NONLINEAR DISPERSIVE WAVES AND MODULATIONAL INSTABILITIES IN PLASMAS

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AHMEDABAD

Śreyo hi jñānam abhyāsāj
jñānād dhyānaṁ viśiṣyate
dhyānāt karmaphalatyāgas
tyāgāc chāntir anantaram

Better indeed is knowledge than the practice (of concentration); better than knowledge is meditation, better than meditation is the renunciation of the fruit of action;
on renunciation (follows) immediately peace.

– Bhagavadgita
Chapter XII.12

To the memory of my father

ADHIKARIMAYUM AMUBA SHARMA

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C E R T I F I C A T E

I hereby declare that the work presented in this thesis is original and has not formed the basis for the award of any degree or diploma by any University or Institution.

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A B S T R A C T

A study of the nonlinear dispersive properties and the turbulent behaviour of plasmas is presented in this thesis. The presence of a small fraction of cold electrons in a plasma is found to affect the spectrum of modulationally unstable Ion-Acoustic Waves. The envelope hole state of the Ion-Acoustic waves is found to be broadened by the presence of random inhomogeneities. In a collisional plasma, obliquely propagating Langmuir waves are modulationally stable and the angle between the directions of propagation and modulation affects the envelope hole state. The plane wave solutions of the Sine-Gordon equation are modulationally unstable, whereas those of the Nonlinear Klein-Gordon equation can be stable or unstable depending on the sign of the cubic nonlinear term. The stability of a generalized nonlinear dispersive wave equation (Hirota equation) and the Boussinesq equation against long-wave modulations and consequent envelope states are investigated. The saturation of the hot beam-plasma instability due to turbulent diffusion and the effect of particle thermal velocities on the nonlinear saturation and energy balance are discussed.

CHAPTER I

INTRODUCTION

The collective oscillations in a plasma make the study of plasmas a fascinating and rich field. In general plasma waves are dispersive in nature. By definition, a dispersive wave of frequency ω and wavenumber k satisfies the condition, $d^2\omega/dk^2 \neq 0$. In general ω can be complex with an imaginary part ω_i that is small compared with the real part ω_r. However, for our work, we have followed the definition of dispersive waves adopted by Whitham (1974), namely

$$\frac{d^2\omega}{dk^2} = \frac{dV_g}{dk} \neq 0; \tag{1.1}$$

with ω real. In Eq.(1.1) $V_g = d\omega/dk$ is the group velocity of the waves. The phase velocities of the dispersive waves depend on the wavenumbers and thus disperse the medium. Also the definition (1.1) ensures that the group velocity V_g is not a constant but a function of the wave-number. The electron plasma or Langmuir waves have the frequencies given by

$$\omega^2 = \omega_p^2 + \frac{3k^2 T_e}{m} , \tag{1.2}$$

where $\omega_p = (4\pi n_o e^2/m)^{1/2}$ is the plasma frequency of electrons of charge e, mass m, in a plasma of average number density n_o, and T_e is the average kinetic energy of the electrons. The ion-acoustic waves have the frequency given by

$$\omega^2 = \frac{k^2 C_s^2}{1 + k^2 \lambda_D^2} , \tag{1.3}$$

where $C_s = (T_e + T_i)^{1/2}/(m + M)^{1/2}$ is the ion-acoustic velocity; T_i is the average kinetic energy of the ions, M the ion mass, and $\lambda_D = (T_e/4\pi n_o e^2)^{1/2}$ is the Debye length. From Eqs. (1.1) - (1.3), it is clear that both the Langmuir and the ion-acoustic waves are dispersive.

Nonlinearity is an inherent feature of plasmas and hence the waves in it, whether in equilibrium or growing, are essentially nonlinear. In an equilibrium plasma the nonlinearity is exhibited in the form of wave steepening and overtaking, whereas in a growing wave the nonlinearity due to the increasing amplitude of the wave is more dominant. The nonlinearity of an equilibrium plasma in one dimension is best illustrated by the equation of motion,

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = F \, , \qquad (1.4)$$

where v is the velocity of the fluid element and F is the self-consistent field. If v is linearized around a constant value v_o, Eq. (1.4) in absence of F has solutions of the form, $v_1 = v_1\,(x - v_o t)$. Thus a sinusoidal perturbation propagates without any change in form. However, the non-linear equation has implicit solutions of the form $v = \mathbf{v}(x-v(x,t)\,t,\ 0)$; so that if the perturbation is initially sinusoidal, the crest moves faster than the trough and this leads to the steepening of those parts of the waveform for which $\partial v/\partial x < 0$. As the wave propagates it steepens more and more, leading eventually to wave breaking or overtaking. In the presence of the field F in Eq. (1.4), the wave breaking may not occur because the nonlinearity can be balanced by a suitable choice of F. For example in the case of ion-acoustic waves using appropriate spacetime scales, Eq. (1.4) can be reduced to (Washimi and Taniuti 1966)

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\partial^3 v}{\partial x^3} = 0. \qquad (1.5)$$

This is the well known Korteweg-de Vries (KdV) equation which was originally derived for the propagation of long waves in water of relatively shallow depth. (Korteweg and de Vries 1895). This equation preserves both nonlinearity and dispersion, and is one of the important equations that describe nonlinear dispersive properties of physical systems. The third space derivative term in Eq. (1.5) represents dispersion and as the wave steepens this term becomes more important so that the nonlinearity can be balanced by the dispersion. Indeed such a balance does occur and consequently Eq. (1.5) has, in a frame moving with speed U, the localized stationary solutions,

$$v = 3U \operatorname{sech}^2\{(U/2)^{1/2}(x - Ut)\}. \qquad (1.6)$$

This localized hump has been found to be stable against perturbations and collisions amongst themselves. Consequently, they have been given the name solitons (Zabusky and Kruskal 1965); this is a property of nonlinear dispersive media. It may be noted that the speed of the soliton is directly proportional to its amplitude whereas the width is inversely proportional to the square root of the amplitude.

In a nonlinear dispersive medium the study of the envelope of waves is important because in such a medium the amplitude of a plane wave, as it propagates, does not remain constant but changes slowly due to the nonlinearity and the dispersion. The slow variations of the amplitude a of the plane wave

$$\phi = a \exp\{i(kx - \omega t)\} + c.c., \tag{1.7}$$

in a nonlinear dispersive medium is described by the equation (Karpman 1967, Karpman and Kruskal 1969)

$$i\frac{\partial a}{\partial \tau} + p\frac{\partial^2 a}{\partial \xi^2} + q|a|^2 a + r a = 0, \tag{1.8}$$

where $\xi = \epsilon\,(x - V_g t)$, $\tau = \epsilon^2 t$. $p = \frac{1}{2}(dV_g/dk)$, q and r in Eq. (1.8) are functions of k and ω, and ϵ is a small expansion parameter appropriate for the system. Because of its close similarity with the Schrödinger equation of quantum mechanics, Eq. (1.8) is generally known as the nonlinear Schrödinger (NS) equation.

The plane waves represented by Eq. (1.7) can be unstable against longwavelength perturbations. This instability is called the modulational instability (Lighthill 1965). The physical origin of this instability can be discussed either by using the analogy of Eq. (1.8) with the Schrödinger equation of quantum mechanics, or through the wave kinetics of the phenomenon; these two are discussed below.

On dividing Eq. (1.8) by p and then defining a proper time variable, the resulting equation can be looked upon as the Schrödinger equation for quasiparticles with the wave-function 'a', in a self-generated potential of strength $q|a|^2/p$. If $pq > 0$, this potential is attractive and these quasiparticles can be 'trapped' in it. This will increase the quasiparticle density $|a|^2$ and hence the strength of the potential. Consequently more quasiparticles will be trapped-thus leading to instability. Because of this trapping the modulational instability is also known as the self-trapping instability (Hasegawa 1970, 1971 and 1975). If $pq < 0$, the equivalent potential is repulsive and the system is stable.

To discuss the wave kinetic picture of the modulational instability, let us consider the plane waves in the medium to be described by the nonlinear dispersion relation,

$$\omega = \omega\left(k, |a|^2\right). \tag{1.9}$$

On Taylor expanding about k_o and $|a_o|^2$ and on retaining the lowest order dispersive and nonlinear terms, we get,

$$\omega = \omega_o + (k-k_o)\frac{\partial \omega}{\partial k_o} + \frac{1}{2}(k-k_o)^2\frac{\partial^2 \omega}{\partial k_o^2} + \left(|a|^2 - |a_o|^2\right)\frac{\partial \omega}{\partial |a_o|^2}, \tag{1.10}$$

where $\omega_o = \omega(k_o)$, $\dfrac{\partial \omega}{\partial k_o} = \left(\dfrac{\partial \omega}{\partial k}\right)_{k=k_o}$,

$$\frac{\partial^2 \omega}{\partial k_o^2} = \left(\frac{\partial^2 \omega}{\partial k^2}\right)_{k=k_o} \text{ and } \frac{\partial \omega}{\partial |a_o|^2} = \left(\frac{\partial \omega}{\partial |a|^2}\right)_{|a|^2=|a_o|^2}.$$

Eq. (1.10) can be put in the operator form by replacing $(k-k_o)$ by $-i\partial/\partial x$ and $(\omega-\omega_o)$ by $i\partial/\partial\tau$. Then operating on 'a' from the left and transforming to a moving frame $\xi = x - V_g\tau$, we get the NS equation

$$i\frac{\partial a}{\partial \tau} + \frac{1}{2}\frac{dV_g}{dk}\frac{\partial^2 a}{\partial \xi^2} - \frac{\partial \omega}{\partial |a_o|^2}\left(|a|^2 - |a_o|^2\right)a = 0 \tag{1.11}$$

A comparison of Eq. (1.11) with Eq. (1.8) shows that the coefficient q of the latter corresponds to the nonlinear frequency shift, $-\partial\omega/\partial|a_o|^2$. Eq. (1.10), which may be written as

$$\omega = \omega(k) - q\left(|a|^2 - |a_o|^2\right), \tag{1.12}$$

shows that the wave phase velocity is proportional to the amplitude. If the plane waves are subjected to a longwave perturbation then for $q < 0$ the waves falling in the crest of the perturbation will move faster than the rest. This will lead to wave compression and energy concentration in front of the crest. The local wave number in this region will consequently increase. Now, the group velocity represents the velocity with which energy is transported in the medium. Therefore if $p < 0$, the energy concentrated in front of the crest will be preferentially deposited on the crest because of the wave number dependence of the group velocity, and this leads to the instability. On the other hand if $p > 0$, the energy will be transported into the depleted region behind the crest. Similar considerations, for thecase $q > 0$, show that the instability occurs only for $p > 0$. Thus the general criterion for modulational instability is $pq > 0$ (Kadomtsev and Karpman 1971; Nishikawa and Liu 1976).

The linear growth rate of this instability can be obtained by expressing the complex amplitude a in terms of two real functions ρ and σ, as

$$a = \rho^{1/2}(\xi,\tau)\exp\{i\sigma(\xi,\tau)\}. \tag{1.13}$$

On further taking

$$\begin{pmatrix}\rho\\ \sigma\end{pmatrix} = \begin{pmatrix}\rho_0\\ \sigma_0\end{pmatrix} + \begin{pmatrix}\rho_1\\ \sigma_1\end{pmatrix}\exp\{i(K\xi-\Omega\tau)\}, \tag{1.14}$$

where Ω is the frequency and K the wavenumber of the perturbation modulating the constant amplitude ρ_0 and the phase σ_0, Eq. (1.8) leads to the dispersion relation

$$\Omega^2 = (pK^2 - q\rho_0)^2 - q^2\rho_0^2. \tag{1.15}$$

From this relation, it is clear that for $pq > 0$, a perturbation with $K < (2q\rho_0/p)^{1/2}$ is unstable. The maximum growth rate, which occurs at $K = (q\rho_0/p)^{1/2}$, is $q\rho_0$.

For the modulationally unstable case, viz., the $pq > 0$ case, the localized stationary solution of Eq.(1.8) is found to be

$$\rho = \rho_0 \operatorname{sech}^2\{(q\rho_0/2p)^{1/2}\xi\}. \tag{1.16}$$

This solution represents the envelope of the waves and propagates with the group velocity V_g. Like the soliton, Eq. (1.16) also represents a localized hump and is called on envelope soliton (Karpman 1967). The amplitude and the speed of the envelope soliton are not related though the width is inversely proportional to the square root of the amplitude, as in the case of the soliton. Some authors refer to the envelope solitons as compression-envelope-solitons.

For the modulationally stable case, i.e., $pq < 0$, the corresponding stationary solution of Eq. (1.8) is

$$\rho = \rho_1[1 - \tilde{a}^2 \operatorname{sech}^2\{(|pq|\rho_1/2p^2)\tilde{a}\xi\}], \tag{1.17}$$

where ρ_1 is a constant representing the asymtotic value of ρ and $\tilde{a}$ is the depth of modulation. Eq. (1.17) represents a depleted region propagating with the velocity V_g and is called the envelope hole (Hasegawa 1975, Karpman 1975a), which is alternatively known as rarefaction-envelope-soliton in contrast to compression-envelope soliton. When the depth of modulation a is unity the envelope hole becomes

$$\rho = \rho_1 \tanh^2\{(|pq|\rho_1/2p^2)\xi\}. \tag{1.18}$$

This may be called the envelope shock.

From the above discussion it is clear that to study the envelope properties of a nonlinear dispersive medium, it is appropriate to first derive the nonlinear Schrödinger equation, Eq. (1.8), for the amplitudes of the plane waves in the medium. Eq. (1.8) has been derived by assuming that the amplitudes vary slowly on distances of the order of wavelength and for times of the order of the oscillation period. This assumption is the central point of the various schemes for studying the envelope properties of waves and these schemes are as follows:

i) The simple expansion around the linear values retaining the nonlinearity and the dispersion to the lowest order.(Karpman 1967 and 1975a, Karpman and Kruskal 1969, Benney and Newell 1967).

ii) The general variational approach of Whitham (Whitham 1965 and 1974). In this scheme the slowly varying wave trains, in a continuous medium, are treated in a manner analogous to the problems treated by the theory of adiabatic invariants in classical mechanics (Landau and Lifshitz 1969).

iii) The reductive perturbation scheme, which is essentially a perturbation method with appropriate scaling of the space and the time (Taniuti and Yajima 1969). In this procedure the scaling of the space-time variables are decided a priori,

iv) The Krylov-Bogoliubov-Mitropolsky (KBM) multiple space-time method, which is also a perturbation theoretic scheme in which the secularities, due to the fast scale variations during the slow scales are systematically annihilated by imposing appropriate conditions (Bogoliubov and Mitropolsky 1961). In this method no prior scaling of the space-time variables is necessary.

