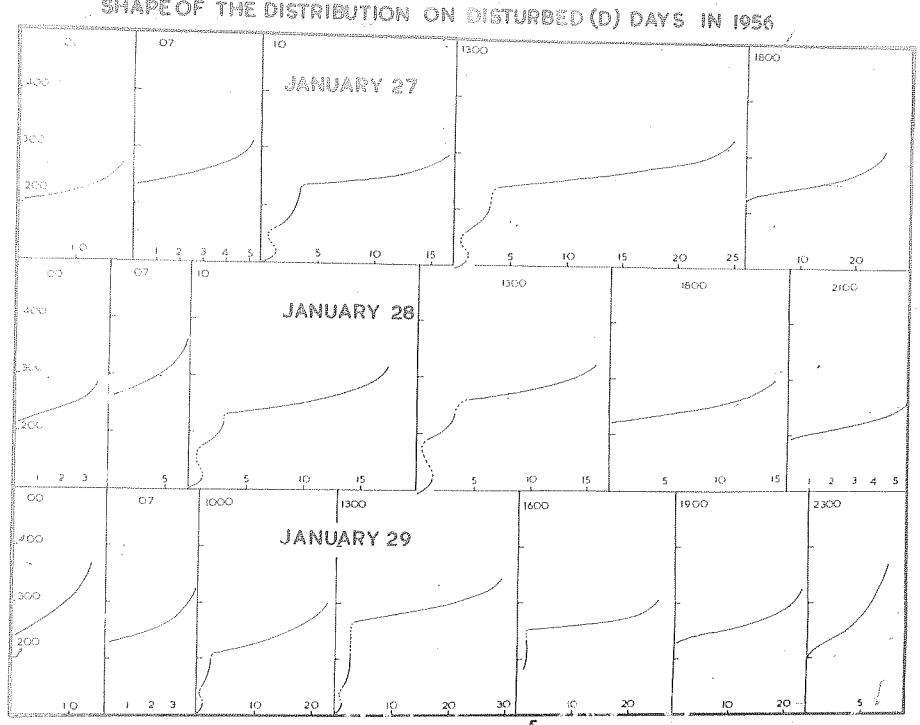


Figure 39

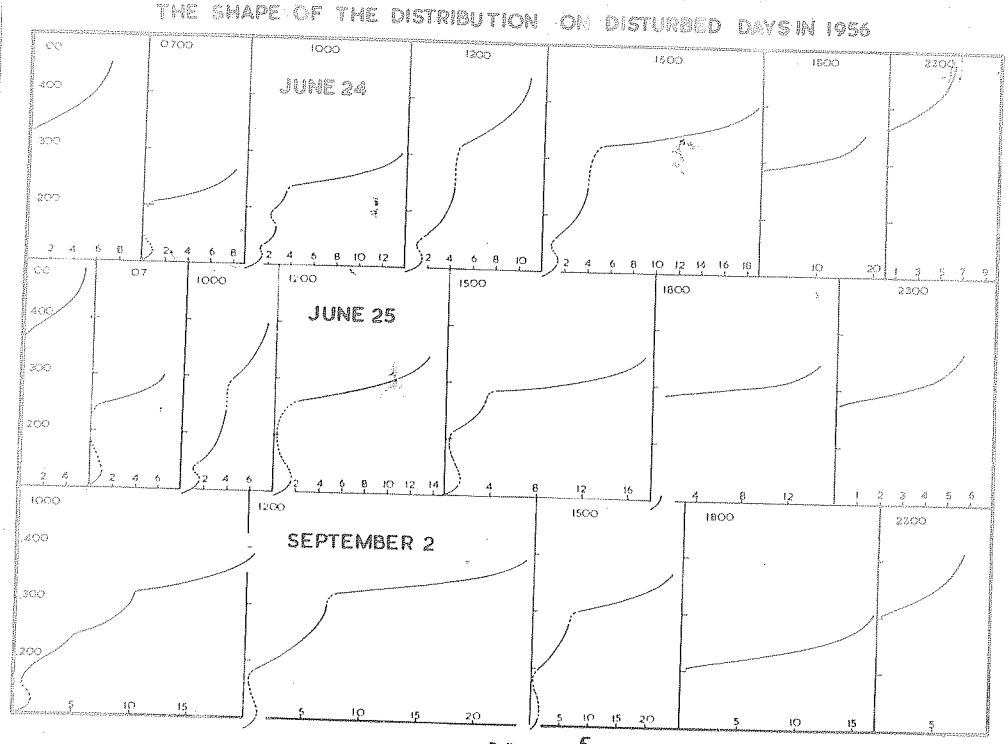


Figure 40

THE SHAPE OF THE DISTRIBUTION ON DISTURBED DAYS IN 1956

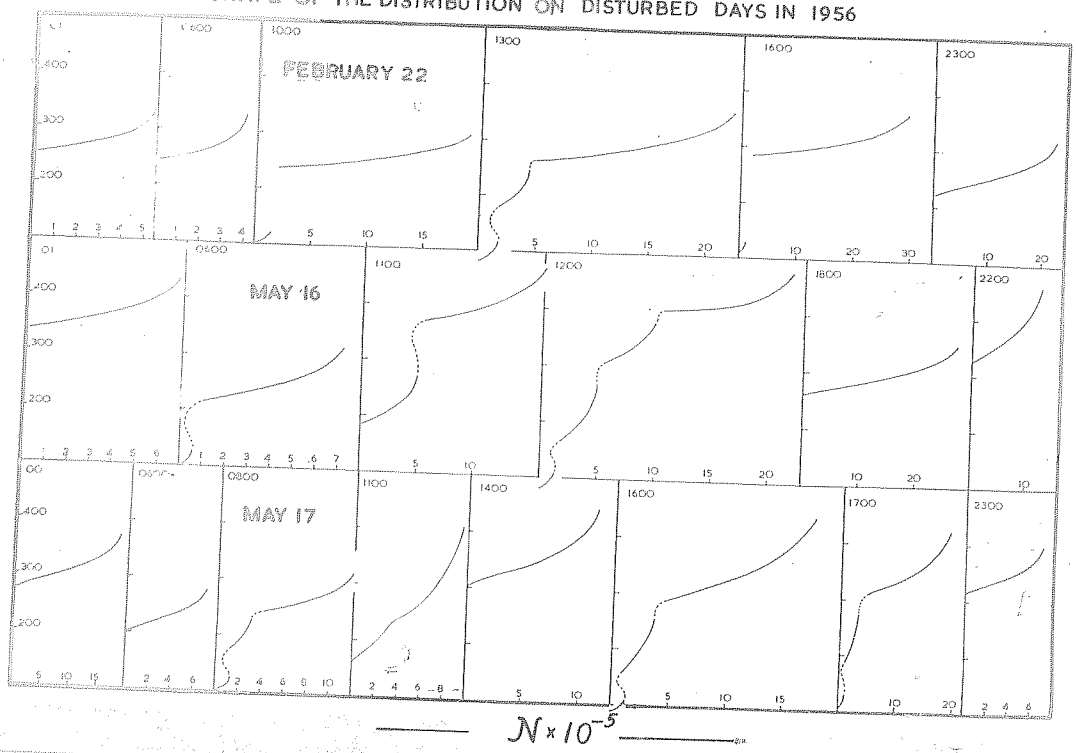


Figure 41

Fig. 42

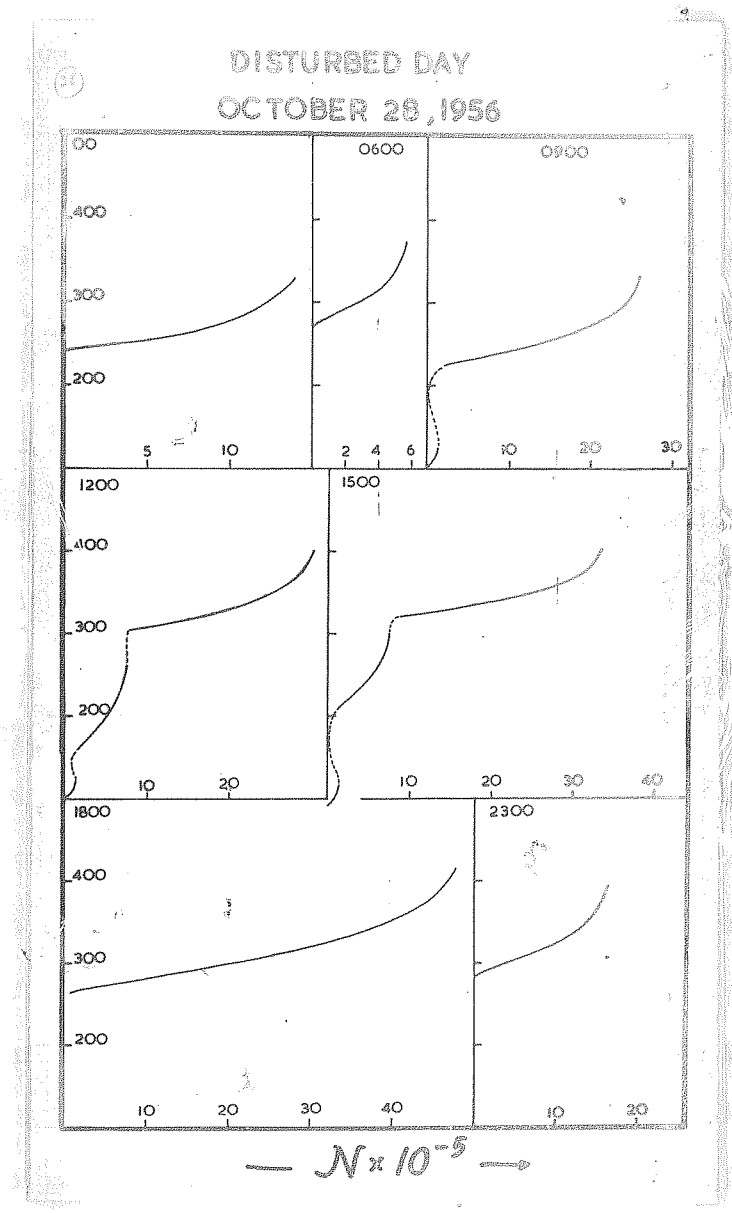

DISTURBED DAY
OCTOBER 28, 1956
00
0600
0900
1200
1500
1800
2300
400
300
200
5
10
2
4
6
10
20
30
10
20
10
20
30
40
10
20
30
40
10
20
— N x 10^-5 →

Figure 43

two quiet days per month in a year of sunspot maximum and a year of sunspot minimum for three stations. Sheriff analysed three quiet days and three disturbed days for the four seasons in 1953 for Ahmedabad. Their results therefore are based on more data but the results were similar in that n_T showed a closer dependence on the sun's zenith angle than N_{max}. Schmerling and Thomas have analysed profiles for one year of sunspot maximum and one year of sunspot minimum. They have considered three months Jan/Dec, March, June/July for a year. Their curves (in their figure 1 on page 613) show some resemblance to parabolicity although they have not allowed for retardation in the lower regions. We do not know however about the parabolicity of the regions from the work of Thomas et al.

