STUDIES ON NONCOMMUTATIVE FIELD THEORIES

A THESIS

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FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

BY

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TO MY PARENTS

Declaration

I hereby declare that the material presented in this thesis is, the result of investigations carried out by me in the School of Physics, University of Hyderabad, under the supervision of Dr. Prasanta K. Panigrahi (currently at the Physical Research Laboratory, Ahmedabad-380 009).

The results reported in the thesis are new to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

In keeping with the general practice of reporting scientific observations due acknowledgement has been made whenever the work described is based on the findings of other investigators.

T. SHREECHARAN

Place: Hyderabad

Date:

Certificate

This is to certify that the work embodied in this thesis entitled “Studies on noncommutative field theories”, has been carried out by Mr. T. Shreecharan, under my supervision and the same has not been submitted in whole or part for a degree in any university.

Dr. PRASANTA K. PANIGRAHI

DEAN

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Abstract

The present thesis is devoted to the study of effect of noncommutativity in space on various aspects of field theory. We first compute the effect of a noncommutative background on the magnetic moment of charged particles in a planar field theory. Specifically, we study this in the context of a Chern-Simons field theory representing anyons. The fact that this theory is relevant for the fractional quantum Hall effect makes this study worthwhile. We then study the symmetry aspects of interacting fermions on the plane through the Slavnov-Taylor identity. The non-Abelian structure of the $U(1)$ noncommutative field theory necessitates the above analysis. This formal study is followed by the computation of the magnetic and electric dipole moments. Several interesting results are obtained from the above analysis. The effect of the noncommutative background on the theories exhibiting phase transition is then analyzed through a non-perturbative approach. In particular, we study the Bardeen-Cooper-Schreiffer (BCS) theory and find the effect of the noncommutative parameter θ on the order parameter. The effect of noncommutativity on the structure of the vacuum is investigated.

This thesis is organized as follows. In chapter 2 we introduce noncommutative Chern-Simons theories coupled to matter fields, bosonic as well as fermionic, in the fundamental representation. We state the interaction vertices necessary for various perturbative calculations. In Chapters 3 and 4, we evaluate the one loop vertex diagrams, for the bosonic and the fermionic cases respectively. These yield various moments for the scalars as well as the fermions. In Chapter 5, the structure of

phase transition in noncommutative BCS theory is studied and the impact of the NC parameter on the BCS vacuum is analyzed. We conclude in the final chapter indicating the open problems and directions for further research work.

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Abbreviations

Abbreviation	Explanation
QFT	Quantum field theory
QED	quantum electrodynamics
NC	Noncommutative
CS	Chern-Simons
UV	Ultraviolet
IR	Infrared
MCS	Maxwell-Chern-Simons
BRST	Becchi-Rouet-Stora-Tyutin
w.r.t	with respect to
1-PI	one particle irreducible
MM	magnetic moment
ST	Slavnov-Taylor
LOFF	Larkin-Ovchinnikov-Fulde-Ferrel

Table 1: *Table of abbreviations and acronyms used in the thesis.*

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Chapter 1

Introduction

The idea of noncommuting variables made their appearance with the advent of quantum mechanics, although it was familiar to the mathematicians dealing with matrices. As is well-known [35], the canonical classical phase space variables, upon quantization, obey,

$$[x^i, x^j] = 0, \quad [p^i, p^j] = 0, \quad [x^i, p^j] = i\hbar\,\delta^{ij}. \tag{1.0.1}$$

The consequence of noncommuting phase space variables results in the Heisenberg uncertainty relation: $\Delta x \Delta p \geq \hbar/2$. The presence of non-zero variance Δx and Δp means that localization of points is not possible in phase space, a situation which led von Neumann to a rigorous study of the "pointless" geometry of the quantum phase space.

In recent times, this noncommutativity of phase space variables has been extended to real space-time coordinates [1, 2]. The previously commuting coordinates are replaced by the Hermitian operators x^i satisfying

$$[x^i, x^j] = i\theta^{ij}. \tag{1.0.2}$$

Here θ^{ij} is a constant real valued antisymmetric matrix with the length dimension two. The immediate consequence of the above replacement is that the space-time coordinates satisfy the uncertainty relation $\Delta x^i \Delta x^j \geq |\theta^{ij}|/2$. From the previous experience of quantum phase space, a space-time point is now replaced with a Planck cell of area $\approx |\theta|$.

In 1947 Snyder [86], taking a cue from Heisenberg, constructed a new set of Lorentz transformations where the underlying space-time structure was noncommuting. This was motivated by the immediate need to tame the divergences that had plagued quantum electrodynamics (QED). It was believed that the presence of a fundamental length scale would provide a natural ultraviolet momentum cut-off [39]. This idea of noncommuting coordinates was later abandoned due to the successful implementation of the renormalization program. Noncommuting geometry resurfaced in mathematics and physics literature in the 1980's [1].

On the mathematical front, noncommutative (NC) geometry has been pioneered by A. Connes [1]. The idea was to study the noncommuting algebra of functions and from this algebra construct a noncommutative version of the Gelfand-Naimark theorem. This enables one to extend various aspects of classical differential geometry to the NC setting. In physics, NC coordinates came into vogue when it was found that certain string theories with background field, naturally gave rise to NC coordinates. Subsequently it was shown by Seiberg and Witten that, the gauge theories on noncommuting coordinates can be mapped to the gauge theories with commutative coordinates. This map goes by the name of Seiberg-Witten map in the literature [83].

This spurred a lot of activity in the physics community. The developments ranged from NC classical mechanics, NC quantum mechanics to string and gauge theories.

1.1 Noncommutative classical mechanics

Noncommutative classical mechanics [3, 33, 36, 79, 80] may sound, at the outset, quite improbable due to the absence of operators, but we will see that postulating a symplectic structure on the phase space of functions, one can consistently construct a NC version of classical physics.

1.1.1 Modification to the Newton's law

The theory under consideration is defined by a set of canonical variables ζ^a, with $a = 1, 2, \cdots, 2n$ with a symplectic structure $\{\zeta^a, \zeta^b\}$. For arbitrary functions of ζ^a one can consistently write

$$\{F, G\} = \{\zeta^a, \zeta^b\} \frac{\partial F}{\partial \zeta^a} \frac{\partial G}{\partial \zeta^b}. \tag{1.1.1}$$

The equation of motion with the above symplectic structure is

$$\dot{\zeta}^a = \{\zeta^a, H\}, \tag{1.1.2}$$

where the Hamiltonian (H) is a function of ζ^a. Similarly for any function F in this phase space, time evolution is given by,

$$\dot{F} = \{F, H\}. \tag{1.1.3}$$

In three dimensional configuration space $\zeta^a = x^i$ for $a = 1, 2, 3$ and $\zeta^a = p_i$ for $a = 4, 5, 6$ and $i = 1, 2, 3$. The new Poisson bracket relations with noncommuting coordinates are

$$\{x^i, x^j\} = \theta^{ij}, \quad \{x^i, p_j\} = \delta^i_j, \quad \{p_i, p_j\} = 0. \tag{1.1.4}$$

For any two arbitrary functions F and G in the phase space, we obtain the following modified Poisson bracket

$$\{F, G\} = \theta^{ij} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^j} + \left[\frac{\partial F}{\partial x^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x^i} \right]. \tag{1.1.5}$$

Let us consider the Hamiltonian

$$H = \frac{p_i p^i}{2m} + V(x),$$

here $V(x)$ is some arbitrary potential. The equations of motion are

$$\dot{x}^i = \frac{p^i}{m} + \theta^{ij} \frac{\partial V}{\partial x^j} \quad \dot{p}_i = -\frac{\partial V}{\partial x^i}, \tag{1.1.6}$$

which can be cast in the form

$$m\ddot{x}^i = -\frac{\partial V}{\partial x_i} + m\theta^{ij} \frac{\partial^2 V}{\partial x^j \partial x_k} \dot{x}_k. \tag{1.1.7}$$

The above equation can be interpreted as the modified Newton's second law due to the noncommutative symplectic structure [79, 80].

1.2 Noncommutative quantum mechanics

Introduction of noncommuting coordinates in physics is natural in the quantum mechanical setting. The commutation relation between the phase space operators that needs to be modified is $[x_i, x_j] = i\theta_{ij}$. This brings in many interesting features as will be explored in the simple below.

1.2.1 The Landau problem

In this section a simple quantum mechanical system is dealt with, which gives rise to NC coordinates. The system that we have in mind is charged particles in a magnetic field [55].

Consider N electrons, whose position and velocities are

$$r_k = (x_k, y_k), \quad v_k = \dot{r}_k, \quad k = 1, 2, \cdots, N. \tag{1.2.1}$$

Electrons experience a constant magnetic field in the z-direction i.e., $\vec{B} = B\hat{z}$. The gauge is chosen such that, the vector potential is $\vec{A}(x_k) = (0, Bx_k)$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. The Lagrangian for such a system is

$$\mathcal{L} = \sum_{k=1}^{N} \frac{1}{2} m v_k^2 + \frac{e}{c} \vec{v}_k \cdot \vec{A}(r_k) - V(r_k) - \sum_{k<l} U(r_k - r_l), \tag{1.2.2}$$

where V is the electron self-energy which can arise from impurity and U is a pair interaction potential between the electrons.

The Hamiltonian is obtained by the standard minimal substitution

$$\mathcal{H} = \sum_{k=1}^{N} \frac{\pi_k^2}{2m} + V(r_k) + \sum_{k<l} U(r_k - r_l). \tag{1.2.3}$$

Here $\pi_k = p_k - eA(r_k)/c$ is the gauge invariant momentum and p_k is the usual canonical momentum, that obeys the usual canonical commutation relations

$$[x_k, p_l^x] = i\hbar\delta_{kl} = [y_k, p_l^y] \quad \text{and} \quad [x_k, y_k] = 0 = [p_k^x, p_l^y]. \tag{1.2.4}$$

It follows that

$$[\pi_k^x, \pi_l^y] = i\hbar \frac{eB}{c} \delta_{kl}. \tag{1.2.5}$$

From the above equation we see that the gauge invariant momenta do not commute in the presence of a magnetic field. This momentum can be written in terms of harmonic oscillator creation and annihilation operators. In the absence of interactions, $V = U = 0$, the energy eigenvalues of the normal ordered Hamiltonian are those of the Landau levels

$$E = \sum_{k=1}^{N} \hbar\omega(n_k + \frac{1}{2}), \quad n_k = 0, 1, 2, \cdots , \tag{1.2.6}$$

where $\omega = eB/mc$, the cyclotron frequency of the classical electron orbits in the magnetic field. The mass gap between Landau levels is the constant $\Delta = \hbar\omega$.

To see how a NC coordinate space emerges from the above Lagrangian, we take the $B \to \infty$ or equivalently $m \to 0$ limit. In this limit Eq. (1.2.2) reduces to

$$\mathcal{L}_0 = \sum_{k=1}^{N} \frac{eB}{c} x_k \dot{y}_k - \sum_{k<l} U(r_k - r_l). \tag{1.2.7}$$

We see that for every k the Lagrangian is of the form $p\dot{q} - h(p, q)$, therefore the following commutation rule is obtained:

$$[x_k, y_k] = i\frac{\hbar c}{eB}. \tag{1.2.8}$$

The above can be identified with the NC parameter as $\theta^{ij} = \epsilon^{ij} \frac{\hbar c}{eB}$. This completes our simple example of the appearance of NC coordinate space.

1.2.2 Two dimensional oscillator

The Hamiltonian for the two dimensional oscillator is given by [44]

$$H = \frac{P_i P_i}{2M} + \frac{M\omega^2 X_i X_i}{2}. \tag{1.2.9}$$

Here X and P satisfy the commutation relations:

$$[X_i, X_j] = -2i\hbar\theta_{ij} \quad [X_i, P_j] = i\hbar\delta_{ij} \quad [P_i, P_j] = 0, \tag{1.2.10}$$

where θ_{ij} is a real $N \times N$ matrix. Instead of working with the noncommuting coordinates X, one can work with commuting coordinates x but with the products between the coordinates being replaced by star product. We will have more to say about star products in the next chapter. For the time being we can perform a change of variable: $X_i = x_i + \theta_{ij} p_j$, $P_i = p_i$. These new variables x, p satisfy the usual commutation relations: $[x_i, x_j] = 0$, $[x_i, p_j] = i\hbar\delta_{ij}$ and $[p_i, p_j] = 0$.

The oscillator Hamiltonian in the new variables becomes

$$\begin{aligned} H &= (1 + M^2\omega^2\theta^2)H_\theta, \quad \text{where} \\ H_\theta &= \frac{1}{2M}\left[p_i p_i + M^2\omega_\theta^2 x_i x_i + 2\theta M^2\omega_\theta^2 \epsilon_{ij} x_i p_j\right]. \end{aligned} \tag{1.2.11}$$

In the above equation

$$\omega_\theta^2 \equiv \frac{\omega^2}{(1 + M^2\omega^2\theta^2)} \tag{1.2.12}$$

and $\theta_{ij} = \theta\epsilon_{ij}$, where ϵ_{ij} is the antisymmetric Levi-Civita tensor and obeys $\epsilon_{ij}\epsilon_{jk} = -\delta_{ik}$. One defines the creation ($a^\dagger$) and annihilation (a) operators such that

$$x_i = \frac{1}{\sqrt{2}}\left(\frac{\hbar}{M\omega_\theta}\right)^{1/2}(a_i^\dagger + a_i), \tag{1.2.13}$$

$$p_i = \frac{i}{\sqrt{2}}(\hbar M\omega_\theta)^{1/2}(a_i^\dagger - a_i). \tag{1.2.14}$$

These operators satisfy the well-known Heisenberg-Weyl algebra

$$[a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij}. \tag{1.2.15}$$

Writing the Hamiltonian in terms of these operators

$$H_\theta = \hbar\omega_\theta(N + I + 2\theta M\omega_\theta J_3) \tag{1.2.16}$$

where $N = a_i^\dagger a_i$, $J_3 = \epsilon_{ij} x_i p_j \hbar/2 = -a_i^\dagger \epsilon_{ij} a_j/2$ and I is the identity operator. If θ_{ij} is not proportional to ϵ_{ij} then, the third term of Eq. (1.2.16) will not be proportional to the angular momentum. It would yield $[H_\theta, J_3] \neq 0$, implying that the rotational invariance is broken.

The NC two dimensional oscillator has a $SU(2)$ symmetry. The generators J_a $(a = 1, 2, 3)$ and the Casimir operator J^2 are

$$J_1 = \frac{1}{2}(a_2^\dagger a_1 + a_1^\dagger a_2), \quad J_2 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2), \quad J_3 = -\frac{i}{2}(a_2^\dagger a_1 - a_1^\dagger a_2), \tag{1.2.17}$$

and

$$J^2 = \sum_{a=1}^{3} J_a J_a = \frac{N}{2}\left(\frac{N}{2}+1\right) \tag{1.2.18}$$

respectively. One can verify that Eq. (1.2.15) implies that $[J_a, J_b] = i\epsilon_{abc}J_c$ and $[J^2, J_a] = 0$. Since the number operator (N) commutes with the generators of the $SU(2)$ algebra (J_a), it follows from Eq. (1.2.16) that the energy eigenvalue problem becomes

$$H_\theta \mid j, m\rangle = \hbar\omega_\theta(n + 1 + 2\theta m M\omega_\theta) \mid j, m\rangle, \tag{1.2.19}$$

where $\mid j, m\rangle$ are the eigenvectors of J^2 and J_3 simultaneously. The eigenvalues of J^2 and J_3 are given by j and m, respectively. In the commutative case ($\theta = 0$), the degeneracy of the nth energy level is $2j+1 = n+1$. Therefore, due to the presence of noncommutativity the degeneracy is lifted.

The angular momentum states $\mid j, m\rangle$ can be obtained using Schwinger's construction for angular momentum in terms of oscillators. The oscillator operators $A^\dagger_\pm, A_\pm$ are defined as

$$A_\pm = \frac{1}{\sqrt{2}}(a_1 \mp ia_2), \quad A^\dagger_\pm = \frac{1}{\sqrt{2}}(a^\dagger_1 \pm ia^\dagger_2) \tag{1.2.20}$$

which satisfy the commutation relations $[A_\alpha, A_\beta] = 0$, $[A^\dagger_\alpha, A^\dagger_\beta]$, $[A_\alpha, A^\dagger_\beta] = \delta_{\alpha\beta}$, where α and β are $+$ or $-$. We can define

$$\mid n_+, n_-\rangle = \frac{(A^\dagger_+)^{n_+}(A^\dagger_-)^{n_-}}{\sqrt{n_+!}\sqrt{n_-!}} \mid 0, 0\rangle, \tag{1.2.21}$$

where $n_\pm$ are semi-positive definite integers, and $A_\pm \mid 0, 0\rangle = 0$, a complete and normalizable set of common eigenstates of the Hermitian operators $N_+ \equiv A^\dagger_+ A_+$ and $N_- \equiv A^\dagger_- A_-$. By construction $[N_+, N_-] = 0$ and $N = N_+ + N_-$, $J_2 = 1/2(N_+ - N_-)$, we can conclude that the common eigenstates of energy and angular momentum can also be denoted as $\mid n_+, n_-\rangle$. The relationship amongst the quantum numbers follows from $n = 2j$ the relations for N and J_3: $2j = n_+ + n_-$, $2m = n_+ - n_-$. Instead of Eq. (1.2.21) we can write

$$\mid j, m\rangle = \frac{(A^\dagger_+)^{(j+m)}(A^\dagger_-)^{(j-m)}}{\sqrt{(j+m)!}\sqrt{(j-m)!}} \mid 0, 0\rangle. \tag{1.2.22}$$

With this the NC two dimensional oscillator has been solved exactly. It is interesting to note that the Lagrangian that gives the Hamiltonian of Eq. (1.2.16) is

$$L = \frac{1}{2}M\dot{\tilde{q}}_l\dot{\tilde{q}}_l - M^2\omega_\theta^2\theta\tilde{q}_l\epsilon_{lk}\dot{\tilde{q}}_k - \frac{M\omega_\theta^2}{2}\tilde{q}_l\tilde{q}_l. \tag{1.2.23}$$

The second term in the above equation describes the interaction of a charged particle with a constant magnetic field B. The components of the vector potential $\mathbf{A}$ are

$$A_l = \frac{M^2\omega_\theta^2\theta c}{e}\epsilon_{lk}\tilde{q}_k, \tag{1.2.24}$$

the magnetic field in turn is given by

$$B = \nabla \times \mathbf{A} = -2\frac{M^2\omega_\theta^2\theta c}{e} = const \tag{1.2.25}$$

here c denotes the speed of light in vacuum. Therefore, the NC two dimensional harmonic oscillator maps into the Landau problem. This also shows that the dynamics of a system in presence of a NC parameter is similar to the commutative case with a magnetic field.

1.2.3 Monopoles and magnetic fields

In our discussion of the NC oscillator we had mentioned that rotational invariance is preserved in two dimensions unlike in higher dimension where rotational invariance is broken. In this section we will take up an example which will show the construction of these generators for higher dimensions and also bring out the similarities between the dynamics in the presence of a magnetic field to that of dynamics in a NC space.

Consider a more generalized commutation relation between the coordinates : $[x^i, x^j] = i\hbar q_\theta \theta^{ij}(x,p)$ [9]. Here θ has been elevated to the status of a field instead of retaining it as a constant tensor. The Jacobi identity

$$[p^i, [x^j, x^k]] + [x^j, [x^k, p^i]] + [x^k, [p^i, x^j]] = 0 \tag{1.2.26}$$

implies that θ is a function of momenta only $\theta(p)$. To further explore the properties of the θ field we examine another Jacobi identity

$$[x^i, [x^j, x^k]] + [x^j, [x^k, x^i]] + [x^k, [x^i, x^j]] = 0, \tag{1.2.27}$$

which gives

$$\frac{\partial\theta^{jk}}{\partial p^i} + \frac{\partial\theta^{ki}}{\partial p^j} + \frac{\partial\theta^{ij}}{\partial p^k} = 0. \tag{1.2.28}$$

It can be noticed that the above equation is similar to the Maxwell equation $\nabla \cdot B = 0$, which can be written in the above form with $F_{ij} = \epsilon_{ijk} B_k$.

Defining the angular momentum operator as $L^i = \epsilon^{ijk} x_j p_k$ the following algebra follows

$$[x^i, L^j] = i\hbar\epsilon^{ijk} x_k + i\hbar q_\theta \epsilon^j_{kl} p^l \theta^{ik}, \quad [p^i, L^j] = i\hbar\epsilon^{ijk} p_k,$$

$$[L^i, L^j] = i\hbar\epsilon^{ijk} L_k + i\hbar q_\theta \epsilon^i_{kl} \epsilon^j_{mn} p^l p^n \theta^{km}(p). \tag{1.2.29}$$

From the above relations it is clear that the $SO(3)$ algebra is broken. Therefore one would believe that in the (x, p) space there are no rotation generators. To restore the algebra one considers the transformation $L^i \to L^i + M^i_\theta(x, p)$:

$$[x^i, L^j] = i\hbar\epsilon^{ijk} x_k, \quad [p^i, L^j] = i\hbar\epsilon^{ijk} p_k, \quad \text{and} \quad [L^i, L^j] = i\hbar\epsilon^{ijk} L_k. \tag{1.2.30}$$

The second commutation relations leads to the position independence criterion $M^j_\theta(x, p) = M^j_\theta(p)$ and the third commutation relation yields $M^i_\theta(p) = \frac{1}{2} q_\theta \epsilon_{jkl} p^i p^l \theta^{kj}(p)$. Substituting these relations in Eq. (1.2.30), we obtain the dual of the Dirac monopole in the momentum space [9, 40]

$$\vec{\Theta}(p) = \frac{g_\theta}{4\pi} \frac{\vec{p}}{p^3} \tag{1.2.31}$$

where g_θ is the dual magnetic charge associated with the Θ field and is related to θ in the following manner: $\theta^{ij} = \epsilon^{ijk} \Theta_k$. Therefore one has

$$\vec{M}_\theta(p) = -\frac{g_\theta q_\theta}{4\pi} \frac{\vec{p}}{p}. \tag{1.2.32}$$

The generalized angular momentum then becomes

$$\vec{L} = (\vec{r} \wedge \vec{p}) - \frac{g_\theta q_\theta}{4\pi} \frac{\vec{p}}{p}. \tag{1.2.33}$$

This angular momentum operator now satisfies the usual angular momentum algebra.

1.2.4 Hydrogen atom

The Coulomb Hamiltonian in terms of the NC coordinates is [22]

$$H = \frac{\hat{p}\hat{p}}{2m} - \frac{Ze^2}{\sqrt{\hat{x}\hat{x}}}. \tag{1.2.34}$$

Similar to a variable change described in the previous section for the two-dimensional oscillator, we perform the change given by

$$x_i = \hat{x}_i + \frac{1}{2\hbar}\theta_{ij}\hat{p}_j \quad \text{and} \quad p_i = \hat{p}_j. \tag{1.2.35}$$

The variables x and p satisfy the usual commutation relations of Eq. (1.0.1). In these new variables the Coulomb potential becomes

$$\begin{aligned} V(r) &= -\frac{Ze^2}{\sqrt{(x_i - \theta_{ij}p_j/2\hbar)(x_i - \theta_{ik}p_k/2\hbar)}} \\ &= -\frac{Ze^2}{r} - Ze^2\frac{x_i\theta_{ij}p_j}{2\hbar r^3} + O(\theta^2) \\ &= -\frac{Ze^2}{r} - Ze^2\frac{\vec{L}\cdot\vec{\theta}}{4\hbar r^3} + O(\theta^2), \end{aligned} \tag{1.2.36}$$

where $\theta_i = \epsilon_{ijk}\theta_{jk}$, $\vec{L} = \vec{r}\times\vec{p}$. Using $(\vec{r}\times\vec{p})\cdot\theta = -\vec{r}\cdot(\vec{\theta}\times\vec{p})$, we can write the modified Coulomb potential as

$$V(r) = -\frac{Ze^2}{r} - \frac{e}{4\hbar}(\vec{\theta}\times\vec{p})\cdot\left(-\frac{Ze\vec{r}}{r^3}\right) + O(\theta^2). \tag{1.2.37}$$

The higher order terms can be neglected since they carry higher powers of θ. To calculate the energy spectrum and the wavefunctions, the NC effects can be treated as perturbations of the commutative theory since the effects are assumed to be small. This also enables one to use the usual wavefunctions and probabilities. Therefore making use of the usual perturbation theory the corrections to the energy levels, to first order in θ, is given by

$$\Delta_{NC}^{H-atom} = \langle nl'jj_z' \mid \frac{Ze^2}{4\hbar}\frac{\vec{L}\cdot\vec{\theta}}{r^3} \mid nljj_z\rangle. \tag{1.2.38}$$

It is worth mentioning that the above expression is similar to spin-orbit coupling, with the role of spin being played by the noncommutative parameter θ.