The major portion of the present thesis deals with the study of envelope properties of some physical states of equilibrium plasmas, and of some plasma-like media. As pointed out earlier, these properties are best studied by deriving the nonlinear Schrödinger equation for the system. In Chapters II - VI, the KBM method has been used because of its mathematical elegance and advantage.

The properties of the envelope of ion-acoustic waves in a two-electron-temperature plasma are investigated in Chapter II. In a plasma with the electrons having a single temperature the ion-acoustic waves with $k\lambda_D > 1.47$ are modulationally unstable (Kakutani and Sugimoto 1974). It is shown here that the presence of a low-temperature electron component strongly modifies this spectrum. Even a small fraction of cold electrons can quench the modulational instability. For different values of the ratio between the densities of the two electron components and their temperatures, the critical values of k which marks the transition between modulational instability and stability is obtained.

In Chapter III the ion-acoustic waves in the presence of random density inhomogeneities are studied. The density fluctuations increase the dispersion and the dissipation of these waves without altering the nonlinearity. For Gaussian density inhomogeneities, we find that the dissipation is small compared with the dispersion. A study of the properly scaled equation governing the system shows that the stability of the system is not affected by the presence of the density fluctuations but the stationary states of the system, i.e., the envelope holes, respond to the increased dispersion by increasing their width.

A study of the Langmuir waves in a collisional plasma is presented in Chapter IV. The collision frequency is properly scaled and the NS equation is derived from the plasma hydrodynamic equations. The Langmuir waves in a two-dimensional plane are modulated at an angle to its direction of propagation. It is found that these waves are stable against oblique modulation. The collisions damp the wave packets of Langmuir waves.

Besides the plasmas, a large number of other physical systems also are nonlinear and dispersive. In the fields of nonlinear optics, theory of surfaces, hydrodynamics, theory of solids, superconductivity, nonlinear quantum field theories, etc., the study of nonlinear dispersive waves is of importance. The Sine-Gordon (SG) equation,

$$\Phi_{tt} - \Phi_{xx} + \sin\Phi = 0, \tag{1.19}$$

describes a number of these systems (Barone et al. 1971). In Chapter V the plane wave solutions of the SG equation is shown to be modulationally unstable. The nonlinear Klein-Gordon (NKG) equation,

$$\phi_{tt} - \phi_{xx} + \phi + \alpha\phi^3 = 0, \tag{1.20}$$

where α is a constant, is also studied in this chapter. The plane wave solutions of the Eq. (1.20) are found to be modulationally unstable for $\alpha < 0$ and modulationally stable for $\alpha > 0$. The results presented in this chapter have interesting implications in the nonlinear field theory of particles (Sharma and Buti 1976a).

The propagation of waves in a one-dimensional nonlinear lattice, e.g., the continuum approximation of the Fermi-Pasta-Ulam problem (Toda 1975) and waves in shallow water under gravity propagating in both directions, are described by the Zabusky-Boussinesq equation,

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - 6\frac{\partial^2 \phi^2}{\partial x^2} - \frac{\partial^4 \phi}{\partial x^4} = 0. \tag{1.21}$$

Moroever, a nonlinear Schrödinger equation and a modified Korteweg-de Vries equation can be unified into a generalized nonlinear dispersive equation (Hirota 1973) of the form,

$$i\frac{\partial \chi}{\partial t} + i3\alpha|\chi|^2\frac{\partial \chi}{\partial x} + \beta\frac{\partial^2 \chi}{\partial x^2} + i\gamma\frac{\partial^3 \chi}{\partial x^3} + \delta|\chi|^2\chi = 0. \tag{1.22}$$

The envelope properties of these two equations have been investigated in Chapter VI. The plane wave solutions of Eq. (1.21) are found to be modulationally unstable for $k > 0.866$. However, the plane wave solutions of Eq. (1.22) can be modulationally stable or unstable depending on the coefficients of the different terms. Both these equations admit envelope soliton or envelope hole solutions. It is well known that the nonlinear Schrödinger equation corresponding to the stable situation can be converted into the Korteweg-de Vries equation. Here it has been shown that the latter can also be converted into the former.

The nonlinearity discussed in Chapter II - VI is the inherent nonlinearity of a plasma in equilibrium. However, the study of nonequilibrium plasmas, i.e., plasmas with unstable waves present in them, are also of great interest. In such plasmas, the waves initially have small amplitudes and hence the system behaves linearly. As the unstable waves grow, their amplitudes are no longer small and hence the system goes over to the nonlinear regime. In the linear regime, the orbits of the plasma particles are not influenced by the waves but as their amplitudes grow, the waves begin to influence the particle orbits. Depending on the spectrum of the growing waves, this leads to two physically distinct processes, viz., particle trapping and diffusion in velocity space (Dupree 1966), as discussed below.

If the spectrum of the waves is narrow the particles can be trapped in the potential of the wave (Bohm and Gross 1949). If an electron is trapped in a wave with amplitude E and wave number k, the trapping time, defined by $\tau_{tr} = (m/ekE)^{1/2}$, characterizes the trapping process. Physically τ_{tr} is the time during which a particle trapped near the bottom of the potential well of the wave will bounce up and down. The corresponding frequency $\omega_B = \tau_{tr}^{-1}$ is called the bounce frequency. Also τ_{tr} signifies the time for which the linearization is valid (Dawson 1961). In case of the growing waves having a broad spectrum, their phase velocities will have a spread $\Delta(\omega/k)$ and the fluctuations have a autocorrelation time τ_{ac} defined as (Davidson 1972) $\tau_{ac} = |k\,\Delta(\omega/k)|^{-1}$. This time scale physically represents the time for which a wave retains its individual waveform. Thus for the particles to be trapped by the wave the necessary condition is, $\tau_{ac} >> \tau_{tr}$. Under this condition, the width of the spectrum can be neglected and thus the trapping process is essentially a coherent process. An equilibrium between the wave and the trapped particles follow the trapping process, thus leading to the saturation of the instability.

For a broad band of growing waves we can have $\tau_{ac} << \tau_{tr}$; in which case a number of waves, around a central wave grow and each of them perturbs the orbits of the particles. The interaction between the waves and the particles in this case is incoherent. This stochasticity is the basis of the Perturbed Orbit Formalism discussed by Dupree (1966), Weinstock (1969), Rudakov and Tsytovich (1971), Benford and Thomson (1972), Cook and Taylor (1973) and Misguich and Balescu (1975). In this formalism, the evolution of the plasma particles, as they interact with the waves, is represented by a diffusion equation in velocity space. This diffusion which arises because of wave-particle interactions, brings about the saturation of the growth and consequent equilibrium sharing of the available free energy.

The beam-plasma instability is a simple but rather interesting phenomenon occuring in plasmas. The waves in this case grow due to the energy fed by the beam particles. A study of the nonlinear saturation of this instability due to the above diffusion process when the particles have finite thermal velocities is presented in Chapter VII. It is found that the particle thermal velocities influence the saturation level and the subsequent energy balance appreciably(Sharma and Buti 1976b)

CHAPTER II

MODULATIONAL INSTABILITY OF ION-ACOUSTIC WAVES IN A TWO-ELECTRON TEMPERATURE PLASMA

II.1 INTRODUCTION

The ion-acoustic waves (IAW) in a plasma arise due to the restoring action of the electron thermal pressure on ion density perturbations. The properties of these waves are therefore functions of electron temperature. In the linear regime, the presence of a small fraction of cold electrons in a plasma of hot electrons and cold ions is found to affect the IAW characteristics appreciably (Jones et al. 1975). Recently Goswami and Buti (1976) have shown that due to decreased dispersion, which results due to the decrease in the Debye length in such a plasma, the ion-acoustic solitons have increased amplitude for a given soliton width.

In a plasma composed of cold ions and hot isothermal electrons, if a small fraction of cold isothermal electrons is introduced, the electron velocity distribution can be represented by a superposition of two Maxwellians. This plasma is referred to as the two-electron temperature (TET) plasma. Such plasmas are not uncommon. The plasma produced by hot cathode discharge are exactly TET plasmas (Oleson and Found 1949, Jones et al. 1975). The plasmas of thermonuclear interest are generally turbulent and have high energy tails, e.g., the interaction of charged particles with localized fields give rise to highly populated super-thermal tails (Morales and Lee 1974). The nonlinear beam-plasma interaction results in high-energy tails. Computer simulations also show the formation of high energy tails (Sudan 1973). The plasma produced by the radio-frequency breakdown in the ELMO confinement device (Krall and Trivelpiece 1973) is also a TET plasma.

The envelope properties of the IAW in a TET plasma are discussed in this chapter. The Krylov-Bogoliubov-Mitropolsky (KBM) perturbation method is used to derive the nonlinear Schrödinger (NS) equation governing the envelope of these waves. The modulational stability of the IAW for different ratios of the densities of the cold and hot electrons and also of their temperatures is studied. The envelope solutions for the different physical states of the plasma are obtained.

II.2 Perturbation Scheme

Consider a one-dimensional plasma in which the electrons are divided into two groups-the hot component with density n_h and temperature T_h, and the cold component with density n_l and temperature T_l. We assume the electrons to be isothermal and neglect the effect of electron inertia. The propagation of IAW in this TET plasma can be described by the following fluid equations:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0, \tag{2.1a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \frac{e}{M}E = 0, \tag{2.1b}$$

$$\frac{\partial E}{\partial x} - 4\pi e(n - n_l - n_h) = 0, \tag{2.1c}$$

$$\frac{\partial n_l}{\partial x} + \frac{e}{T_l}n_l E = 0, \tag{2.1d}$$

$$\frac{\partial n_h}{\partial x} + \frac{e}{T_h}n_h E = 0, \tag{2.1e}$$

where e and M are the charge and the mass of a proton, T_l is the average kinetic energy of the cold electrons and T_h that of the hot electrons; n and E being the ion density and the electric field respectively.

The quantities describing the system, may be expanded around the unperturbed values as

$$\begin{bmatrix} E \\ u \\ n \\ n_l \\ n_h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ n_o \\ n_{lo} \\ n_{ho} \end{bmatrix} + \epsilon \begin{bmatrix} E_1 \\ u_1 \\ n_1 \\ n_{l1} \\ n_{h1} \end{bmatrix} + \epsilon^2 \begin{bmatrix} E_2 \\ u_2 \\ n_2 \\ n_{l2} \\ n_{h2} \end{bmatrix} + \ldots \qquad (2.2)$$

The charge neutrality demands that

$$n_o = n_{lo} + n_{ho}.$$

Let us take a monochromatic plane wave for E_1, i.e.,

$$E_1 = a \exp(i\psi) + \bar{a} \exp(-i\psi), \qquad (2.3)$$

where a is the amplitude, $\bar{a}$ its complex conjugate and $\psi = kx - \omega t$ is the phase; k being the wavenumber and ω the frequency. The quantities other than the zero-order quantities in the expansion (2.2) depend on x and t only through a, $\bar{a}$ and ψ . On substituting Eq. (2.2) into Eq. (2.1), we obtain the ϵ-order equations, whose solutions are

$$u_1 = \frac{ie}{M\omega}(a \exp(i\psi) - \bar{a} \exp(-i\psi)),$$

$$n_1 = \frac{ien_ok}{M\omega^2}(a \exp(i\psi) - \bar{a} \exp(-i\psi)),$$

$$n_{l1} = \frac{ien_{lo}}{T_l k}(a \exp(i\psi) - \bar{a} \exp(-i\psi)), \qquad (2.4)$$

$$n_{h1} = \frac{ien_{ho}}{T_h k}(a \exp(i\psi) - \bar{a} \exp(-i\psi)),$$

From Eqs. (2.1a) – (2.1e) of order ϵ we obtain,

$$\mathcal{L}(E_1) = 0, \tag{2.5}$$

where the operator $\mathcal{L}$ is defined as

$$\mathcal{L} \equiv k\frac{\partial^3}{\partial\psi^3} + \left\{\frac{\omega_{pi}^2 k}{\omega^2} - \frac{4\pi e^2}{k}\left(\frac{n_{l0}}{T_l} + \frac{n_{h0}}{T_h}\right)\right\}\frac{\partial}{\partial\psi}; \tag{2.6}$$

$\omega_{pi} = (4\pi n_o e^2/M)^{1/2}$ being the ion plasma frequency. If we define the effective temperature of the electrons as

$$T_{eff} = \frac{n_o T_h T_l}{n_{l0} T_h + n_{h0} T_l},$$

and the corresponding effective Debye length λ_{Deff} as

$$\lambda_{Deff} = \frac{T_{eff}}{4\pi n_o e^2},$$

then from Eqs. (2.3) and (2.5), we get the linear dispersion relation

$$D(k,\omega) \equiv -k + \frac{\omega_{pi}^2 k}{\omega^2} - \frac{1}{k\lambda_{Deff}^2} = 0. \tag{2.7}$$

The complex amplitude a is a slowly varying function of x and t through the relations

$$\begin{aligned} \frac{\partial a}{\partial t} &= \epsilon A_1(a,\bar{a}) + \epsilon^2 A_2(a,\bar{a}) + \cdots, \\ \frac{\partial a}{\partial x} &= \epsilon B_1(a,\bar{a}) + \epsilon^2 B_2(a,\bar{a}) + \cdots, \end{aligned} \tag{2.8}$$

and their complex conjugates. The functions A_1, B_1; A_2, B_2; ..

are yet unknown and are to be determined from the condition that the perturbation scheme envisaged by Eqs. (2.2) and (2.8) are free from secularities. This is the essence of the KBM method used here. This method belongs to the class of perturbation schemes based on the multiple space-time scales where the secularity arising from the fast scales are systematically annihilated (Jackson 1960, Bogoliubov and Mitropolsky 1961, Frieman 1963 and Sandri 1963).