Thomas et al (1958) have given the curves of h_m for E, F1 and F2 regions. They have also given curves for N_m and n_T. They find that the n_T is not the same in the three seasons. January to September ratio is maximum (about two) in a year of high sunspot number, as well as in a year of low sunspot number. We give the curves showing the variation of h_m and h_o on <u>individual</u> days. The lower curve in figure indicates the variation in h_o while the upper curve indicates the variation of h_m. The difference shows the variation in τ. The n_T variation is shown in a separate figure ( ). It can be observed that n_T does not remain at the same value in the four seasons. There is ratio of two from October to January in a year of

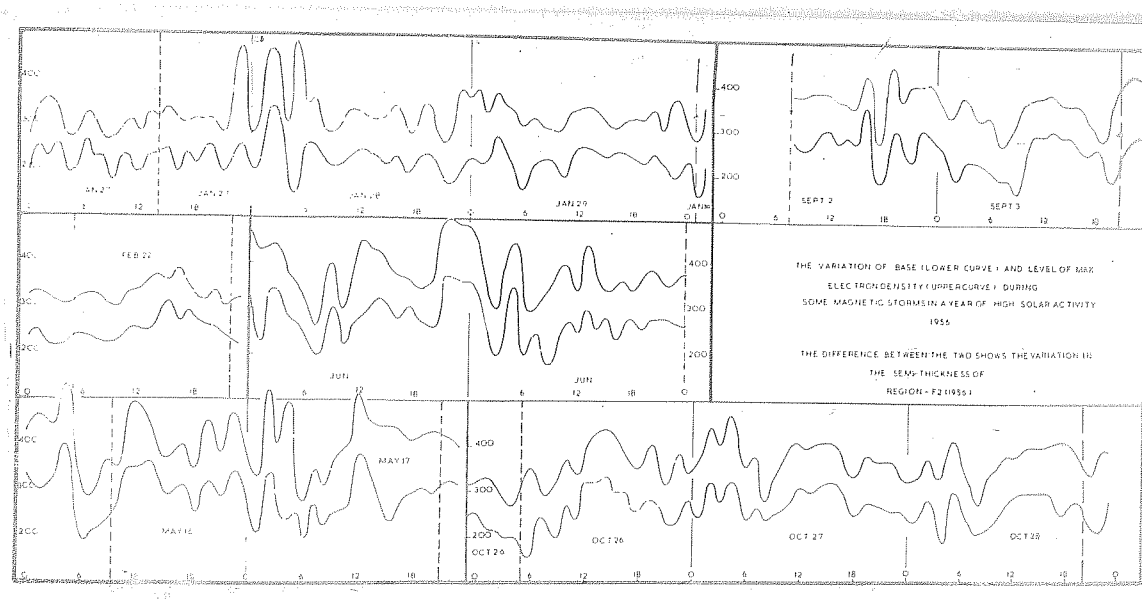


Figure 44

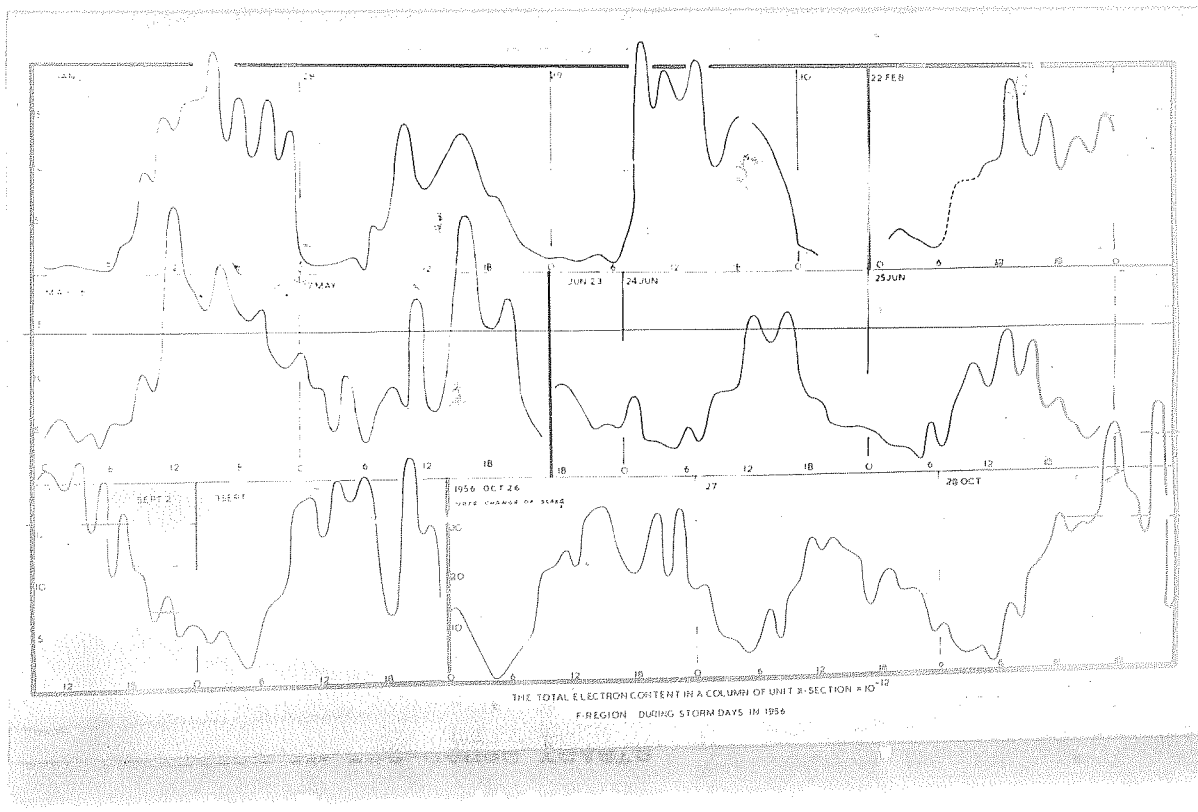


Figure 45

sunspot maximum, 1956. This is in agreement with the results reported for Watheroo, Huancayo and College by Ratcliffe while it is contrary to those of White and Wachtel (1949) who reported results for Washington in 1945. We shall summarise the differences between the quiet and disturbed day patterns. This comparative sketch is based on individual days observations and hence their generality cannot be taken for granted as they do not represent average behaviour.

Quiet type variation	Disturbed type variation.
(a) Variation in h_o, h_m and τ	
h_o and h_m have a smooth daily variation about their mean value. The changes of h_o and h_m are in phase.	h_o and h_m fluctuate more violently. The range of variation in τ is much larger on a disturbed day. This is explained by some workers by saying that the ionosphere is more expanded. During night hours the changes of h_o and h_m sometimes go opposite in phase.
(b) Shape of the distribution	
Normally no deviations from parabolicity. There are occasionally one or two cases	More deviations from parabolicity to Chapman-like distribution.
(c) Variation in n_T	
Although the curves are given for individual days it can be seen that the average behaviour of n_T will be faithful to the sun's zenith angle.	Variation during the storm is not as smooth as it is on a quiet day. Range of variation is much larger.