1.3 Space-time symmetries

Poincaré invariance of the theory is of utmost importance for a consistent construction of field theories. Using commutation relations for the coordinates given in Eq. (1.0.2), we notice that it violates Lorentz invariance since only the spatial coordinates are taken to be noncommuting. This is done for the reasons of protecting the unitarity of the theory [46]. The obvious question then that springs to mind is "How can we justify the construction of consistent field theories with a constant theta?".

In the next section we present the construction of Snyder [86] which states "It is usually assumed that space-time is a continuum. This assumption is not required by Lorentz invariance. In this paper we give an example of a Lorentz invariant discrete space-time".

In the subsequent section, we will present new Lorentz transformations, which respect the constant character of the NC parameter.

1.3.1 Snyder construction

Special theory of relativity is based on the invariance of $S^2 = c^2t^2 - x^2 - y^2 - z^2$ from one inertial frame to another. Here the variables x, y, z, and t take on a continuum of values simultaneously. Elevating these variables to the status of operators, it is assumed that the spectra of the space-time coordinate operators are invariant under Lorentz transformations. The main point to be understood is that this continuum of values is not the only solution and there does exist a Lorentz invariant space-time in which there is a natural unit of length.

It is this introduction of a natural unit of length that forces the noncommutativity of the coordinates. To find the explicit form of the operators x, y, z, t that possess Lorentz invariant spectra we consider the homogeneous quadratic form

$$-\eta^2 = \eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2, \tag{1.3.1}$$

here η's are assumed to be real variables. The coordinates are defined as

$$x = ia\left[\eta_4\frac{\partial}{\partial\eta_1} - \eta_1\frac{\partial}{\partial\eta_4}\right], \quad y = ia\left[\eta_4\frac{\partial}{\partial\eta_2} - \eta_2\frac{\partial}{\partial\eta_4}\right],$$

$$z = ia\left[\eta_4\frac{\partial}{\partial\eta_3} - \eta_3\frac{\partial}{\partial\eta_4}\right], \quad t = \frac{ia}{c}\left[\eta_4\frac{\partial}{\partial\eta_0} - \eta_0\frac{\partial}{\partial\eta_4}\right], \tag{1.3.2}$$

where a is the natural unit of length, and c is the velocity of light. The operators x, y, z, t are assumed to be Hermitian operators. From the above equation it can be shown that the spectrum of the above operators can be positive, negative or zero. The operator t, has a continuous spectrum, from plus infinity to minus infinity.

The transformations leaving Eq. (1.3.1) and η_4 invariant are covariant Lorentz transformations on the variables η_1, η_2, η_3, and η_0. When the transformed variables η_1', η_2', η_3', and η_0' are substituted in Eq. (1.3.2) it is found that x, y, z, and t undergo contravariant Lorentz transformation. The new operators x', y', z' and t' that are formed by replacing η_1, η_2, η_3, and η_0 in Eq. (1.3.2) by η_1', η_2', η_3', and η_0' are linear expressions with real constant coefficients in x, y, z and t and are Hermitian operators.

Other physical operators can be defined as,

$$L_x = i\hbar\left[\eta_3\frac{\partial}{\partial\eta_2} - \eta_2\frac{\partial}{\partial\eta_3}\right], L_y = i\hbar\left[\eta_1\frac{\partial}{\partial\eta_3} - \eta_3\frac{\partial}{\partial\eta_1}\right], L_z = i\hbar\left[\eta_2\frac{\partial}{\partial\eta_1} - \eta_1\frac{\partial}{\partial\eta_3}\right],$$

$$M_x = i\hbar\left[\eta_0\frac{\partial}{\partial\eta_1} + \eta_1\frac{\partial}{\partial\eta_0}\right], M_y = i\hbar\left[\eta_0\frac{\partial}{\partial\eta_2} + \eta_2\frac{\partial}{\partial\eta_0}\right], M_z = i\hbar\left[\eta_0\frac{\partial}{\partial\eta_3} + \eta_3\frac{\partial}{\partial\eta_0}\right]. \quad (1.3.3)$$

Here L_x, L_y, L_z, M_x, M_y and M_z are the infinitesimal elements of three dimensional Lorentz transformation and commute with the quadratic form S^2. It can be seen that L_x, L_y, L_z, M_x, M_y and M_z do not involve η_4 and as a consequence leave Eq. (1.3.1) invariant. Thus from the above facts it is clear that the usual assumptions about the continuous nature of space-time are not necessary for Lorentz invariance. The operators defined in Eq. (1.3.1) and Eq. (1.3.2) have forty-five commutators, out of which only six differ from the ordinary ones. These are

$$[x, y] = \frac{ia^2}{\hbar}L_z, \quad [t, x] = \frac{ia^2}{\hbar c}M_z,$$

$$[y, z] = \frac{ia^2}{\hbar}L_x, \quad [t, y] = \frac{ia^2}{\hbar c}M_y, \quad (1.3.4)$$

$$[z, x] = \frac{ia^2}{\hbar}L_y, \quad [t, x] = \frac{ia^2}{\hbar c}M_x.$$

1.3.2 Constant θ and Lorentz transformations

As we have shown above the Snyder method is able to preserve Lorentz invariance but not Poincaré. A way out of this quagmire is to postulate that the NC parameter itself is space-time dependent. However, such theories have not been fully understood, furthermore in this thesis we have taken our noncommutativity parameter to be constant hence, we will not venture into these proposals any further.

The Lorentz transformation of the coordinates: $x'^i = \Lambda^i_j x^j$ is not compatible with the algebra, defined in Eq. (1.0.2), since it requires θ^{ij} to transform as a second rank tensor *i.e.*, $\theta^{ij} = \Lambda^i_k \Lambda^j_l \theta^{kl}$. We have taken the NC parameter to be a constant, therefore it makes little sense to accept that it transforms under Lorentz transformation. To get around this trouble an interesting solution was provided in [16].

Similar to the commuting coordinates introduced for the Coulomb problem, we denote new coordinates as x_c

$$x^i_c = x^i + \frac{1}{2\hbar}\theta^{ij} p_j. \tag{1.3.5}$$

These new coordinates are commuting and obey the algebra defined in Eq. (1.0.1). Since time is not an operator in ordinary quantum mechanics noncommutativity can be restricted to only the spatial coordinates. We take x_c transform as $x^\mu_c = \Lambda^\mu_\nu x^\nu_c$ that leaves the interval $s^2 = \eta_{\mu\nu} x^\mu_c x^\nu_c$ invariant if $\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$. Further the momentum four vector p^μ transforms as an usual Lorentz vector $p^\mu = \Lambda^\mu{}_\nu p^\nu$. Using Eq. (1.3.5), the NC coordinates transform as

$$\begin{aligned} x^{\mu\prime} &= x^{\mu\prime}_c - \frac{1}{2\hbar}\theta^{\mu\nu} p'_\nu \\ &= \Lambda^\mu{}_\nu x^\nu_c - \frac{1}{2\hbar}\theta^{\nu\rho}\Lambda_\rho{}^\sigma p_\sigma. \end{aligned} \tag{1.3.6}$$

Using Eq. (1.3.5) once again, the final expression for the Lorentz transformation for the NC coordinates are given by

$$x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu + \frac{1}{2\hbar}\Lambda^\mu{}_\nu \theta^{\nu\rho} p_\rho - \frac{1}{2\hbar}\theta^{\mu\nu}\Lambda_\nu{}^\rho p_\rho. \tag{1.3.7}$$

The first thing to be noticed about the above transformation is that in the limit $\theta^{\mu\nu} \to 0$ we recover the ordinary Lorentz transformation for the coordinates.

The square of the invariant length for the commutative coordinate x_c^μ is $s^2 = \eta_{\mu\nu} x_c^\mu x_c^\nu$.

$$s_{nc}^2 = x^\mu x_\mu + \frac{1}{\hbar}\theta_{\mu\nu}\hat{x}^\mu p^\nu + \frac{1}{4\hbar^2}\theta^{\mu\alpha}\theta_{\mu\beta}p_\alpha p^\beta. \tag{1.3.8}$$

It is easy to verify that s_{nc}^2 is left invariant by the noncommutative Lorentz transformation Eq. (1.3.7).

Chapter 2

Noncommutative gauge theories

2.1 Introduction

In this chapter we outline the construction of gauge theories in the NC setting. Of central importance is the NC star product or the Moyal product [70, 48]. The convenience of star product lies in the fact that it preserves the operator character of the coordinates but treating them as ordinary classical variables. In the next section we derive the Moyal product and subsequently we show that under the star operation only the $U(N)$ gauge group closes and the others such as $SU(N)$, $SO(N)$ etc, do not [66]. After that we present the construction of Wess and collaborators [57] where it is indeed possible to go to other gauge groups including $U(N)$. Then in Section IV we introduce NC Chern-Simons (CS) theory and present our motivation for the study of magnetic moment (MM) for the scalars as well spinors, that form chapters 3 and 4 respectively.

2.2 Star product

Noncommutativity of coordinates is incorporated by postulating the commutation relation of Eq. (1.0.2). One can see that these are operators. Instead of working with operators, taking a cue from developments in phase space formulation of quantum mechanics, one can work with functions but by deforming the multiplication rule between these functions. This deformed product happens to be the star product [48, 70]. Below we give a derivation of the star product for the NC coordinate operators.

Weyl-Moyal correspondence associates a function to every operator valued object

$$\hat{W}(\hat{X}) \longleftrightarrow \Phi(x). \tag{2.2.1}$$

Now defining

$$\begin{aligned} \hat{\Phi}(\hat{X}) &= \int_k d\kappa\, e^{i\,k\,\hat{X}}\, \phi(k) \\ \phi(k) &= \int dx\, e^{-i\,k\,x}\, \Phi(x), \end{aligned} \tag{2.2.2}$$

where k and x are real variables. The multiplication of two Weyl operators is given by

$$\begin{aligned} \hat{W}_1(\hat{X})\hat{W}_2(\hat{X}) &= \int_k \int_p dk\, dp\, e^{i\,k\,\hat{X}}\, \phi_1(k)\, e^{i\,p\,\hat{X}}\, \phi_2(p) \\ &= \int_k \int_p dk\, dp\, e^{i\,(k+p)\,\hat{X} - \frac{1}{2} k_\mu p_\nu [\hat{X}_\mu, \hat{X}_\nu]}\, \phi_1(k)\, \phi_2(p). \end{aligned} \tag{2.2.3}$$

The BCH formula has been used in obtaining the second equation. This gives the correspondence

$$\hat{W}_1(\hat{X})\hat{W}_2(\hat{X}) \longleftrightarrow (\Phi \star \Phi)\,(x). \tag{2.2.4}$$

Therefore we finally have the expression for the star product:

$$(\Phi \star \Phi)\,(x) \equiv \left[e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu^y \partial_\nu^z}\, \Phi(y)\Phi(z) \right]_{y=z=x}. \tag{2.2.5}$$

With the star/Moyal product having been defined the NC coordinate commutator can be written as

$$(x^i \star x^j - x^j \star x^i) = i\theta^{ij}. \tag{2.2.6}$$

This commutation relation with the star product put in, is called the Moyal bracket.

2.3 Properties of star product

In this section we enumerate a few properties of the star product.

1. An important relation deals with the exponentials. This relation is useful in finding the Feynman rules in momentum space.

$$\begin{aligned} e^{ikx} \star e^{ipx} &= e^{i(k+p)x} e^{-\frac{i}{2}k\times p}, \quad \text{where} \\ k \times p &\equiv k^\mu p^\nu \theta_{\mu\nu}. \end{aligned} \tag{2.3.1}$$

2. Associativity:

$$[(f \star g) \star h](x) = [f \star (g \star h)](x), \tag{2.3.2}$$

this is very easy to prove if we can go to the momentum space and then making use of the relation given in Eq. (2.3.1).

3. Star products under the integral

$$\int d^4x (f \star g)(x) = \int d^4x (g \star f)(x) = \int d^4x (f \cdot g)(x). \tag{2.3.3}$$

This can be proved by going to the momentum space and using Eq. (2.3.1). The x integration is performed yielding a delta function $\delta^4(k+p)$. Since θ is antisymmetric the exponential vanishes and hence we get

$$\int d^4x (f \star g)(x) = \int d^4k \tilde{f}(k) \tilde{g}(-k) = \int d^4x (f \cdot g)(x). \tag{2.3.4}$$

This particular property of the star product has important consequences for field theories on NC spaces. It tells that the kinetic part of the action is same as its commutative counterpart. Hence, only the interaction term of the action is affected by the star product. Therefore, one can think of NC field theories as ordinary commutative theory but with a highly non-local interaction.

4. From Eq. (2.3.3), we can obtain the cyclic property of the star products:

$$\int d^4x (f_1 \star f_2 \star \cdots \star f_n)(x) = \int d^4x (f_n \star f_1 \star \cdots \star f_{n-1})(x). \tag{2.3.5}$$

5. Complex conjugation.

$$(f \star g)^* = g^* \star f^*. \tag{2.3.6}$$

It is clear that if f is a real function then $f \star f$ is also real.

2.4 Noncommutative Schrödinger equation

In the first chapter we had presented various quantum mechanical potential problems wherein the technique of solving for the spectrum mainly relied in going to a new coordinate variable that satisfied the usual commutative relations: $[x^i, x^j] = 0$. This redefinition of variables looks ad-hoc at first sight but, with the use of star product we show that such a redefinition of the coordinates is quite natural. This shift is similar to the Bopp's shift of the phase space variables, obtained in the deformation quantization formalism of quantum mechanics. For this we start with the NC Schrödinger equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \left[\frac{p^2}{2m} + V(x)\right] \star \psi(x,t). \tag{2.4.1}$$

For the sake of generality we have retained the general form of the potential. We have already pointed out that quadratic terms are unaffected hence, only the potential term is modified due to the $\star$-product. Let us evaluate $V(x) \star \psi(x)$

$$\begin{aligned}
&= e^{\frac{i}{2}\theta^{ij}\partial_i^x\partial_j^y} V(x)\psi(y)|_{x=y} \\
&= V(x)\psi(y) + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\right)^n \partial_{i_1}^x \cdots \partial_{i_n}^x \theta^{i_1 j_1} \cdots \theta^{i_n j_n} \\
&\quad \partial_{j_1}^y \cdots \partial_{j_n}^y V(x)\psi(y)|_{x=y}.
\end{aligned}$$

Replacing $\partial_{j_k}^y = \partial/\partial y^{j_k} \equiv ip_{j_k}^y/\hbar$, and introducing the notation $\tilde{p}_{i_k}^y = \theta_{i_k j_k} p^{y j_k}$, we get

$$\partial_{i_1}^x \cdots \partial_{i_n}^x V(x)\tilde{p}_{i_1}^y \cdots \tilde{p}_{i_n}^y \psi(y) = (i/\hbar)^n \int d^D k\, e^{ikx} V(k)(ik\tilde{p}^y)^n \psi(y). \tag{2.4.2}$$

In the above expression we have gone to the momentum space. Summing over n we get

$$\begin{aligned}
V(x) \star \psi(x) &= \int d^D k e^{ikx} e^{-\frac{i}{2\hbar}k\tilde{p}^y} V(k)\psi(y)|_{x=y} \\
&= V\left[x - \frac{\tilde{p}^x}{2\hbar}\right]\psi(x).
\end{aligned} \tag{2.4.3}$$

The above equation shows clearly that we can work with operator valued coordinates or using the star product one can work with the classical variables.

2.5 Noncommutative gauge theories

Construction of gauge theories in the NC setting has been studied quite extensively [2]. The main reason seems to be that it exhibits many interesting features that are absent in commutative theories. For example one immediate thing that one notices is that due to the noncommuting nature of the fields the $U(1)$ theory itself has a structure very similar to that of its corresponding non-abelian commutative theory. Then we have have the interesting feature of the UV/IR mixing [67, 68]. Finally it must be mentioned that the renormalizability of such theories has been and still is quite challenging and remains a open problem [27].

2.5.1 Only $U(N)$ gauge groups

In this subsection it is shown that under a star operation only the $U(N)$ gauge algebra closes and the others do not [66]. Consider the $U(N)$ algebra whose generators, X, Y, are anti-Hermitean matrices: $\overline{X^t} = -X$. The bar stands for complex conjugation. The crucial observation for the proof is the following property of the Moyal product,

$$\overline{(X \star Y)^t} = \overline{Y^t} \star \overline{X^t}. \tag{2.5.1}$$

Using the ordinary rules for the transpose of matrices,

$$(X \star Y)^t = Y^t X^t + \frac{i}{2}\theta^{ij}\partial_j Y^t \partial_i X^t - \frac{1}{8}\theta^{ij}\theta^{kl}\partial_j\partial_l Y^t \partial_i\partial_k X^t + \cdots \tag{2.5.2}$$

The higher order terms are obvious. Now applying the complex conjugation and renaming the indices of θ

$$\overline{(X \star Y)^t} = \overline{Y^t X^t} + \frac{i}{2}\theta^{ij}\partial_i \overline{Y^t}\partial_j \overline{X^t} - \frac{1}{8}\theta^{ij}\theta^{kl}\partial_i\partial_k \overline{Y^t}\partial_j\partial_l \overline{X^t} + \cdots = \overline{Y^t} \star \overline{X^t}. \tag{2.5.3}$$

Taking into account $\overline{X^t} = -X$ and $\overline{Y^t} = -Y$ yields,

$$\begin{aligned}
\overline{[X, Y]^t_\star} &= \overline{(X \star Y)^t} - \overline{(Y \star X)^t} \\
&= \overline{Y^t} \star \overline{X^t} - \overline{X^t} \star \overline{Y^t} \\
&= Y \star X - X \star Y = -[X, Y]_\star.
\end{aligned} \tag{2.5.4}$$

This shows that the algebra $U(N)$ is closed under the Moyal commutator.

Now turning to the algebras of $SO(N)$, $SU(N)$ and $Sp(N)$. It is first shown that for $N = 2$ these algebras do not close w.r.t the Moyal commutator. The counter examples for both $SO(2)$ and $Sp(2)$ are given by the formulae,

$$X = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}, \tag{2.5.5}$$

and the counter example for $SU(2)$ is

$$X = \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} Y = \begin{pmatrix} i\beta & 0 \\ 0 & -i\beta \end{pmatrix}. \tag{2.5.6}$$

Here α and β are coordinates on the manifold chosen so that $\theta^{\alpha\beta} \neq 0$. This can always be done unless $\theta = 0$ and the Moyal product coincides with the ordinary multiplication of matrix-valued functions. With X and Y as given above one can easily compute the Moyal commutator since all derivatives of order higher than one vanish. The result for both counter examples is

$$[X, Y]_* = i\theta^{\alpha\beta} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.5.7}$$

This matrix has a non-vanishing trace. Since the Lie algebras of $SO(2), SU(2)$ and $Sp(2)$ consist of traceless matrices, it can be concluded that they are not closed under the Moyal commutator. This conclusion is also valid for $SO(N), SU(N)$ and $Sp(N)$ for arbitrary N because they contain $SO(2), SU(2)$ and $Sp(2)$ as their subgroups.

2.6 The Munich construction

In this section we present the technique of constructing NC gauge theories developed by Wess and his collaborators [57, 62]. The goal of this approach is to consider noncommutative theories as effective theories. The essential ingredient, in their technique, is that the fields and the gauge transformations do not form a Lie algebra instead satisfy the enveloping algebra. This enables one to consider gauge theories that are genuinely non-Abelian. Unlike the usual method where the algebra does not close under star operation hence have to be restricted to only $U(N)$ theories [66]. This technique relies on the idea of covariant coordinates which has also been discovered in [56]

2.6.1 Covariant coordinates

Consider fields $\psi(\hat{x})$ as elements of the associative algebra $\mathcal{A}_x$. Under an infinitesimal gauge transformation they transform as

$$\delta\psi(\hat{x}) = i\alpha(\hat{x})\psi(\hat{x}). \tag{2.6.1}$$

This transformation is covariant. The gauge parameter $\alpha(\hat{x})$ is also an element of $\mathcal{A}_x$. If they are matrix valued then the transformation is non-Abelian in nature. Now we assume that the coordinates are invariant under the gauge transformation: $\delta\hat{x} = 0$. Multiplication of a field on the left is not a covariant operation:

$$\delta(\hat{x}^i\psi) = i\hat{x}^i\alpha(\hat{x})\psi, \tag{2.6.2}$$

and in general the RHS is not equal to $i\alpha(\hat{x})\hat{x}^i\psi$. Taking cue from ordinary gauge theory one can introduce covariant coordinates X [56]such that

$$\delta(X^i\psi) = i\alpha X^i\psi, \tag{2.6.3}$$

$\delta(X^i) = i[\alpha, X^i]$. To find the relation between X^i and $\hat{x}^i$, an ansatz of the form $X^i = \hat{x}^i + A^i(\hat{x})$, is chosen. Here $A^i(\hat{x}) \in \mathcal{A}_x$. One can notice that the expression for the covariant coordinate resembles that of the ordinary derivative plus a gauge potential that gives us the covariant derivative. The transformation of the gauge potential can be obtained from Eq. (2.6.2):

$$\delta A^i = i[\alpha, A^i] - i[\hat{x}^i, \alpha]. \tag{2.6.4}$$

2.6.2 Gauge transformations

In this subsection the explicit expression for the gauge transformation is derived.

The commutator $[x^i, .]$ in the transformation of the gauge potential, Eq. (2.6.4), acts as a derivation on the elements of $\mathcal{A}_x$. Since, coordinates coordinate do not commute this can be written as a derivative on elements $f \in \mathcal{A}_x$:

$$[x^i, f] = i\theta^{ij}\partial_j f. \tag{2.6.5}$$

The derivative acts as a derivation on $\mathcal{A}_x$. *i. e.*, $\partial_j(fg) = (\partial_j f)g + f(\partial_j g)$ and on the coordinates as: $\partial_j x^i \equiv \delta^i_j$. The RHS of Eq. (2.6.5) is a derivation since θ is constant. The transformation of the gauge field can be written as

$$\delta A^i = \theta^{ij}\partial_j\alpha + i[\alpha, A^i]. \tag{2.6.6}$$

The gauge potential $\hat{A}$ of noncommutative Yang-Mills is introduced by defining $A^i \equiv \theta^{ij}\hat{A}_j$. The transformation law then for the gauge field $\hat{A}_j$:

$$\delta\hat{A}_j = \partial_j\alpha + i[\alpha, \hat{A}_j]. \tag{2.6.7}$$

As already pointed out in the beginning of this chapter the usage of star products is more convenient. Therefore we can represent the elements of $\mathcal{A}_x$ by functions of the classical variables x^i. In terms of the star product and the classical variables Eq. (2.6.5) becomes

$$x^i * f - f * x^i = i\theta^{ij}\partial_j f, \tag{2.6.8}$$

where $f(x)$ is now a function and $\partial_j f = \partial f/\partial x^j$ is the ordinary derivative. This follows directly from the Moyal-Weyl product. It is interesting to note the form of the covariant coordinates written in terms of $\hat{A}$:

$$\hat{X}^i = \xi^i + \theta^{ij}\hat{A}_j. \tag{2.6.9}$$

Before we end this section we must point out that this technique of covariant coordinates is quite general and can be applied to a variety of NC coordinates scenarios namely

- Lie algebra structure: $[\hat{x}^i, \hat{x}^j] = iC^{ij}_k\hat{x}^k$.
- Quantum space structure: $\hat{x}^i\hat{x}^j = q^{-1}R^{ij}_{kl}\hat{x}^k\hat{x}^l$.