Eqs. (2.1a) – (2.1e) to order ϵ^2, yield,

$$\mathcal{L}(E_2) = \frac{2e\omega_{pi}^2}{M}\left(\frac{3k^2}{\omega^4} - \frac{1}{k^2V^4}\right)a^2\exp(2i\psi) + \left(-\frac{\partial D}{\partial \omega}A_1 + \frac{\partial D}{\partial k}B_1\right)\exp(i\psi) + \text{c.c.} \tag{2.9}$$

where

$$V^4 = \frac{n_o T_\ell^2 T_h^2}{M^2(n_{\ell o}T_h^2 + n_{ho}T_\ell^2)} .$$

In Eq. (2.9) the terms proportional to exp $(i\psi)$ give rise to resonant secularity because of the linear dispersion relation (2.7). This resonant secularity can be annihilated by the condition

$$A_1 + V_g B_1 = 0 , \tag{2.10}$$

where

$$V_g = -\frac{\partial D/\partial k}{\partial D/\partial \omega},$$

is the group velocity of the plane waves. The secular-free solution of Eq. (2.9) is then given by

$$E_2 = \frac{i e \omega_{pi}^2}{3M}\left(\frac{3k}{\omega^4} - \frac{1}{k^3 V^4}\right) a^2 \exp(2i\Psi)$$
$$+ b(a,\bar{a}) \exp(i\Psi) + \text{c.c.} + C_1(a,\bar{a}), \tag{2.11}$$

where b and C_1 are constants with respect to Ψ. Also we have from Eq. (2.1b) to order ϵ^2,

$$\frac{\partial u_2}{\partial \Psi} = -\frac{i e^2 \omega_{pi}^2}{3M^2\omega}\left(\frac{3k}{\omega^4} - \frac{1}{k^3V^4} + \frac{3k}{\omega_{pi}^2\omega^2}\right) a^2 \exp(2i\Psi)$$
$$+ \left(-\frac{eb}{M\omega} + \frac{ieA_1}{M\omega^2}\right)\exp(i\Psi) + \text{c.c.} - \frac{eC_1}{M\omega}.$$

On integrating this equation we notice that the last term gives rise to secularity in the expression for u_2. Therefore, for the solution to be free of any secularity, we must take $C_1 = 0$; in which case, the secular-free solution is

$$u_2 = -\frac{e^2 k}{6M^2\omega}\left\{\omega_{pi}^2\left(\frac{3}{\omega^4} - \frac{1}{k^4V^4}\right) + \frac{3}{\omega^2}\right\} a^2 \exp(2i\Psi)$$
$$+ \frac{e}{M\omega}\left(ib + \frac{1}{\omega}A_1\right)\exp(i\Psi) + \text{c.c.} + C_2(a,\bar{a}). \tag{2.12}$$

Similarly we get, from Eqs. (2.1a), (2.1d) and (2.1.e) to order ϵ^2,

$$n_2 = -\frac{n_0 e^2 k^2}{6M^2\omega^2}\left\{\omega_{pi}^2\left(\frac{3}{\omega^4}-\frac{1}{k^4V^4}\right)+\frac{9}{\omega^2}\right\}a^2\exp(2i\psi)$$

$$+\frac{n_0 e k}{M\omega^2}\left(ib+\frac{2}{\omega}A_1+\frac{1}{k}B_1\right)\exp(i\psi)$$

$$+\text{c.c.}+C_3(a,\bar{a}),$$

(2.13)

$$n_{\ell 2} = -\frac{n_{\ell 0}e^2}{6}\left\{\frac{\omega_{pi}^2}{MT_\ell}\left(\frac{3}{\omega^4}-\frac{1}{k^4V^4}\right)+\frac{3}{k^2T_\ell^2}\right\}a^2\exp(2i\psi)$$

$$-\frac{e n_{\ell 0}}{T_\ell k}\left(\frac{1}{k}B_1-ib\right)\exp(i\psi)+\text{c.c.}+C_4(a,\bar{a}),$$

and

(2.14)

$$n_{h2} = -\frac{n_{h0}e^2}{6}\left\{\frac{\omega_{pi}^2}{MT_h}\left(\frac{3}{\omega^4}-\frac{1}{k^4V^4}\right)+\frac{3}{k^2T_h^2}\right\}a^2\exp(2i\psi)$$

$$-\frac{e n_{h0}}{T_h k}\left(\frac{1}{k}B_1-ib\right)\exp(i\psi)+\text{c.c.}+C_5(a,\bar{a})$$

(2.15)

The constants C_2, C_3, C_4 and C_5 are determined from the conditions for the removal of secularities in Eq. (2.1.a)-(2.1e) to order ϵ^3. These five conditions involve the four unknowns C_2, C_3, C_4 and C_5, and another constant that would occur in the expression for E_3. On eliminating the latter we are left with four equations in four unknowns. From these conditions we find

$$C_2 = \frac{1}{MV_g}\left\{-\frac{T_h}{n_{h0}}\frac{y_2}{y_1} + \frac{e^2}{M}\left(\frac{1}{\omega^2} - \frac{M}{T_h k^2}\right)\right\} a\bar{a} + C_{20}, \tag{2.16}$$

$$C_3 = -\left\{\left(1 + \frac{n_{\ell 0} T_h}{n_{h0} T_\ell}\right)\frac{y_2}{y_1} - \frac{e^2 n_{\ell 0}}{T_\ell k^2}\left(\frac{1}{T_\ell} - \frac{1}{T_h}\right)\right\} a\bar{a} + C_{30}, \tag{2.17}$$

$$C_4 = -\left\{\frac{n_{\ell 0} T_h}{n_{h0} T_\ell}\frac{y_2}{y_1} - \frac{e^2 n_{\ell 0}}{T_\ell k^2}\left(\frac{1}{T_\ell} - \frac{1}{T_h}\right)\right\} a\bar{a} + C_{40}, \tag{2.18}$$

and

$$C_5 = -\frac{y_2}{y_1} a\bar{a} + C_{50}, \tag{2.19}$$

where

$$y_1 = -MV_g^2\left(1 + \frac{n_{\ell 0} T_h}{n_{h0} T_\ell}\right) + \frac{n_0 T_h}{n_{h0}}, \tag{2.20}$$

$$y_2 = -MV_g^2 \frac{e^2 n_{\ell 0}}{T_\ell k^2}\left(\frac{1}{T_\ell} - \frac{1}{T_h}\right) + \frac{e^2 n_0}{M}\left\{\left(\frac{1}{\omega^2} - \frac{M}{T_h k^2}\right) + \frac{2kV_g}{\omega^3}\right\}. \tag{2.21}$$

C_{20}, C_{30}, C_{40} and C_{50} are independent of a, $\bar{a}$ and ψ, i.e., they are absolute constants.

From Eqs. (2.1a) – (2.1e) to order ϵ^3, we obtain,

$$\mathcal{L}(E_3) = -\frac{4\pi e}{\omega^2}\frac{\partial^2}{\partial t\partial x}(n_1u_2+n_2u_1) + \frac{4\pi e n_0}{\omega^2}\frac{\partial^2}{\partial x^2}(u_1u_2)$$

$$+\frac{4\pi e^2}{k^2}\frac{\partial}{\partial x}\left\{\left(\frac{n_{\ell 1}}{T_\ell}+\frac{n_{h1}}{T_h}\right)E_2+\left(\frac{n_{\ell 2}}{T_\ell}+\frac{n_{h2}}{T_h}\right)E_1\right\}$$

$$+\frac{1}{\epsilon}\left\{-\frac{4\pi e}{\omega^2}\frac{\partial^2 n_2}{\partial t^2}-\frac{\omega_{pi}^2}{\omega^2}\frac{\partial E_2}{\partial x}-\frac{4\pi e}{\omega^2}\frac{\partial^2}{\partial t\partial x}(n_1u_1)\right.$$

$$+\frac{4\pi e n_0}{\omega^2}\frac{\partial}{\partial x}\left(u_1\frac{\partial u_1}{\partial x}\right)+\frac{1}{k^2\lambda_{Deff}^2}\frac{\partial E_2}{\partial x}-\frac{1}{k^2}\frac{\partial^3 E_2}{\partial x^3}$$

$$\left.+\frac{4\pi e^2}{k^2}\frac{\partial}{\partial x}\left\{\left(\frac{n_{\ell 1}}{T_\ell}+\frac{n_{h1}}{T_h}\right)E_1\right\}+\frac{4\pi e}{k^2}\frac{\partial^2 n_2}{\partial x^2}\right\}$$

$$+\frac{1}{\epsilon^2}\left\{-\frac{\omega_{pi}^2}{\omega^2}\frac{\partial E_1}{\partial x}+\frac{1}{k^2\lambda_{Deff}^2}\frac{\partial E_1}{\partial x}-\frac{1}{k^2}\frac{\partial^3 E_1}{\partial x^3}\right.$$

$$\left.-\frac{4\pi e}{\omega^2}\frac{\partial^2 n_1}{\partial t^2}+\frac{4\pi e}{k^2}\frac{\partial^2 n_1}{\partial x^2}\right\}. \qquad (2.22)$$

As indicated above the terms proportional to exp $(i\psi)$ on the right hand side of Eq. (2.22) will give rise to resonant secularity in the solution for E_3. The condition for the removal of this resonant secularity is found to be

$$i(A_2+V_gB_2)+P\left(B_1\frac{\partial B_1}{\partial a}+\bar{B}_1\frac{\partial B_1}{\partial \bar{a}}\right)$$

$$+Q|a|^2a+Ra=0, \qquad (2.23)$$

where

$$P = \frac{1}{2}\frac{dV_g}{dk} = -\frac{3M^2\omega^5}{8\pi n_o e^2 T_{e\,\mathrm{eff}} k^4}, \tag{2.24}$$

$$Q = \frac{1}{\partial D/\partial\omega}\frac{e^2\omega_{pi}^2 k^3}{6M^2\omega^4}\left\{\omega_{pi}^2\left(\frac{3}{\omega^4}-\frac{1}{k^4V^4}\right)\left(3-\frac{\omega^4}{k^4V^4}\right)\right.$$

$$\left. + \frac{3M^3\omega^4}{n_o k^6}\left(\frac{n_{lo}}{T_l^3}+\frac{n_{ho}}{T_h^3}\right)+\frac{15}{\omega^2}+\frac{6M^2\omega}{e^2k}\left(\frac{y_3y_2}{y_1}+y_4\right)\right\},$$

and (2.25)

$$R = \frac{1}{\partial D/\partial\omega}\frac{\omega_{pi}^2 k^2}{\omega^3}\left\{\frac{\omega}{n_o k}C_{30}+2C_{20}-\frac{M\omega^3}{n_o k^3}\left(\frac{C_{40}}{T_l}+\frac{C_{50}}{T_h}\right)\right\} \tag{2.26}$$

with

$$y_3 = -\left(1+\frac{n_{lo}T_l}{n_{ho}T_h}\right)\frac{\omega}{n_o k} - \frac{2T_h}{Mn_{ho}V_g}+\frac{M\omega^3}{n_oT_hk^3}\left(1+\frac{n_{lo}T_h^2}{n_{ho}T_l^2}\right),$$

and (2.27)

$$y_4 = -\frac{e^2 n_{lo}}{n_o T_l}\left(\frac{1}{T_l}-\frac{1}{T_h}\right)\frac{\omega}{k^3}+\frac{2e^2}{M^2V_g}\left(\frac{1}{\omega^2}-\frac{M}{T_hk^2}\right)$$

$$+\frac{e^2Mn_{lo}}{n_oT_l^2}\left(\frac{1}{T_l}-\frac{1}{T_h}\right)\frac{\omega^3}{k^5}, \tag{2.28}$$

and y_1, y_2 are defined by Eqs. (2.20) - (2.21).

II.3 Envelope Equation

The slow variations of a with respect to space and time are governed by the conditions (2.10) and (2.23). In Eq. (2.8), which defines the quantities A_1, B_1, A_2, B_2,..., we can introduce multiple space and time scales to convert these conditions into differential equations governing the evolution of the amplitude a.

On defining the new space and time variables as

$$t_2 = \epsilon t_1 , \quad t_1 = \epsilon t ,$$
$$x_2 = \epsilon x_1 , \quad x_1 = \epsilon x , \tag{2.29}$$

we can interpret the quantities A_1, B_1, A_2, B_2 as

$$A_1 = \frac{\partial a}{\partial t_1} , \qquad B_1 = \frac{\partial a}{\partial x_1} ,$$

$$A_2 = \frac{\partial a}{\partial t_2} - \frac{A_1}{\epsilon} , \qquad B_2 = \frac{\partial a}{\partial x_2} - \frac{B_1}{\epsilon} , \tag{2.30}$$

$$A_1 \frac{\partial A_1}{\partial a} + \bar{A}_1 \frac{\partial A_1}{\partial \bar{a}} = \frac{\partial^2 a}{\partial t_1^2} , \qquad B_1 \frac{\partial B_1}{\partial a} + \bar{B}_1 \frac{\partial B_1}{\partial \bar{a}} = \frac{\partial^2 a}{\partial x_1^2} .$$

With these definitions Eq. (2.10) becomes

$$\frac{\partial a}{\partial t_1} + V_g \frac{\partial a}{\partial x_1} = 0 .$$

This equation indicates that in the slow scale t_1 and x_1, the amplitude a propagates with the group velocity V_g without any change of form. Using thedefinitions given by Eq.(2.29), Eq.(2.23) can be rewritten as

$$i\left(\frac{\partial a}{\partial t_2}+V_g\frac{\partial a}{\partial x_2}\right)+P\frac{\partial^2 a}{\partial x_1^2}+Q|a|^2a+Ra=0. \quad (2.31)$$

On using the coordinate transformation

$$\begin{aligned}\xi &= \epsilon(x-V_g t)=x_1-V_g t_1=\frac{1}{\epsilon}(x_2-V_g t_2),\\ \tau &= t_2=\epsilon t_1=\epsilon^2 t.\end{aligned} \quad (2.32)$$

Eq. (2.31) becomes

$$i\frac{\partial a}{\partial \tau}+P\frac{\partial^2 a}{\partial \xi^2}+Q|a|^2a+Ra=0, \quad (2.33)$$

which is the Nonlinear Schrödinger equation governing the envelope of the IAW.