The ratio of n_T varies from season to season. The ratio is about 2 from October to January. In 1954 the April to January ratio was 3 while the October to January ratio in 1956 was 2.5.

(d) Variation in iso-dense levels

the distance between consecutive iso-dense levels remains close. The F1 is a region of low density gradient & and F2 are regions of higher gradient.

F2 is the densest region.

Upper regions are pulled upward and lower regions are pulled downward causing the valley between regions to increase. The distance between consecutive density levels increases during a storm causing a fall in the gradient throughout or diluting the ionosphere at the same time dilating it.

Conclusions

We shall conclude by summarising the points brought out in the investigation reported in the thesis :

(a) The application of Gaussian quadratures to the evaluation of true heights of reflection of radio waves provides a consistent and sound technique which is mathematically rigorous and physically satisfactory. The integral

$$\int \frac{h'(f)\,df}{(f_N^2 - f^2)^{1/2}}$$

could be replaced by the mean value of a discrete sum of five height ordinates. The five point summation is exact enough for all ionospheric standards.

(b) The distribution of electrons over Ahmedabad is satisfactorily parabolic in most cases and a few instances of deviation to a Chapman-like distribution are noticed.

(c) The result in (b) lends meaning to the allowance for retardation due to the passage of the pulse through lower

ionized regions and for the total electron content in a unit column of the ionosphere.

(d) The effect of the geomagnetic field on the passage of electromagnetic waves through the ionosphere is essentially a retardation and this retardation at Ahmedabad is not more than 0.2%. The correction at Delhi is not more than 9%. This serves as an upper limit to the errors in neglecting the field at all Indian ionospheric stations. The corresponding value for S-E England is about 19.4%.

(e) Routine analysis of data on the lines suggested in the thesis could now be done in a fairly quick way with the help of a Facit computing machine and the distribution corrected for the effect of the geomagnetic field could be studied to provide the bases for an accurate knowledge of the physical processes in the ionosphere.

(f) The distribution of electrons over Ahmedabad and the nature of variation of the parameters characterising the distribution have been studied with a sampling from the years 1954 to 1956. Unfortunately one cannot draw very general conclusions from the study reported here due to the small size of the sample and hence the results have been presented graphically.

(g) A magnetic storm produces perturbations in the ionosphere and the pathological changes could be studied with the help of the iso-dense levels from the onset of the storm to the stage when normalcy is restored. Besides the wellknown

effects the storm induces a fall in the density gradient in the ionosphere as evidenced by the separation of consecutive iso-dense levels. Parabolicity of regions is slightly disturbed and there is tendency toward a Chapman-like distribution during magnetic storm. The changes in the nature of the variation of parameters of the distribution have been studied. With more analysis one could draw the general conclusions about the effect of a storm in the same way as the general features could be precisely appreciated on quiet days also.

Acknowledgments

The author would like to express his gratitude to Professor K.R.Ramanathan, Professor Vikram Sarabhai and Professor Vachaspati for having extended to him the hospitality of the Physical Research Laboratory, Ahmedabad.

The work reported in the thesis was done under the supervision of Professor Ramanathan. He wishes to record his indebtedness to Professor Ramanathan for introducing him to the subject, for his valuable guidance and helpful discussions during the progress of the work.

Finally, it is a pleasant duty to thank the Ministry of Scientific Research and Cultural Affairs of the Government of India for the award of a Senior Research Scholarship for the period of work.

Table 4

(Form B).

PHYSICAL RESEARCH LABORATORY

Vertical Distribution of Electron-Density in the Ionosphere.

Determination of the parameters characterising the Distribution and the Group Retardation in the Wave due to the passage through lower ionized Regions

Station: _____ Critical Frequency: _____ Mc/s.

Date: _____ Time : _____ Region: _____

The Ionogram Virtual Height vs Frequency.	f/fc	ψ (f/fc)	The parameters by Appleton-Beynon method: ho h_M t n	Group Retardation (f/fb)	dh'=t.ϕ		Corrected Virtual Height.
					ϕ (f/fb)	dh'	h'- dh'

Table 5

(Form A)

PHYSICAL RESEARCH LABORATORY

Vertical Distribution of electron density in the ionosphere
Determination of True Heights of reflection of vertically indident Radio Waves.

Station: _____ Critical Frequency: _____ Mc/s.

Date : _____ Time: _____ Region: _____

Probing Freq. in Mc/s.	Height Ordinates in the numerical integration h'(f1)	h'(f2)	h'(f3)	h'(f4)	h'(f5)	$\Sigma h'(fi)$	h(f)	N in 10^5 per cc.
1	0.156	0.454	0.707	0.891	0.988			0.124
2	0.313	0.908	1.414	1.782	1.975			0.496
3	0.469	1.362	2.121	2.673	2.963			1.116
4	0.626	1.816	2.828	3.564	3.951			1.984
5	0.782	2.270	3.536	4.455	4.939			3.100
6	0.938	2.724	4.243	5.346	5.926			4.464
7	1.095	3.178	4.950	6.237	6.914			6.076
8	1.251	3.632	5.657	7.128	7.902			7.936
9	1.408	4.086	6.364	8.019	8.889			10.044
10	1.564	4.540	7.071	8.910	9.877			12.400
11	1.720	4.994	7.778	9.801	10.865			15.004
12	1.877	5.448	8.485	10.692	11.852			17.856

..2.

(Form A).