2.7 Noncommutative Chern-Simons theory

In 2+1 dimensions conventional Maxwell electrodynamics can be modified by the presence of a Chern-Simons (CS) term [34, 82]. The latter violates parity and time reversal invariance [77] and can provide a gauge field, a gauge invariant mass. The $U(1)$ Maxwell-Chern-Simons (MCS) Lagrangian is given by

$$S = \int d^3x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{M}{2} \epsilon_{\mu\nu\rho} A^{\mu} \partial^{\nu} A^{\rho} + \bar{\psi} i \not{D} \psi - m \bar{\psi} \psi \right]. \tag{2.7.1}$$

Here the covariant derivative is $\partial_\mu - ieA_\mu$ and unlike 3+1 dimensional QED here the gauge coupling e is not dimensionless. Its mass dimension in natural units is $[\text{mass}]^{1/2}$. The Bianchi identity is given by $\partial_\mu {}^*F^\mu = 0$, where ${}^*F^\mu = 1/2\epsilon^{\mu\nu\rho} F_{\nu\rho}$. The equation of motion for the gauge field is given by

$$\left[\Box + M^2 \right]^* F^{\mu} = M \left[g^{\mu\nu} - \epsilon^{\mu\nu\rho} \frac{\partial_\rho}{M} \right] J_\nu, \tag{2.7.2}$$

where $J_\nu = -e\bar{\psi}\gamma_\nu\psi$. It is worth mentioning that unlike the conventional massive gauge non-invariant theory, the transverse vector field ${}^*F^\mu$ satisfies linear equations. The two polarizations modes are determined by the signs of the mass M and are independent degrees of freedom in the absence of interaction.

Pure CS theory has also attracted considerable attention in the context of quantum Hall effect [37, 87] and Knot theory [91]. CS theory can have dramatic consequences when coupled to matter fields. It gives rise to particles called anyons, particles whose statistics are intermediate to that of fermions and bosons [90]. The emergence of anyons can be seen by coupling the CS term to the matter current $J^\mu = (\rho, \vec{J})$. In terms of components we have

$$\rho = MB, \quad J^i = M\epsilon^{ij} E_j. \tag{2.7.3}$$

The first expression in the above equation gives the relation of the charge density to the magnetic field. Therefore a CS term fixes a magnetic flux to the electric charge. The second expression gives the conservation of charge-flux since, the time derivative of the first expression

$$\dot{\rho} = M\dot{B} = M\epsilon^{ij}\partial_i \dot{A}_j \tag{2.7.4}$$

along with current conservation equation: $\dot{\rho} + \partial_i J^i = 0$ implies

$$J^i = -M\epsilon^{ij}\dot{A}_j + \epsilon^{ij}\partial_j\chi, \tag{2.7.5}$$

where $\chi = MA_0$. Thus this attachment of a magnetic flux to the charge is an explicit realization of anyons. For many years anyons were considered merely a theoretical construct. Recently it has been reported that anyons can indeed be experimentally observed [17].

Another feature of the CS term that is worth pointing out is that one cannot write down such a term in 3+1 dimensions. The CS term is quite intrinsic and a very natural object in 2+1 dimensions. It is known that the CS term can be generated by one-loop effect in the presence of massive fermions even though one does include it at the tree level [6, 30, 23, 77]. A CS type term is also generated in the effective action of charged particles in a magnetic field [72]. It was later on noticed via explicit loop calculations that even though a CS term can be generated by one-loop fermionic effects, the CS coefficient (M) does not get radiative contributions at the two loop level, either for the Abelian or in the non-Abelian theory. This peculiar feature in the case of Abelian theory goes by the name of Coleman-Hill theorem [29]. It must be pointed out that this theorem is valid at zero temperature and at finite temperature it does not hold. There have been attempts at generalizing this theorem to the non-Abelian case [14]. But we will not be concerned about these aspects in this thesis. It must however be mentioned that at finite temperature the coefficient of the CS term does get modified [6].

In one-loop the vacuum polarization and the self energy do not receive corrections in pure CS theory. Vertex gets modified by pure CS gauge field. This is expected since the CS field is known to alter the statistics of the particles interacting with the same. The immediate question that arises is how does one give the scalar particles a magnetic moment in 2+1? The answer can be found if one notices the Gordon decomposition for the fermions

$$\gamma_\mu = \frac{1}{2m}[\mathcal{P}_\mu + i\epsilon_{\mu\nu\lambda}\mathcal{K}^\nu\gamma^\lambda]. \tag{2.7.6}$$

With $\mathcal{K}$ being the momentum transfer of the incoming p and the outgoing q fermions. The presence of the antisymmetric object is responsible for the MM. This antisymmetric Levi-Civita tensor can also be provided to the scalars coupled to a CS gauge field thereby leading to the MM for the bosons in 2+1.

The $U(1)$ NCCS action is given by [47]

$$S_{CS} = \frac{M}{2}\int d^3x \epsilon^{\mu\nu\rho}\left[A_\mu \star \partial_\nu A_\rho + \frac{2ie}{3}A_\mu \star A_\nu \star A_\rho\right]. \tag{2.7.7}$$

The covariant gauge fixing term is

$$S_{GF} = -\frac{1}{2\xi}\int d^3x\, \partial_\mu A^\mu \star \partial_\nu A^\nu. \tag{2.7.8}$$

The new feature of the NCCS action is the presence of the non-linear $(A_\mu \star A_\nu \star A_\rho)$ term, leading to self-interaction amongst the photons; this is similar to the commutative non-Abelian version of the theory [7, 58]. It is therefore natural to expect that the one loop contributions arising from the above action will be similar to that of the commutative non-Abelian version of the above theory.

In this thesis we take up the study of the NCCS to which matter fields have been added. We specially concentrate on the explicit evaluation of the one-loop vertex integrals leading to the magnetic moment of the bosons and the fermions. The bosonic and the fermionic actions are

$$S_{Bosonic} = \int d^3x\, (D_\mu\phi^\dagger \star D^\mu\phi - m\,\phi^\dagger \star \phi) \tag{2.7.9}$$

and

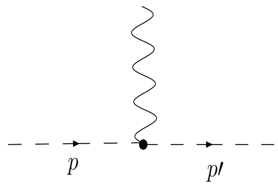
$$S_{Dirac} = \int d^3x\, \bar{\psi} \star (i\not{D} - m)\psi, \tag{2.7.10}$$

respectively. In both the above cases matter fields are taken to be in the fundamental representation. Therefore the expression for the covariant derivative acquires the form: $D_\mu\phi = \partial_\mu\phi - ieA_\mu \star \phi$.

It has already been pointed out that the kinetic part of the action is same as their commutative counterpart thereby the propagators are same in both the theories. The gauge field $(G_{\mu\nu}(p))$, Bosonic $(D(p))$, and the Fermionic $(S(p))$ propagators are given by

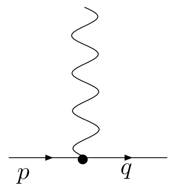
$$iG_{\mu\nu}(p) = -\frac{1}{M}\epsilon_{\mu\nu\rho}\frac{p^\rho}{p^2}, \tag{2.7.11}$$

$$iD(p) = \frac{i}{p^2 - m^2 + i\epsilon}, \tag{2.7.12}$$



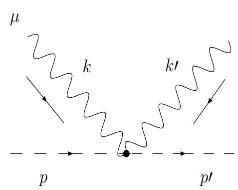
$$\equiv ie(p+p')_\mu \exp\left[\tfrac{i}{2}p \times p'\right].$$

Figure 2.1: *The boson-photon vertex.*



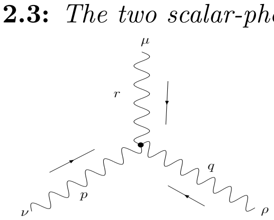
$$\equiv ie\gamma_\mu \exp\left[\tfrac{i}{2}p \times q\right].$$

Figure 2.2: *The fermion-photon vertex.*



$$\equiv 2ie^2 g^{\mu\nu} \exp\left[\tfrac{i}{2}p \times p'\right] \cos\left[(k \times k')/2\right].$$

Figure 2.3: *The two scalar-photon vertex.*



$$\equiv 2ieM\epsilon^{\nu\mu\rho} \sin\left[(p \times r)/2\right].$$

Figure 2.4: *The three gauge boson vertex.*

and

$$iS(p) = \frac{i(\not{p}+m)}{p^2 - m^2 + i\epsilon}, \tag{2.7.13}$$

respectively. The interaction vertices are depicted in the figures given below [12]

Chapter 3

Chern-Simons scalar QED

3.1 Introduction

The possibility of particles carrying fractional angular momentum on a plane is, by now, well accepted [17, 90]. The role of CS term [82] in inducing fractional spin has been carefully investigated [89]. Field theoretically, it has been shown in 2+1 dimensions that, one can calculate fractional angular momentum eigenvalues of single particle states. Furthermore, Polyakov showed that, the interaction of scalar particles with the CS gauge field leads to the transmutation of a boson into a spinning particle [75]. An interesting consequence of this is the appearance of spin MM for the bosons, not possible in 3+1 dimensions. Although not present at the tree level, the boson spin is induced at the one-loop level, leading to a MM for the bosons [61]. The existence of MM leads to unusual planar dynamics, as shown for scalars and spinors in the context of MCS electrodynamics [60, 42]. Therefore, the MM of anyons has been studied extensively [26, 41].

Recently, various aspects of NC theories with a CS term have been under the scrutiny of a number of authors [10, 31, 64, 28, 47]. Mainly since they have many interesting connections with other areas of physics and mathematics. NCCS theory and its variants have been quite useful in explaining the filling fraction of the electrons in the lowest Landau level [88]. Keeping this as well as the fact that, a spin magnetic moment can play an important role in the planar dynamics, we compute the magnetic moment of scalar particles in the context of noncommutative scalar QED in 2+1 dimension with a tree level CS term.

The chapter is organized as follows. In the following section, we give the explicit evaluation of the non-planar integrals that will be needed for the loop calculations of this chapter as well the subsequent chapter on fermionic magnetic moment. In Section III, the vertex contributions arising from all the diagrams at one-loop level are computed. We concentrate on the parity odd gauge invariant pieces, since the same lead to magnetic moment type interactions.

3.2 Calculation of nonplanar integrals

In this section we evaluate a non-planar integral. It is typical of the integrals that appear in NC loop calculations. The same method will be used to calculate the vertex amplitudes. Consider a non-planar integral of the type

$$\int \frac{d^3k}{(2\pi)^3}\frac{k_\mu}{[k^2-\Delta^2]^3}e^{ik\times\mathcal{P}}. \tag{3.2.1}$$

Introducing an auxiliary variable for the numerator, *i.e.*,

$$k_\mu = -i\frac{\partial}{\partial z_\mu}e^{ikz} \tag{3.2.2}$$

and also noting that

$$\frac{1}{[k^2-\Delta^2]^3} = \frac{1}{2}\frac{\partial^2}{\partial(\Delta^2)^2}\frac{1}{[k^2-\Delta^2]}, \tag{3.2.3}$$

the integral to be evaluated, Eq. (3.2.1),

$$\begin{aligned}
&= -\frac{1}{2}\int_0^\infty d\alpha\,\alpha^2 e^{-\alpha\,\Delta^2}\frac{\partial}{\partial z_\mu}\int\frac{d^3k}{(2\pi)^3}e^{-\alpha k^2+ikz+ik\times\mathcal{P}}\\
&= -\frac{1}{2}\int_0^\infty d\alpha\,\alpha^2 e^{-\alpha\,\Delta^2}\frac{\partial}{\partial z_\mu}\int\frac{d^3k}{(2\pi)^3}\,e^{-\alpha\left[k-\frac{i(\tilde{\mathcal{P}}+z)}{2\alpha}\right]^2-\frac{(\tilde{\mathcal{P}}+z)^2}{4\alpha}}\\
&= \frac{1}{2}\frac{\tilde{\mathcal{P}}_\mu}{(2\sqrt{\pi})^3}\int_0^\infty d\alpha\,\alpha^{-1/2}\,e^{-\alpha\,\Delta^2-\frac{\tilde{\mathcal{P}}^2}{4\alpha}}.
\end{aligned} \tag{3.2.4}$$

In obtaining the above expression we have used Schwinger's parametrization:

$$\frac{1}{[k^2-\Delta^2]} = \int_0^\infty d\alpha\, e^{-\alpha\,(k^2-\Delta^2)}. \tag{3.2.5}$$

Using the standard expression for the integral representation for the modified Bessel function of the second kind:

$$\int_0^\infty dx\, x^{\nu-1}\, e^{-\gamma x - \frac{\beta}{x}} = 2\left[\frac{\beta}{\gamma}\right]^{\nu/2} K_\nu(2\sqrt{\beta\gamma}), \tag{3.2.6}$$

the final expression for the integral to be evaluated becomes

$$\int \frac{d^3k}{(2\pi)^3}\frac{k_\mu}{[k^2-\Delta^2]^3}e^{ik\times\mathcal{P}} = \frac{1}{2}\frac{\tilde{\mathcal{P}}_\mu}{(2\sqrt{\pi})^3}\left[\frac{|\tilde{\mathcal{P}}|}{2\,|\Delta|}\right]^{1/2} K_{1/2}\left[|\tilde{\mathcal{P}}||\Delta|\right]. \tag{3.2.7}$$

Similarly other types of integrals also arise in the evaluation of the vertex amplitude. These also can be evaluated in a similar fashion as shown above. For the sake of completeness a lengthy list of non-planar integrals and their solutions has been provided in the appendix.

3.3 Induced magnetic moment

In this section, we evaluate various scalar one-loop diagrams contributing to the vertex, upto first order in θ. The calculations have been broken up into different subsections, corresponding to different diagrams, for the sake of convenience.

3.3.1 Boson-photon vertex contribution

The contribution to the vertex arising from the diagram shown in Fig. (3.1), which is also present in the commutative case, can be written in the form

$$\Gamma^1_\mu = -e^2\int\frac{d^3q}{(2\pi)^3}\frac{(p+p'-2q)_\mu(2p-q)_\nu(2p'-q)_\rho G^{\nu\rho}(q)}{[(p-q)^2-m^2][(p'-q)^2-m^2]}\, e^{-iq\times\mathcal{K}}e^{\frac{i}{2}p\times p'}, \tag{3.3.1}$$

where $\mathcal{K}_\mu \equiv (p'-p)_\mu$. The above can be simplified to yield

$$\Gamma^1_\mu = -\frac{4e^2}{M}\int\frac{d^3q}{(2\pi)^3}\frac{\epsilon^{\nu\rho\alpha}p_\nu {p'}_\rho q_\alpha(p+p'-2q)_\mu}{q^2[(p-q)^2-m^2][(p'-q)^2-m^2]}\, e^{-iq\times\mathcal{K}}e^{\frac{i}{2}p\times p'}. \tag{3.3.2}$$

The loop integral can be evaluated in the standard manner. After combining the denominators and shifting the integration variable we get

$$\Gamma^1_\mu = -\frac{8e^2}{M}\int_0^1 dx\int_0^x dy\int\frac{d^3q}{(2\pi)^3}\frac{\epsilon^{\nu\rho\alpha}p_\nu p'_\rho\tilde{q}_\alpha[(2x-1)p_\mu+(1-2x+2y)p'_\mu-2\tilde{q}_\mu]}{[\tilde{q}^2-\omega_1^2]^3}\, e^{-i\tilde{q}\times\mathcal{K}}e^{-\frac{i}{2}(1-2y)p\times p'}, \tag{3.3.3}$$

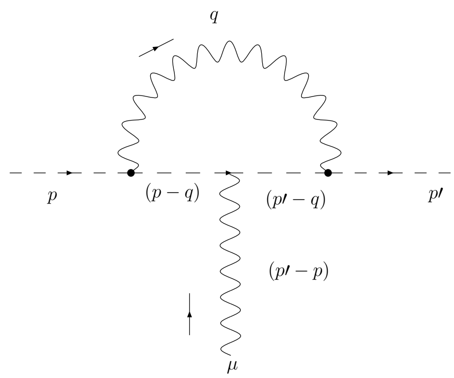


Figure 3.1: *scalar-gauge field vertex.*

where $\omega_1^2 = (1-y)^2 m^2 - (1-x)(x-y)\mathcal{K}^2$ and $q = \bar{q} + (x-y)p' + (1-x)p$. For the sake of notational simplicity, we continue to denote the new integration variable $\tilde{q}$ as q in this, as well as later calculations. In solving the above integrals, we retain only the $q_\mu q_\alpha$ term, since only this term gives magnetic moment type interaction. The momentum integral yields

$$\Gamma^1_\mu = -\frac{8ie^2\epsilon_{\mu\nu\rho}p^\nu p'^\rho}{(2\sqrt{\pi})^3 M}\int_0^1 dx \int_0^x dy e^{-\frac{i}{2}(1-2y)p\times p'}\left[\frac{|\widetilde{\mathcal{K}}|}{2|\omega_1|}\right]^{1/2} K_{1/2}(|\widetilde{\mathcal{K}}||\omega_1|). \tag{3.3.4}$$

The parametric integrals can be handled in an elegant manner by going to a particular frame of reference: the rest frame of the scalar particle, where $p \times p' = 0$. Also, we take $\theta^{0i} = 0$, since it is known that space-time noncommutativity violates unitarity [46]. Using

$$K_{\pm 1/2}(z) = \sqrt{\frac{\pi}{2}}\frac{e^{-z}}{\sqrt{z}}, \tag{3.3.5}$$

and retaining terms first order in θ from the above expansion, we get

$$\Gamma^1_\mu = -\frac{ie^2\epsilon_{\mu\nu\rho}\mathcal{P}^\nu\mathcal{K}^\rho}{4\pi M}\left[\frac{1}{m} - \frac{|\widetilde{\mathcal{K}}|}{2}\right]. \tag{3.3.6}$$

It must be mentioned that, the above expression is obtained in the $\mathcal{K}^2 \to 0$ limit. Furthermore, we have replaced p and p' using the relations for $\mathcal{K}_\mu$ and $\mathcal{P}_\mu = {p'}_\mu + p_\mu$.

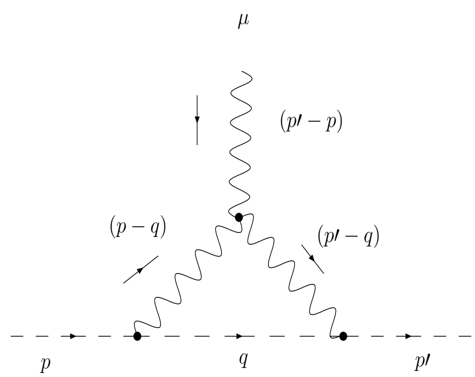


Figure 3.2: *The three photon vertex contribution.*

3.3.2 Three-photon vertex contribution

Here, we deal with the three-gluon contribution shown in Fig. (3.2), to the NC vertex:

$$\begin{aligned}\Gamma^2_\mu &= -2ie^2M\int\frac{d^3q}{(2\pi)^3}\frac{(p'+q)_\rho(p-q)_\lambda G^{\nu\alpha}(p-q)\epsilon_{\alpha\mu\beta}G^{\beta\rho}(p'-q)}{(q^2-m^2)}\sin\left[\frac{(p-q)\times(p'-p)}{2}\right]e^{\frac{i}{2}q\times\mathcal{K}},\\ &= \frac{2ie^2}{M}\int\frac{d^3q}{(2\pi)^3}\frac{\epsilon^{\nu\alpha\lambda}\epsilon_{\mu\alpha\beta}\epsilon^{\beta\rho\delta}(p'+q)_\rho(p-q)_\lambda(p+q)_\nu(p'-q)_\delta}{(q^2-m^2)(p-q)^2(p'-q)^2}\sin\left[\frac{p\times p'-q\times\mathcal{K}}{2}\right]e^{\frac{i}{2}q\times\mathcal{K}}\end{aligned} \quad (3.3.7)$$

The above vertex Γ^2_μ, can be written in terms of planar and non-planar contributions in the form,

$$\Gamma^2_\mu = \frac{4e^2}{M}\int\frac{d^3q}{(2\pi)^3}\frac{\epsilon^{\nu\alpha\lambda}p_\lambda p'_\alpha q_\nu q_\mu}{(q^2-m^2)(p-q)^2(p'-q)^2}[e^{\frac{i}{2}p\times p'}-e^{-\frac{i}{2}p\times p'}e^{iq\times\mathcal{K}}]. \quad (3.3.8)$$

In obtaining the above expression, we have simplified the numerator using the standard ϵ manipulations. As before, combining the denominators and shifting the integration variable we get

$$\Gamma^2_\mu = \frac{8e^2}{M}\int_0^1 dx\int_0^x dy\int\frac{d^3q}{(2\pi)^3}\frac{\epsilon^{\nu\alpha\lambda}p_\lambda p'_\alpha q_\nu q_\mu}{(q^2-\omega_2^2)^3}[e^{\frac{i}{2}p\times p'}-e^{iq\times\mathcal{K}}e^{\frac{i}{2}(1-2y)p\times p'}], \quad (3.3.9)$$

where $\omega_2^2 = m^2y^2-(x-y)(1-x)\mathcal{K}^2$. The vertex can be separated as: $\Gamma^2_\mu = \Gamma^{2P}_\mu + \Gamma^{2NP}_\mu$. The planar part can be simplified:

$$\Gamma^{2P}_\mu = -\frac{ie^2}{4\pi M}\int_0^1 dx\int_0^x dy\frac{\epsilon_{\mu\alpha\lambda}p'^\alpha p^\lambda}{my}. \quad (3.3.10)$$

It can be noticed that the above planar contribution has a logarithmic divergence. The non-planar contribution can be written in the form

$$\Gamma^{2NP}_\mu = \frac{4ie^2\epsilon_{\nu\alpha\lambda}g^{\mu\nu}}{M(2\sqrt{\pi})^3}\int_0^1 dx\int_0^x dy\left[\frac{|\widetilde{\mathcal{K}}|}{2|\omega_2|}\right]^{1/2}K_{1/2}(|\widetilde{\mathcal{K}}||\omega_2|). \quad (3.3.11)$$

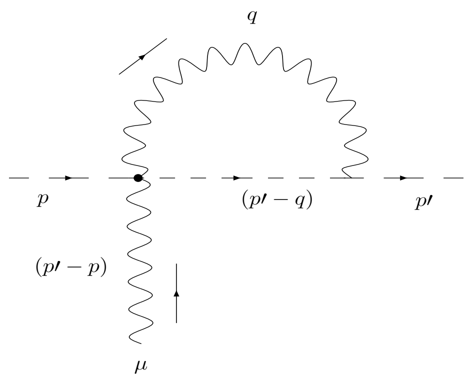


Figure 3.3: *The two photon vertex.*

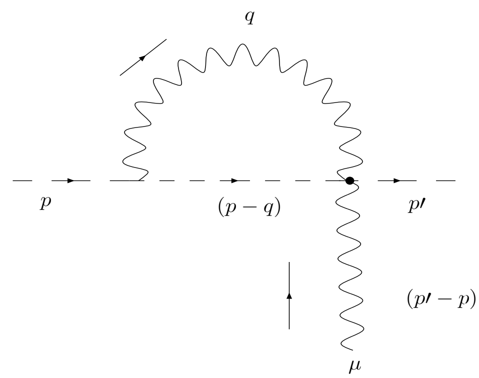


Figure 3.4: *The two photon vertex.*

On expanding the Bessel function and retaining contribution linear in the NC parameter, we see that the log divergence from the planar piece exactly cancels a similar divergence from the non-planar contribution. Hence, the 3-photon vertex is divergence free. Such a cancellation of divergences stemming from the planar and non-planar contributions has been noted in the photon self-energy calculation in 3+1 dimensions [32]. The contribution to the vertex can be combined into a compact form:

$$\Gamma^2_\mu = \frac{ie^2}{4\pi M}\epsilon_{\mu\nu\rho}\mathcal{P}^\nu\mathcal{K}^\rho\frac{|\widetilde{\mathcal{K}}|}{4}. \tag{3.3.12}$$