For the sake of convenience, we now introduce dimensionless quantities; length is normalized to the effective Debye length λ_{Deff}, time to ω_{pi}^{-1}, velocities to the effective ion-acoustic velocity $C_{seff}=(T_{eff}/M)^{1/2}$, electric field by $(T_{eff}/e\,\lambda_{Deff})$ and densities by the density of the hot electron component n_{ho}. Also we define the ratios α and β as

$$\alpha=\frac{n_{lo}}{n_{ho}} \quad \text{and} \quad \beta=\frac{T_l}{T_h}$$

In the rest of the chapter all the quantities are normalized as above. Eq. (2.24) - (2.26) then become

$$P=-\frac{3\beta^2}{2(\alpha+\beta)^2}\frac{\omega^5}{k^4}, \quad (2.34)$$

$$Q = -\frac{(\alpha+\beta)\omega^3(\sigma_1+\sigma_2 k^2+\sigma_3 k^4+\sigma_4 k^6+\sigma_5 k^8+3k^{10})}{12\beta k^2(3+3k^2+k^4)}, \tag{2.35}$$

and

$$R = -\frac{1}{2}\omega C_{30} - \left\{\frac{(\alpha+\beta)}{\beta(1+\alpha)}\right\}^{1/2} k C_{20} + \frac{\beta(1+\alpha)\omega^3}{2(\alpha+\beta)k^2}\left(C_{50}+\frac{C_{40}}{\beta}\right), \tag{2.36}$$

where

$$\sigma_1 = 12 + \frac{6\alpha(1-\beta)(1+\alpha)}{\beta(\alpha+\beta)} - \frac{12(1+\alpha)(\alpha+\beta^2)}{(\alpha+\beta)^2}$$

$$+ \frac{6\mu_1}{(\alpha+\beta)^2} + \frac{3\mu_2}{\beta^2(\alpha+\beta)^3} - \frac{3(1+\alpha)\mu_3}{\beta^2(\alpha+\beta)^3},$$

$$\sigma_2 = 51 + \frac{6\alpha(1-\beta)(1+\alpha)}{\beta(\alpha+\beta)} - \frac{54(1+\alpha)(\alpha+\beta^2)}{(\alpha+\beta)^2}$$

$$+ \frac{24\mu_1}{(\alpha+\beta)^2} + \frac{9\mu_2}{\beta^2(\alpha+\beta)^3} + \frac{3(1+\alpha)^2(\alpha+\beta^2)^2}{(\alpha+\beta)^4},$$

$$\sigma_3 = 84 + \frac{36\mu_1}{(\alpha+\beta)^2} - \frac{180(1+\alpha)(\alpha+\beta^2)}{(\alpha+\beta)^2}$$

$$+ \frac{9\mu_2}{\beta^2(\alpha+\beta)^3} + \frac{(1+\alpha)^2(\alpha+\beta^2)^2}{(\alpha+\beta)^4},$$

$$\sigma_4 = 66 + \frac{24\mu_1}{(\alpha+\beta)^2} + \frac{3\mu_2}{\beta^2(\alpha+\beta)^3} - \frac{30(1+\alpha)(\alpha+\beta^2)}{(\alpha+\beta)^2},$$

$$\sigma_5 = 24 + \frac{6\mu_1}{(\alpha+\beta)^2} - \frac{6(1+\alpha)(\alpha+\beta^2)}{(\alpha+\beta)^2},$$

$$\mu_1 = 2\alpha^2\beta - 4\alpha^2 + 2\alpha\beta^2 - 2\beta^2 - 6\alpha\beta,$$

$$\mu_2 = (1+\alpha)^2(\alpha+\beta)^2(\alpha+\beta^3) - 5(\alpha+\beta)^3\beta^2$$
$$+2(1+\alpha)\beta^2(2\alpha^2\beta^2+\alpha\beta^3+3\alpha^2+6\alpha\beta^2+3\beta^3+\alpha),$$

and

$$\mu_3 = \alpha^3(1+\alpha) + 2\alpha^2(1+\alpha)\beta + \alpha(1-\alpha)\beta^2$$
$$+\alpha^2(5+\alpha)\beta^3 + 2\alpha\beta^4 + \beta^5.$$

From Eqs. (2.34) - (2.35), we get,

$$PQ = \frac{\beta\omega^8\chi}{8(\alpha+\beta)k^6(3+3k^2+k^4)}, \qquad (2.37)$$

with

$$\chi = \sigma_1 + \sigma_2 k^2 + \sigma_3 k^4 + \sigma_4 k^6 + \sigma_5 k^8 + 3k^{10}. \qquad (2.38)$$

In the limit $\alpha \to 0$, i.e., in the absence of the cold electron component, the expressions (2.34) - (2.38) reduce to those obtained by Kakutani and Sugimoto (1974).

II.4 Modulational Instability and Envelope Solutions

The NS equation, i.e., Eq. (2.33) governs the evolution of the envelope of the plane IAW. In order to study the envelope behaviour of these waves we express the complex amplitude a in terms of two real functions ρ and σ (Hasegawa 1975), namely,

$$a = \rho^{1/2}(\xi,\tau)\exp\{i\sigma(\xi,\tau)\}. \qquad (2.39)$$

Eq. (2.33) can then be separated into its real and imaginary parts as

$$\frac{\partial\rho}{\partial\tau} + 2P\frac{\partial}{\partial\xi}\left(\rho\frac{\partial\sigma}{\partial\xi}\right) = 0, \qquad (2.40)$$

and

$$\frac{\partial\sigma}{\partial\tau} + P\left(\frac{\partial\sigma}{\partial\xi}\right)^2 + \frac{P}{4\rho^2}\left(\frac{\partial\rho}{\partial\xi}\right)^2 - \frac{P}{2\rho}\frac{\partial^2\rho}{\partial\xi^2} - Q\rho = 0. \qquad (2.41)$$

In obtaining Eqs. (2.40) and (2.41), the R-term in Eq.(2.33) has been eliminated by using the transformation $a \rightarrow a\exp(iR\tau)$. Now if we linearize Eqs. (2.40) and (2.41) as

$$\begin{pmatrix}\rho\\ \sigma\end{pmatrix} = \begin{pmatrix}\rho_0\\ \sigma_0\end{pmatrix} + \begin{pmatrix}\rho_1\\ \sigma_1\end{pmatrix}\exp\{i(K\xi-\Omega\tau)\}, \qquad (2.42)$$

we obtain the dispersion relation,

$$\Omega^2 = P^2K^4 - 2PQ\rho_0K^2 = (PK^2-Q\rho_0)^2 - Q^2\rho_0^2, \qquad (2.43)$$

which shows that there is no instability if $PQ < 0$. On the other hand if $PQ > 0$, the perturbations with wavenumber

$K < (2Q\rho_0/P)^{1/2}$ are unstable and the mode with $K = (Q\rho_0/P)^{1/2}$ grows fastest with the growth rate $Q\rho_0$.

From Eq. (2.37) it is clear that $PQ > 0$ only if $\chi > 0$. For different values of α and β, the critical wavenumbers k_c for modulational instability can then be obtained from the equation $\chi = 0$. The variation of the critical wavenumber squared, k_c^2, with the different values of α and β is depicted in the Table.

The localized stationary solutions of the NS equation in the form given by Eqs. (2.40) and (2.41) for the modulationally stable and unstable cases can be obtained as follows (Hasegawa 1975). For a localized solution, i.e. with a single hump for example, we require that $\rho = |a|^2$ be bounded between the extremum value ρ_s and the asymtotic value ρ_D. For stationary ρ, i.e., $\partial\rho/\partial\tau = 0$, Eq. (2.40) can be integrated to give

$$\rho\frac{\partial\sigma}{\partial\xi} = C(\tau), \tag{2.44}$$

$C(\tau)$ being a function of τ alone. Moreover Eq. (2.41) can be rewritten as

$$\frac{\partial\sigma}{\partial\tau} + P\left(\frac{\partial\sigma}{\partial\xi}\right)^2 = \frac{1}{4}\frac{d}{d\rho}\left\{2Q\rho^2 + \frac{P}{\rho}\left(\frac{\partial\rho}{\partial\xi}\right)^2\right\} \equiv f(\xi) \tag{2.45}$$

From Eqs. (2.44) and (2.45) we get

$$\frac{d^2C}{d\tau^2}\bigg/\frac{dC^2}{d\tau} = \frac{1}{\rho^2}\frac{d\rho}{d\xi} = \text{const.}$$

TABLE

Critical Wavenumber (k_c^2) for Modulational Instability as function of α and β

β \ α	0.0	0.1	0.2	0.3	0.4	0.5	0.7	0.9
0.01	2.163	*	*	*	*	*	*	*
0.05	2.163	*	*	*	*	*	*	*
0.10	2.163	*	*	*	*	*	*	*
0.15	2.163	*	*	*	*	*	*	*
0.20	2.163	*	*	*	*	*	*	*
0.25	2.163	*	*	*	*	*	*	*
0.30	2.163	0.313	*	*	*	*	*	*
0.35	2.163	1.671	0.217	0.044	0.008	*	*	*
0.40	2.163	1.874	1.433	0.264	0.087	0.039	0.007	*
0.45	2.163	1.941	1.646	1.215	0.293	0.116	0.036	0.012
0.50	2.163	1.959	1.713	1.410	0.980	0.284	0.076	0.032
0.55	2.163	1.956	1.727	1.471	1.167	0.720	0.133	0.056
0.60	2.163	1.943	1.718	1.483	1.226	0.916	0.213	0.084
0.65	2.163	1.927	1.698	1.471	1.237	0.977	0.318	0.115
0.70	2.163	1.908	1.673	1.448	1.225	0.990	0.419	0.145
0.75	2.163	1.890	1.646	1.419	1.200	0.979	0.477	0.172
0.80	2.163	1.872	1.618	1.387	1.169	0.955	0.496	0.191
0.85	2.163	1.855	1.590	1.354	1.135	0.923	0.492	0.201
0.90	2.163	1.839	1.564	1.321	1.098	0.887	0.473	0.202
0.95	2.163	1.825	1.539	1.288	1.062	0.848	0.446	0.196

*indicates modulational stability for all k.

Since for a localized solution, we cannot have $\rho^{-2}d\rho/d\xi$ = const., we conclude that

$$C(\tau) = \text{const.} = C_1, \text{ say.}$$

Eq. (2.44) can then be integrated to give

$$\sigma = \int \frac{C_1}{\rho} d\xi + A(\tau),$$

with $A(\tau)$ as constant of integration. Since $\partial\sigma/\partial\xi$ is a function of ξ only, it follows from Eq. (2.45) that $\partial\sigma/\partial\tau$ is a function of ξ only. Consequently

$$\frac{\partial\sigma}{\partial\tau} = \frac{dA}{d\tau} = \Lambda, \text{ say},$$

and then

$$\sigma = \int \frac{C_1}{\rho} d\xi + \Lambda\tau. \tag{2.46}$$

On substituting Eq. (2.46) into Eq. (2.45) we get

$$\left(\frac{d\rho}{d\xi}\right)^2 = -\frac{2Q}{P}\rho^3 + \frac{4\Lambda}{P}\rho^2 + \frac{C_2}{P}\rho - 4C_1^2. \tag{2.47}$$

If $PQ > 0$, i.e., when the waves are modulationally unstable, Eqs. (2.46) - (2.47) can be integrated to give the following localized solution:

$$\rho = \rho_s \operatorname{sech}^2\left\{\left(\frac{Q\rho_s}{2P}\right)^{1/2}\xi\right\}, \quad \text{with } \rho_s = \frac{2\Lambda}{Q}, \tag{2.48}$$
$$\sigma = \Lambda\tau.$$

This is an envelope soliton. If $PQ < 0$, i.e. when the waves are modulationally stable, Eqs. (2.46) - (2.47) can be integrated to give

$$\rho = \rho_1\left[1 - \tilde{\alpha}^2 \operatorname{sech}^2\left\{\left(\frac{|PQ|\rho_1}{2P^2}\right)^{1/2}\tilde{\alpha}\xi\right\}\right],$$

$$\sigma = \sin^{-1}\left\{\frac{\tilde{\alpha}\tanh\{(|PQ|\rho_1/2P^2)^{1/2}\tilde{\alpha}\xi\}}{[1 - \tilde{\alpha}^2 \operatorname{sech}^2\{(|PQ|\rho_1/2P^2)^{1/2}\tilde{\alpha}\xi\}]^{1/2}}\right\} + \Lambda\tau + \frac{C_1}{\rho_1}\xi,$$

where

$$\tilde{a}^2 = \frac{\rho_1 - \rho_s}{\rho_s} \leqslant 1. \qquad (2.50)$$

The solution (2.49) is an envelope hole and represents a region of depletion in the wave intensity. The depth of depletion or modulation is given by $\tilde{a}$, as defined by Eq. (2.50).

II.5 Conclusions and Discussion

If the electrons in a plasma can be divided into hot and cold groups the behaviour of the ion-acoustic waves are drastically changed by the relative abundance of the two groups and their temperatures. We have discussed the properties of such a plasma by considering the cold electrons to be a fraction of the hot electrons, and for various ratios of their temperatures. The variation of the critical wave-number k_c^2 with α and β are shown in the Table. When $\alpha = 0$, i.e., the cold electron component is absent, $k_c^2 = 2.163$. For nonzero α, the waves are stable for all k until a critical value of β is reached. Below this critical β, the presence of the cold electron component stabilizes the wave. This critical β increases with α. For a given α, as β increases, k_c^2 reaches a maximum and then decreases slowly.

As discussed in Chapter I, the modulational instability and consequent envelope states are due to the balance of the dispersion and nonlinearity. As seen from Eqs. (2.34) and (2.35), the dispersion, given by P, and the nonlinearity, given by Q, are functions of α and β. The sign of the nonlinear term Q can change for different values of k. The functional dependence of P and Q on α and β are different and hence the dispersion and nonlinearity vary differently. Consequently, for a given α and β, the critical wavenumber for modulational instability changes. Similarly the characteristics of the envelope states also change.

CHAPTER III

EFFECT OF RANDOM INHOMOGENEITIES ON NONLINEAR ION-ACOUSTIC WAVES

III.1 Introduction

The ion-acoustic waves in a plasma give rise to interesting phenomena, viz., solitons, modulational instability, envelope solitons and envelope holes. Weakly nonlinear longwavelength ion-acoustic waves in a homogeneous plasma is governed by the KdV equation and admits soliton solutions (Washimi and Taniuti 1966). In the presence of a magnetic field both the slow and fast ion-acoustic modes are described by the KdV equation (Pokroev and Stepanov 1973, Tagare and Sharma 1976). The envelope characteristics of ion-acoustic waves are governed by the NS equation. In a collisionless system these waves are modulationally unstable for $k\lambda_D > 1.47$. The modification of the spectrum of unstable waves due to collisions have been discussed by Buti (1976) and that due to electron inertia and ion temperature by Chan and Seshadri (1975).

The effect of random fluctuations is of great importance for systems like fluids and plasmas. Plasmas, in general are turbulent and consequently can be treated as random media. The propagation of nonlinear waves in these random media has been an active field of investigation. The effect of random fluctuations in plasmas is of special interest to the study of nonlinear dispersive waves because both the dispersion and dissipation of the system are affected. This influences the balance between the nonlinearity and the dispersion of the waves and consequently their stationary states, such as solitons, etc. In plasmas the nonlinear interaction among waves in the presence of random inhomogeneities has been studied by averaging the equations over the inhomogeneities (Tamoikin and Fainshtein 1972 and 1976). It has been shown that fluctuations can lead to the collapse of magnetohydrodynamic shock waves (Akhiezer et al. 1971).

The ion-acoustic waves in the presence of weak random inhomogeneities has been studied by Tamoikin and Fainshtein (1973). By averaging the relevant equations over the random inhomogeneities, they obtained a Korteweg-de Vries-Burgers type equation. A study of the stationary solutions of this equation shows that the random electron density fluctuations introduce oscillations behind the shock-front. Here we study the envelope properties of the ion-acoustic waves in the presence of random inhomogeneities in the electron concentration. As in Chapter II we use the Krylov-Bogoliubov-Mitropolsky method. First we obtain the linear ion-acoustic modes in such a medium and then study the nonlinear states of these modes.