Station: _____

Date: _____ Time : _____

Critical Frequency: ____ Mc/s

Region: _____

Probing Freq. in Mc/s.	Height Ordinates in the numerical integration h'(f1)	h'(f2)	h'(f3)	h'(f4)	h'(f5)	$\sum h'(fi)$	True Height in km.	N in 10^5 per cc.
13	2.033	5.902	9.192	11.583	12.840			20.956
14	2.190	6.356	9.899	12.474	13.828			24.30
15	2.346	6.810	10.607	13.365	14.816			27.90
16	2.502	7.264	11.314	14.256	15.803			31.74
17	2.659	7.718	12.021	15.147	16.791			35.84
18	2.815	8.172	12.728	16.038	17.779			40.18
19	2.972	8.626	13.435	16.929	18.766			44.76
20	3.128	9.080	14.142	17.820	19.754			49.60
1	3.284	9.534	14.849	18.711	20.742			54.68
2	3.441	9.988	15.556	19.602	21.729			60.02
3	3.597	10.442	16.263	20.493	22.717			65.60
4	3.754	10.896	16.970	21.384	23.705			71.42
[illegible]	3.910	11.350	17.678	22.275	24.692			77.50

Table 6

APPENDIX 3

Table of values of the function $f/2f_c \cdot \log_e\left(\frac{f_c+f}{f_c-f}\right)$ for values of f/f_c in the interval $0 \leq f/f_c \leq 0.99$.

f/f_c	Ψ	f/f_c	Ψ	f/f_c	Ψ	f/f_c	Ψ
0.00	0.00	0.26	0.07	0.51	0.29	0.75	0.73
0.01	0.00	0.27	0.07	0.52	0.30	0.76	0.76
0.02	0.00	0.28	0.08	0.53	0.31	0.77	0.79
0.03	0.00	0.29	0.09	0.54	0.33	0.78	0.81
0.04	0.00	0.30	0.09	0.55	0.34	0.79	0.85
0.05	0.00	0.31	0.10	0.56	0.35	0.80	0.88
0.06	0.00	0.32	0.10	0.57	0.37	0.81	0.91
0.07	0.00	0.33	0.10	0.58	0.38	0.82	0.95
0.08	0.01	0.34	0.12	0.59	0.40	0.83	0.99
0.09	0.01	0.35	0.13	0.60	0.42	0.84	1.03
0.10	0.01	0.36	0.14	0.61	0.43	0.85	1.07
0.11	0.01	0.37	0.14	0.62	0.45	0.86	1.11
0.12	0.01	0.38	0.15	0.63	0.47	0.87	1.16
0.13	0.02	0.39	0.16	0.64	0.48	0.88	1.21
0.14	0.02	0.40	0.17	0.65	0.50	0.89	1.27
0.15	0.02	0.41	0.18	0.66	0.52	0.90	1.33
0.16	0.03	0.42	0.19	0.67	0.54	0.91	1.39
0.17	0.03	0.43	0.20	0.68	0.56	0.92	1.46
0.18	0.03	0.44	0.21	0.69	0.58	0.93	1.54
0.19	0.04	0.45	0.22	0.70	0.61	0.94	1.63
0.20	0.04	0.46	0.23	0.71	0.63	0.95	1.74
0.21	0.04	0.47	0.24	0.72	0.65	0.96	1.87
0.22	0.05	0.48	0.25	0.73	0.68	0.97	2.03
0.23	0.05	0.49	0.26	0.74	0.70	0.98	2.25
0.24	0.06	0.50	0.27	0.		0.99	2.62
0.25	0.06						

These values are taken from Booker & Seaton (1940) and are given correct to the second place of decimal. This function $\Psi(f/f_c)$ is equal to 1 + Booker's $\varphi(f/f_c)$

Table 7

APPENDIX 4

Table of values of the function $\varphi(f/f_c) = f/2f_c \cdot \log_e\left(\frac{f_c+f}{f_c-f}\right) - 1$

for values of f/f_c in the interval $1 \leq f/f_c \leq 1.99$

f/f_c	φ	f/f_c	φ	f/f_c	φ	f/f_c	φ
1.00		1.25	0.373	1.50	0.207	1.75	0.137
1.01	1.678	1.26	0.362	1.51	0.203	1.76	0.135
1.02	1.354	1.27	0.352	1.52	0.199	1.77	0.133
1.03	1.170	1.28	0.342	1.53	0.196	1.78	0.131
1.04	1.045	1.29	0.333	1.54	0.192	1.79	0.129
1.05	0.950	1.30	0.324	1.55	0.189	1.80	0.127
1.06	0.874	1.31	0.316	1.56	0.186	1.81	0.126
1.07	0.812	1.32	0.307	1.57	0.182	1.82	0.124
1.08	0.759	1.33	0.300	1.58	0.179	1.83	0.122
1.09	0.714	1.34	0.292	1.59	0.176	1.84	0.120
1.10	0.674	1.35	0.285	1.60	0.173	1.85	0.119
1.11	0.639	1.36	0.279	1.61	0.170	1.86	0.118
1.12	0.608	1.37	0.272	1.62	0.167	1.87	0.116
1.13	0.580	1.38	0.266	1.63	0.165	1.88	0.114
1.14	0.554	1.39	0.260	1.64	0.162	1.89	0.113
1.15	0.531	1.40	0.254	1.65	0.159	1.90	0.112
1.16	0.510	1.41	0.249	1.66	0.157	1.91	0.110
1.17	0.490	1.42	0.243	1.67	0.155	1.92	0.109
1.18	0.471	1.43	0.238	1.68	0.152	1.93	0.107
1.19	0.455	1.44	0.233	1.69	0.150	1.94	0.106
1.20	0.439	1.45	0.229	1.70	0.148	1.95	0.105
1.21	0.424	1.46	0.224	1.71	0.145	1.96	0.104
1.22	0.410	1.47	0.220	1.72	0.143	1.97	0.102
1.23	0.397	1.48	0.215	1.73	0.141	1.98	0.101
1.24	0.385	1.49	0.211	1.74	0.139	1.99	0.100

These values are the same as given in page 87 of Booker and Seaton : Physical review, Vol. 57, 1940.

APPENDIX 5

Calculation of the group refractive index and the quantity $\mu'.t$

To calculate the group refractive index we shall have to compute the phase refractive index first from the general formula of magneto-ionic theory of Appleton and Hartree. We shall briefly outline the mode of computation.