3.3.3 Two-photon vertices

The two photon vertex amplitude in Fig. (3.3) can be written in the form

$$\Gamma^3_\mu = \frac{2e^2}{M}\int\frac{d^3q}{(2\pi)^3}\frac{\epsilon^{\nu\rho\lambda}g_{\mu\nu}(2p'-q)_\rho q_\lambda}{q^2[(p'-q)^2-m^2]}\cos\left[\frac{q\times\mathcal{K}}{2}\right]e^{\frac{i}{2}p\times p'}e^{-\frac{i}{2}q\times\mathcal{K}}, \tag{3.3.13}$$

which yields,

$$\Gamma^3_\mu = \frac{2e^2}{M}\int_0^1 dx\int\frac{d^3q}{(2\pi)^3}\frac{\epsilon_{\mu\rho\lambda}p'^\rho q^\lambda}{(q^2-\omega_3^2)^2}\left[e^{\frac{i}{2}p\times p'}+e^{-iq\times\mathcal{K}}e^{-\frac{i}{2}(2x-1)p\times p'}\right]. \tag{3.3.14}$$

In obtaining the above expression we have redefined the integration variable by $q=\bar{q}+xp'$ and defined $\omega_3^2 = {p'}^2x^2$. It is clear that the planar contribution is zero and only the non-planar integral survives:

$$\Gamma^{3NP}_\mu = \frac{2e^2\epsilon_{\mu\rho\lambda}p'^\rho\widetilde{\mathcal{K}}^\lambda}{(2\sqrt{\pi}^3)}\int_0^1 dx\, e^{-\frac{i}{2}(2x-1)p\times p'}\left[\frac{|\widetilde{\mathcal{K}}|}{2|\omega_3|}\right]^{-1/2}K_{-1/2}(|\widetilde{\mathcal{K}}||\omega_3|), \tag{3.3.15}$$

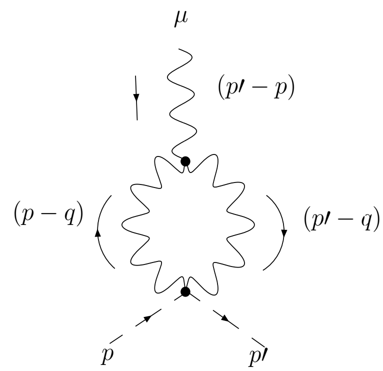


Figure 3.5: *The two and three photon vertex.*

which in the rest frame gives the final answer in the form

$$\Gamma_\mu^{3NP} = \frac{e^2 \epsilon_{\mu\nu\rho} p'^\nu \widetilde{\mathcal{K}}^\rho}{4\pi M} \left[\frac{1}{|\widetilde{\mathcal{K}}|} - \frac{m}{2} \right]. \tag{3.3.16}$$

Similarly for the other two photon vertex [Fig. (3.4)] we get,

$$\Gamma_\mu^{4NP} = -\frac{e^2 \epsilon_{\mu\nu\rho} p^\nu \widetilde{\mathcal{K}}^\rho}{4\pi M} \left[\frac{1}{|\widetilde{\mathcal{K}}|} - \frac{m}{2} \right]. \tag{3.3.17}$$

3.3.4 Two-photon and three-photon vertex

This last subsection deals with the two photon and three gluon vertex. Similar to the contribution of Fig. (3.2) the contribution from this diagram is purely due to NC nature of the action. Calling the contribution from this diagram as Γ_μ^5:

$$\Gamma_\mu^5 = \frac{4ie^2}{M} \int \frac{d^3q}{(2\pi)^3} \frac{g_{\alpha\beta} \epsilon^{\alpha\nu\lambda} \epsilon_{\mu\nu\rho} \epsilon^{\rho\beta\delta} (p' - q)_\delta (p - q)_\lambda}{(p' - q)^2 (p - q)^2} \cos\left[\frac{(p' - q) \times (p - q)}{2} \right] \sin\left[\frac{(p' - p) \times (p' - q)}{2} \right] e^{\frac{i}{2} p \times p'}. \tag{3.3.18}$$

The standard manipulations give

$$\Gamma_\mu^5 = -\frac{2ie^2}{M} \int \frac{d^3q}{(2\pi)^3} \frac{\epsilon_{\mu\nu\lambda}}{(p' - q)^2 (p - q)^2} \left[\frac{\mathcal{P}^\lambda \mathcal{K}^\nu}{2} + \mathcal{K}^\lambda q^\nu \right] \sin[p \times p' - q \times \mathcal{K}] e^{\frac{i}{2} p \times p'}. \tag{3.3.19}$$

Proceeding as before we get

$$\Gamma_\mu^5 = \frac{2ie^2}{M} \int_0^1 dx \int \frac{d^3q}{(2\pi)^3} \frac{\epsilon_{\mu\nu\lambda} \mathcal{K}^\lambda q^\nu}{(q^2 - \omega_5^2)^2} \sin[q \times \mathcal{K}] e^{\frac{i}{2} p \times p'}, \tag{3.3.20}$$

where $\omega_5^2 = x(x-1)\mathcal{K}^2$. Performing the momentum integration, with $q = \bar{q} + p + x\mathcal{K}$, one gets

$$\Gamma_\mu^5 = -\frac{2e^2\epsilon_{\mu\nu\lambda}\mathcal{K}^\lambda\widetilde{\mathcal{K}}^\nu}{(2\sqrt{\pi})^3 M}\int_0^1 dx\left[\frac{|\widetilde{\mathcal{K}}|}{2|\omega_5^2|}\right]^{-1/2} K_{-1/2}(|\widetilde{\mathcal{K}}||\omega_5^2|)e^{\frac{i}{2}p\times p'}. \tag{3.3.21}$$

Upon simplification, the contribution from this vertex diagram turns out to be

$$\Gamma_\mu^5 = \frac{e^2\epsilon_{\mu\nu\rho}\mathcal{K}^\nu\widetilde{\mathcal{K}}^\rho}{4\pi M|\widetilde{\mathcal{K}}|}. \tag{3.3.22}$$

Combining the various vertices at first order in θ, one gets

$$\Gamma_\mu = -\frac{ie^2\epsilon_{\mu\nu\rho}\mathcal{P}^\nu\mathcal{K}^\rho}{4\pi M}\left[\frac{1}{m} - \frac{3|\widetilde{\mathcal{K}}|}{4}\right] + \frac{e^2\epsilon_{\mu\nu\rho}\mathcal{K}^\nu\widetilde{\mathcal{K}}^\rho}{4\pi M}\left[\frac{2}{|\widetilde{\mathcal{K}}|} - \frac{m}{2}\right]. \tag{3.3.23}$$

The above vertex contributions, as can be noticed, is separates into real and imaginary parts. The real part results due to the appearance of the θ dependent spin type term, unlike the other term where only the magnitude of θ appears. It can be seen from the above expression that, the theta independent term of the first piece arises due to the original vertex diagram which yielded a finite value to the parity odd part of the vertex function present in the commutative theory [60]. This parity odd term can couple to the external magnetic field and hence it was interpreted as the magnetic moment for the scalar particles. The present term receives a finite NC correction due to the appearance of non-planar integrals. This correction depends on the value of the NC parameter and hence can also be interpreted as a correction to the MM structure. The real piece of the vertex function is interesting because, the parity odd spin term couples not only to the external fields but it also couples to the NC parameter.

3.4 Discussion

In conclusion, we have evaluated the NC vertex diagrams at one loop level, up to first order in θ, for the scalar particles. The non-planar contributions brought in corrections to the spin structure and also coupled the external field to the θ tensor. It is worth noting that, the NC contribution to the imaginary part of the vertex (the part responsible for the MM in the commutative case) does not depend on the mass of the fermion. Realizing that, the $|\tilde{\mathcal{K}}|$ acts as a derivative on the magnetic field, one can infer that the presence of this term in the Hamiltonian (with a structure similar as the coupling of spin to the magnetic field) will generate additional precession of a charged particle in a in-homogenous magnetic field. It is also straightforward to see that, the force experienced by the particles will be different as compared to the commutative case. Hence, a sufficiently strongly varying magnetic field may make this effect experimentally verifiable, even though the NC parameter is small. The fact that, NC theories are more apt for condensed matter systems like fractional Hall effect where scalar CS theories appear, makes our result quite exciting and potentially amenable to verification. Considering the real part of the vertex, it can be seen that it does not contribute to the MM interaction since we get a term of the type $\mathcal{K}.A$. This is due to the fact that noncommutativity is restricted only to the spatial components i.e., $\theta^{0i} = 0$. It has been shown recently that, the MM for scalar matter fields in NC MCS can lead to the formation of bound states on plane [43]. Hence, the implication of these loop corrections needs careful investigation.

Chapter 4

Fermionic matter with Chern-Simons coupling

4.1 Introduction

This chapter is devoted to the study of MM of fermions coupled to a CS gauge field. The MM calculation for the fermions coupled to the commutative non-Abelian and the Abelian CS term was carried out in [20, 21, 41]. The calculation for the Abelian version of the theory is rather straightforward, whereas evaluation of the vertex diagram for the non-Abelian case is non-trivial. This is due to the self interaction of the gauge fields leading to a three-gauge boson vertex. The authors of [20], make use of the BRST symmetry of the generating function to set up a Slavnov-Taylor (ST) identity leading to a considerable simplification in extracting the MM. In this chapter we will follow [20] because, even though we consider a $U(1)$ NCCS theory, due to noncommutativity it has a three-gauge boson term in the action.

The chapter is organized as follows. In the next section we evaluate the Abelian vertex diagram. Then we show that the NCCS action has a $\star$-BRST symmetry. This is then utilized to derive the NC ST identity in section 4. The various Feynman amplitudes that are contained in the ST identity, like the fermion self-energy and the composite ghost-gluon vertices are calculated in section 5 and section 6.

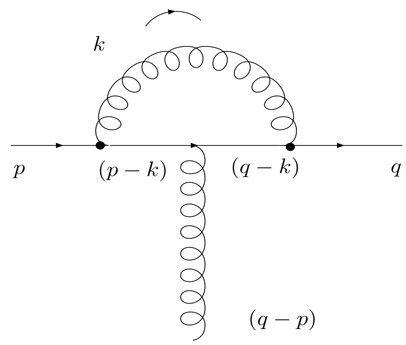


Figure 4.1: *The abelian vertex diagram.*

4.2 The Abelian vertex

In Fig. (4.1) we depict the Abelian type vertex. The vertex amplitude is given by

$$\Gamma_\mu = e^{\frac{i}{2}p\times q}\frac{e^2}{M}\int\frac{d^3k}{(2\pi)^3}\epsilon_{\lambda\sigma\rho}\frac{k^\rho}{k^2}\bar{u}(q)\frac{\gamma^\sigma(\not{q}+\not{k}+m)\gamma_\mu(\not{p}+\not{k}-m)\gamma^\lambda}{[(q+k)^2-m^2][(p+k)^2-m^2]}u(p)e^{ik\times\mathcal{K}}. \tag{4.2.1}$$

In the above $\mathcal{K} = (q-p)$. Here the gamma matrices are defined as $\gamma^0 = \sigma_2, \gamma^1 = i\sigma_3, \gamma^3 = i\sigma_1$, and $g_{\mu\nu}$ is taken to be diag(1,-1,-1) same as that adopted in Ref. [34]. The above integral can be simplified using the identity $\gamma_\mu\gamma_\nu = g_{\mu\nu} - i\epsilon_{\mu\nu\rho}\gamma^\rho$ and the mass-shell condition: $(\not{p}-m)u(p) = 0$ and $\bar{u}(q)(\not{q}-m) = 0$. Defining $\mathcal{P} = (q+p)$ we have

$$\Gamma_\mu = \frac{-e^2}{M}e^{\frac{i}{2}p\times q}\int\frac{d^3k}{(2\pi)^3}\left[\frac{4(m\gamma_\mu - q_\mu - \mathcal{P}_\mu)}{(2q\cdot k+k^2)(2p\cdot k+k^2)} + \frac{\gamma_\mu\not{k}}{k^2(2p\cdot k+k^2)} + \frac{\not{k}\gamma_\mu}{k^2(2q\cdot k+k^2)} + \frac{4\epsilon_{\lambda\sigma\rho}k^\rho q^\sigma p^\lambda\gamma_\mu}{k^2[(q+k)^2-m^2][(p+k)^2-m^2]}\right]e^{ik\times\mathcal{K}} \tag{4.2.2}$$

Before we proceed to calculate the above loop integrals, a few points are worth mentioning. Similar to the bosonic case we restrict to NC spatial coordinates. Terms proportional to k_μ which were zero in the commutative case, due to symmetry arguments cannot be dropped because of a momentum dependent phase $\exp(ik \times \mathcal{K})$. Also the last term of the above integral which was zero, in the commutative case, because of the appearance of terms proportional to $\epsilon_{\lambda\sigma\rho}p^\rho p^\lambda q^\sigma$ and $\epsilon_{\lambda\sigma\rho}q^\rho p^\lambda q^\sigma$, upon integration are non-zero. Writing the above integral as $\Gamma_\mu = \Gamma_\mu^{(1)} + \Gamma_\mu^{(2)} + \Gamma_\mu^{(3)} + \Gamma_\mu^{(4)}$, for the sake of convenience, and performing the standard noncommutative loop integral yields

$$\Gamma_\mu^{(1)} = -\frac{ie^2}{M\sqrt{\pi^3}}\, e^{\frac{i}{2}p\times q}\int_0^1 dx \left\{ (m\gamma_\mu + x\mathcal{K}_\mu + p_\mu - \mathcal{P}_\mu)\left[\frac{|\tilde{\mathcal{K}}|}{2|\Delta_1|}\right]^{1/2} K_{1/2}(|\tilde{\mathcal{K}}\Delta_1|) \right. \\ \left. -\frac{i\tilde{\mathcal{K}}_\mu}{2}\left[\frac{2|\Delta_1|}{|\tilde{\mathcal{K}}|}\right]^{1/2} K_{-1/2}(|\tilde{\mathcal{K}}\Delta_1|)\right\}, \qquad (4.2.3)$$

$$\Gamma_\mu^{(2)} = -\frac{ie^2\gamma_\mu}{4M\sqrt{\pi^3}}\, e^{\frac{i}{2}p\times q}\int_0^1 dx\, e^{-\frac{i}{2}x\mathcal{P}\times\mathcal{K}} \left\{ \frac{i\,\tilde{\not{\mathcal{K}}}}{2}\left[\frac{2|\Delta_2|}{|\tilde{\mathcal{K}}|}\right]^{1/2} K_{-1/2}(|\tilde{\mathcal{K}}\Delta_2|) \right. \\ \left. - \not{p}\,x\left[\frac{|\tilde{\mathcal{K}}|}{2|\Delta_2|}\right]^{1/2} K_{1/2}(|\tilde{\mathcal{K}}\Delta_2|)\right\}, \qquad (4.2.4)$$

$$\Gamma_\mu^{(3)} = -\frac{ie^2}{4M\sqrt{\pi^3}}\, e^{\frac{i}{2}p\times q}\int_0^1 dx e^{-\frac{i}{2}x\mathcal{P}\times\mathcal{K}} \left\{ \frac{i\,\tilde{\not{\mathcal{K}}}}{2}\left[\frac{2|\Delta_3|}{|\tilde{\mathcal{K}}|}\right]^{1/2} K_{-1/2}(|\tilde{\mathcal{K}}\Delta_3|) \right. \\ \left. - \not{q}\,x\left[\frac{|\tilde{\mathcal{K}}|}{2|\Delta_3|}\right]^{1/2} K_{1/2}(|\tilde{\mathcal{K}}\Delta_3|)\right\}\gamma_\mu, \qquad (4.2.5)$$

$$\Gamma_\mu^{(4)} = \frac{ie^2\gamma_\mu}{4M\sqrt{\pi^3}}\, e^{\frac{i}{2}p\times q}\int_0^1 dx\int_0^x dy\, e^{-\frac{i}{2}(1-y)\mathcal{P}\times\mathcal{K}}\, i\epsilon_{\lambda\sigma\rho}\mathcal{P}^\lambda\mathcal{K}^\sigma\tilde{\mathcal{K}}^\rho\left[\frac{|\tilde{\mathcal{K}}|}{2|\Delta_4|}\right]^{1/2} K_{1/2}(|\tilde{\mathcal{K}}\Delta_4|). \quad (4.2.6)$$

Here and in what follows, a tilde over a momentum indicates that it is contracted with the NC parameter θ *i.e.*, $\tilde{\mathcal{K}}^\mu \equiv \theta^{\mu\nu}\mathcal{K}_\nu$. Furthermore we have abbreviated $\Delta_1^2 = (x\mathcal{K}-p)^2$, $\Delta_2^2 = (px)^2$, $\Delta_3^2 = (qx)^2$ and $\Delta_4^2 = m^2(1-y)^2 - (x-y)(1-x)\mathcal{K}^2$. In solving the above integrals we have used

$$\int_0^\infty dx\, x^{\nu-1}e^{-\gamma x-\beta/x} = 2(\beta/\gamma)^{\nu/2}K_\nu[2\sqrt{\beta\gamma}], \qquad (4.2.7)$$

where K_ν is the modified Bessel function of the second kind. Using

$$K_{\pm 1/2}(z) = \sqrt{\frac{\pi}{2}}\frac{e^{-z}}{\sqrt{z}} \qquad (4.2.8)$$

the above integrals can be cast in the following form

$$
\begin{aligned}
\Gamma^{(1)}_\mu &= -\frac{ie^2}{2\pi M}\, e^{\frac{i}{2}p\times q}\int_0^1 dx \left\{ \frac{(m\gamma_\mu + x\mathcal{K}_\mu + p_\mu - \mathcal{P}_\mu)}{|\Delta_1|} - \frac{i\tilde{\mathcal{K}}_\mu}{|\tilde{\mathcal{K}}|} \right\} e^{-|\tilde{\mathcal{K}}\Delta_1|} \\
\Gamma^{(2)}_\mu &= -\frac{ie^2\gamma_\mu}{4\pi M}\, e^{\frac{i}{2}p\times q}\int_0^1 dx\, e^{-\frac{i}{2}x\mathcal{P}\times\mathcal{K}} \left\{ \frac{i\, \tilde{\not{\mathcal{K}}}}{2|\tilde{\mathcal{K}}|} - \frac{\not{p}}{2|\Delta_2|} \right\} e^{-|\tilde{\mathcal{K}}\Delta_2|} \\
\Gamma^{(3)}_\mu &= -\frac{ie^2}{4\pi M}\, e^{\frac{i}{2}p\times q}\int_0^1 dx\, e^{-\frac{i}{2}x\mathcal{P}\times\mathcal{K}} \left\{ \frac{i\, \tilde{\not{\mathcal{K}}}}{2|\tilde{\mathcal{K}}|} - \frac{\not{q}}{2|\Delta_3|} \right\} \gamma_\mu e^{-|\tilde{\mathcal{K}}\Delta_3|} \\
\Gamma^{(4)}_\mu &= -\frac{ie^2\gamma_\mu}{4\pi M}\, e^{\frac{i}{2}p\times q}\int_0^1 dx \int_0^x dy \frac{-i\epsilon_{\lambda\sigma\rho}\mathcal{P}^\lambda\mathcal{K}^\sigma\tilde{\mathcal{K}}^\rho e^{-\frac{i}{2}(1-y)\mathcal{P}\times\mathcal{K}}e^{-|\tilde{\mathcal{K}}\Delta_4|}}{\sqrt{m^2(1-y)^2-(x-y)(1-x)\mathcal{K}^2}}.
\end{aligned}
\tag{4.2.9}
$$

The parametric integrals can be solved elegantly by going over to the rest frame of the electron. Retaining terms to only first order in θ we get

$$
\begin{aligned}
\Gamma^{(1)}_\mu &= -\frac{ie^2}{2\pi M}\int_0^1 dx \left\{ (m\gamma_\mu + x\mathcal{K}_\mu + p_\mu - \mathcal{P}_\mu)(\frac{1}{|\Delta_1|} - |\tilde{\mathcal{K}}|) - i\tilde{\mathcal{K}}_\mu(\frac{1}{|\tilde{\mathcal{K}}|} - |\Delta_1|) \right\} \\
\Gamma^{(2)}_\mu &= -\frac{ie^2\gamma_\mu}{8\pi M}\int_0^1 dx \left\{ i\, \tilde{\not{\mathcal{K}}}(\frac{1}{|\tilde{\mathcal{K}}|} - |\Delta_2|) - \not{p}(\frac{1}{|\Delta_2|} - |\tilde{\mathcal{K}}|) \right\} \\
\Gamma^{(3)}_\mu &= -\frac{ie^2}{8\pi M}\int_0^1 dx \left\{ i\, \tilde{\not{\mathcal{K}}}(\frac{1}{|\tilde{\mathcal{K}}|} - |\Delta_3|) - \not{q}(\frac{1}{|\Delta_3|} - |\tilde{\mathcal{K}}|) \right\} \gamma_\mu \\
\Gamma^{(4)}_\mu &= -\frac{ie^2}{8\pi M}\int_0^1 dx \int_0^x dy \frac{-i\epsilon_{\lambda\sigma\rho}\mathcal{P}^\lambda\mathcal{K}^\sigma\tilde{\mathcal{K}}^\rho}{\sqrt{m^2(1-y)^2-\mathcal{K}^2(x-y)(1-x)}}.
\end{aligned}
\tag{4.2.10}
$$

Solving the integrals in the low momentum transfer limit *i.e.*, $\mathcal{K}^2 = 0$ and making use of the three dimensional analogue of Gordon's decomposition

$$
\gamma_\mu = \frac{1}{2m}[\mathcal{P}_\mu + i\epsilon_{\mu\nu\lambda}\mathcal{K}^\nu\gamma^\lambda], \tag{4.2.11}
$$

the amplitude for the NC abelian type vertex can be written in the form

$$
\Gamma_\mu = \frac{ie^2}{4\pi M}\left[\gamma_\mu - \frac{i\gamma_\mu\epsilon_{\lambda\sigma\rho}\mathcal{P}^\lambda\mathcal{K}^\sigma\tilde{\mathcal{K}}^\rho}{m} - \frac{i\epsilon_{\mu\nu\lambda}\mathcal{K}^\nu\gamma^\lambda}{m} + \frac{i\tilde{\mathcal{K}}_\mu}{|\tilde{\mathcal{K}}|} - 2im\tilde{\mathcal{K}}_\mu - i\epsilon_{\mu\nu\lambda}\mathcal{K}^\nu\gamma^\lambda|\tilde{\mathcal{K}}| \right]. \tag{4.2.12}
$$

This completes the calculation of the NC abelian type vertex. As can be seen that the extra θ dependent contributions vanish smoothly in the $\theta \to 0$ limit. This is because this vertex contribution does not have any divergences.

4.3 BRST symmetry

In this section we show the existence of a $\star$-BRST symmetry for the NCCS action. This is a global symmetry whose associated charge is nilpotent. Just like the local gauge invariance of the leads to the Ward identities, the symmetry under $\star$-BRST transformation will yield a NC version of the ST identity. This identity will form the topic of the subsequent section.

The NCCS action with fermionic matter fields is given by

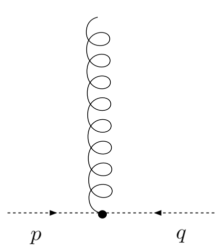
$$S_{CS} = \frac{M}{2}\int d^3x \left[\epsilon^{\mu\nu\rho} A_\mu \star \partial_\nu A_\rho + \frac{2ie}{3} A_\mu \star A_\nu \star A_\rho\right], \; S_{Dirac} = \int d^3x\, \bar{\psi} \star (i \not{D} - m)\psi, \quad (4.3.1)$$
$$S_{Ghost} = \frac{1}{2}\int d^3x\, \partial^\mu \bar{c} \star D^A_\mu c, \qquad S_{GF} = -\frac{1}{2\xi}\int d^3x\, \partial_\mu A^\mu \star \partial_\nu A^\nu.$$

The covariant derivatives are $D_\mu \psi = \partial_\mu \psi - igA_\mu \star \psi$ and $D^A_\mu c = \partial_\mu c - ig[A_\mu, c]_{MB}$ and $c(x)$ are the ghost fields. S_{Ghost} is the ghost action. Usually ghosts decouple in case of the Abelian commutative theories, but in noncommutative theories ghost terms cannot be integrated, similar to the non-Abelian theory. It is worth mentioning here that in [32], it was shown that for NC $U(N)$ CS theory, in the axial gauge, ghosts completely decouple, furthermore it was shown that there is no UV/IR mixing. The presence of the ghost term in the action leads to a new vertex between the gauge fields and the ghosts. This is depicted in Fig. (4.2).