III.2 Plasma Equations in a Randomly Inhomogeneous Medium

Let us consider a plasma in which the electrons have average kinetic energy T_e and the ions are cold. The ion density n, the fluid velocity v and the electric potential φ for such a plasma are described by the equations

$$\begin{aligned} &\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0, \\ &\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{e}{M}\frac{\partial \varphi}{\partial x}, \\ &-\frac{\partial^2 \varphi}{\partial x^2} = 4\pi e(n_e - n). \end{aligned} \tag{3.1}$$

The response of the electrons to the ion-acoustic waves is described by the Boltzmann distribution

$$n_e = n_o \exp(-e\varphi/T_e), \tag{3.2}$$

where n_o is the average number density.

In the presence of random density inhomogeneities $\delta n(x)$, the ion density n can be expressed as

$$n = n_0 + \delta n(x) + \langle \tilde{n}(x,t) \rangle + n'(x,t)$$
$$\equiv N_0 + \langle \tilde{n}(x,t) \rangle + n'(x,t) , \qquad (3.3)$$

where the brackets indicate averaging over $\delta n(x)$. Tilde denotes the average wave fluctuation, whereas prime denotes its deviation due to the wave. The equilibrium density N_o is free of the wave fluctuations but has the random inhomogeneities embedded in it. Consequently,

$$\langle N_0 \rangle = \langle n_0 + \delta n(x) \rangle = n_0 .$$

The ele tric potential φ and the fluid velocity v can similary be written as

$$\varphi = \langle \tilde{\varphi}(x,t) \rangle + \varphi'(x,t),$$
$$v = \langle \tilde{v}(x,t) \rangle + v'(x,t). \qquad (3.4)$$

On considering the density fluctuations to be weak, we define the small parameter μ as

$$\mu = \frac{|\delta n(x)|}{n_0} .$$

Also for the weakly nonlinear system

$$\frac{e \langle \tilde{\varphi}_{max} \rangle}{T_e} \sim \frac{\langle \tilde{n} \rangle}{n_0} \sim \frac{\langle \tilde{v} \rangle}{C_s} \sim \mu^2 ,$$

where C_s is the ion-acoustic velocity.

Now on averaging equations (3.1) over the random inhomogeneities, the ion-acoustic waves are found to be described by the equation (Tamoikin and Fainshtein 1973),

$$\frac{\partial}{\partial t}\langle\tilde{\varphi}\rangle + C_s\left(1+\frac{e}{T_e}\langle\tilde{\varphi}\rangle\right)\frac{\partial}{\partial x}\langle\tilde{\varphi}\rangle + \frac{1}{2}C_s\lambda_D^2\frac{\partial^3}{\partial x^3}\langle\tilde{\varphi}\rangle$$
$$+\alpha_1\frac{\partial^3}{\partial x\partial t^2}\langle\tilde{\varphi}\rangle - \alpha_2\frac{\partial^3}{\partial x^2\partial t}\langle\tilde{\varphi}\rangle + \delta\frac{\partial^2}{\partial x\partial t}\langle\tilde{\varphi}\rangle = 0, \quad (3.5)$$

where ω_{pi} is the ion plasma frequency and λ_D the Debye length. The parameters α_1, α_2 and δ are defined as

$$\alpha_1 = \frac{C_s}{3\omega_{pi}^2}\left\langle\left(\frac{\delta n}{n_0}\right)^2\right\rangle\frac{\overline{L^2}}{\lambda_D^2}, \quad \alpha_2 = C_s\alpha_1,$$

$$\text{and} \quad \delta = \frac{1}{3}\bar{L}\left\langle\left(\frac{\delta n}{n_0}\right)^2\right\rangle.$$

For a given correlation function f(x) of the random density inhomogeneities, the quantities $\bar{L}$ and $\overline{L^2}$ are determined by

$$\bar{L} = \int_0^\infty f(x)\,dx \quad \text{and} \quad \overline{L^2} = \int_0^\infty x\,f(x)\,dx.$$

Thus $\bar{L}$ and $\overline{L^2}$ are the integral scales of the inhomogeneities. In general the random fluctuations can be represented by a Gaussian distribution (Landau and Lifshitz 1974). Hence we take

$$f(x) = \frac{1}{(2\pi l^2)^{1/2}}\exp\left(-\frac{x^2}{2l^2}\right),$$

where l^2 is the mean square scale-length of the random fluctuations. Then the integral scales defined above are given by

$$\bar{L} = \int_0^{\infty} f(x)dx = \frac{1}{2} \quad \text{and} \quad \overline{L^2} = \int_0^{\infty} x f(x) dx = \frac{\ell}{2(8\pi)^{1/2}}$$

On normalizing lengths by the Debye length λ_D, the time by ω_{pi}^{-1}, the electric potential ϕ by the characteristic potential (T_e/e), α_1 by $(\lambda_D/\omega_{pi}^2)$, α_2 by $(\lambda_D^2/\omega_{pi})$ and δ by (λ_D/ω_{pi}), equation (3.5) reduces to

$$\frac{\partial}{\partial t}\langle\tilde{\varphi}\rangle + (1+\langle\tilde{\varphi}\rangle)\frac{\partial}{\partial x}\langle\tilde{\varphi}\rangle + \frac{1}{2}\frac{\partial^3}{\partial x^3}\langle\tilde{\varphi}\rangle$$

$$+\alpha_1\frac{\partial^3}{\partial x\partial t^2}\langle\tilde{\varphi}\rangle - \alpha_2\frac{\partial^3}{\partial x^2\partial t}\langle\tilde{\varphi}\rangle + \delta\frac{\partial^2}{\partial x\partial t}\langle\tilde{\varphi}\rangle = 0. \tag{3.6}$$

III.3 Linear Ion-Acoustic Waves

The linear ion-acoustic waves can be represented by the plane waves

$$\langle\tilde{\varphi}\rangle = a\exp(i\psi) + \text{c.c.}, \tag{3.7}$$

where $\psi = kx - \omega t$ denotes the phase. The dispersion relation governing these waves is, from eq. (3.6),

$$-\omega + k - \frac{1}{2}k^3 - \alpha_1 k\omega^2 - \alpha_2 k^2\omega - i\delta k\omega = 0. \tag{3.8}$$

Among the contributions from the random inhomogeneities,

α_1 and α_2 terms are of the same order and contribute to the dispersion of the waves, whereas the δ term is a dissipative term. The ratio of the dissipation to the dispersion term is $\sim (kl)^{-1}$ which is small for $kl \gg 1$; i.e., for wavelengths much smaller than the characteristic scalelength.

The dispersion relation (3.8) can be solved by writing $\omega = \omega_r + i\,\omega_i$. The two modes of oscillations are given by

$$\omega_{r1} = k\left(1 - \frac{1}{2}k^2 - \alpha_1 k^2 - \alpha_2 k^2\right) \qquad (3.9a)$$

and

$$\omega_{r2} = -\frac{1}{\alpha_1 k}\left(1 + \alpha_1 k^2 + \alpha_2 k^2\right), \qquad (3.9b)$$

with the corresponding damping rates

$$\omega_{i1} = -\delta k^2\left(1 - \frac{1}{2}k^2 - 2\alpha_1 k^2 - 2\alpha_2 k^2\right)$$

and

$$\omega_{i2} = -\frac{\delta}{\alpha_1}\left(1 - \alpha_1 k^2\right).$$

The solution ω_{r1} is the ion-acoustic mode and is weakly damped by the random inhomogeneities. The other solution ω_{r2} is entirely due to the random inhomogeneities and is heavily damped with the damping rate ω_{i2}. This mode is similar to the usual thermal noise in a plasma due to the particle thermal motions.

III.4 Nonlinear Ion-Acoustic Waves

The envelope properties of ion-acoustic waves are governed by the NS equation. To study the effect of the random density fluctuations on these properties we apply the KBM method to (3.6). As shown above the contribution of the dissipative term is small compared to that of the dispersive terms, and hence we take δ to be of order ϵ^2, i.e.,

$$\delta = \epsilon^2 \bar{\delta},$$

where ϵ is the smallness parameter. For convenience we write $\phi = \langle \tilde{\Phi} \rangle$ and make the expansion

$$\phi = \epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots, \tag{3.10}$$

where ϕ_1 is the plane wave described by the dispersion relation

$$D(k,\omega) \equiv -\omega + k - \frac{1}{2}k^3 - \alpha_1 k\omega^2 - \alpha_2 k^2 \omega = 0. \tag{3.11}$$

Since the mode given by Eq.(3.9b) is heavily damped, we will discuss the evolution of the ion-acoustic mode of Eq. (3.9a). The variations of the amplitude a of the plane wave representing this mode are given by Eq. (2.8).

On defining the operator

$$\mathcal{L} \equiv \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^3}{\partial x^3} + \alpha_1 \frac{\partial^3}{\partial x \partial t^2} - \alpha_2 \frac{\partial^3}{\partial x^2 \partial t}, \tag{3.12}$$

and substituting the expansion (3.10) in Eq. (3.6), we get, to order ϵ,

$$\mathcal{L}(\phi_1) = 0.$$

This yields the dispersion relation (3.11). The equation to order ϵ^2 is

$$\mathcal{L}(\phi_2) = -ika^2\exp(2i\psi) - \left(-\frac{\partial D}{\partial\omega}A_1 + \frac{\partial D}{\partial k}B_1\right)\exp(i\psi) + \text{C.C.} \tag{3.13}$$

The terms, proportional to exp $(i\psi)$ onthe right hand side of Eq. (3.13) give rise to resonant singularity in the solution of ϕ_2 and the condition for its removal is simply

$$A_1 + V_g B_1 = 0, \tag{3.14}$$

where V_g is the group velocity, namely

$$V_g = -\frac{\partial D/\partial k}{\partial D/\partial\omega} = 1 - \frac{3}{2}k^2 - 3\alpha_1 k^2 - 3\alpha_2 k^2. \tag{3.15}$$

The secular-free solution of Eq.(3.13) is

$$\phi_2 = \frac{k}{6(k-\omega)}a^2\exp(2i\psi) + b(a,\bar{a})\exp(i\psi) + \text{C.C.} + c(a,\bar{a}), \tag{3.16}$$

where $b(a, \bar{a})$ and $c(a, \bar{a})$ are constants with respect to ψ.

To order ϵ^3, Eq. (3. .6) can be written as

$$\mathcal{L}(\phi_3) = -\frac{\partial}{\partial x}(\phi_1\phi_2) - \delta\frac{\partial^2\phi_1}{\partial x\partial t} - \frac{1}{\epsilon}\left\{\mathcal{L}(\phi_2) + \frac{\partial}{\partial x}\phi_1^2\right\} - \frac{1}{\epsilon^2}\mathcal{L}(\phi_1). \tag{3.17}$$

On collecting the ψ independent terms in Eq. (3.17), we get the condition for the removal of secularity:

$$\frac{\partial C}{\partial a}(A_1+B_1)+B_1\bar{a}+c.c.=0.$$

On using Eq. (3.14), this can be immediately integrated to give

$$C=\frac{1}{V_g-1}a\bar{a}+\beta,$$

where β is an absolute constant.

The terms proportional to exp ($\pm i\psi$) on the right side of Eq.(3.17), however, give rise to the resonant secularity in the solution for ϕ_3. This resonant secularity can be removed by the condition

$$i(A_2+V_gB_2)+\frac{1}{2}\frac{dV_g}{dk}\left(B_1\frac{\partial B_1}{\partial a}+\bar{B}_1\frac{\partial B_1}{\partial\bar{a}}\right)$$

$$+\frac{k}{\partial D/\partial\omega}\left\{\frac{k}{6(k-\omega)}+\frac{1}{V_g-1}\right\}a^2\bar{a}+\frac{k(\beta-i\bar{\delta}(\omega))}{\partial D/\partial\omega}a=0,$$

which on writing

$$P=\frac{1}{2}\frac{dV_g}{dk}$$

$$=-\frac{1}{2\,\partial D/\partial\omega}\left\{3k+2\alpha_2\omega\right. \tag{3.19}$$

$$\left.+\frac{2}{\partial D/\partial\omega}\left(2(\alpha_1\omega+\alpha_2k)+\frac{\alpha_1k}{\partial D/\partial\omega}\right)\left(1-\frac{3}{2}k^2-\alpha_1\omega^2-2\alpha_2k\omega\right)\right\},$$

$$Q=\frac{k}{\partial D/\partial\omega}\left\{\frac{k}{6(k-\omega)}+\frac{1}{V_g-1}\right\}, \tag{3.20}$$

and

$$R = \frac{k}{\partial D/\partial\omega}(\beta - i\delta\omega), \tag{3.21}$$

with

$$\frac{\partial D}{\partial\omega} = -(1 + 2\alpha_1 k\omega + \alpha_2 k^2)$$

can be reduced to

$$i(A_2 + V_g B_2) + P\left(B_1\frac{\partial B_1}{\partial a} + \bar{B}_1\frac{\partial B_1}{\partial \bar{a}}\right) + Q|a|^2 a + Ra = 0. \tag{3.22}$$

On using the definitions given by Eqs. (2.29), (2.30) and (2.32), Eq. (3.22) reduces to

$$i\frac{\partial a}{\partial\tau} + P\frac{\partial^2 a}{\partial\xi^2} + Q|a|^2 a + Ra = 0, \tag{3.23}$$

which is the nonlinear Schrödinger equation for the system under consideration.

The R-term in Eq. (3.23) can be eliminated by the transformation $a \rightarrow a \exp(iR\tau)$. It may be noted that the effect of the dissipative term is manifested through this term only. By appropriate choice of the constant p, we can choose this damping to be negligible for the time scales of our interest. On keeping upto the first order terms only in α_1 and α_2 we find that

$$PQ = -\frac{3}{2}(1 - 6\alpha_1 k^2 - 5\alpha_2 k^2); \tag{3.24}$$

so that $PQ < 0$. Consequently the ion-acoustic waves in the presence of weak random inhomogeneities are stable against longwave modulations.

To discuss the envelope stationary states of the ion-acoustic waves, we express the complex amplitude a as in Eq. (2.39). The stationary solution is the envelope hole given by Eq. (2.49), where P and Q are given by Eqs. (3.19) and (3.20) respectively. Thus the ion-acoustic wave may be represented asymtotically as

$$\phi \sim \rho_1^{1/2}\left[1-\tilde{a}^2 \operatorname{sech}^2\left\{(|QP|\rho_1/2P^2)^{1/2}\tilde{a}\,\xi\right\}\right] \times \exp\{i(kx-\omega t+\sigma)\}, \quad (3.25)$$

where the phase change σ is given by Eq. (2.49) with P and Q of Eqs. (3.19) and (3.20).