The phase refractive index is given by

$$\mu^2 = 1 - \frac{2x(1-x)}{D}$$

the notation has been explained in the body of the thesis. The group refractive index is best computed accurately from the formula

$$(\mu\mu' - \mu^2) D = 2\left[1 - \mu^2 - x^2 + \frac{1}{3}(1-\mu^2)(1-x^2) y^2 \cos^2\theta\right]$$

Chinn (1955) has calculated the value of $\mu'.t$ for some stations. We shall give the mode of computation of $\mu'.t$ for Ahmedabad and Delhi.

We first compute the value of f_H given by the formula $He/2\pi mc$. Here H is the total field strength at the given place This could be evaluated at a place for which we know the magnetic latitude λ from the formula

$$H^2 = M^2/r^3 \cdot (1 + 3\sin^2\lambda)$$

or if we know the angle of dip at that place ϕ we can get it from $\sin^2\lambda = \sin^2\phi/4 - 3\sin^2\phi$. Here M is the dipole moment of the earth's magnetic field and r is the radius of the earth ($r =$ 6370 km). H is also tabulated for different places by magnetic

observatories. It is also occasionally published in the Journal of Geophysical Research. Since we need f_H at certain heights above the earth the value of the field at that particular height can be computed from the inverse cube law of field variation. H at any given height h = H at ground level x $(r/r+h)^3$. Similarly, f_H also is given by the inverse cube law.

Having evaluated f_H we can compute the different values of $y = f_H/f$ for different values of f. We next compute $y \cos\theta$, and $y \sin\theta$ where θ is the propagation angle. Denoting $y \cos\theta$ by y_L and $y \sin\theta$ by y_T we compute in order

f_H, y_L, y_T, y_L^2, y_T^2, y_T^4, $4y_L^2$

x, x^2, $1-x$, $(1-x)^2$, $(1-x^2)$, $2(1-x)$, $2x(1-x)$

$S = \sqrt{y_T^4 + 4y_L^2(1-x)^2}$

$D = 2(1-x) + S - y_T^2$

μ^2, $1-\mu^2$,

Then we compute

$$(\mu\mu' - \mu^2)\, D = 2\left[1-\mu^2-x^2 + \frac{1}{S}(1-\mu^2)(1-x^2)\, y_L^2\right]$$

from which we get $\mu\mu'$ and μ' finally.

Then we compute the quantity $\mu'.t$ where $t^2 = 1-x$

We could then evaluate the integral

$$\int_0^x \mu'\, dx = 2\int_{t=(1-x)^{1/2}}^{1} \mu'.t\, dt = G(x)$$, say, for different values of x and tabulate them. We used numerical methods for integration. The results were checked by two methods : Simpson's rule and trapezium rule.

μ' can be computed from Jackson's method (1956) as outlined in the text using the formula for numerical differentiation:

$$\partial f/\partial x = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$

if with unequal intervals,

$$\partial f/\partial x = \frac{1}{2}\left\{ \frac{f(x+h_1) - f(x)}{h_1} + \frac{f(x) - f(x-h_2)}{h_2} \right\}$$

$\partial f/\partial y$ is obtained from the family of curves $f(x, y)$

Δx should be very small, particularly when x is near 1.

Shinn (1955) has given a table of values of $(10^3 . \mu' . t)$. One can get by interpolation the values for other stations from his values. We first vary ϕ and get the values. Then we vary f_H and then get the relevant values. Weierstrass has shown that given a set of discrete values of a function and its arguments one can fit the polynomial approximation to the function. If a function say F(x) is known by a set of values corresponding to a given set of abscissae, then the value of the function for an abscissa x is given by the formula due to Lagrange:

Argument	Function
ξ	$f(\xi)$
x_1	$f(x_1)$
x_2	$f(x_2)$
x_3	$f(x_3)$
⋮	⋮
x_j	$f(x_j)$
⋮	⋮
x_n	$f(x_n)$

If $f(x)$ is the value of the function for $\xi = x$, then

$$f(x) = \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_j)\cdots(x-x_n)}{(x-x_j)(x_j-x_1)(x_j-x_2)\cdots(x_j-x_k)\cdots(x_j-x_n)} f(x_j)$$

Expanding we get,

$$\frac{f(x)}{(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)} = \frac{f(x_1)}{(x-x_1)(x_1-x_2)\cdots(x_1-x_n)}$$

$$+ \frac{f(x_2)}{(x-x_2)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)}$$

$$+ \frac{f(x_3)}{(x-x_3)(x_3-x_1)(x_3-x_2)\cdots(x_3-x_n)}$$

$$+ \cdots\cdots\cdots\cdots$$

$$+ \frac{f(x_j)}{(x-x_j)(x_j-x_1)(x_j-x_2)\cdots(x_j-x_n)}$$

$$+ \cdots\cdots\cdots\cdots$$

$$+ \frac{f(x_n)}{(x-x_n)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})}$$

APPENDIX VI

We give in this appendix the relative values of

$$G_0(x) = \int_0^x \mu' \, dx = \int_0^x \frac{dx}{(1-x)^{1/2}}, \text{ in the field-free case}$$

and $$G_H(x) = \int_0^x \mu' dx = 2\int_{t=(1-x)^{1/2}}^{1} (\mu' . t) \, dt, \text{ when field is considered}$$

for different values of $x = f_N^2/f^2$.

It should be noted that the height is given by the value of the integral

$$h'(f/f_N) = \int_0^{f_N^2/f^2=1} \mu' . \left(dz/dx\right) . dx$$

and therefore that the contribution to the height comes from the total value of the integral when the integration is taken from $f/f_N = 0$ to $f/f_N = 1$, so that the maximum and minimum values of the error indicated in the column at the extreme right represent only the limits since the height is obtained as the total value of the integral. We shall give the values in the decreasing order of error that is

England - Delhi - Ahmedabad.

in the form of tables overleaf. The difference has been shown graphically also earlier.