Replacing the gauge parameter of the $U(1)$ gauge transformations, by $i\,g\,c(x)$ we get

$$\hat{\delta}_B A_\mu(x) = D_\mu c(x) = \partial_\mu c(x) - ig[A_\mu(x), c(x)], \quad (4.3.2)$$
$$\hat{\delta}_B \psi(x) = ig\, c(x) \star \psi(x). \quad (4.3.3)$$

In the above equations $\hat{\delta}_B$ is the BRST operator that enforces the BRST transformations, and



$$\equiv -2eq^\mu \sin[(p \times q)2].$$

Figure 4.2: *The ghost-photon vertex.*

is independent of the space-time coordinates. Another important property of $\hat{\delta}_B$ is that it is a Grassmanian object which satisfies the rule

$$\hat{\delta}_B(F \star G) = (\hat{\delta}_B) \star G + (-1)^{|F|} F \star (\hat{\delta}_B G), \tag{4.3.4}$$

where $|F|$ is the ghost number of the field F. The BRST transformation of the ghost field can be derived from the nilpotency condition of the transformation: $\hat{\delta}_B^2 = 0$. This gives the BRST transformations for the fields as

$$\hat{\delta}_B c(x) = ig\, c(x) \star c(x), \hat{\delta}_B \bar{c}(x) = \mathcal{B}(x),\ \hat{\delta}_B \mathcal{B}(x) = 0. \tag{4.3.5}$$

In the above we have introduced a field $\mathcal{B}$, and is called the Nakanishi-Lautrup multiplier field. The use of this field is that, the ghost and the gauge fixing parts of the action can be written in a unified manner

$$S_{Ghost} + S_{GF} = \bar{c}(x) \star [\mathcal{B}(x)\xi/2 + \partial_\mu A^\mu(x)]. \tag{4.3.6}$$

We now show that the BRST transforms of the various fields, equations (4.3.2), (4.3.3), and (4.3.5) are nilpotent; $\hat{\delta}_B^2 = 0$. For the gauge field $A_\mu(x)$ we have

$$\begin{aligned} \hat{\delta}_B^2 A_\mu(x) &= \hat{\delta}_B(\hat{\delta}_B A_\mu) = \hat{\delta}_B(\partial_\mu c - ig[A_\mu, c]) \\ &= ig\partial_\mu(c \star c) - ig\hat{\delta}_B A_\mu \star c - igA_\mu \star \hat{\delta}_B c \\ &+ ig\hat{\delta}_B c \star A_\mu - igc \star \hat{\delta}_B A_\mu = 0, \end{aligned} \tag{4.3.7}$$

from the rule given Eq. (4.3.4). Similarly for the other fields we have

$$\begin{aligned} \hat{\delta}_B^2 \psi(x) = ig\hat{\delta}_B[c \star \psi] = ig\hat{\delta}_B c \star \psi - igc \star \hat{\delta}_B \psi = 0, \\ \hat{\delta}_B^2 c(x) = ig\hat{\delta}_B[c \star c] = ig\hat{\delta}_B c \star c - igc \star \hat{\delta}_B c = 0, \\ \hat{\delta}_B \bar{c}(x) = \hat{\delta}_B \mathcal{B} = 0, \\ \hat{\delta}_B^2 \mathcal{B} = 0. \end{aligned}$$

Thus we have shown in this section how to obtain the $\star$-BRST symmetry for the CS spinor action.

4.4 Noncommutative Slavnov-Taylor identity

Under a $\star$-BRST variation, $\hat{\delta}_B$, the fields transform as

$$\hat{\delta}_B A_\mu = D_\mu c = \partial_\mu c - ig[A_\mu, c];\ \hat{\delta}_B c = ig(c \star c);\ \hat{\delta}_B \bar{c} = \mathcal{B};$$
$$\hat{\delta}_B \mathcal{B} = 0;\ \hat{\delta}_B \psi = igc \star \psi;\ \hat{\delta}_B \bar{\psi} = ig\bar{\psi} \star c. \tag{4.4.1}$$

The generating functional with the source terms is given by $Z = Z[J_\mu, \eta, \bar{\eta}, \omega, \bar{\omega}, \rho, \bar{\rho}, \mathcal{J}_\mu]$. Now

$$Z = \int [D\Phi] \exp\left\{i(S + S_{so} + S_{comp})\right\}, \tag{4.4.2}$$

with $\Phi = (A, \bar{\psi}, \psi, \bar{c}, c, \mathcal{B})$ denoting the fields participating in the action.

$$S_{so} = \int d^3x \left(J_\mu \star A^\mu + \bar{\eta} \star \psi + \bar{\psi} \star \eta + \bar{\omega} \star c + \bar{c} \star \omega + \mathsf{H} \star \mathcal{B}\right) \tag{4.4.3}$$

are the source terms for the fields, and

$$S_{comp} = \int d^3x \left(\mathcal{J}_\mu \star \hat{\delta}_B A^\mu + \bar{\rho} \star \hat{\delta}_B \psi + \hat{\delta}_B \bar{\psi} \star \rho + \alpha \star \hat{\delta}_B c\right) \tag{4.4.4}$$

are the source terms for the composite BRST variations, that linearize the ST identity [59]. In the above, source terms $\eta, \bar{\eta}, \omega, \bar{\omega}$ and $\mathcal{J}_\mu$ are Grassmann sources while $J_\mu, H, \rho, \bar{\rho}, \alpha$ are bosonic.

Under a BRST redefinition of the fields we have $S' \to S + \hat{\delta}_B S$, $S'_{so} \to S_{so} + \hat{\delta}_B S_{so}$, $S'_{comp} \to S_{comp} + \hat{\delta}_B S_{comp}$. Since S_{comp} is already a BRST variation of the fields $\hat{\delta}_B S_{comp} = 0$, from the nilpotency of the transformation. Therefore the BRST redefined partition function is

$$Z' = e^{i[S + \hat{\delta}_B S + S_{so} + \hat{\delta}_B S_{so} + S_{comp}]}. \tag{4.4.5}$$

The invariance of the generating functional, $\hat{\delta}_B Z = Z' - Z = 0$ then yields the Ward identity: $i[\hat{\delta}_B S_{so}]Z = 0$. Where

$$\hat{\delta}_B S_{so} = \int d^3x [J_\mu \star \hat{\delta}_B A^\mu + \hat{\delta}_B \bar{\psi} \star \eta - \bar{\eta} \star \hat{\delta}_B \psi - \bar{\omega} \star \hat{\delta}_B c + \hat{\delta}_B \bar{c} \star \omega + \mathsf{H} \star \hat{\delta}_B \mathcal{B}]. \tag{4.4.6}$$

Since $\hat{\delta}_B \mathcal{B} = 0$ and $\hat{\delta}_B \bar{c} = \mathcal{B}$, the ST identity becomes

$$\int d^3t [J_\mu \star \hat{\delta}_B A^\mu + \hat{\delta}_B \bar{\psi} \star \eta - \bar{\eta} \star \hat{\delta}_B \psi - \bar{\omega} \star \hat{\delta}_B c + \mathcal{B} \star \omega] Z. \tag{4.4.7}$$

The above identity can be written in terms of the functional derivatives acting on the generating function. The usefulness of the sources for the composite BRST variations becomes transparent here. In terms of the functional derivatives the ST identity can be written as

$$\int d^3t \left[J_\mu(t) \star \frac{\delta}{\delta \mathcal{J}_\mu(t)} - \bar{\eta}(t) \star \frac{\delta}{\delta \bar{\rho}(t)} + \frac{\delta}{\delta \rho(t)} \star \eta(t) - \bar{\omega}(t) \star \frac{\delta}{\delta \alpha(t)} + \frac{\delta}{\delta \mathsf{H}(t)} \star \omega(t) \right] W = 0. \quad (4.4.8)$$

In the above expression we have taken $Z = \exp(iW)$. To obtain the relationship between various Greens functions we will follow the standard technique of differentiating the above ST identity w.r.t various sources and later setting them to zero. Acting $\delta/\delta\bar{\eta}(x)$,

$$\int d^3t \left[J_\mu(t) \star \frac{\delta^2}{\delta\bar{\eta}(x)\delta \mathcal{J}_\mu(t)} + \bar{\eta}(t) \star \frac{\delta^2}{\delta\bar{\eta}(x)\delta \bar{\rho}(t)} + \frac{\delta^2}{\delta\bar{\eta}(x)\delta \rho(t)} \star \eta(t) - \delta^3(t-x) \frac{\delta}{\delta\bar{\rho}(t)} \right.$$
$$\left. - \bar{\omega}(t) \star \frac{\delta^2}{\delta\bar{\eta}(x)\delta \alpha(t)} + \frac{\delta^2}{\delta\bar{\eta}(x)\delta \mathsf{H}(t)} \star \omega(t) \right] W = 0. \quad (4.4.9)$$

Taking the functional derivative w.r.t $\delta/\delta\eta(y)$ yields

$$\int d^3t \left[J_\mu(t) \star \frac{\delta^3}{\delta\eta(y)\delta\bar{\eta}(x)\delta \mathcal{J}_\mu(t)} + \frac{\delta^3}{\delta\eta(y)\delta\bar{\eta}(x)\delta \rho(t)} \star \eta(t) - \frac{\delta^2}{\delta\bar{\eta}(x)\delta\rho(t)} \delta^3(t-y) \right.$$
$$- \delta^3(t-x) \frac{\delta^2}{\delta\eta(y)\delta\bar{\rho}(t)} - \bar{\eta}(t) \star \frac{\delta^3}{\delta\eta(y)\delta\bar{\eta}(x)\delta\bar{\rho}(t)} - \bar{\omega}(t) \star \frac{\delta^3}{\delta\eta(y)\delta\bar{\eta}(x)\delta\alpha(t)}$$
$$\left. + \frac{\delta^3}{\delta\eta(y)\delta\bar{\eta}(x)\delta \mathsf{H}(t)} \star \omega(t) \right] W = 0. \quad (4.4.10)$$

Finally taking the derivative w.r.t $\delta/\delta\omega(z)$ we obtain

$$\int d^3t \left[J_\mu(t) \star \frac{\delta^4}{\delta\omega(z)\delta\eta(y)\delta\bar{\eta}(x)\delta \mathcal{J}_\mu(t)} + \frac{\delta^4}{\delta\omega(z)\delta\eta(y)\delta\bar{\eta}(x)\delta \rho(t)} \star \eta(t) - \frac{\delta^3}{\delta\omega(z)\delta\bar{\eta}(x)\delta\rho(t)} \delta^3(t-y) \right.$$
$$- \delta^3(t-x) \frac{\delta^3}{\delta\omega(z)\delta\eta(y)\delta\bar{\rho}(t)} - \bar{\eta}(t) \star \frac{\delta^4}{\delta\omega(z)\delta\eta(y)\delta\bar{\eta}(x)\delta\bar{\rho}(t)} + \bar{\omega}(t) \star \frac{\delta^4}{\delta\omega(z)\delta\eta(y)\delta\bar{\eta}(x)\delta\alpha(t)}$$
$$\left. + \frac{\delta^3}{\delta\eta(y)\delta\bar{\eta}(x)\delta \mathsf{H}(t)} \delta^3(t-z) + \frac{\delta^4}{\delta\omega(z)\delta\eta(y)\delta\bar{\eta}(x)\delta \mathsf{H}(t)} \star \omega(t) \right] W = 0. \quad (4.4.11)$$

Now setting the external sources to zero and integrating out the delta functions we have the final expression for the ST identity

$$\left[\frac{\delta^3}{\delta\omega(z)\delta\eta(y)\delta\bar{\rho}(x)} + \frac{\delta^3}{\delta\omega(z)\delta\bar{\eta}(x)\delta\rho(y)} - \frac{\delta^3}{\delta\eta(y)\delta\bar{\eta}(x)\delta \mathsf{H}(z)} \right] W = 0. \quad (4.4.12)$$

The above identity can also written in the form

$$\left[\frac{1}{\xi} \partial_\mu \langle \psi(x)\bar{\psi}(y) A^\mu(z) \rangle + ig \langle \bar{c}(z)\bar{\psi}(y) c(x) \star \psi(x) \rangle - ig \langle \bar{c}(z)\psi(x)\bar{\psi}(y) \star c(y) \rangle \right] = 0. \quad (4.4.13)$$

To obtain the 1-PI expression from the connected Greens function, Legendre transformation has to be taken:

$$\mathbb{\Gamma}[A_\mu, \psi, \bar{\psi}, c, \bar{c}, \mathcal{B}] = W[J_\mu, \bar{\eta}, \eta, \bar{\omega}, \omega, \mathsf{H}] - \int d^3x\, S_{so}. \tag{4.4.14}$$

Similar to the commutative theory, the composite operators remain inactive in the above transformation. Hence we have,

$$\frac{\delta W}{\delta \bar{\rho}(x)} = \frac{\delta \Gamma}{\delta \bar{\rho}(x)}, \qquad \frac{\delta W}{\delta \rho(x)} = \frac{\delta \Gamma}{\delta \rho(x)}. \tag{4.4.15}$$

With the above transformation the 1-PI ST identity becomes

$$\begin{aligned}\frac{1}{\xi}\frac{\partial}{\partial z_\mu}\int d^3u\, d^3v\, d^3w \left[iG_{\mu\nu}(z-w)\, iS(x-u)\, \frac{\delta^3 \mathbb{\Gamma}}{\delta\bar{\psi}(u)\delta\psi(v)\delta A_\nu(w)}\, iS(v-y)\right] \\ + \int d^3u\, d^3v \left[\frac{\delta^3\Gamma}{\delta\psi(u)\delta c(v)\delta\bar{\rho}(w)}\, iS(u-y)\, iD(v-z) \right. \\ \left. + iS(x-u)\, \frac{\delta^3\Gamma}{\delta\bar{\psi}(u)\delta c(v)\delta\rho(w)}\, iD(v-z)\right].\end{aligned} \tag{4.4.16}$$

where $\mathbb{\Gamma}(u,v,w)$ is the 1-PI part of the fermion-gluon vertex function, $\Gamma_{\bar{\rho}}(x,u,v)$ and $\Gamma_\rho(u,y,v)$ are the composite ghost-gluon vertex functions arising from the source terms $\bar{\rho}$ and ρ respectively. Also D denotes the ghost propagator. In order to obtain the above identity we have made use of the chain rule for the various sources, *e.g.*,

$$\frac{\delta}{\delta\omega(x)} = \int d^3u\, \frac{\delta c(u)}{\delta\omega(x)}\frac{\delta}{\delta c(u)}. \tag{4.4.17}$$

The above identity in the momentum space acquires the form

$$\frac{1}{\xi}\mathcal{K}^\mu\, iG_{\mu\nu}(\mathcal{K})\, iS(p)\, i\mathbb{\Gamma}^\nu(p,q,\mathcal{K})\, iS(q) + [i\Gamma_{\bar{\rho}}(p,q,\mathcal{K})\, iS(q) - iS(p)\, i\Gamma_\rho(p,q,\mathcal{K})]\; iD(\mathcal{K}) = 0. \tag{4.4.18}$$

$\mathcal{K}$ is the same as defined earlier in the text. We have also used the same functional form for the propagators as well as for the vertices to avoid cluttering of notation. Noting that the longitudinal part of gauge field receives no quantum correction even in the noncommutative theory, *i.e.*,

$$\mathcal{K}^\mu\, iG_{\mu\nu}(\mathcal{K}) = \mathcal{K}^\mu\, iG^{(0)}_{\mu\nu}(\mathcal{K}) = -i\xi\frac{\mathcal{K}_\nu}{\mathcal{K}^2}, \tag{4.4.19}$$

and using the general form of the full ghost propagator

$$iD(\mathcal{K}) = \frac{1}{\mathcal{K}^2\left[1 + \Sigma_g(\mathcal{K}^2)\right]}, \tag{4.4.20}$$

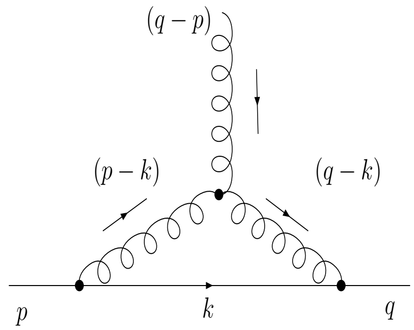


Figure 4.3: *The non-abelian type vertex diagram.*

we obtain the required ward identity

$$\mathcal{K}^{\mu}\mathbb{\Gamma}_{\mu}(p,q,\mathcal{K})\left[1+\Sigma_{g}(\mathcal{K}^{2})\right]+\Gamma_{\bar{\rho}}(p,q,\mathcal{K})S^{-1}(q)+S^{-1}(p)\Gamma_{\rho}(p,q,\mathcal{K}). \tag{4.4.21}$$

Expanding the above identity up to one-loop order by using

$$\begin{aligned}
&\mathbb{\Gamma}_{\mu}(p,q,\mathcal{K})=\left[\gamma_{\mu}+g^{2}\mathbb{\Gamma}_{\mu}^{(1)}\right]\exp[\frac{i}{2}p\times q]+\mathcal{O}(g^{4}),\quad \Sigma_{g}(\mathcal{K}^{2})=g^{2}\Sigma_{g}^{(1)}(\mathcal{K}^{2})+\mathcal{O}(g^{4}),\\
&S^{-1}(p)=\not{p}-m-g^{2}\Sigma^{(1)}(p)+\mathcal{O}(g^{4}),\quad \Gamma_{\bar{\rho}}(p,q,\mathcal{K})=\left[I+g^{2}\Gamma_{\bar{\rho}}^{(1)}(p,q,\mathcal{K})\right]\exp[\frac{i}{2}\mathcal{K}\times q]+\mathcal{O}(g^{4}),\\
&\Gamma_{\rho}(p,q,\mathcal{K})=-\left[I+g^{2}\Gamma_{\rho}^{(1)}(p,q,\mathcal{K})\right]\exp[\frac{i}{2}\mathcal{K}\times p]+\mathcal{O}(g^{4}),
\end{aligned} \tag{4.4.22}$$

the required one-loop ST identity:

$$\begin{aligned}
\mathcal{K}^{\mu}\mathbb{\Gamma}_{\mu}^{(1)}(p,q,\mathcal{K})=&-(\not{q}-\not{p})\Sigma_{g}^{(1)}(\mathcal{K}^{2})-i\left[\Sigma^{(1)}(q)-\Sigma^{(1)}(p)\right]\exp[-ip\times q]\\
&+g^{2}\left[\Gamma_{\rho}^{(1)}(p,q,\mathcal{K})(\not{q}-m)-(\not{p}-m)\Gamma_{\bar{\rho}}^{(1)}(p,q,\mathcal{K})\right]\exp[-ip\times q].
\end{aligned} \tag{4.4.23}$$

In obtaining the above expression we have dropped terms of the form $\mathcal{K}^{\mu}\gamma_{\mu}$ due to the on-shell condition requirement. Therefore we see that the non-abelian type three photon vertex Fig. (4.3), is now equivalent to evaluating the ghost self-energy, fermion self-energy and the composite vertex diagrams. Before we go on to explicitly calculate the individual diagrams in later section it must

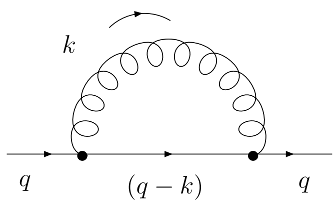


Figure 4.4: *Fermion self-energy one.*

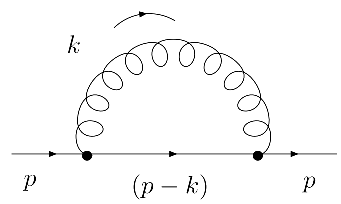


Figure 4.5: *Fermion self-energy two.*

be noted that the ghost self-energy is zero for the pure CS case. In the next section we calculate the fermion self-energy.

4.5 Fermion self-energy

In this section we will calculate the self-energy of the fermion. From the ST identity it is clear that we need to consider two separate fermion self-energy diagrams, these are depicted in the figures (4.4) and (4.5). The contribution from these diagrams turns out to be the same as the commutative case since the Moyal phases at the vertices cancel out.