The width of the envelope hole is defined as

$$\Delta = \frac{1}{\rho_1\tilde{a}}\left|\frac{P}{Q}\right| \simeq \frac{3k^2}{2\rho_1\tilde{a}}\left\{1+2\alpha_1(2-3k^2)+\alpha_2(4-7k^2)\right\}.$$

In the absence of the random inhomogeneities the width is simply

$$\Delta_0 = \frac{3k^2}{2\rho_1\tilde{a}},$$

so that the ratio of these two widths is

$$\frac{\Delta}{\Delta_0} = 1+2\alpha_1(2-3k^2)+\alpha_2(4-7k^2). \quad (3.26)$$

Since both α_1 and α_2 are real positive, and $k^2 \ll 1$,

$$\Delta > \Delta_0,$$

and hence the random inhomogeneities increase the width of the envelope hole state.

III.5 Discussion

The modulational instability of longwave ion-acoustic waves is not affected by the presence of the weak random inhomogeneities. However the stationary envelope states of the waves are affected as depicted by Eq. (3.26), i.e. the inhomogeneities increase the width of the envelope state.

Physically this can be explained by looking into the process of formation of the envelope hole. The balance between the nonlinearity and dispersion leads to the formation of a hole of a certain width. As seen above, the random inhomogeneities contribute to the dispersion of the waves. Hence the balancing of the same nonlinearity against this increased dispersion will lead to the formation of an envelope hole with increased width.

In a homogeneous plasma the ion-acoustic waves with $k\lambda_D > 1.47$ are modulationally unstable (Kakutani and Sugimoto 1974) and the presence of collisions modifies the spectrum of the unstable waves (Buti 1976). The present analysis is restricted to the longwavelengths, $k^2\lambda_D^2 \ll 1$, and for this region of the spectrum it is evident from Eq. (3.24) that ion-acoustic waves are modulationally stable in the presence as well as in the absence of random inhomogeneities.

CHAPTER IV

MODULATIONAL STABILITY OF OBLIQUELY PROPAGATING LANGMUIR WAVES IN COLLISIONAL PLASMAS

IV.1 Introduction

The study of the nonlinear properties of Langmuir waves is of increasing interest because of its relevance to the problems of the relativistic beam and laser heating of plasmas, and to the strong plasma turbulence theory (Nishikawa et al. 1975, and Morales and Lee 1975). The longtime behaviour of Langmuir waves is governed by the nonlinear Schrödinger (NS) equation and in a collisionless plasma they are found to be modulationally stable (Asano et al. 1969). Using the reductive perturbation technique, the NS equation for Langmuir waves has been derived from the isothermal plasma fluid equations by Asano et al. (1969) and from the plasma kinetic equations by Ichikawa et al. (1972). In the latter work the effects of the resonant wave-particles were neglected. The resonant particles moving with the group velocity of the wave are expected to give rise to the nonlinear Landau damping. This interaction introduces a nonlocal-nonlinear term, which makes the Langmuir waves modulationally unstable (Ichikawa and Taniuti 1973). Starting from the adiabatic plasma fluid equations, Kakutani and Sugimoto (1974) derived the NS equation for these waves by using the KBM method. Moreover, Zakharov (1972) showed that the strong Langmuir turbulence is described by the NS equation coupled to a wave equation for the associated low frequency Ion-Acoustic oscillations. In this case, the Langmuir solitons, which normally do not interact among themselves, emit sound waves and coalesce together (Abdulloev et al. 1974).

The influence of collisions on the nonstationary evolution of the electroacoustic waves has been discussed by Gurovich and Karpman (1970) and on the Langmuir waves, when the perturbation propagates with near sound speed, by Karpman (1975b). The effect on the modulational stability and stationary states was not discussed by these authors. The effect of collisions on the modulational instability of ion-acoustic waves has been discussed by Buti (1976) by using the KBM method. The spectrum of the modulationally unstable ion-acoustic waves is found to be drastically modified by the collisions.

The oblique modulation of the ion-acoustic waves has been discussed by Kakoand Hasegawa (1975). The modulationally stable ion-acoustic waves are found to become unstable when modulated at an angle to the direction of the phase velocity.

The Langmuir oscillations are rapid processes and we can associate a fast space-time scale to it. Over these rapid variations are superimposed slow variations of the amplitudes and these are characterized by slow space-time scales. The presence of weak collisions in the system is not of much significance during the initial stages where the system is governed by the fast space-time scale and these collisions may be neglected in this regime. However it may be of interest to study the effects of collisions during the slow space-time scale.

In this chapter, we investigate the modulational stability of Langmuir waves in a weakly collisional plasma when modulated at an angle to the direction of the phase velocity. The KBM method is used for the present study.

IV.2 Nonlinear Schrödinger Equation

Consider a two-dimensional warm collisional plasma with the ions providing the neutralizing background. If the electron fluid is taken to be isothermal and the wave propagates in the x-y plane, its dynamics is governed by the equations

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{u}) = 0,$$
$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + \frac{T_e}{mn}\nabla n + \frac{e}{m}\vec{E} + \nu\vec{u} = 0,$$
$$\nabla\cdot\vec{E} - 4\pi e(n_0 - n) = 0, \qquad (4.1)$$
$$\frac{\partial \vec{E}}{\partial t} - 4\pi e n\vec{u} = 0,$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$, $\vec{u} = (u_x, u_y)$ and $\vec{E} = (E_x, E_y)$. In this set of equations the first three are the familiar continuity equation, momentum-balance equation and Poisson's equation respectively. The fourth equation is

$$\nabla\times\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J},$$

with the magnetic field $\vec{B}$ taken to be zero and the current density $\vec{J} = ne\vec{u}$. The first and the last equations of the set (4.1) imply that the Poisson's equation holds good for all times, provided it is valid initially. Eqs. (4.1) can be expressed in the dimensionless form on normalizing the lengths to the Debye length λ_D, the time to the inverse of electron plasma frequency, ω_p^{-1}, the electron number density n to the average density n_o, the electron fluid velocity $\vec{u}$ to the characteristic sound speed C_s, the electric field $\vec{E}$ to the characteristic field $(T_e/e\lambda_D)$ and the collision frequency ν to ω_p. We choose the first, second and fourth equations of the set (4.1) to describe the system in terms of the three functions n, $\vec{u}$ and $\vec{E}$. From these three equations, $\vec{E}$ can be eliminated. So for a two-dimensional system our basic equations become

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n u_x) + \frac{\partial}{\partial y}(n u_y) = 0, \tag{4.2}$$

$$\frac{\partial^2 u_x}{\partial t^2} + \frac{\partial}{\partial t}\left(u_x \frac{\partial u_x}{\partial x}\right) + \frac{\partial}{\partial t}\left(u_y \frac{\partial u_x}{\partial y}\right) + \frac{\partial}{\partial t}\left(\frac{1}{n}\frac{\partial n}{\partial x}\right) + n u_x + \epsilon\tilde{\nu}\frac{\partial u_x}{\partial t} = 0, \tag{4.3}$$

$$\frac{\partial^2 u_y}{\partial t^2} + \frac{\partial}{\partial t}\left(u_x \frac{\partial u_y}{\partial x}\right) + \frac{\partial}{\partial t}\left(u_y \frac{\partial u_y}{\partial y}\right) + \frac{\partial}{\partial t}\left(\frac{1}{n}\frac{\partial n}{\partial y}\right) + n u_y + \epsilon\tilde{\nu}\frac{\partial u_y}{\partial t} = 0, \tag{4.4}$$

where $\tilde{\nu}$ is the scaled collision frequency, $\nu = \epsilon\tilde{\nu}$; this guarantees the collisions to be weak.

Considering a weakly nonlinear system we take the following expansions in powers of the small parameter ϵ.

$$
\begin{aligned}
n &= 1+\epsilon n_1+\epsilon^2 n_2+\cdots\ , \\
u_x &= \epsilon u_{x1}+\epsilon^2 u_{x2}+\cdots\ , \\
u_y &= \epsilon u_{y1}+\epsilon^2 u_{y2}+\cdots\ ,
\end{aligned}
\tag{4.5}
$$

and choose n_1 to be the monochromatic plane wave

$$
n_1 = a\exp(i\psi)+\bar{a}\exp(-i\psi), \tag{4.6}
$$

where $\psi = kx - \omega t$. The complex amplitude a is a slowly varying function of x and t, defined by Eq. (2.8), but does not depend on y. Thus the system under consideration is not strictly two-dimensional. In Eqs. (4.5), n_1, u_{x1}, u_{y1}, n_2, u_{x2}, u_{y2}, ... are functions of x and t only through a, $\bar{a}$ and ψ ; and are functions of y only through ψ .

From Eqs. (4.2) - (4.4) to order ϵ and from Eq. (4.6) we get

$$
\begin{aligned}
u_{x1} &= \frac{k_x\omega}{k^2}a\exp(i\psi)+\text{c.c.}\ , \\
u_{y1} &= \frac{k_y\omega}{k^2}a\exp(i\psi)+\text{c.c.}\ ,
\end{aligned}
\tag{4.7}
$$

with the linear dispersion relation of the system

$$
D(k,\omega) \equiv -\omega^2+k^2+1 = 0, \tag{4.8}
$$

where $k^2 = k_x^2+k_y^2$.

Now let us consider Eqs. (4.2) - (4.4) to order ϵ^2. From these equations we eliminate u_{x2}, u_{y2} and then substitute, into the resulting equation, for n_1, u_{x1} and u_{y1}, alongwith the definitions given by Eq. (2.8). Then we get

$$\frac{\partial^2 n_2}{\partial t^2} - \frac{\partial^2 n_2}{\partial x^2} - \frac{\partial^2 n_2}{\partial y^2} + n_2 = -2(2k^2+3)\exp(2i\psi) + (2i\omega A_1 + i\tilde{\nu}\omega a + 2ik_x B_1)\exp(i\psi) + \text{c.c.} \tag{4.9}$$

The resonant secularity arising from the term with $\exp(i\psi)$ and its complex conjugate can be removed by imposing the condition

$$A_1 + V_{gx} B_1 + \frac{1}{2}\tilde{\nu} a = 0, \tag{4.10}$$

where the group velocity along the x-axis is defined as

$$V_{gx} = -\frac{\partial D/\partial k_x}{\partial D/\partial \omega} = \frac{k_x}{\omega}.$$

The secular-free solution of Eq. (4.9) then is

$$n_2 = \frac{2}{3}(2k^2+3)a^2\exp(2i\psi) + b(a,\bar{a})\exp(i\psi) + \text{c.c.} + \alpha(a,\bar{a}), \tag{4.11}$$

where $b(a, \bar{a})$ is complex function of a and $\bar{a}$, but a constant with respect to ψ. From Eqs. (4.2) - (4.4) to order ϵ^2 and Eq. (4.11), we get

$$u_{x2} = \frac{k_x\omega(4k^2+3)}{3k^2}a^2\exp(2i\psi) + \left\{-\frac{ik_x(k^2+2)}{k^4}A_1 - \frac{i\omega}{k^2}B_1 + \frac{k_x\omega}{k^2}b - \frac{i\tilde{\nu}k_x\omega^2}{k^4}a\right\}\exp(i\psi) + \text{c.c.} - \frac{2k_x\omega}{k^2}a\bar{a}, \tag{4.12a}$$

$$u_{y2} = \frac{k_y\omega(4k^2+3)}{3k^2}a^2\exp(2i\psi)$$
$$+\left\{-\frac{ik_y(k^2+2)}{k^4}A_1+\frac{k_y\omega}{k^2}b-\frac{i\tilde{\nu}k_y\omega^2}{k^4}a\right\}\exp(i\psi)$$
$$+\text{C.C.}-\frac{2k_y\omega}{k^2}a\bar{a}. \tag{4.12b}$$

In order to determine α we consider Eq. (4.2) to order ϵ^3. The condition for the removal of the secularity in this equation is

$$\frac{\partial\alpha}{\partial a}A_1+\frac{\partial\alpha}{\partial\bar{a}}\bar{A}_1=0,$$

which implies that α is an absolute constant.

Now let us consider Eqs. (4.2) - (4.4) to order ϵ^3. From these equations u_{x3} and u_{y3} can be eliminated and the expressions for n_1, u_{x1}, u_{y1}, n_2, u_{x2} and u_{y2} substituted. In the resulting equation for n_3, the coefficient of the exp ($i\psi$) term gives rise to resonant secularity, which is removed by the condition

$$i(A_2+V_{gx}B_2)+\frac{1}{2\omega}\left\{\left(B_1\frac{\partial B_1}{\partial a}+\bar{B}_1\frac{\partial B_1}{\partial\bar{a}}\right)-\left(A_1\frac{\partial A_1}{\partial a}+\bar{A}_1\frac{\partial A_1}{\partial\bar{a}}\right)\right\}$$
$$+\frac{\tilde{\nu}}{2\omega}\left\{\frac{k^2+2}{k^4}A_1+\frac{2k_x\omega}{k^2}B_1-\frac{1}{2}V_{gx}\left(B_1+\bar{a}\frac{\partial B_1}{\partial\bar{a}}+a\frac{\partial B_1}{\partial a}\right)\right\}$$
$$-\frac{i\tilde{\nu}}{2}\left(a\frac{\partial b}{\partial a}+\bar{a}\frac{\partial b}{\partial\bar{a}}-b\right)-\frac{k^2}{6\omega}(8k^2+9)|a|^2a$$
$$+\frac{1}{2\omega}\left\{\alpha+\tilde{\nu}^2\left(\frac{\omega^2}{k^2}-\frac{1}{4}\right)\right\}a=0. \tag{4.13}$$

The function $b(a, \bar{a})$ occuring in (4.13) is the complementary function of the differential equation (4.9). In the present case this function cannot be determined uniquely and we choose

$$b = a\frac{\partial b}{\partial a} + \bar{a}\frac{\partial b}{\partial \bar{a}},$$

so that $b = \text{constant}\,(a\,\bar{a})^{1/2}$. On using Eq. (4.10) into Eq. (4.13), we get,

$$i(A_2 + V_{gx}B_2) + \frac{1}{2\omega}(1 - V_{gx}^2)\left(B_1\frac{\partial B_1}{\partial a} + \bar{B}_1\frac{\partial B_1}{\partial \bar{a}}\right)$$
$$- \frac{\tilde{\nu} V_{gx}}{4\omega}\left(-B_1 + a\frac{\partial B_1}{\partial a} + \bar{a}\frac{\partial B_1}{\partial \bar{a}}\right)$$
$$- \frac{k^2}{6\omega}(9 + 8k^2)|a|^2 a + \frac{1}{2\omega}\left(\alpha + \frac{1}{4}\tilde{\nu}^2\right)a = 0. \qquad (4.14)$$

From Eqs. (2.8) and (4.10) we get

$$a\frac{\partial B_1}{\partial a} + \bar{a}\frac{\partial B_1}{\partial \bar{a}} = B_1,$$

so that with the definitions (2.29), (2.30) and (2.32), Eq. (4.14) becomes

$$i\frac{\partial a}{\partial \tau} + P\frac{\partial^2 a}{\partial \xi^2} + Q|a|^2 a + Ra = 0, \qquad (4.15)$$

where

$$P = \frac{1}{2}\frac{dV_{gx}}{dk_x} = \frac{1 + k_y^2}{2\omega^3}, \quad Q = -\frac{k^2}{6\omega}(9 + 8k^2) \qquad (4.16)$$

and

$$R = \frac{1}{2\omega}\left(\alpha + \frac{1}{4}\tilde{\nu}^2\right).$$

Eq. (4.15) is the NS equation describing the envelope of the Langmuir waves in the x-direction, i.e., at an angle to the direction of the phase velocity, in a collisional plasma. For the collisionless one-dimensional case, i.e., $\tilde{\nu} = 0$ and $k_x = k$, $k_y = 0$, Eq. (4.15) reduces to that obtained by Asano et al. (1969). The R term is due to the integration constant and the collisions, and is absent in the work of Asano et al. (1969). This term is not of much significance and can be eliminated by the transformation $a \rightarrow a \exp(iR\tau)$.