Table 8

$G_o(x) - G_H(x)$ for England

f	x	$G_o(x)$	$G_H(x)$	$G_o - G_H$	% error & sign
$1.f_H$	0.2	0.2111	0.2039	0.0072	3.53%
	0.4	0.4508	0.4154	0.0354	7.85%
1.15 Mc/s	0.6	0.7351	0.6450	0.0901	12.26%
	0.8	1.1056	0.9230	0.1826	16.51%
	1.0	2.0000	2.3870	-0.3870	-19.35%
$2.f_H$	0.2	0.2111	0.2061	0.0050	2.37%
	0.4	0.4508	0.4257	0.0251	5.57%
2.3 Mc/s	0.6	0.7351	0.6654	0.0697	9.48%
	0.8	1.1056	0.9481	0.1575	14.25%
	1.0	2.0000	2.2690	-0.2690	-13.45%
$4.f_H$	0.2	0.2111	0.2077	0.0034	1.61%
	0.4	0.4508	0.4337	0.0171	3.79%
4.6 Mc/s	0.6	0.7351	0.6865	0.0486	6.61%
	0.8	1.1056	0.9869	0.1187	10.74%
	1.0	2.0000	2.2667	-0.2667	-13.34%
$8.f_H$	0.2	0.2111	0.2094	0.0017	0.08%
	0.4	0.4508	0.4413	0.0095	2.11%
9.2 Mc/s	0.6	0.7351	0.7063	0.0288	3.92%
	0.8	1.1056	1.0288	0.0768	6.95%
	1.0	2.0000	2.1928	-0.1928	- 9.64%

Table 9

$G_o(x) - G_H(x)$ for Delhi

	x	$G_o(x)$	$G_H(x)$	$G_o - G_H$	% error
f_H	0.2	0.2111	0.2071	0.0040	1.89%
	0.4	0.4508	0.4322	0.0176	3.90%
	0.6	0.7351	0.7063	0.0288	3.92%
	0.8	1.1056	1.0655	0.0401	3.63%
	1.0	2.0000	2.1832	-0.1832	-9.16%
$2f_H$	0.2	0.2111	0.2079	0.0032	1.5%
	0.4	0.4508	0.4361	0.0147	3.3%
	0.6	0.7351	0.7010	0.0341	4.6%
	0.8	1.1056	1.0518	0.0538	4.9%
	1.0	2.0000	2.1383	-0.1383	-6.9%
$4f_H$	0.2	0.2111	0.2088	0.0023	1.09%
	0.4	0.4508	0.4396	0.0112	2.48%
	0.6	0.7351	0.7046	0.0305	4.15%
	0.8	1.1056	1.0469	0.0587	5.3%
	1.0	2.0000	2.1027	-0.1027	-5.14%
$8f_H$	0.2	0.2111	0.2099	0.0012	0.6%
	0.4	0.4508	0.4439	0.0069	1.53%
	0.6	0.7351	0.7151	0.0200	2.8%
	0.8	1.1056	1.0599	0.0457	4.13%
	1.0	2.0000	2.0858	-0.0858	-4.3%

Table 10a

$G_0(x) - G_H(x)$ for Ahmedabad

	x	$G_0(x)$	$G_H(x)$	G_0-G_H	% error
$1f_H$	0.2	0.2111	0.2089	0.0022	1.04%
	0.4	0.4508	0.4398	0.0110	2.44%
1.1 Mc/s	0.6	0.7351	0.7161	0.0190	2.58%
	0.8	1.1056	1.0942	0.0114	1.03%
	1.0	2.0000	2.1242	-0.1242	-6.21%
$2f_H$	0.2	0.2111	0.2992	0.0019	0.09%
	0.4	0.4508	0.4405	0.0103	2.28%
2.2 Mc/s	0.6	0.7351	0.7118	0.0233	3.17%
	0.8	1.1056	1.0785	0.0271	2.45%
	1.0	2.0000	2.1036	-0.1036	-5.18%
$4f_H$	0.2	0.2111	0.2095	0.0016	0.08%
	0.4	0.4508	0.4428	0.0080	1.77%
4.4 Mc/s	0.6	0.7351	0.7147	0.0204	2.77%
	0.8	1.1056	1.0683	0.0373	3.4 %
	1.0	2.0000	2.0760	-0.0760	-3.8 %
$8f_H$	0.2	0.2111	0.2100	0.0011	0.05%
	0.4	0.4508	0.4455	0.0053	1.17%
8.8 Mc/s	0.6	0.7351	0.7200	0.0151	2.05%
	0.8	1.1056	1.0717	0.0339	3.07%
	1.0	2.0000	2.0531	-0.0531	-2.66%

We have given above the values for $G_H(x)$ for $f = f_H$, $2f_H$, $4f_H$ and $8f_H$ so that comparison of errors at the three places can be directly made. The other values of $G_H(x)$ are given overleaf.

Table 10b

$G_H(x)$ for Ahmedabad

f_N^2/f^2 x	$1f_H$	$2f_H$	$3f_H$	$4f_H$	$5f_H$
0.2	0.2089	0.2092	0.2094	0.2095	0.2096
0.4	0.4398	0.4405	0.4417	0.4428	0.4434
0.6	0.7161	0.7118	0.6978	0.7147	0.7156
0.8	1.0942	1.0785	1.0712	1.0683	1.0663
1.0	2.1242	2.1036	2.0884	2.0760	2.0668

	$6f_H$	$7f_H$	$8f_H$	$9f_H$	$10f_H$
0.2	0.2097	0.2098	0.2100	0.2101	0.2102
0.4	0.4443	0.4448	0.4455	0.4461	0.4467
0.6	0.7173	0.7185	0.7200	0.7216	0.7233
0.8	1.0669	1.0680	1.0717	1.0730	1.0757
1.0	2.0611	2.0556	2.0531	2.0522	2.0521