The amplitude for the first self-energy diagram Fig. (4.4) (the second is exactly same with the momentum factors appropriately put in) is

$$\Sigma(q) = \int \frac{d^3k}{(2\pi)^3} ie\gamma^\nu e^{\frac{i}{2}(q-k)\times q} \frac{i(\not{q} - \not{k} + m)}{[(q-k)^2 - m^2]} ie\gamma^\mu e^{\frac{i}{2}q\times(q-k)} \left[-\frac{1}{M}\epsilon_{\mu\nu\rho}\frac{k^\rho}{k^2}\right]. \tag{4.5.1}$$

The numerator can be simplified as

$$\begin{aligned} \text{Num}[\Sigma(q)] &= \gamma^\nu(\not{q} - \not{k} + m)\gamma^\mu \epsilon_{\mu\nu\rho}k^\rho \\ &= \left[\gamma^\nu\gamma^\alpha\gamma^\mu(q_\alpha - k_\alpha) + m\gamma^\nu\gamma^\mu\right]\epsilon_{\mu\nu\rho}k^\rho \\ &= \left[2ik^2 - 2iq\cdot k + 2im\not{k}\right]. \end{aligned} \tag{4.5.2}$$

Making use of the relation $2q \cdot k = (q^2 - m^2) + k^2 - [(q-k)^2 - m^2]$ in the numerator we get

$$\begin{aligned} \mathrm{Num}[\Sigma(q)] &= ik^2 - i(q^2 - m^2) + i[(q-k)^2 - m^2] + 2im \not{k} \\ &= ik^2 + i[(q-k)^2 - m^2] + 2im \not{k}. \end{aligned} \tag{4.5.3}$$

where for the last equation we have used the on-shell condition $q^2 = m^2$. Similarly the numerator for the the other self-energy contribution is

$$\mathrm{Num}[\Sigma(p)] = ik^2 + i[(p-k)^2 - m^2] + 2im \not{k}. \tag{4.5.4}$$

The combined contribution from both the diagrams can be simplified to

$$\Sigma(q) - \Sigma(p) = C \int \frac{d^3k}{(2\pi)^3} \frac{2k \cdot (q-p)}{[(q-k)^2 - m^2][(p-k)^2 - m^2]} \left[1 + \frac{\not{k}}{k^2}\right]. \tag{4.5.5}$$

Where $C = (ie)^2/M$. We solve the integrals separately for the sake of convenience and denote them as I_1 and I_2.

$$I_1 = \int \frac{d^3k}{(2\pi)^3} \frac{k_\mu}{[(q-k)^2 - m^2][(p-k)^2 - m^2]}. \tag{4.5.6}$$

Combining the denominators and shifting the integration variable we get

$$I_1 = \int_0^1 dx \int \frac{d^3k}{(2\pi)^3} \frac{k_\mu + [xq_\mu + (1-x)p_\mu]}{[k^2 - \omega_1^2]^2}. \tag{4.5.7}$$

Here $\omega_1^2 = [xq + (1-x)p]^2$ and $\tilde{k} = k - [xq + (1-x)p]$, but we continue to call $\tilde{k}$ as k for the sake of convenience. The k_μ integration is zero. Performing the momentum integration we get

$$I_1 = \frac{i}{8\pi} \int_0^1 dx \frac{xq + (1-x)p}{\sqrt{x^2\mathcal{K}^2 - x\mathcal{K}^2 + m^2}}. \tag{4.5.8}$$

Solving the above integral and putting all the factors the contribution from I_1 becomes

$$I_1 = 2(q-p)^\mu C \frac{i\mathcal{P}_\mu}{16\pi\mathcal{K}} \ln\left[\frac{1 + \mathcal{K}/2m}{1 - \mathcal{K}/2m}\right] \quad . \tag{4.5.9}$$

The integral I_2 is

$$I_2 = \int \frac{d^3k}{(2\pi)^3} \frac{\not{k} k_\mu}{k^2[(q-k)^2 - m^2][(p-k)^2 - m^2]}. \tag{4.5.10}$$

Following the standard procedure we need to solve the integral

$$I_2 = \gamma^\alpha \int_0^1 dx \int_0^{1-x} dy \int \frac{d^3k}{(2\pi)^3} \frac{A}{[k^2 - \omega_2^2]^3}, \tag{4.5.11}$$

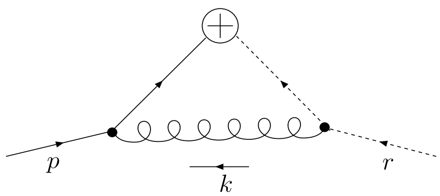


Figure 4.6: *The first composite vertex.*

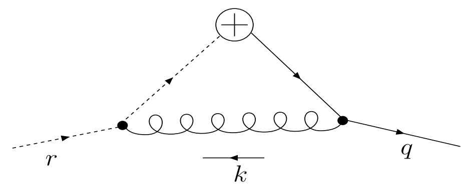


Figure 4.7: *The second composite vertex.*

where $\omega_2^2 = [xq + yp]^2$ and $\tilde{k} = k - [xq + yp]$. Furthermore the numerator $A = k_\alpha k_\mu + (xq_\mu + yp_\mu)k_\alpha + (xq_\mu + yp_\mu)k_\mu + (xq_\alpha + yp_\alpha)(xq_\mu + yp_\mu)$. It can be seen that the k_α and k_μ integrals will not contribute. Performing the momentum integrals one gets

$$I_2 = \frac{i\gamma^\alpha}{16\pi}\int_0^1 dx \int_0^{1-x} dy \left[\frac{g_{\alpha\mu}}{\sqrt{m^2(x+y)^2 - \mathcal{K}^2 xy}} - \frac{x^2 q_\alpha q_\mu + y^2 p_\alpha p_\mu + xy(q_\alpha p_\mu + p_\alpha q_\mu)}{(\sqrt{m^2(x+y)^2 - \mathcal{K}^2 xy})^3}\right]. \tag{4.5.12}$$

The solution of the above integrals give

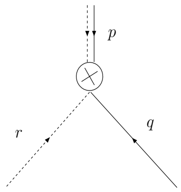
$$\begin{aligned} I_2 = \frac{i\gamma^\alpha}{16\pi}&\left\{\frac{g_{\alpha\mu}}{\mathcal{K}}\ln\left[\frac{1+\mathcal{K}/2m}{1-\mathcal{K}/2m}\right] - \frac{\mathcal{K}_\alpha\mathcal{K}_\mu}{\mathcal{K}^3}\ln\left[\frac{1+\mathcal{K}/2m}{1-\mathcal{K}/2m}\right]\right. \\ &\left. +(q_\alpha p_\mu + p_\alpha q_\mu)\left(\frac{4m}{\mathcal{K}^2(\mathcal{K}^2 - 4m^2)} + \frac{1}{\mathcal{K}^3}\ln\left[\frac{1+\mathcal{K}/2m}{1-\mathcal{K}/2m}\right]\right)\right\}. \end{aligned} \tag{4.5.13}$$

Remembering that the expression is sandwiched between the spinors $\bar{u}(q)$ and $u(p)$ we see that using the on-shell condition the contribution from I_2 turns out to be

$$I_2 = \frac{i}{16\pi}\frac{\gamma_\mu}{\mathcal{K}}\ln\left[\frac{1+\mathcal{K}/2m}{1-\mathcal{K}/2m}\right] + \frac{im\mathcal{P}_\mu}{16\pi}\left(\frac{4m}{\mathcal{K}^2(\mathcal{K}^2 - 4m^2)} + \frac{1}{\mathcal{K}^3}\ln\left[\frac{1+\mathcal{K}/2m}{1-\mathcal{K}/2m}\right]\right). \tag{4.5.14}$$

4.6 The composite diagrams

The composite vertex diagrams that contribute to the magnetic moment are given in figures (4.6) and (4.7). To calculate the amplitude for the composite vertex diagrams we need the NC Feynman rule for the vertex. This is shown in Fig. (4.8). Note that in the figures of the composite vertices we have denoted the momentum $\mathcal{K} = r$. With the above vertex rule the amplitude for Fig. (4.6)



$$\equiv -e\exp\left[-\tfrac{i}{2}\mathcal{K}\times q\right].$$

Figure 4.8: *The tree level composite vertex.*

reads

$$\Gamma^{(1)}_{\rho} = \bar{u}(q)\left[-\frac{(ie)^3}{M}\right]e^{-\frac{i}{2}p\times q}\int\frac{d^3k}{(2\pi)^3}\frac{(\not{p}+\not{k}+m)\gamma^{\mu}\epsilon_{\mu\nu\rho}k^{\rho}(\mathcal{K}-k)^{\nu}}{[(p+k)^2-m^2]k^2(k-\mathcal{K})^2}(\not{q}-m)(e^{ip\times k}-e^{iq\times k})u(p). \tag{4.6.1}$$

Similarly the amplitude corresponding to the second diagram, Fig. (4.7), is

$$\Gamma^{(1)}_{\bar{\rho}} = \bar{u}(q)\left[\frac{(ie)^3}{M}\right]e^{-\frac{i}{2}p\times q}\int\frac{d^3k}{(2\pi)^3}(\not{p}-m)\frac{(k+\mathcal{K})^{\mu}\epsilon_{\mu\nu\rho}k^{\rho}\gamma^{\mu}(\not{q}+\not{k}+m)}{[(q+k)^2-m^2]k^2(k+\mathcal{K})^2}(e^{-ip\times k}-e^{-iq\times k})u(p). \tag{4.6.2}$$

Combining both the expressions, using gamma matrix algebra and making use of

$$\gamma^{\mu}\epsilon_{\mu\nu\rho}k^{\rho} = -\frac{1}{2i}(\gamma_{\nu}\not{k}-\not{k}\gamma_{\nu}); \tag{4.6.3}$$

the contribution from the composite vertex diagrams becomes

$$\begin{aligned}\Gamma^{(1)}_{c} &= \left[\frac{(ie)^3}{2iM}\right]e^{-\frac{i}{2}p\times q}(q^{\mu}-p^{\mu})\bar{u}(q)\left\{\int\frac{d^3k}{(2\pi)^3}\frac{(\not{k}+\not{p}+m)(\gamma_{\mu}\not{k}-\not{k}\gamma_{\mu})(\not{q}-m)}{[(p+k)^2-m^2]k^2(k-\mathcal{K})^2}(e^{ip\times k}-e^{iq\times k})\right.\\ &\left.+\int\frac{d^3k}{(2\pi)^3}\frac{(\not{p}-m)(\not{k}\gamma_{\mu}-\gamma_{\mu}\not{k})(\not{k}+\not{q}+m)}{[(q+k)^2-m^2]k^2(k+\mathcal{K})^2}(e^{-ip\times k}-e^{-iq\times k})\right\}.\end{aligned} \tag{4.6.4}$$

The numerators of the above two integrals can be simplified and cast in a more illuminating manner. For example taking the numerator of the first integral in the above equation

$$\begin{aligned}\mathrm{Num\,A} &= (\not{k}+\not{p}+m)(\gamma_{\mu}\not{k}-\not{k}\gamma_{\mu})\\ &= (\not{k}\gamma_{\mu}\gamma_{\alpha}-\not{k}\not{k}\gamma_{\mu})+(\not{p}+m)(\gamma_{\mu}\not{k}-\not{k}\gamma_{\mu})\\ &= 2\,(\not{k}k_{\mu}-\not{k}\not{k}\gamma_{\mu})+(\not{p}+m)(\gamma_{\mu}\not{k}-\not{k}\gamma_{\mu})\\ &= 2\,[\not{k}k_{\mu}+(\not{p}+m)(k_{\mu}-\not{k}\gamma_{\mu})-\not{k}\not{k}\gamma_{\mu}]\\ &= 2\left[\not{k}k_{\mu}+(\not{p}+m)(k_{\mu}-\not{k}\gamma_{\mu})-k^2\gamma_{\mu}\right].\end{aligned} \tag{4.6.5}$$

Where, in obtaining the above expression we have made use of the relations $\gamma_\mu\gamma_\alpha = 2g_{\mu\alpha} - \gamma_\alpha\gamma_\mu$. Similarly for the other numerator we get

$$\begin{aligned} \mathrm{Num\,B} &= (\not{k}\gamma_\mu - \gamma_\mu \not{k})(\not{k} + \not{q} + m) \\ &= 2\left[\not{k}k_\mu + (k_\mu - \gamma_\mu \not{k})(\not{q} + m) - k^2\gamma_\mu\right]. \end{aligned} \tag{4.6.6}$$

Using the expressions of numerator A and B from equations (4.6.5) and (4.6.6) respectively, in Eq. (4.6.4) we get

$$\Gamma_c^{(1)} = \left[\frac{(ie)^3}{iM}\right] e^{-\frac{i}{2}p\times q}(q^\mu - p^\mu)\bar{u}(q)\left\{\int \frac{d^3k}{(2\pi)^3}\left[\frac{\not{k}k_\mu + (\not{p} + m)(k_\mu - \not{k}\gamma_\mu)}{[(p+k)^2 - m^2]k^2(k-\mathcal{K})^2}\right.\right.$$
$$\left. - \frac{\gamma_\mu}{[(p+k)^2 - m^2](k-\mathcal{K})^2}\right](\not{q} - m)(e^{ip\times k} - e^{iq\times k}) + \int \frac{d^3k}{(2\pi)^3}(\not{p} - m)$$
$$\left.\left[\frac{\not{k}k_\mu + (k_\mu - \gamma_\mu \not{k})(\not{q} + m)}{[(q+k)^2 - m^2]k^2(k+\mathcal{K})^2} - \frac{\gamma_\mu}{[(q+k)^2 - m^2]k^2(k+\mathcal{K})^2}\right](e^{-ip\times k} - e^{-iq\times k})\right\} u(p). \tag{4.6.7}$$

For the sake of convenience we will solve the above amplitude in parts first we consider the contributions from the pure γ_μ terms. We will call the contribution from this as I_1

$$\begin{aligned} I_1 &= \left[\frac{2(ie)^3}{iM}\right] e^{-\frac{i}{2}p\times q}\bar{u}(q)(q^\mu - p^\mu)\left\{\int \frac{d^3k}{(2\pi)^3}\frac{(m\gamma_\mu - q_\mu)}{(k^2 + 2k.p)(k-\mathcal{K})^2}(e^{-ik\times p} - e^{-ik\times q})\right. \\ &\quad \left. + \int \frac{d^3k}{(2\pi)^3}\frac{(m\gamma_\mu - p_\mu)}{(k^2 + 2q.k)(k+\mathcal{K})^2}(e^{ik\times p} - e^{ik\times q})\right\} u(p) \end{aligned} \tag{4.6.8}$$

Solving the above integrals is now a standard matter. Using Feynman parametric integral we get for the above equation

$$I_1 = C_1\bar{u}(q)\left\{\int_0^1 dx \int \frac{d^3k_A}{(2\pi)^3}\frac{(m\gamma_\mu - q_\mu)}{[k_A^2 - \omega_A^2]^2}P_A + \int_0^1 dx \int \frac{d^3k_B}{(2\pi)^3}\frac{(m\gamma_\mu - q_\mu)}{[k_B^2 - \omega_B^2]^2}P_B\right\} u(p), \tag{4.6.9}$$

Where $C_1 = [2(ie)^3/iM]\exp(-\frac{i}{2}p\times q)(q^\mu - p^\mu)$, $k_A = k + x(p+\mathcal{K}) - \mathcal{K}$, $k_B = k + x(q-\mathcal{K}) + \mathcal{K}$, $\omega_A = [x(p+\mathcal{K}) - \mathcal{K}]^2 - \mathcal{K}^2(1-x)$, and $\omega_B = [x(q-\mathcal{K}) + \mathcal{K}]^2 - \mathcal{K}^2(1-x)$. Furthermore, for the phases associated with the integrals, we have used $[\exp(-ik_A\times p)\exp(-i(1-x)q\times p) - \exp(-ik_A\times q)\exp(ip\times q)] \equiv P_A$ and $[\exp(ik_B\times p)\exp(-iq\times p) - \exp(ik_B\times q)\exp(i(1-x)p\times q)] \equiv P_B$. The momentum integrals are easy to evaluate, the relevant integrals are tabulated in Appendix A, and yield

$$I_1 = C_1 \frac{2i}{(2\sqrt{\pi})^3} \bar{u}(q) \int_0^1 d^3x (m\gamma_\mu - q_\mu) \left[\sqrt{\frac{|\tilde{p}|}{2|\omega_A|}} K_{1/2}(|\tilde{p}||\omega_A|) e^{i(1-x)p\times q} \right.$$
$$\left. - \sqrt{\frac{|\tilde{q}|}{2|\omega_A|}} K_{1/2}(|\tilde{q}||\omega_A|) e^{ip\times q} \right] + \int_0^1 d^3x (m\gamma_\mu - p_\mu) \left[\sqrt{\frac{|\tilde{p}|}{2|\omega_B|}} K_{1/2}(|\tilde{p}||\omega_B|) e^{ip\times q} \right.$$
$$\left. - \sqrt{\frac{|\tilde{q}|}{2|\omega_B|}} K_{1/2}(|\tilde{q}||\omega_B|) e^{i(1-x)p\times q} \right] u(p). \qquad (4.6.10)$$

The parametric integrals are solved by going to the Fermion rest frame which results in the phases present in the above integral becoming unity. This is because $p \times q = 0$. The integral I_1 then becomes

$$I_1 = \frac{iC_1}{8\pi} \bar{u}(q) \left\{ \int_0^1 d^3x (m\gamma_\mu - q_\mu) \left[\frac{\exp(-|\tilde{p}||\omega_A|)}{|\omega_A|} - \frac{\exp(-|\tilde{q}||\omega_A|)}{|\omega_A|} \right] \right.$$
$$\left. + \int_0^1 d^3x (m\gamma_\mu - p_\mu) \left[\frac{\exp(-|\tilde{p}||\omega_B|)}{|\omega_B|} - \frac{\exp(-|\tilde{q}||\omega_B|)}{|\omega_B|} \right] \right\} u(p). \qquad (4.6.11)$$

Expanding the exponentials to first order in θ and performing the parametric integral we get

$$I_1 = (ie)^3 (q^\mu - p^\mu) \frac{(|\tilde{q}| - |\tilde{p}|)}{4\pi M} \bar{u}(q) [2m\gamma_\mu - (q_\mu + p_\mu)] u(p) \quad . \qquad (4.6.12)$$

The other terms of the numerator can be simplified as follows:

$$\begin{aligned} \mathrm{Num\,A} &= [\not{k} k_\mu + (\not{p} + m)(k_\mu - \not{k}\gamma_\mu)](\not{q} - m) \\ &= (\gamma_\alpha\gamma_\beta k^\alpha q^\beta - m\not{k})k_\mu + (\gamma_\alpha\gamma_\beta p^\alpha q^\beta - m^2)k_\mu - (\not{p} + m)\not{k}\gamma_\mu(\not{q} - m) \\ &= \underbrace{2q\cdot k k_\mu + (2p\cdot q - 2m^2)k_\mu}_{A2} - \underbrace{2m\not{k} k_\mu}_{A3} - \underbrace{(\not{p} + m)\not{k}\gamma_\mu(\not{q} - m)}_{A4} \end{aligned} \qquad (4.6.13)$$

Similarly the contribution from the other composite vertex can be cast as

$$\mathrm{Num\,B} = \underbrace{2p\cdot k k_\mu + (2p\cdot q - 2m^2)k_\mu}_{B2} - \underbrace{2m\not{k} k_\mu}_{B3} - \underbrace{(\not{p} - m)\gamma_\mu\not{k}(\not{q} + m)}_{B4} \qquad (4.6.14)$$

For the sake of convenience we have grouped the terms. Making use of the relations $2q\cdot k = \mathcal{K}^2 - (k - \mathcal{K})^2 + (k^2 + 2k\cdot p)$ in $A2$ and $2p\cdot k = \mathcal{K}^2 - (k + \mathcal{K})^2 + (k^2 + 2k\cdot q)$ in $B2$ respectively, and $\mathcal{K}^2 = (q-p)^2 = 2m^2 - 2q\cdot p$, integral I_2 can be written as

$$I_2 = \left[\frac{(ie)^3}{iM}\right] e^{-\frac{i}{2}p\times q}(q^\mu - p^\mu)\bar{u}(q)\left\{\int \frac{d^3k}{(2\pi)^3}\left[\frac{k_\mu}{k^2(k-\mathcal{K})^2} - \frac{k_\mu}{k^2(k^2+2k\cdot p)}\right]\right.$$
$$\left.(e^{-ik\times p} - e^{-ik\times q}) + \int \frac{d^3k}{(2\pi)^3}\left[\frac{k_\mu}{k^2(k+\mathcal{K})^2} - \frac{k_\mu}{k^2(k^2+2k\cdot q)}\right](e^{ik\times p} - e^{ik\times q})\right\} u(p). \quad (4.6.15)$$

We will not elaborate further on the explicit procedure of solving the above integral but merely state the contribution from I_2:

$$I_2 = \frac{(ie)^3}{16\pi M}(q^\mu - p^\mu)(q_\mu + p_\mu)(|\tilde{q}| - |\tilde{p}|). \quad (4.6.16)$$

Terms $A3$ and $B3$ lead to the integral I_3

$$I_3 = \left[\frac{-2m(ie)^3}{iM}\right] e^{-\frac{i}{2}p\times q}(q^\mu - p^\mu)\bar{u}(q)\left\{\int \frac{d^3k}{(2\pi)^3}\left[\frac{\not{k}k_\mu}{k^2(k^2+2k\cdot p)(k-\mathcal{K})^2}\right]\right.$$
$$\left.(e^{-ik\times p} - e^{-ik\times q}) + \int \frac{d^3k}{(2\pi)^3}\left[\frac{\not{k}k_\mu}{k^2(k^2+2k\cdot q)(k+\mathcal{K})^2}\right](e^{ik\times p} - e^{ik\times q})\right\} u(p). \quad (4.6.17)$$

Combining the denominators in the standard manner and defining the contributions from $A3$ and $B3$ as $k = k_{A3} - (x+y)p + yq$ and $k = k_{B3} - (x+y)q + yp$ respectively. With this redefinition of the momentum variables

$$\begin{aligned} I_3 &= C_3 e^{-\frac{i}{2}p\times q}\bar{u}(q)\left\{\int_0^1 dx\int_0^{1-x} dy\int \frac{d^3k}{(2\pi)^3}\left[\frac{(\not{k}_{A3} - mx)[k_{\mu A3} - (x+y)p_\mu + yq_\mu]}{[k_{A3}^2 - \omega_{A3}^2]^3}\right] P_{A3}\right. \\ &+ \left.\int_0^1 dx\int_0^{1-x} dy\int \frac{d^3k}{(2\pi)^3}\left[\frac{(\not{k}_{B3} - mx)[k_{\mu B3} - (x+y)q_\mu + yp_\mu]}{[k_{B3}^2 - \omega_{B3}^2]^3}\right] P_{B3}\right\} u(p). \end{aligned} \quad (4.6.18)$$

Where $C_3 = [-4m(ie)^3/iM](q^\mu - p^\mu)$, $\omega_{A3}^2 = (xp - y\mathcal{K})^2 - y\mathcal{K}^2$, $\omega_{B3}^2 = (xq + y\mathcal{K})^2 - y\mathcal{K}^2$, $P_{A3} = [\exp(-ik_{A3}\times p)\exp(-iyq\times p) - \exp(-ik_{A3}\times p)\exp(i(x+y)p\times q)]$ and $P_{B3} = [\exp(ik_{B3}\times p)\exp(-i(x+y)q\times p) - \exp(ik_{B3}\times q)\exp(iyp\times q)]$. Going to the fermion rest frame we have

$$I_3 = C_3\bar{u}(q)\left\{\int_0^1 dx\int_0^{1-x} dy\int \frac{d^3k}{(2\pi)^3}\left[\frac{(\not{k}_{A3} - mx)[k_{\mu A3} - (x+y)p_\mu + yq_\mu]}{[k_{A3}^2 - \omega_{A3}^2]^3}\right]\right.$$
$$(e^{-ik_{A3}\times p} - e^{-ik_{A3}\times q}) + \int_0^1 dx\int_0^{1-x} dy\int \frac{d^3k}{(2\pi)^3}$$
$$\left.\left[\frac{(\not{k}_{B3} - mx)[k_{\mu B3} - (x+y)q_\mu + yp_\mu]}{[k_{B2}^2 - \omega_{B3}^2]^3}\right](e^{ik_{B3}\times p} - e^{ik_{B3}\times q})\right\} u(p). \quad (4.6.19)$$

It can be noticed that we need to evaluate three different types of integrals. That is depending on whether k occurs in the numerator or not, with the denominators being same. We first consider integral independent of k_{A3}. Denoting this integral by $I_3(0)$, we get

$$\begin{aligned} I_3(0) &= \int_0^1 dx \int_0^{1-x} dy \left\{ \int \frac{d^3k}{(2\pi)^3} \frac{mx[(x+y)p_\mu - yq_\mu]}{[k_{A3}^2 - \omega_{A3}^2]^3} P_1 - \int \frac{d^3k}{(2\pi)^3} \frac{mx[yp_\mu - (x+y)q_\mu]}{[k_{B3}^2 - \omega_{B3}^2]^3} P_2 \right\} \\ &= -\frac{i}{32\pi} \frac{(p_\mu + q_\mu)}{6} (|\tilde{q}|^2 - |\tilde{p}|^2) \quad . \end{aligned} \tag{4.6.20}$$

where we have defined $P_1 = [\exp(-ik_{A3} \times p) - \exp(-ik_{A3} \times q)]$ and $P_2 = [\exp(ik_{B3} \times p) - \exp(ik_{B3} \times q)]$. Also the factor C_3 is understood to be present in the above integrals as well as those given below. The above result for the integrals has been calculated in the low momentum transfer limit. Similarly the integrals with two momenta in the numerator gives

$$\begin{aligned} I_3(kk) &= \int_0^1 dx \int_0^{1-x} dy \left\{ \int \frac{d^3k}{(2\pi)^3} \frac{\not{k}_{A3} k_{\mu A3}}{[k_{A3}^2 - \omega_{A3}^2]^3} P_1 + \int \frac{d^3k}{(2\pi)^3} \frac{\not{k}_{B3} k_{\mu B3}}{[k_{B3}^2 - \omega_{B3}^2]^3} P_2 \right\} \\ &= \frac{iC_3\gamma^\nu}{96\pi} \left[\frac{\tilde{p}_\mu \tilde{p}_\nu}{|\tilde{p}|} - \frac{\tilde{q}_\mu \tilde{q}_\nu}{|\tilde{q}|} - g_{\mu\nu}(|\tilde{q}| - |\tilde{p}|) \right] - \frac{iC_3 m\gamma^\nu}{32\pi} [\tilde{p}_\mu \tilde{p}_\nu - \tilde{q}_\mu \tilde{q}_\nu] \quad . \end{aligned} \tag{4.6.21}$$

Finally we solve the integrals with a single k in the numerator

$$\begin{aligned} I_3(k) = &\int_0^1 dx \int_0^{1-x} dy \int \frac{d^3k}{(2\pi)^3} \left[(yq_\mu - (x+y)p_\mu) \frac{\not{k}_{A3}}{[k_{A3}^2 - \omega_{A3}^2]^3} P_{A3} - mx \frac{k_{\mu A3}}{[k_{A3}^2 - \omega_{A3}^2]^3} P_{A3} \right] \\ + &\int_0^1 dx \int_0^{1-x} dy \int \frac{d^3k}{(2\pi)^3} \left[(yp_\mu - (x+y)q_\mu) \frac{\not{k}_{B3}}{[k_{B3}^2 - \omega_{B3}^2]^3} P_{B3} - mx \frac{k_{\mu B3}}{[k_{B3}^2 - \omega_{B3}^2]^3} P_{B3} \right] . \end{aligned} \tag{4.6.22}$$

Performing the momentum integrals

$$\begin{aligned} I_3(k) &= \int_0^1 dx \int_0^{1-x} dy (yq_\mu - (x+y)p_\mu)\gamma^\nu \left[-\frac{1}{2} \frac{\tilde{p}_\nu}{(\sqrt{2\pi})^3} \sqrt{\frac{|\tilde{p}|}{2|\omega_{A3}|}} K_{1/2}(|\tilde{p}||\omega_{A3}|) \right. \\ &+ \left. \frac{1}{2} \frac{\tilde{q}_\nu}{(\sqrt{2\pi})^3} \sqrt{\frac{|\tilde{q}|}{2|\omega_{A3}|}} K_{1/2}(|\tilde{p}||\omega_{A3}|) \right] \\ &- \int_0^1 dx \int_0^{1-x} dy\, mx \left[-\frac{1}{2} \frac{\tilde{p}_\mu}{(\sqrt{2\pi})^3} \sqrt{\frac{|\tilde{p}|}{2|\omega_{A3}|}} K_{1/2}(|\tilde{p}||\omega_{A3}|) \right. \\ &+ \left. \frac{1}{2} \frac{\tilde{q}_\mu}{(\sqrt{2\pi})^3} \sqrt{\frac{|\tilde{q}|}{2|\omega_{A3}|}} K_{1/2}(|\tilde{p}||\omega_{A3}|) \right] \\ &+ \int_0^1 dx \int_0^{1-x} dy \left[(yp_\mu - (x+y)q_\mu)\gamma^\nu [\omega_{A3} \to \omega_{B3}] - mx[\omega_{A3} \to \omega_{B3}] \right] . \end{aligned} \tag{4.6.23}$$

In the above equation $\omega_{A3} \to \omega_{B3}$ in parenthesis means that the expression is same as that given previously in the equation but with ω_{A3} being replaced by ω_{B3}. Expanding the bessel functions to first order in θ and after some algebra the parametric integrals that need to be calculated are

$$I_3(k) = \int_0^1 dx \int_0^{1-x} dy \Big[2\gamma^\nu (\tilde{q}_\nu |\tilde{q}| - \tilde{p}_\nu |\tilde{p}|)(p_\mu - q_\mu)(x+y) - \frac{\gamma^\nu}{m}(\tilde{q}_\nu - \tilde{p}_\nu)(p_\mu - q_\mu) \\ - \frac{2y\gamma^\nu}{mx}(\tilde{q}_\nu - \tilde{p}_\nu)(p_\mu - q_\mu) \Big]. \quad (4.6.24)$$

The parametric integrals give

$$I_3(k) = \left[\frac{2(\gamma \cdot \tilde{q}|\tilde{q}| - \gamma \cdot \tilde{p}|\tilde{p}|)}{3} + \frac{(\gamma \cdot \tilde{q} - \gamma \cdot \tilde{p})}{m} - \int_0^1 dx \frac{(\gamma \cdot \tilde{q} - \gamma \cdot \tilde{p})}{mx} \right] (p_\mu - q_\mu). \quad (4.6.25)$$

We have left the last integral in the above expression as it is, for the time being, since it is divergent.