IV.3 Modulational Stability and Envelope Holes

The stability of the envelope of Langmuir waves against oblique longwave perturbations can be studied in the same manner as in Chapter II. We express a as in Eq. (2.39) and the resulting equations, i.e., Eqs. (2.40) - (2.41) are perturbed as in Eq. (2.42).

From Eq. (4.16) we find that

$$PQ = -\frac{k^2(1+k_y^2)(9+8k^2)}{12\,\omega^2},$$

so that for all k, $PQ < 0$. As shown in Chapter II, modulations are unstable when $PQ > 0$. Hence the Langmuir waves in two-dimensions in a collisional plasma are stable against oblique modulations. The corresponding localized stationary solutions are the envelope holes, Eq. (2.49), with P and Q given by Eq. (4.16). The collisions do not affect the shape of the envelope hole but the obliqueness of the modulations affects its width.

IV.4 Conclusions and Discussion

The Langmuir waves in a one-dimensional collisionless plasma are known to be modulationally stable (Asano et al. 1969). We find the collisions and obliqueness of the perturbation do not change this stability. Using the definitions given by Eq. (2.30), Eq. (4.10) can be written as

$$\frac{\partial a}{\partial t_1} + V_{gx}\frac{\partial a}{\partial x_1} + \frac{1}{2}\tilde{\nu}a = 0.$$

Transforming to a frame moving with velocity V_{gx} and then integrating, we get

$$a = a(t_1 = 0)\exp\left(-\frac{1}{2}\tilde{\nu}t_1\right).$$

Thus the envelope, which propagates with the group velocity V_{gx}, is damped by the collisions in this space-time scale. However, in the space-time scale defined by Eq. (2.32), it is seen from Eq. (4.15) that the collisions have insignificant effect.

The width of the envelope hole, given by Eq. (2.49), is defined as

$$\Delta = \left(\frac{2}{\rho_1} \left| \frac{P}{Q} \right| \right)^{1/2} \frac{1}{\tilde{\alpha}} . \tag{4.17}$$

For the Langmuir waves modulated obliquely, from Eq. (4.16), we have

$$\left| \frac{P}{Q} \right| = \frac{3(1 + k^2 \sin^2 \theta)}{k^2(9 + 8k^2)} , \tag{4.18}$$

where $\tan\theta = k_y/k_x$. If the wave is modulated parallel to the phase velocity, then

$$\left| \frac{P}{Q} \right|_{\theta=0} = \frac{3}{k^2(9 + 8k^2)} . \tag{4.19}$$

From Eqs. (4.17) - (4.19), it can be seen that the ratio of the widths for the oblique and parallel cases is

$$\frac{\Delta(\theta)}{\Delta(\theta=0)} = (1 + k^2 \sin\theta)^{1/2} .$$

Thus the width of the envelope hole increases with the angle between the direction of propagation and modulation. This effect is due to the change in the dispersion of the system.

CHAPTER V

ENVELOPE SOLITONS AND HOLES FOR SINE-GORDON AND NONLINEAR KLEIN-GORDON EQUATIONS

V.1 Introduction

The Sine-Gordon (SG) equation is one of the few nonlinear equations that can be solved exactly in various important cases. This equation has travelling wave solutions called kinks (Barone et al. 1971). Also it can be factorized into a scattering problem involving the desired solution and an evolution equation for the eigen-functions (Ablowitz et al. 1973 and Lamb 1974). Then the inverse scattering method (Zakharov and Shabat 1972) may be directly applied to obtain the solution of an initial value problem. The SG equation is found to admit an exact doubly periodic solution, which in the limiting case corresponds to the kinks (Ben-Abraham 1976).

The evolution of a variety of physical system is governed by the equation

$$\Phi_{tt} - \Phi_{xx} + V'(\Phi) = 0, \qquad (5.1)$$

where $V'(\Phi)$ is a nonlinear function of Φ and may be taken as the derivative of a potential energy $V(\Phi)$ (Barone et al. 1971, Whitham 1974). For $V(\Phi) = -\cos\Phi$, Eq. (5.1) becomes the Sine-Gordon (SG) equation, namely,

$$\Phi_{tt} - \Phi_{xx} + \sin\Phi = 0, \qquad (5.2)$$

and when $V(\Phi) = \Phi^2/2 + \alpha\,\Phi^4/4$, we get the nonlinear Klein-Gordon (NKG) equation,

$$\Phi_{tt} - \Phi_{xx} + \Phi + \alpha\Phi^3 = 0. \qquad (5.3)$$

In Eqs. (5.2) and (5.3) the variables Φ, x and t, and the constant α have been made dimensionless by appropriate normalizations. Eq. (5.3) corresponds to a small amplitude expansion of Eq.(5.2) when $\alpha = -1/6$. The NKG equation describes the many-body behaviour of elementary particles (Schiff 1951). Many physical systems represented by Eq.(5.2) are described by Barone et al. (1971) and Whitham (1974). This equation arises in the study of Josephson junctions, theory of surfaces, dislocations in crystals, ferromagnetic materials, Laser pulse propagation, etc. Also based on Eq. (5.2), Perring and Skyrme (1962) have discussed the strong interation among elementary particles. A lucid discussion of the relevance of Eqs. (5.2) and (5.3) to quantum field theory is given in the recent review by Rajaraman (1975).

The SG and NKG equations play an important role in the nonlinear field theories, classical as well as quantum. The localized solutions of these equations have particle like properties such as localized position and velocity. A classical field theory for the interaction of elementary particles can be constructed by using the SG equation as the model (Caudrey et al. 1975). Recently there have been several attempts (Goldstone and Jackiw 1975, Rajaramam 1975 and, Gervais and Neveu 1976) to use the known properties of the classical theories as a starting point for the investigation of the quantum theories. The correspondence of such theories with the direct theories have been obtained, e.g., the Sine-Gordon theory has been found to be equivalent to the massive Thirring model (Coleman 1975) in particle theory.

The modulational instability of the NKG equation, Eq. (5.3), has been studied by Asano et al. (1969) by using the reductive perturbation method (Taniuti and Yajima 1969). In this chapter, we study the modulational instability and then obtain the localized stationary solutions of Eqs. (5.2) and (5.3). For this purpose we use the KBM method to obtain the nonlinear Schrödinger (NS) equation which describes the slow variations of the amplitudes of the plane wave solutions of these equations.

V.2 Sine-Gordon Equation

In order to study the envelope properties of the SG equation, we use the transformation $\phi = \tan(\Phi/4)$; Eq. (5.2) then reduces to

$$(1+\phi^2)(\phi_{tt}-\phi_{xx}+\phi)-2\phi(\phi_t^2-\phi_x^2+\phi^2)=0. \qquad (5.4)$$

The NS equation describing the envelope of the plane wave solutions of this equation can be derived as follows.

Let us consider a perturbation solution of Eq.(5.4) of the form

$$\phi = \epsilon\phi_1(a,\bar{a},\psi)+\epsilon^2\phi_2(a,\bar{a},\psi)+\cdots, \qquad (5.5)$$

where a is the complex amplitude and $\psi = kx - \omega t$ is the phase. The scalar ϕ depends on x and t only through a, $\bar{a}$ and ψ. The slow variations of the amplitudes are defined by Eq. (2.8). The lowest order (order ϵ), Eq. (5.4) has the plane wave solution given by

$$\phi_1 = a\exp(i\psi)+\text{c.c.},$$

where ω and k satisfy the linear dispersion relation

$$D(\omega,k)\equiv -\omega^2+k^2+1=0. \qquad (5.6)$$

The equation to order ϵ^2 is

$$\omega^2\frac{\partial^2\phi_2}{\partial\psi^2}-k^2\frac{\partial^2\phi_2}{\partial\psi^2}+\phi_2=2i\omega\left(A_1+\frac{k}{\omega}B_1\right)\exp(i\psi)+\text{c.c.} \qquad (5.7)$$

The terms on the right hand side give rise to resonant secularity and this is removed by the condition

$$A_1 + V_g B_1 = 0,$$

where $V_g = k/\omega$, is the group velocity. The secular free solution of Eq. (5.7) is then given by

$$\phi_2 = b(a,\bar{a})\exp(i\psi) + c.c.$$

where $b(a,\bar{a})$ is a function of a and $\bar{a}$, but is constant with respect to ψ . Eq. (5.4) to order ϵ^3 may similarly be written down; the condition for the removal of the resonant secularity in this case is simply

$$i(A_2+V_gB_2)+\frac{1}{2}\frac{dV_g}{dk}\left(B_1\frac{\partial B_1}{\partial a}+\bar{B}_1\frac{\partial B_1}{\partial \bar{a}}\right)+\frac{4}{\omega}|a|^2a=0. \quad (5.8)$$

On using the definitions given in Eqs. (2.29), (2.30) and (2.32), Eq. (5.8) reduces to

$$i\frac{\partial a}{\partial \tau}+P\frac{\partial^2 a}{\partial \xi^2}+Q|a|^2a=0, \quad (5.9)$$

where

$$P=\frac{1}{2}\frac{dV_g}{dk}=\frac{1}{2\omega^3} \text{ and } Q=\frac{4}{\omega}. \quad (5.10)$$

Eq. (5.9) is the NS equation governing the envelope properties of the plane wave solutions of Eq. (5.4).

We investigate the stability of Eq. (5.9) against longwavelength perturbations in the same manner as in Chapter II. Since according to Eq. (5.10), $PQ = 4/\omega^4 > 0$, we find that the perturbations with $K < (4\omega \rho_0^{1/2})$ are unstable. These equations have envelope soliton solutions given by Eq. (2.30) with P and Q given by Eq.(5.10).

V.3 Nonlinear Klein-Gordon Equation

The NS equation describing the envelope of the plane wave solutions of the NKG equation, Eq. (5.3), can be obtained in the same manner as above by using the KBM method. On considering a perturbation expansion of the form of Eq. (5.5), we find that the plane wave solutions of Eq. (5.3) satisfy the dispersion relation given by Eq. (5.6). The condition for the removal of the resonant secularity in Eq. (5.3) to order ϵ^3 yields the NS equation, given by Eq. (5.9), with the coefficients P and Q given by

$$P = \frac{1}{2\omega^3} \quad \text{and} \quad Q = -\frac{3\alpha}{2\omega}. \tag{5.11}$$

These coefficients are identical to the ones obtained by Asano et al. (1969). If $\alpha < 0$, the plane wave solutions of the NKG equation are modulationally unstable and have envelope soliton solutions, as given by Eq. (2.48). On the other hand if $\alpha > 0$, these plane waves are modulationally stable and have the envelope hole solutions of the form given by Eq. (2.49) where P and Q are as defined by Eq.(5.11).

As pointed out earlier, Eq. (5.3) corresponds to the small amplitude expansion of Eq. (5.2) when $\alpha = -1/6$. On putting this value of α in Eq. (5.11) we get $PQ = 1/(8\omega^4) > 0$ and thus under this approximation also the plane wave solutions of the SG equation are modulationally unstable and hence the equation admits envelope solitons.

V.4 Discussion

Here we have studied the envelope properties of the Sine-Gordon equation by transforming it into the NS equation which admits envelope solitons. The envelope solitons discussed here are the bound states of $n(= k/K)$ quasiparticles represented by the plane waves in contrast to the two kink or soliton bound states which have been given various names: 'mesons' in the nonlinear field theory of elementary particles (Perring and Skyrme 1962), '0π' pulses in the self-induced transparency problems (Lamb 1971), 'bions' (Caudrey et al. 1973) and 'doublets' (Rajaraman 1975). In comparison with the doublets, these envelope solitons may be called 'multiplets'. In the nonlinear field theory of elementary particles, the kinks correspond to extended or dressed particles and the mesons or bions or doublets are the bound states of two such particles. The constituents of the multiplets in our case are the plane waves which unlike the kinks do not carry the effect of the nonlinearity and hence may correspond to 'bare' particles.

CHAPTER VI

MODULATIONAL INSTABILITY AND ENVELOPE SOLUTIONS OF NONLINEAR DISPERSIVE WAVE EQUATIONS

VI.1 Introduction

The nonlinear Schrödinger (NS) equation governs a variety of phenomena, e.g., the self-focussing and self-modulation of plane waves, the propagation of heat pulses in solids, the propagation of a number of plasma waves e.g., Langmuir, Ion-Acoustic and Magnetosonic waves, etc. (Scott et al. 1973). The modified Korteweg-de Vries (KdV) equation on the other hand arises in the study of acoustic waves in anharmonic lattices, Alfv̌en waves in a collisionless plasma, etc. (Jeffrey and Kakutani 1972). Both these equations can be unified into the following generalized nonlinear dispersive wave equation (Hirota 1973)

$$i\frac{\partial \chi}{\partial t} + i3\alpha|\chi|^2\frac{\partial \chi}{\partial x} + \beta\frac{\partial^2 \chi}{\partial x^2} + i\gamma\frac{\partial^3 \chi}{\partial x^3} + \delta|\chi|^2\chi = 0, \quad (6.1)$$

where α, β, γ and δ are real constants. This equation, which is known as the Hirota equation, reduces to a NS equation for $\alpha = \gamma = 0$, and to a modified KdV equation for $\beta = \delta = 0$. For the case $\alpha\beta = \gamma\delta$, by using a rather heuristic approach, the exact envelope soliton solutions of Eq. (6.1) were obtained by Hirota (1973).