	$11f_H$	$12f_H$	$16f_H$	$18f_H$	$20f_H$	$G_0(x)$	
0.2	0.2102	0.2103	0.2105	0.2106	0.2108	2111	·0003
0.4	0.4472	0.4474	0.4482	0.4489	0.4495	4508	·0013
0.6	0.7241	0.7251	0.7268	0.7283	0.7302	7351	·0049
0.8	1.0770	1.0786	1.0819	1.0846	1.0873	1.1056	+·0183
1.0	2.0497	2.0466	2.0425	2.0363	2.0317	2.0000	−·0317

REFERENCES

Abel, N.H. 1881 Oeuvres, Christiania, 1, 11.
Appleton, E.V. 1925 Nature, March 7.
Appleton, E.V. 1928 Proc. Phys. Soc., London, 41, 43.
Appleton, E.V. 1930 Proc. Phys. Soc., London, 42, 321.
Appleton, E.V. 1937 Proc. Roy. Soc.,London, A162, 451.
Appleton, E.V. & Barnett 1925 Electrician, 398.
Appleton,&E.V. & Barnett 1925 Proc. Roy. Soc. London, A109, 621.
Appleton, E.V. & Beynon W.J.G. 1940 Proc. Phys. Soc., London, 52, 518.
Beynon, W.J.G. 1947 Proc. Phys. Soc.,London, 59, 97.
Beynon, W.J.G. & Thomas,J.O., 1956 JATP, 9, 184.
Booker, H.G. & Seaton, S.L. 1940 Phys. Rev., 57, 88.
Breit, G. & Tuve, M.A. 1926 Phys. Rev. 28, 571.
Budden, K.G. 1955 Physics of the Ionosphere, p.332. London : Physical Society.
Chapman, S. 1931 Proc. Phys. Soc. London, 43, 26; 433.
Darwin, C.G. 1934 Proc. Roy. Soc.,London, A146, 17.
Darwin, C.G. 1943 Proc. Roy. Soc. London, A182, 152.
de Groot, W. 1930 Phil.Mag. 10, 521.
Duncan, R.A. 1956 Australian J. Phys. 9, 436.
Duncan, R.A. 1958 J. Geophys. Res. 63, 491.
Eccles, 1912 Proc. Roy. Soc.,London, A87, 79.
Friedman, B. 1950 New York University, Math. Dept. Report EM-17.
Hartree, D.R. 1952 Numerical Analysis, Oxford : Univ. Press,164.
Jackson, J.E. 1956 J. Geophys. Res. 61, 107.
Kelso, J.M. 1952 J. Geophys. Res. 57, 357.
Kelso, J.M. 1954 JATP, 5, 11.
Kelso, J.M. 1957 JATP, 10, 103.
King, G.A.M. 1954 JATP, 5, 245.
King, G.A.M. 1956 JATP, 8, 184.
Kopal, Z. 1955 Numerical Analysis, London, Chapman & Hall, 352.
Kotadia, A.R. 1958 Ph.D. thesis, Gujarat Univ. (Unpublished).
Kraus, E. 1950 New York Univ. Math. Dept. Report EM-6.
Kraus, E. 1953 J. Geophys. Res. 58, 551.
Larmor, Sir J. 1924 Phil. Mag. 48, 1025.
Manning, L.A. 1947 Proc. I.R.E. 35, 1203.
Mitra, S.N. 1952 The Upper Atmosphere, Calcutta : The Asiatic Society.
Mehler, F.G. 1864 J. Reine. Angewandte Math, 63, 152.
Manning, L.A. 1953 J. Geophys. Res. 58, 117.
Mirick & Hentschel 1929 Proc. IRE, 17, 1034.
Martyn, D.F. 1956 Australian J. Phys. 9, 161.
Pekeris, C.L. 1940 Terr. Mag. Atmos. Elect. 45, 205.
Picard, E. 1904 Comptes Rendus, 139, 245.
Ratcliffe, J.A. 1939 Proc. Phys. Soc. 51, 747.
Ratcliffe, J.A. 1951 J. Geophys. Res. 56, 463; 487.
Ratcliffe, J.A. & others 1956 Phil. Trans. A248, 621.
Rastogi, R.G. 1956 Ph.D. thesis, Gujarat Univ. (Unpublished).
Rydbeck, O.E.H. 1940 Phil. Mag. 30, 282.

Rydbeck, O.E.H. 1942 J.Appl. Phys. 13, 577.
Rydbeck, O.E.H. 1942 J. Geophys. Res. 215, Vol. 47, 219.
Rydbeck, O.E.H. 1942 Trans. Chalmers Univ. Tech. Goteborg, No.3
Rydbeck, O.E.H. 1944 Trans. Chalmers Univ. Tech. Goteborg, No.34
Smith-Rose, R.L. & Barfield 1926 Proc. Roy. Soc. London, A 110, 580. and A 116, 682 (1927)
Schmerling, E.R. 1958 JATP, 12, 8.
Schmerling, E.R. & Thomas J.O. 1956 Phil. Trans. A 248, 75.
Sheriff, R.M. 1956 JATP, 8, 91.
Shinn, D.H. 1953 J. Geophys. Res. 58, 416.
Shinn, D.H. 1955 Physics of the Ionosphere, London, Physical Society, p. 402.
Shinn, D.H. & Whale, H.A. 1952 JATP, 2, 85.
Szego, G. 1939 Orthogonal Polynomials, New York, American Mathematical Society.
Thomas, J.O. and others 1958 JATP, 12, 46.
Thomas, J.O. and Robbins, A.J. 1958 JATP, In press.
Volterra, V. 1913 Lecons sur les Equations Integrales (Paris).
Weierstrass, Karl 1903 Werke (Berlin), 2, 1.
Whale, H.A. 1951 JATP, 1, 244.
White, G.R. & Wachtel, I.I. 1949 J.Geophys. Res. 54, 239.

Goubau, G. 1934 Hochfrequenz. Electroakust. 44, 17.
Millinton, G. 1938 Proc. Phys. Soc. London, 50, 561.
Poeverlien, H. 1951 Z. angew. Phys. 3, 135.
Chatterjee, B. 1953 J. Geophys. Res. 58, 353.