For obtaining the final contribution of the composite ghost-gluon vertex, terms $A4$ and $B4$ can be simplified using gamma matrix algebra

$$A4 = (\not{p} + m)\not{k}\gamma_\mu(\not{q} - m) = \underbrace{\not{p}\not{k}\gamma_\mu\not{q}}_{\boxed{1}} - \underbrace{m\not{p}\not{k}\gamma_\mu}_{\boxed{2}} + \underbrace{m\not{k}\gamma_\mu\not{q}}_{\boxed{3}} - \underbrace{m^2\not{k}\gamma_\mu}_{\boxed{4}}. \quad (4.6.26)$$

The individual terms can be simplified to give

$$\begin{aligned}
\boxed{1} &= 4k\cdot pq_\mu - 2m\not{k}q_\mu - 4k\cdot qp_\mu + 2mk\cdot q\gamma_\mu + 2p\cdot q\not{k}\gamma_\mu - 2mk\cdot p\gamma_\mu + 2m\not{k}p_\mu - m^2\not{k}\gamma_\mu \\
\boxed{2} &= 2mk\cdot p\gamma_\mu - 2m\not{k}p_\mu + m^2\not{k}\gamma_\mu \\
\boxed{3} &= 2m\not{k}q_\mu - 2mk\cdot q\gamma_\mu + m^2\not{k}\gamma_\mu \\
\boxed{4} &= m^2\not{k}\gamma_\mu. \qquad (4.6.27)
\end{aligned}$$

Collecting all the contributions

$$A4 = (4k\cdot pq_\mu - 4k\cdot qp_\mu) + (4m\not{k}p_\mu - 4mk\cdot p\gamma_\mu) + 2(p\cdot q - m^2)\not{k}\gamma_\mu. \quad (4.6.28)$$

Similarly for $B4$ we have

$$B4 = (4k\cdot qp_\mu - 4k\cdot pq_\mu) + (4m\not{k}q_\mu - 4mk\cdot q\gamma_\mu) + 2(p\cdot q - m^2)\gamma_\mu\not{k}. \quad (4.6.29)$$

From the above we can see that only two types of integrals occur in the above equations

$$I_{4A} = C_4\bar{u}(q)\int\frac{d^3k}{(2\pi)^3}\left[\frac{k_\mu}{k^2(k^2+2k\cdot p)(k-\mathcal{K})^2}\right](e^{ip\times k}-e^{iq\times k})u(p),$$
$$I_{4B} = C_4\bar{u}(q)\int\frac{d^3k}{(2\pi)^3}\left[\frac{k_\mu}{k^2(k^2+2k\cdot q)(k+\mathcal{K})^2}\right](e^{-ip\times k}-e^{-iq\times k})u(p). \tag{4.6.30}$$

We have defined $C_4 = [(ie)^3/iM]\exp(-ip\times q/2)(q^\mu - p^\mu)$. It is now a easy matter to solve the integrals. It can be noticed that we have single k in the numerator so we can expect divergent integrals. These integrals can be read of from the integrals provided in the appendix. From the expressions of the various non-planar integrals calculated above we notice that the divergent pieces occur in the terms that do not contribute to the MM.

4.7 Discussion

We can see from the calculation of the vertex diagrams that similar to the vertex corrections of QED the contributions can be broken into several pieces. There exists purely θ dependent contribution to the MM namely terms of the type $\tilde{\mathcal{K}}$. Furthermore there exists terms of the type $\mathcal{K}_\mu$ with theta dependent coefficients. It can be noticed that they do not satisfy the Ward identity. It must be mentioned that they arise from the composite diagrams.

It is known that in the NC scenario, unitarity may not be preserved. The problems occurring in the BRST analysis is probably rooted in the above cause. In a physical gauge, like $A_3 = 0$ the three gluon interaction drops out, indicating that the violation of the Ward identity encountered in the above calculation may be a gauge artefact.

Chapter 5

Phase structure of noncommutative field theories

5.1 Introduction

Noncommutative theories have occupied a great deal of interest in recent years. We have already pointed out various reasons in the first chapter. In this chapter we study the phase structure of NC theories. To be precise we study the NC version of the BCS theory. Phase transitions in NC theories are specially intriguing and challenging due to the fact that these theories exhibit UV/IR mixing. In many cases the presence of infrared singularities leads to phase transitions. A further aspect worth mentioning is that these infrared singularities do not arise due to massless propagating fields but rather due to loop effects. The effect of these singularities on the phase transition in the context of $\lambda\phi^4$ theories has been studied. It was shown that the due to noncommutativity there is a transition to a non-uniform striped phase [5, 11, **?**, 50].

In this chapter, we concentrate on a non-relativistic noncommutative field theory at finite density. We adopt a non-perturbative approach, through an appropriate Bogoliubov transformation to find the stable vacuum in the presence of the four fermion contact interaction. It is found that a LOFF type ansatz is ideal for the same purpose. As will be seen, the presence of the noncommuting parameter, yields a non-trivial ground state, even for a single species of fermions. It is worth pointing out that, for the commutative case, the chemical potential difference between the two species of fermions was responsible for LOFF type instability.

5.2 Noncommutative gap equation

The NC version of the BCS Hamiltonian is

$$\mathcal{H} = \psi_r^\dagger(x) \left[-\frac{\nabla^2}{2m} - \mu \right] \psi_r(x) - \frac{g}{2}\, \psi_r^\dagger(x) \star \psi_s^\dagger(x) \star \psi_s(x) \star \psi_r(x). \tag{5.2.1}$$

Here g is the coupling constant and μ the chemical potential and r is the spinor index and can take values $\pm 1/2$. In momentum space the interaction has the form

$$\mathcal{H}_{int} = \frac{g}{2} \int \prod_{i=1}^{4} \frac{d^3 k_i}{(2\pi)^3}\, e^{-i(k_1+k_2-k_3-k_4)x} e^{i/2 \sum_{i<j} k_i \times k_j}. \tag{5.2.2}$$

A suitable trial wavefunction can be constructed for the LOFF state as

$$\mid \Omega\rangle = e^{\lambda(B^\dagger - B)} \mid 0\rangle. \tag{5.2.3}$$

Here the operators appearing in the exponential are defined as

$$B^\dagger = \int d^3k\, \psi_r^\dagger(k + \frac{q}{2})\, r\, \psi_{-r}^\dagger(-k + \frac{q}{2})\, f(k) \tag{5.2.4}$$

Here $f(k)$ is the condensate function. It can be noticed from the above definition of the operators that the particles have a relative total momentum with respect to each other and is characterized by the vector q. Now using the above operators we can cast the particle creation and annihilation operators in terms of the quasi-particle creation and destruction operators:

$$\begin{bmatrix} \psi_r(k) \\ \psi_{-r}^\dagger(-k+q) \end{bmatrix} = \begin{pmatrix} \cos f(k - q/2) & 2r \sin f(k-q/2) \\ -2r \sin f(k-q/2) & \cos f(k-q/2) \end{pmatrix} \begin{bmatrix} \widetilde{\psi}_r(k) \\ \widetilde{\psi}_{-r}^\dagger(-k+q) \end{bmatrix}. \tag{5.2.5}$$

In the above the tilde operators denote the quasi-particle operators and are constructed such that $\widetilde{\psi}_r(k) \mid \Omega\rangle = 0$.

One can in principle have a more general operator where in the definition of $B^\dagger$ the condensate function is taken to be complex $f^*(x)$. With this definition of $B^\dagger$, the transformation matrix for the quasi-particle operators can be written as

$$\begin{bmatrix} \psi_r(k) \\ \psi^\dagger_{-r}(-k+q) \end{bmatrix} = \begin{bmatrix} \cos|f(k-q/2)| & \frac{2rf^*(k-q/2)\sin|f(k-q/2)|}{|f(k-q/2)|} \\ -\frac{2rf(k-q/2)\sin|f(k-q/2)|}{|f(k-q/2)|} & \cos|f(k-q/2)| \end{bmatrix} \begin{bmatrix} \overline{\psi}_r(k) \\ \overline{\psi}^\dagger_{-r}(-k+q) \end{bmatrix} \tag{5.2.6}$$

The free energy is obtained by taking the vacuum expectation value (VEV) of the operators $\langle\psi^\dagger\psi^\dagger\rangle_\Omega$ and $\langle\psi\psi\rangle_\Omega$ in the LOFF state. We calculate $\langle\psi^\dagger_r(k_1)\psi^\dagger_s(k_2)\rangle_\Omega$ (From now on we will drop the subscript Ω in the VEV expressions and it is to be understood that the state is the LOFF state) going over to the quasi-particle operators, the expectation yields

$$\begin{aligned} \langle\psi^\dagger_r\psi^\dagger_s\rangle &\equiv \int \frac{d^3k_1}{\sqrt{(2\pi)^3}}\frac{d^3k_2}{\sqrt{(2\pi)^3}} \langle[\cos f(-k_1+q/2)\widetilde{\psi}^\dagger_r(k_1) + 2r\sin f(-k_1+q/2)\widetilde{\psi}_{-r}(-k_1+q)] \\ &\quad [\cos f(-k_2+q/2)\widetilde{\psi}^\dagger_r(k_2) + 2s\sin f(-k_2+q/2)\widetilde{\psi}_{-s}(-k_2+q)]\rangle, \\ &= \int \frac{d^3k_1}{\sqrt{(2\pi)^3}}\frac{d^3k_2}{\sqrt{(2\pi)^3}} 2r\sin f(-k_1+q/2)\cos f(-k_2+q/2)\langle\widetilde{\psi}_{-r}(-k_1+q)\widetilde{\psi}^\dagger_s(k_2)\rangle, \\ &= \int \frac{d^3k_1}{\sqrt{(2\pi)^3}}\frac{d^3k_2}{\sqrt{(2\pi)^3}} 2r\sin f(-k_1+q/2)\cos f(-k_2+q/2)\delta_{-r,s}\delta^3(k_1+k_2-q). \end{aligned} \tag{5.2.7}$$

With $k_1 = P'/2 + Q'$ and $k_2 = P'/2 - Q'$, the expectation value can be written in the form

$$-\int \frac{d^3P'}{\sqrt{(2\pi)^3}}\frac{d^3Q'}{\sqrt{(2\pi)^3}} 2r\cos f(P'/2+Q'-q/2)\sin f(P'/2-Q'-q/2)\delta_{-r,s}\delta^3(P'-q). \tag{5.2.8}$$

Next we compute the expectation value of $\langle\psi_r(k_3)\psi_s(k_4)\rangle_\Omega$.

$$\begin{aligned} \langle\psi_r\psi_s\rangle &\equiv \int \frac{d^3k_3}{\sqrt{(2\pi)^3}}\frac{d^3k_4}{\sqrt{(2\pi)^3}} \langle[\cos f(k_3-q/2)\widetilde{\psi}_r(k_3) + 2r\sin f(k_3-q/2)\widetilde{\psi}^\dagger_{-r}(-k_3+q)] \\ &\quad [\cos f(k_4-q/2)\widetilde{\psi}_r(k_2) + 2s\sin f(k_4-q/2)\widetilde{\psi}^\dagger_{-s}(-k_4+q)]\rangle \\ &= \int \frac{d^3k_3}{\sqrt{(2\pi)^3}}\frac{d^3k_4}{\sqrt{(2\pi)^3}} 2s\cos f(k_3-q/2)\sin f(k_4-q/2)\langle\widetilde{\psi}_r(k_3)\widetilde{\psi}^\dagger_{-s}(-k_4+q)\rangle \\ &= \int \frac{d^3k_3}{\sqrt{(2\pi)^3}}\frac{d^3k_4}{\sqrt{(2\pi)^3}} 2s\cos f(k_3-q/2)\sin f(k_4-q/2)\delta_{r,-s}\delta^3(k_3+k_4-q). \end{aligned} \tag{5.2.9}$$

Redefining the momentum as shown above, with $k_3 = P/2 + Q$ and $k_4 = P/2 - Q$:

$$-\int \frac{d^3P}{\sqrt{(2\pi)^3}}\frac{d^3Q}{\sqrt{(2\pi)^3}} 2s\cos f(P/2+Q-q/2)\sin f(P/2-Q-q/2)\delta_{r,-s}\delta^3(P-q). \tag{5.2.10}$$

Performing the P and P' integrations in equations 5.2.8 and 5.2.10, the expectation value of the interacting Hamiltonian becomes

$$\langle \mathcal{H}_{int} \rangle = \frac{g}{4} \int \frac{d^3Q}{\sqrt{(2\pi)^3}} \frac{d^3Q'}{\sqrt{(2\pi)^3}} \sin 2f(Q) \sin 2f(Q') e^{iq\times(Q+Q')/2}. \tag{5.2.11}$$

Similarly the kinetic term acquires the form

$$\langle \mathcal{H}_0 \rangle = \int d^3Q \, 4 \left[\epsilon(Q+q/2)^2 - \mu\right] \sin^2 f(Q), \tag{5.2.12}$$

where $\epsilon(Q + q/2) = (Q + q/2)^2/2m$. Minimizing the free energy with respect to the condensate function $f(p)$ viz $\delta\langle\mathcal{H}\rangle/\delta f(p) = 0$, gives

$$4[\epsilon(p+q/2) - \mu] \sin 2f(p) = -g \int \frac{d^3Q}{(2\pi)^3} \cos 2f(p) \sin 2f(Q) \cos\left[\frac{q\times p}{2}\right] \cos\left[\frac{q\times Q}{2}\right]. \tag{5.2.13}$$

In the above equation we have retained the real part of the phase and have dropped the sine terms since they are odd functions of Q. Dividing by $\cos 2f(p)$ we obtain

$$\tan 2f(p) = \frac{\Delta(q)\cos(q\times p/2)}{[\epsilon(p+q/2) - \mu]}; \tag{5.2.14}$$

where

$$\Delta(q) = -\frac{g}{4} \int \frac{d^3Q}{(2\pi)^3} \sin 2f(Q) \cos\left[\frac{q\times Q}{2}\right]. \tag{5.2.15}$$

The gap equation is then

$$\Delta(q) = -\frac{g}{4} \int \frac{d^3Q}{(2\pi)^3} \frac{\Delta(q)\cos^2(q\times Q/2)}{\sqrt{(\epsilon(Q+q/2)-\mu)^2 + \Delta^2(q)\cos^2(q\times Q/2)}}. \tag{5.2.16}$$

Expanding the cosine in terms of series to the first nontrivial order in θ

$$1 = -\frac{g}{4} \int \frac{d^3Q}{(2\pi)^3} \left[\frac{1}{\sqrt{(\epsilon(Q)-\mu)^2+\Delta^2}} - \frac{(q\times Q)^2}{4\sqrt{(\epsilon(Q)-\mu)^2+\Delta^2}}\right]. \tag{5.2.17}$$

It must be noted that the form of the gap equation depends crucially on the choice of Δ. For example, instead of defining Δ as in Eq. (5.2.15), if we choose

$$\Delta(q,p) = -\frac{g}{4} \int \frac{d^3Q}{(2\pi)^3} \sin 2f(Q) \cos\left[\frac{q\times(Q+p)}{2}\right]; \tag{5.2.18}$$

the gap equation then has the form

$$\begin{aligned}\Delta(q,p) &= -\frac{g}{4}\int\frac{d^3Q}{(2\pi)^3}\frac{\Delta(q,Q)}{\sqrt{(\epsilon(Q+q/2)-\mu)^2+\Delta^2(q,Q)}}\cos\left[\frac{q\times(Q+p)}{2}\right] \\ &= -\frac{g}{4}\int\frac{d^3k}{(2\pi)^3}\frac{\Delta(q,k-q/2)}{\sqrt{(\epsilon(k)-\mu)^2+\Delta^2(q,k-q/2)}}\cos\left[\frac{q\times(k+p)}{2}\right], \qquad (5.2.19)\end{aligned}$$

where $k \equiv Q + q/2$. A few points are in order here. We could have separated the phase $\cos[q \times p/2]$ since it does not depends on the integration variable. Then defining the gap as $\Delta(q,p) = \tilde{\Delta}(q)\cos[q \times p/2]$, we see that Eq. (5.2.19) reduces to that of Eq. (5.2.16). But we find it advantageous to retain the above form of the equation.

5.3 Solution of the gap equation

The solution of the gap equation can be found following the technique presented in [73]. It can be seen from the definition of $\Delta(q,p)$ that it is an even function hence, the ansatz we take is of the form $\Delta + p^2\delta$ [73]. In principle we could have started out with a more general ansatz like $\Delta(q,p) = \Delta + p^2\delta + (p\cdot q)\delta_1 + q^2\delta_2 + (p\cdot\theta)\delta_3$, but as already noted since $\Delta(q,p)$ is an even function of p terms linear in p can be dropped. As for the q^2 dependent term, it does not depend on the integration variable and can be ignored in the contribution to the gap. Keeping in mind the above considerations we have

$$\Delta + p^2\delta = -\frac{g}{4}\int\frac{d^3k}{(2\pi)^3}\frac{(\Delta+k^2\delta)(1-[(q\times p)^2+(q\times k)^2]/8)}{\sqrt{(\epsilon(k)-\mu)^2+(\Delta+k^2\delta)^2}}, \qquad (5.3.1)$$

In the above expression we have ignored the $2(q \times r)(q \times p)$ term from the cosine expansion. Performing the angular integrations and retaining terms to order k^2 in the gap equation

$$\Delta + p^2\delta = -\frac{g}{8\pi^2}\int_0^\infty dk\frac{k^2\Delta[1-(q\times p)^2/8]}{\sqrt{(\epsilon(k)-\mu)^2+(\Delta+k^2\delta)^2}}. \qquad (5.3.2)$$

The above integral can be solved analytically. Computing first the pure Δ integral i.e., setting $p^2 = 0$ we get

$$1 = -\frac{g}{8\pi^2}\int_0^\infty dk\frac{k^2}{\sqrt{(\epsilon(k)-\mu)^2+(\Delta+k^2\delta)^2}}. \qquad (5.3.3)$$

Noting that the above integral can be cast in the form

$$J_\alpha(x,y) = \int_0^\infty dt \frac{t^\alpha}{\sqrt{(t-1)^2 + (x+yt)^2}}, \tag{5.3.4}$$

the integral becomes

$$1 = -\frac{gmk_F}{4\pi^2} J_{1/2}(x,y); \tag{5.3.5}$$

Where $k_F = \sqrt{2m\mu}$, $x = \Delta/\mu$, and $y = k_F^2\delta/\mu$. For δ we get

$$\delta = \frac{gk_F m\Delta}{4\pi^2} \frac{(q \times n_p)^2}{8} J_{1/2}(x,y). \tag{5.3.6}$$

Here n_p denotes the unit vector in the direction of p. It can be seen that in the absence of noncommutativity ($\theta = 0$) δ becomes zero and the expression reduces to that of the ordinary BCS gap equation. Making use of the relation

$$J_\alpha(x,y) = -\frac{\pi}{\sin\pi\alpha}(1+y^2)^{-1/2}\left(\frac{1+x^2}{1+y^2}\right)^{\alpha/2} P_\alpha(-z). \tag{5.3.7}$$

where $z = (1-xy)/\sqrt{(1+x^2)(1+y^2)}$, and setting $y = 0$ in the above expression yields

$$\frac{1}{k_F a} = (1+x^2)^{1/4} P_{1/2}(-1/\sqrt{1+x^2}). \tag{5.3.8}$$

The Legendre polynomial has a logaritmic singularity as $z \to 0$. The expression for the leading term is given by [38]

$$P_\alpha(z) = \frac{\sin(\alpha\pi)}{\pi}\left[\ln\left(\frac{z+1}{2}\right) + \gamma + 2\psi(\alpha+1) + \pi\cot(\alpha\pi)\right]; \tag{5.3.9}$$

where γ is the Euler-Mascheroni constant and ψ is the derivative of the log of the gamma function. The usual gap of the BCS theory is obtained for small values of $k_F a$:

$$\Delta = \frac{8}{e^2}\mu\exp\left(-\frac{\pi}{2k_F|a|}\right). \tag{5.3.10}$$

To find out the effect of θ on the gap we need to solve for δ from Eq. (5.3.6) we get

$$\delta = -(ak_F\mu x)\frac{(q \times n_p)^2}{8}(1+x^2)^{1/4} P_{1/2}(-1/\sqrt{1+x^2}). \tag{5.3.11}$$

It must be pointed out that the three vectors available to us θ, p and q are mutually orthogonal. This feature that the gap depends on the angles between the vectors has also been reported in [19].

5.4 Conclusions

In this chapter we have studied the effect of noncommutative interaction on the BCS pairing mechanism. It can be noticed that due to the presence of noncommutativity the pairing energy decreases as compared to the ordinary BCS theory. Furthermore, it also supports the existence of a finite momentum condensate just like the LOFF type of pairing. The crucial difference however is that, LOFF pairing is possible in the presence of two species of fermions, whereas in our case just single species is enough. It must also be mentioned that when the NC parameter is set to zero the non-zero total momentum of the pair tends to zero leading to the usual BCS type of pairing.

Chapter 6

Summary and future prospects

In conclusion, in this thesis we have studied the effect of noncommutativity on physical quantities both from perturbative and non-perturbative points of view. the perturbative treatment was carried out in anyonic planar field theories. The correction to magnetic moment was calculated at the one-loop level for a bosonic theory. It is worth pointing out that complex scalar field theories can carry a magnetic moment in a planar field theory. The magnetic moment coupling arises due to induced spin in a Chern-Simons field theory. In a noncommutative theory corrections were obtained at one-loop level. It was found that the presence of this dimensional parameter θ can lead to novel type of coupling of the matter field with magnetic field. In particular, a coupling sensitive to inhomogeneity of the magnetic field was identified. The planar spin dynamics and motion of charged particles in a non-uniform magnetic field will be affected.

Keeping in mind the usefulness of noncommutative CS theory to quantum Hall effect, this effect may find physical applications. We then proceeded to relativistic fermionic field theories and studied the BRST symmetry of the same with a CS coupling. The fact that $U(1)$ noncommutative theory is structurally quite analogous to non-Abelian theoires, necessitates the above analysis. The above study was also required in the context of our calculation of the one-loop correction to the fermionic moments. Presence of composite vertex made this calculation, rather tedious, which was accomplished with the help of the above mentioned symmetry analysis. Differences between 3+1 and 2+1 dimensional theories were highlighted.

We then studied the effect of noncommutativity on the vacuum structure of the BCS theory. This was carried out using a non-perturbative approach in the context of a Gorkov type four-Fermi coupling instead of the original BCS coupling. The θ parameter significantly affected the non-perturbative vacuum. It was found that, instead of the original Cooper-paired ground state, the θ-term prefers a LOFF type vacuum, where the paired particles can have non-zero center of mass momentum. The vacuum structure was analyzed in detail for momenta close to the Fermi surface and the similarity of the non-commutative theory with theories having derivative coupling was pointed out.

A number of directions can be envisaged where further studies can be profitable carried out. The precise connection of the noncommutative theories studied here with anyonic theories relevant to quantum Hall effect should be investigated in order to physically test the effect of noncommutative parameter as particle or quasi-particle dynamics. Since finite temperature and non-zero θ have similar effects. The effect of finite temperature and chemical potential in these theories also needs investigation. It should be noted that in the noncommutative theories, one-loop finite temperature effects leads to interesting physics in these theories.

The effect of finite temperature on the aforementioned BCS theory in a noncommutative background is bound to throw light on the phase structure of this theory. The possibility of a quantum phase transition in this model, where θ and temperature can both play significant roles needs to be explored. The dynamics of quasi-particles and collective modes should also be studied carefully for this effect of θ on them.

Appendix A

Important Integrals

Below we list a few frequently encountered nonplanar integrals that arise in the nonplanar loop calculations.

1.

$$\int \frac{d^3q}{(2\pi)^3} \frac{e^{\pm iq\times\mathcal{K}}}{[q^2-\omega^2]^2} = \frac{2i}{(2\sqrt{\pi})^3} \left[\frac{|\widetilde{\mathcal{K}}|}{2|\omega|}\right]^{1/2} K_{1/2}(|\widetilde{\mathcal{K}}||\omega|). \tag{A.0.1}$$

2.

$$\int \frac{d^3q}{(2\pi)^3} \frac{q_\mu}{[q^2-\omega^2]^2}\, e^{\pm iq\times\mathcal{K}} = \mp\frac{\widetilde{\mathcal{K}}_\mu}{(2\sqrt{\pi})^3} \left[\frac{|\widetilde{\mathcal{K}}|}{2|\omega|}\right]^{-1/2} K_{-1/2}(|\widetilde{\mathcal{K}}||\omega|). \tag{A.0.2}$$

3.

$$\int \frac{d^3q}{(2\pi)^3} \frac{q_\mu q_\nu e^{\pm iq\times\mathcal{K}}}{[q^2-\omega^2]^2} = -\frac{2i}{(2\sqrt{\pi})^3} \left[\frac{\widetilde{\mathcal{K}}_\mu\widetilde{\mathcal{K}}_\nu}{4} \left[\frac{|\widetilde{\mathcal{K}}|}{2|\omega|}\right]^{-3/2} K_{-3/2}(|\widetilde{\mathcal{K}}||\omega|) \right.$$
$$\left. -\frac{g_{\mu\nu}}{2} \left[\frac{|\widetilde{\mathcal{K}}|}{2|\omega|}\right]^{-1/2} K_{-1/2}(|\widetilde{\mathcal{K}}||\omega|)\right]. \tag{A.0.3}$$

4.

$$\int \frac{d^3q}{(2\pi)^3} \frac{e^{\pm iq\times\mathcal{K}}}{[q^2-\omega^2]^3} = -\frac{i}{(2\sqrt{\pi})^3} \left[\frac{|\widetilde{\mathcal{K}}|}{2|\omega|}\right]^{3/2} K_{3/2}(|\widetilde{\mathcal{K}}||\omega|). \tag{A.0.4}$$

5.

$$\int \frac{d^3q}{(2\pi)^3} \frac{q_\mu}{[q^2-\omega^2]^3}\, e^{\pm iq\times\mathcal{K}} = \pm\frac{1}{2}\frac{\widetilde{\mathcal{K}}_\mu}{(2\sqrt{\pi})^3} \left[\frac{|\widetilde{\mathcal{K}}|}{2|\omega|}\right]^{1/2} K_{1/2}(|\widetilde{\mathcal{K}}||\omega|). \tag{A.0.5}$$

6.

$$\int \frac{d^3q}{(2\pi)^3}\frac{q_\mu q_\nu e^{\pm iq\times\mathcal{K}}}{[q^2-\omega^2]^3} = \frac{i}{(2\sqrt{\pi})^3}\left[\frac{\widetilde{\mathcal{K}}_\mu\widetilde{\mathcal{K}}_\nu}{4}\left[\frac{|\widetilde{\mathcal{K}}|}{2|\omega|}\right]^{-1/2} K_{-1/2}(|\widetilde{\mathcal{K}}||\omega|) - \frac{g_{\mu\nu}}{2}\left[\frac{|\widetilde{\mathcal{K}}|}{2|\omega|}\right]^{1/2} K_{1/2}(|\widetilde{\mathcal{K}}||\omega|)\right]. \tag{A.0.6}$$

Bibliography

[1] A. Connes, Noncommutative geometry, Academic Press, 1994; G. Landi, An introduction to noncommutative spaces and their geometries, Springer 1997; J. Madore, An introduction to noncommutative geometry and its physical applications, Cambridge University Press 1999.

[2] M. R. Douglas, N. Nekrasov, *Noncommutative field theory*, Rev. Mod. Phys. **73** (2001) 977; R. J. Szabo, *Quantum field theory on noncommutative spaces*, Phys. Rept. **378** (2003) 207 hep-th/0109162; H. O. Girotti, *Noncommutative quantum field theories*, hep-th/0301237.

[3] C. Acatrinei, *Comments on noncommutative particle dynamics*, hep-th/0106141.

[4] L. Alvarez-Gaumé, J. M. F. Labastida, A. V. Ramallo, *A note on perturbative Chern-Simons theory*, Nucl. Phys. **B 334** (1990) 103.

[5] J. Ambjorn, S. Catterall, *Stripes from (noncommutative) stars*, Phys. Lett. **B 549** (2002) 253.

[6] K. S. Babu, A. Das, P. K. Panigrahi, *Derivative expansion and the induced Chern-Simons term at finite temperature in 2+1 dimensions*, Phys. Rev. **D 36** (1987) 3725.

[7] D. Bak, K. Lee, J. H. Park, *Chern-Simons theories on the noncommutative plane*, Phys. Rev. Lett. **87** (2001) 030402; V. P. Nair, A. P. Polychronakos, *Level quantization for the*

noncommutative Chern-Simons theory, Phys. Rev. Lett. **87** (2001) 030403; M. M. Sheikh-Jabbari, *A note on noncommutative Chern-Simons theories*, Phys. Lett. **B 510**, (2001) 247.

[8] J. Bellissard, A. van Elst, H. Schulz-Baldes, *The noncommutative geometry of the quantum Hall effect*, J. Math. Phys. **35** (1994) 5373.

[9] A. Bérard, H. Mohrbach, *Monopole and Berry phase in momentum space in noncommutative quantum mechanics*, Phys. Rev. **D 69** (2004) 127701.

[10] A. A. Bichl, J. M. Grimstrup, V. Putz, M. Schweda, *Perturbative Chern-Simons theory on noncommutative R^3*, J. High Energy Phys. **07** (2000) 046.

[11] W. Bietenholz, F. Hofheinz, J. Nishimura, *Simulating noncommutative firld theory*, `hep-lat/0209021`.

[12] The figures in this thesis have been generated using JaxoDraw: D. Binosi, L. Theussl, *JaxoDraw: A graphical user interface for drawing Feynman diagrams*, Comput. Phys. Commun. **161** (2004) 76.

[13] L. Bonora, M. Salizzoni, *Renormalization of noncommutative U(N) gauge theories*, Phys. Lett. **B 504** (2001) 80.

[14] F. T. Brandt, A. Das, J. Frenkel, *Absence of higher order corrections to the non-abelian topological mass term*, Phys. Rev. **D 63** (2001) 085015.

[15] I. L. Buchbinder, V. A. Krykhtin, *One-loop renormalization of general noncommutative Yang-Mills field model coupled to scalar and spinor fields*, Int. J. Mod. Phys. **A 18** (2003) 3057.

[16] X. Calmet *Space-time symmetries of noncommutative spaces*, Phys. Rev. **D 71** (2005) 85012.

[17] F. E. Camino, W. Zhou, V. J. Goldman, *Realization of a Laughlin quasiparticle interferometer: Observation of fractional statistics*, Phys. Rev. **B 72** (2005) 075342.

[18] P. Castorina, D. Zappala, *Nonuniform symmetry breaking in noncommutative $\lambda\phi^4$ theory*, Phys. Rev. **D 68** (2003) 65008.

[19] P. Castorina, G. Riccobene, D. Zappala, *Inhomogeneous chiral symmetry breaking in noncommutative four fermion interactions*, Phys. Rev. **D 69** (2004) 105024.

[20] M. Chaichian, W. F. Chen, *Inconsistency of naive dimensional and quantum correction to non-Abelian Chern-Simons-matter theory reexamined*, Phys. Rev. **D 58** (1997) 125004.

[21] M. Chaichian, W. F. Chen, V. Ya. Fainberg, *On quantum corrections to Chern-Simons spinor electrodynamics*, Eur. Phys. J. **C 5** (1998) 545.

[22] M. Chaichian, M. M. Sheikh-Jabbari, A. Tureanu, *Hydrogen atom spectrum and the Lamb shift in Noncommutative QED*, Phys. Rev. Lett. **86** (2001) 2716.

[23] B. Chandrasekhar, P. K. Panigrahi, *Finite temperature effects on the induced Chern-Simons term in noncommutative geometry*, J. High Energy Phys. **03** (2003) 015.

[24] G. H. Chen, Y. S. Wu, *One-loop shift in noncommutative Chern-Simons coupling*, Nucl. Phys. **B 593** (2001) 562.

[25] W. Chen, G. W. Semenoff, Y.-S. Wu, *Two-loop analysis of non-Abelian Chern-Simons theory*, Phys. Rev. **D 46** (1992) 5521.

[26] C. Chou, V. P. Nair, A. P. Polychronakos,, *On the electromagnetic interactions of anyons*, Phys. Lett. **B 304** (1993) 105.

[27] I. Chepelev, R. Roiban, *Renormalization of quantum field theories on noncommutative R^d, I. scalars*, J. High Energy Phys. **0005** (2000) 037; I. Chepelev, R. Roiban, *Convergence heorem for noncommutative Feynman graphs and renormalization*, J. High Energy Phys. **0103** (2001) 001.

[28] G. H. Chen, Y. S. Wu, *One-loop shift in noncommutative Chern-Simons coupling*, Nucl. Phys. **B 593** (2001) 562.

[29] S. Coleman, B. Hill, *No more corrections to the topological mass term in QED in three dimensions*, Phys. Lett. **B 159** (1985) 184.

[30] C.-S. Chu, *Induced Chern-Simons and WZW action in noncommutative spacetime*, Nucl. Phys. **B 580** (2000) 352.

[31] A. Das, S. Perez, *Anomalous magnetic moment of electron in Chern-Simons QED*, Phys. Lett. **B 581** (2004) 182.

[32] A. Das, M. M. Sheikh-Jabbari, *Absence of higher order corrections to concommutative Chern-Simons coupling*, J. High Energy Phys. **0106** (2001) 028.

[33] A. A. Deriglazov, *Poincare covariant mechanics on noncommutative space*, J. High Energy Phys. **0303** (2003) 021.

[34] S. Deser, R. Jackiw, S. Templeton *Topologically massive gauge theories*, Ann. Phys. **140** (1982) 372; S. Deser, R. Jackiw, S. Templeton, *Three-dimensional massive gauge theories*, Phys. Rev. Lett. **48** (1982) 975.

[35] P. A. M. Dirac, Principles of Quantum Mechanics, Oxford University Press, 1947.

[36] A. E. F. Djemai, *On noncommutative classical mechanics*, `hep-th/0309034`.

[37] G. V. Dunne, *Aspects of Chern-Simons theory*, `hep-th/9902115`; A. Khare, *Fractional statistics and Chern-Simons field theory in 2+1 dimenions*, `hep-th/9908027`.

[38] A. Erdélyi, Higher Transcendental Functions, Vol I, McGraw-Hill, 1953.

[39] T. Filk, *Divergences in a field theory on quantum space*, Phys. Lett. **B 376** (1996) 53.

[40] Z. Fang, et. al. *The anomalous Hall effect and magnetic monopoles in momentum space*, Nature. **302** (2003) 92.

[41] G. Gat, R. Ray, *Anomalous magnetic moment of anyons*, Phys. Lett. **B 340** (1994) 162.

[42] Y. Georgelin, J. C. Wallet, *Anomalous magnetic moment for scalars and spinors in Maxwell-Chern-Simons electrodynamics* Phys. Rev. **D 50** (1994) 6610.

[43] S. Ghosh, *Noncommutativity in Maxwell-Chern-Simons-Matter theory simulates Pauli magnetic coupling*, Mod. Phys. Lett. **A 20** (2005) 1227.

[44] H. O. Girotti, *Noncommutative quantum mechanics*, Am. J. Phys. **72** (2004) 608.

[45] G. Giavarini, C. P. Martin, F. Ruiz Ruiz, *Chern-Simons as the large-mass limit of topologically massive Yang-Mills theory* Nucl. Phys. **B 381** (1992) 222.

[46] J. Gomis, T. Mehen, *Space-Time noncommutative field theories and unitarity*, Nucl. Phys. **B 591** (2000) 265; O. Aharony, J. Gomis, T. Mehen, *On theories with light-like noncommutativity*, J. High Energy Phys. **0009** (2000) 023; N. Seiberg, L. Susskind, N. Toumbas, *Strings in background electric field, space/time noncommutativity and a new noncritical string theory*, J. High Energy Phys. **06** (2000) 021; L. Alvarez Gaumé, J. L. F. Barbón, R. Zwicky, *Remarks on time-space noncommutative field theories*, J. High Energy Phys. **0105** (2001) 057; A. Bassetto, L. Griguolo, G. Nardelli, F. Vian, *On the unitarity of quantum gauge theories on non-commutative spaces*, J. High Energy Phys. **0107** (2001) 008.

[47] N. Grandi, G. A. Silva, *Chern-Simons action in noncommutative space*, Phys. Lett. **B 507** (2001) 345.

[48] H. J. Groenewold, *On the principles of elementary quantum mechanics*, Physica **12** (1946) 405.

[49] H. Grosse, T. Krajewski, R. Wulkenhaar, *Renormalization of noncommutative Yang-Mills theories: A simple example*, `hep-th/0001182`.

[50] S. Gubser, S. L. Sondhi, *Phase structure of non-commutative scalar field theories*, Nucl. Phys. **B 605** (2001) 395.

[51] C. R. Hagen, P. Panigrahi, S. Ramaswamy, *Still more corrections to the topological mass*, Phys. Rev. Lett. **61** (1988) 389.

[52] M. Hayakawa, *Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on R**4*, hep-th/9912167 and *Perturbative analysis on infrared aspects of noncommutative QED on R**4*, Phys. Lett. **B 478** (2000) 394.

[53] S. V. Hellerman, M. V. Raamsdonk, *Quantum Hall physics equals noncommutative field theory?*, J. High Energy Phys. **10** (2001) 039.

[54] E. Iltan, *The noncommutative effects on the dipole moments of Fermions in the standard model*, J. High Energy Phys. **0305** (2003) 065.

[55] R. Jackiw, *Physical instances of noncommuting coordinates*, Nucl. Phys. Proc. Suppl. **108** (2002) 30; R. J. Szabo, *Magnetic backgrounds and noncommutative field theory*, Int. J. Mod. Phys. **A 19** (2004) 1837.

[56] R. Jackiw, S.-Y. Pi, *Covariant coordinate transformations on noncommutative space*, Phys. Rev. Lett. **88** (2002) 111603.

[57] B. Jurcǒ S. Schraml, P. Schupp, J. Wess, *Enveloping algebra-valued gauge transformations for non-Abelian gauge groups on noncommutative spaces*, Eur. Phys. J. **C 17** (2000) 521; B. Jurcǒ, L. Möller, S. Schraml, P. Schupp, J. Wess, *Construction of non-Abelian gauge theories on noncommutative spaces*, Eur. Phys. J. **C 21** (2001) 383; B. Jurcǒ, P. Schupp, J. Wess, *Nonabelian noncommutative gauge fields and Seiberg-Witten map*, Mod. Phys. Lett. **A 16** (2001) 343; B. Jurcǒ, P. Schupp, J. Wess, *Nonabelian noncommutative gauge theory via noncommutative extra dimensions*, Nucl. Phys. **B 604** (2001) 148.

[58] K. Kaminsky, Y. Okawa, H. Ooguri, *Quantum aspects of Seiberg-Witten map in noncommutative Chern-Simons theory*, Nucl. Phys. **B 663** (2003) 33; K. Kaminsky, *On the quantum equivalence of commutative and noncommutative Chern-Simons theories at higher orders*, Nucl. Phys. **B 679** (2004) 189.

[59] H. Kluberg-Stern, J. B. Zuber, *Ward identities and some clues to the renormalization of gauge-invariant operators*, Phys. Rev. **D 12** (1975) 467.

[60] I. I. Kogan, *Induced magnetic moment for anyons*, Phys. Lett. **B 262** (1991) 83; J. Stern, *Topological action at a distance and the magnetic moment of point-like anyons*, Phys. Lett. **B 265**, (1991) 119.

[61] I. I. Kogan, G. W. Semenoff, *Fractional spin, magnetic moment and the short range interactions of anyons*, Nucl. Phys. **B 368**, (1992) 718.

[62] J. Madore, S. Schraml, P. Schupp, J. Wess, *Gauge theory on noncommutative spaces*, Eur. Phys. J. **C 16** (2000) 161.

[63] C. P. Martin, *Dimensional regularisation of Chern-Simons field-theory*, Phys. Lett. **B 241** (1990) 513.

[64] C. P. Martin, *Computing noncommutative Chern-Simons theory radiative corrections on the back of an envelope*, Phys. Lett. **B 515** (2001) 185.

[65] C. P. Martin, *The BRS invariance of noncommutative U(N) Yang-Mills theory at the one-loop level*, Nucl. Phys. **B 598** (2001) 348.

[66] K. Matsubara, *Restrictions on gauge groups in noncommutative gauge theory*, hep-th/0003294.

[67] A. Matusis, L. Susskind, N. Toumbas, *The IR/UV connection in the non-commutative gauge theories*, J. High Energy Phys. **0012** (2000) 002.

[68] S. Minwalla, M. V. Raamsdonk, N. Seiberg, *Noncommutative perturbative dynamics*, J. High Energy Phys. **02** (2000) 020; M. V. Raamsdonk, N. Seiberg, *Comments on noncommutative perturbative dynamics*, J. High Energy Phys. **03** (2000) 035.

[69] A. Mishra, H. Mishra, *Chiral symmetry breaking, color superconductivity, and color neutral quark matter: A variational approach*, Phys. Rev. **D 69** (2004) 014014.

[70] J. E. Moyal, *Quantum mechanics as a statistical theory*, Proc. Cambridge Phil. Soc. **45** (1949) 99.

[71] P. K. Panigrahi, S. Roy, W. Scherer, *Rotational anomaly and fractional spin in the gauged CP^1 nonlinear sigma model with the Chern-Simons term*, Phys. Rev. Lett. **61** (1988) 2827.

[72] P. K. Panigrahi, R. Ray, B. Sakita, *Effective Lagrangian for a system of nonrelativistic fermions in 2+1 dimensions coupled to an electromagnetic field: Application to anyonic superconductors*, Phys. Rev. **B 42** (1990) 4036 and references therein.

[73] T. Papenbrock, G. F. Bertsch, *Pairing in low-density Fermi gases*, Phys. Rev. **C 59** (1999) 2052.

[74] R. D. Pisarski, S. Rao, *Topologically massive chromodynamics in the perturbative regime*, Phys. Rev. **D 32** (1985) 2081.

[75] A. M. Polyakov, *Fermi-Bose transmutation induced by gauge fields*, Mod. Phys. Lett. **A 3** (1988) 325.

[76] M. E. Peskin, D. V. Schroeder, An introduction to quantum field theory, Addison-Wesley 1995.

[77] A. N. Redlich, *Parity violation and gauge noninvariance of the effective gauge field action in three dimensions*, Phys. Rev. **D 29** (1984) 2366; A. N. Redlich, *Gauge noninvariance and parity nonconservation of three-dimensional Fermions*, Phys. Rev. Lett. **52** (1984) 18.

[78] I. F. Riad, M. M. Sheikh-Jabbari, *Noncommutative QED and anomalous dipole moments*, J. High Energy Phys. **08** (2000) 045.

[79] J. M. Romero, J. A. Santiago, J. D. Vergara, *Newton's second law in a noncommutative space*, Phys. Lett. **A 310** (2003) 9.

[80] J. M. Romero, J. D. Vergara, *The Kepler problem and noncommutativity*, Mod. Phys. Lett. **A 18** (2003) 1673.

[81] M. Saroush, *BRST quantization of noncommutative gauge theories*, Phys. Rev. **D 67** (2003) 105005.

[82] J. Schonfeld, *A mass term for three dimensional gauge fields*, Nucl. Phys. **B**, 185 (1981) 157.

[83] N. Seiberg, E. Witten, *String theory and noncommutative geometry*, J. High Energy Phys. **9901** (1999) 032.

[84] G. W. Semenoff, *Canonical quantum field theory with exotic statistics*, Phys. Rev. Lett. **61** (1998) 517.

[85] S. Ghosh, *Pauli term, anyons, "Cooper Pair", ... or noncommutative Maxwell-Chern-Simons*, hep-th/0407086.

[86] H. S. Snyder, *Quantized spacetime*, Phys. Rev. **71** (1947) 38; *The electromagnetic field in quantized spacetime*, Phys. Rev. **72** (1947) 68.

[87] M. Stone, Quantum Hall effect, World Scientific 1992.

[88] L. Susskind, *The quantum Hall fluid and noncommutative Chern-Simons theory*, hep-th/0101029; V. P. Nair, A. P. Polychronakos, *On level quantization for the noncommutative Chern-Simons Theory*, Phys. Rev. Lett. **87** (2001) 030403; B. Morariu, A. P. Polychronakos, *Finite noncommutative Chern-Simons with a Wilson line and the quantum Hall effect*, J. High Energy Phys. **0107** (2001) 006.

[89] C. H. Tze, *Manifold splitting regularization, self-linking, twisting, writhing numbers of space-time ribbons and Polyakov's Fermi-Bose transmutations*, Int. J. Mod. Phys. **A 3** (1988) 1959.

[90] F. Wilczek, Fractional statistics and anyon superconductivity, World Scientific, 1990.

[91] E. Witten, *Topological quantum field theory*, Comm. Math. Phys. **117** (1988) 353; E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989) 351.