The propagation of waves, in one-dimensional nonlinear lattices e.g., continuum approximation of the Fermi-Pasta-Ulam problem (Zabusky equation) and in shallow water under gravity propagating in both directions (Toda 1975), are described by the nonlinear wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - 6\frac{\partial^2 \phi^2}{\partial x^2} - \frac{\partial^4 \phi}{\partial x^4} = 0, \quad (6.2)$$

where ϕ, x and t are normalized to the quantities appropriate to the particular problem. This is the well known Boussinesq equation (Boussinesq 1872). The exact-N-soliton solution of this equation was also obtained by Hirota (1973).

Both the equations (6.1) and (6.2) are nonlinear and their plane wave solutions are dispersive, and hence they describe nonlinear dispersive media. As discussed in Chapter I, such media can give rise to modulational instability and consequent localized envelope states. In this chapter, we study these properties of Eqs. (6.1) and (6.2) by using the KBM method.

VI.2 Nonlinear Schrödinger Equation

Let us first consider Eq. (6.1). To obtain the NS equation describing the systems governed by this equation, we use the KBM method as in Chapter II. The solution to Eq. (6.1) can then be written as

$$\chi = \epsilon \chi_1(a,\bar{a},\psi) + \epsilon^2 \chi_2(a,\bar{a},\psi) + \cdots , \tag{6.3}$$

where χ_1 is chosen to be the monochromatic plane wave given by

$$\chi_1 = a \exp(i\psi) + \bar{a} \exp(-i\psi). \tag{6.4}$$

Here a is the complex amplitude, $\psi = kx - \omega t$ is the phase factor and $\bar{a}$ is the complex conjugate of a. In Eq. (6.3), χ_1, χ_2, χ_3, ... are functions of x and t only through a, $\bar{a}$ and ψ. The complex amplitude a is a slowly varying function of x and t as given by Eq. (2.8).

On substituting Eq. (6.3) into Eq. (6.1), we get equations of different orders in ϵ. The equation to order ϵ gives the linear dispersion relation, namely

$$D(k,\omega) \equiv \omega - \beta k^2 + \gamma k^3. \tag{6.5}$$

To order ϵ^2, Eq. (6.1) contains terms proportional to

$\exp(\pm i\psi)$ which give rise to resonant secularity. The condition for the removal of this secularity is

$$A_1 + V_g B_1 = 0, \tag{6.6}$$

where $V_g = 2\beta k - 3\gamma k^2$, is the group velocity of the plane waves. The secular free solution can then be written as

$$\chi_2 = b(a,\bar{a})\exp(i\psi) + \text{c.c.} + \chi_{20}(a,\bar{a}), \tag{6.7}$$

where $b(a,\bar{a})$ and $\chi_{20}\,(a,\bar{a})$ are constants with respect to ψ.

The equation to order ϵ^2, which is obtained by using Eqs. (6.4) and (6.8) in Eq. (6.1), has two sources of secularities: the resonant secularity arising from $\exp(\pm i\psi)$ terms, and the second one due to ψ independent terms which become proportional to ψ on integration. The condition for the removal of the latter secularity determines χ_{20} to be an absolute constant. The resonant secularity however is removed by the condition

$$i(A_2 + V_g B_2) + \frac{1}{2}\frac{dV_g}{dk}\left(\bar{B}_1\frac{\partial B_1}{\partial \bar{a}} + B_1\frac{\partial B_1}{\partial a}\right) - 3(\alpha k - \delta)|a|^2 a = 0 \tag{6.8}$$

Defining

$$P_1 = \frac{1}{2}\frac{dV_g}{dk} = \beta - 3\gamma k \quad \text{and} \quad Q_1 = -3(\alpha k - \delta), \tag{6.9}$$

and using Eqs. (2.29) and (2.30), Eq. (6.8) can be written as

$$i\left(\frac{\partial a}{\partial t_2} + V_g\frac{\partial a}{\partial x_2}\right) + P_1\frac{\partial^2 a}{\partial x_1^2} + Q_1|a|^2 a = 0.$$

This, on introducing the new variables as in Eq. (2.32), reduces to the familiar NS equation:

$$i\frac{\partial a}{\partial \tau} + P_1\frac{\partial^2 a}{\partial \xi^2} + Q_1|a|^2 a = 0. \tag{6.10}$$

Exactly similar analysis is carried out for Eq. (6.2) The linear dispersion relation for the plane wave solutions in this case is given by

$$D(k,\omega) \equiv -\omega^2 + k^2 - k^4.$$

To order ϵ^2, secular free solution of Eq. (6.2) is

$$\phi_2 = \frac{2a^2}{k^2}\exp(2i\psi) + c(a,\bar{a})\exp(i\psi) + \text{c.c.} + \phi_{20}(a,\bar{a})$$

From the secularity removal condition for Eq. (6.2) (to order ϵ^4), $\phi_{20}(a,\bar{a})$ is found to be

$$\phi_{20} = \frac{12}{V_g^2 - 1}a\bar{a} + \mu,$$

where μ is an absolute constant. As before the condition for the removal of the resonant secularity in Eq. (6.2) to order ϵ^3 yields the NS equation.

$$i\frac{\partial a}{\partial \tau} + P_2\frac{\partial^2 a}{\partial \xi^2} + Q_2|a|^2 a + Ra = 0, \tag{6.11}$$

with

$$P_2 = \frac{1}{2}\frac{dV_g}{dk} = -\frac{k^2(3-2k^2)}{2\omega(1-k^2)},$$

$$Q_2 = \frac{3(12-17k^2+12k^4)}{\omega(3-4k^2)} \quad \text{and} \quad R = -6\mu k^2. \tag{6.12}$$

6.3 Modulational Instability and Envelope Solutions

The equations (6.10) and (6.11) describe how the amplitudes of the plane wave solutions of the equations (6.1) and (6.2) respectively will evolve according to their dispersion, determined by P_i and nonlinearity, determined by Q_i ($i = 1$ and 2). If $P_i Q_i > 0$, perturbations with $K < K_c = (2Q_i \rho_o / P_i)^{1/2}$ are unstable and grow with the maximum growth rate $\Gamma = Q_i \rho_o$ for $K = (Q_i \rho_o / P_i)^{1/2}$, where $\rho_o = |\phi_o|^2$ is the initial intensity. However, if $P_i Q_i < 0$, perturbations of all wavelengths are stable. From Eq. (6.9), we find that

$$P_1 Q_1 = -3(\beta - 3\gamma k)(\alpha k - \delta).$$

So that modulational instability of Eq. (6.1) is decided by the values of α, β, γ and δ. For $\alpha = \gamma = 0$, when Eq. (6.1) becomes a NS equation itself, $P_1 Q_1 = 3\ \beta\delta$; which in confirmation with earlier results is unstable for $\beta\delta > 0$. For $\beta = \delta = 0$, Eq. (6.1) reduces to the modified KdV equation with $P_1 Q_1 = 9\alpha\gamma k^2$; this unlike KdV equation is unstable provided $\alpha\gamma > 0$. For nonvanishing α, β, γ, δ however, modulational instability can arise only if $k > k^*$ where

$$k^* = \left[(\alpha\beta + 3\gamma\delta) \pm \left\{(\alpha\beta + 3\gamma\delta)^2 - 12\alpha\beta\gamma\delta\right\}^{1/2}\right] / 6\alpha\gamma,$$

For k^* to be real we must have either $\alpha\beta\gamma\delta < 0$ or $\gamma\delta > 0$ with $(\alpha\beta + 3\gamma\delta)^2 > 12\alpha\beta\gamma\delta$.

If $\alpha = \delta = 0$, Eq.(6.1) is linear and dispersive and if $\beta = \gamma = 0$, Eq. (6.1) is nonlinear but dispersionless.

Evidently in both these cases we cannot obtain the NS equation.

In case of Eq. (6.2), Eq. (6.12) gives

$$P_2Q_2 = -\frac{3}{2}\frac{(3-2k^2)(12-17k^2+12k^4)}{(1-k^2)^2(3-4k^2)} = -\frac{3}{2}\frac{f(k)}{(1-k^2)^2(3-4k^2)^2}$$

where $f(k) = 96\,k^8 - 352\,k^6 + 510\,k^4 - 369\,k^2 + 108$. Thus Eq. (6.2) is modulationally unstable for $f(k) < 0$. The critical wavenumber for modulational instability is found to be $k_c = 0.866$. Thus all waves with $k > k_c$ are modulationally unstable.

Having settled the question of modulational stability of the equation (6.1) and (6.2), the corresponding envelope solutions are obtained immediately. If $P_iQ_i > 0$, i.e., the unstable case, the localized stationary solutions are the envelope solitons given by Eq. (2.48) with P_i and Q_i given by Eqs. (6.9) and (6.12). And if $P_iQ_i < 0$ ($P_iQ_i = -|P_iQ_i|$), i.e., the stable case, the localized solutions are the envelope holes as given by Eq. (2.49).

Now the localized stationary solutions of Eqs. (6.1) and (6.2) may be easily obtained from these considerations. For example, in Eq. (6.1) the cases a) $\alpha = \gamma = 0$, β and δ both positive, and b) $\beta = \delta = 0$, α and γ both positive, admit envelope soliton solutions. In case of Eq. (6.2), the stationary solution is an envelope soliton for $k > 0.866$

and an envelope hole otherwise.

VI.4 Conclusions and Discussion

We have shown that the amplitudes of the plane wave solutions of Eq. (6.1) are governed by the NS equation and that these plane waves can be modulationally unstable. When $\beta = \delta = 0$ Eq. (6.1) reduces to the modified KdV equation, which has soliton solutions. And Eq. (6.10) with $\beta = \delta = 0$ describes the evolution of the amplitudes of the plane wave solutions of the modified KdV equation. The plane waves are modulationally unstable for $\alpha \gamma > 0$ and stable otherwise, and give rise to envelope soliton or envelope hole solutions respectively.

In the case of the Boussinesq equation, plane waves with $k > 0.886$ are unstable against perturbations with $K < (2Q_2 \rho_0/P_2)^{1/2}$, and consequently give rise to envelope solitons. When one of these two conditions is violated, it is stable and hence has envelope hole solutions.

Equations (6.1) and (6.2) describe nonlinear dispersive media. In such media we can study two types of phenomana. One is the dynamics of the wave form itself and the other that of the wave envelope. Here we have studied the relation between these two phenomena.

The KdV equation or the modified KdV equation describes the dynamics of the wave form itself. These equations admit stationary solutions called solitons. The KdV equation can be obtained only for a specific type of nonlinear dispersive medium and hence solitons are properties of such restricted media only.

On the other hand one can always obtain a plane wave solution of the system of equation describing a nonlinear dispersive medium. The slow variations of the amplitudes of the plane waves due to the nonlinearity and dispersion are described by the NS equation, provided the lowest nonlinearity is cubic or less. If the medium is modulationally unstable it has envelope soliton solutions and if stable it has envelope hole solutions. Eqs. (6.1) and (6.2), whose corresponding NS equations are given by Eqs. (6.10) and (6.11) respectively, are examples of this fact.

It has been shown that the NS equation, if modulationally stable, can be converted into the KdV equation (Taniuti and Yajima 1969). In the present analysis, however, we have shown that the modified KdV equation can also lead to the NS equation which may be modulationally stable or unstable depending on the constants α and γ.

CHAPTER VII

NONLINEAR SATURATION OF HOT BEAM-PLASMA INSTABILITY

VII.1 Introduction

The evolution of beam-plasma systems, linear as well as nonlinear, has been recently studied extensively In most of the theoretical studies the model chosen is that of a monoenergetic beam traversing through a cold plasma. In a real physical situation however neither the beam is monoenergetic nor the plasma is cold. The laboratory experiments correspond mostly to the case where the plasma and the beam particles have small but finite thermal velocities (Carr et al. 1973 and, Mizuno and Tanaka 1972). Moreover a delta function beam broadens during the evolution of the instability (Shapiro 1963 and Sugawara et al. 1976). The influence of such thermal velocities on the linear growth of the instability and on the subsequent nonlinear processes, like the saturation and the energy transfer, is discussed in the present chapter.

In the linear regime the beam and the plasma temperature effects on beam instability for the high temperature case were discussed by O'Neil and Malmberg (1968), and Briggs (1971). They found that for high beam temperature ($v_b \gg n_{ob}u/n_{op}$) the growth rate decreases as $1/v_b^2$ and for high plasma temperature ($v_p \gg u$).the instability can be excited only by the resistive medium effect rather than the reactive medium effect; v_b and v_p being the thermal speeds and n_{ob} and n_{op} the equilibrium densities of the beam and the plasma electrons respectively, and u is the streaming velocity of the electrons.

The nonlinear stabilization of the beam-plasma instability due to particle trapping has been discussed, among others, by Onishchenko et al. (1970) and Drummond et al. (1970). The effect of beam temperature was included in the work of Onishchenko et al. (1970). The saturation of the linear growth due to the diffusive interaction between the waves and the particles was discussed by Manheimer (1971) and Gupta (1972). The experimental results of Apel (1969) support such a situation. The influence of the weak thermal motion, both of beam and plasma, in such a situation is studied here by using the Perturbed Orbit Formalism, which was originally formulated by Dupree (1966) and later on generalized by various authors (Weinstock 1969, Rudakov and Tsytovich 1971, Benford and Thomson 1972, Cook and Taylor 1973 and Misguich and Balescu 1975).

The complete dispersion relation, linear as well as nonlinear, is derived and the effects of thermal motion on the linear growth and on the oscillation frequency are obtained. The diffusion coefficient governing the nonlinear interaction between the waves and the particles is derived using the Perturbed Orbit Formalism. We get the saturation level by solving the complete dispersion relation. Finally the redistribution of energy during the wave particle interaction and some associated problems are discussed.

VII.2 Dispersion Relation

Let us consider a collisionless beam plasma system in which the plasma electrons are initially distributed as

$$f_{op}(v) = \frac{1}{(2\pi v_p)^{1/2}} \exp(-v^2/2v_p^2), \qquad (7.1)$$

and the beam electrons, which stream through the stationary plasma with a velocity u, are governed by the distribution function

$$f_{ob}(v) = \frac{1}{(2\pi v_b)^{1/2}} \exp\{-(v-u)^2/2v_b^2\}. \tag{7.2}$$

The system is taken to be isotropic, homogeneous, fieldfree and unbounded. Moreover the ions simply provide the neutralizing background. The electrostatic behaviour of such a system is described by the Vlasov equation:

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{e}{m}E\frac{\partial}{\partial v}\right) f_\alpha(x,v,t) = 0. \tag{7.3}$$

In order to study the nonlinear behaviour of the system we will use the perturbed orbit formalism. On following Dupree's technique as used by Gupta (1972) for the beam-plasma system, we write

$$f_\alpha(x,v,t) = \langle f_\alpha(v,t)\rangle + \sum_k e^{ikx} f_{\alpha k}(v,t) \tag{7.4}$$

and

$$E = \sum_k E_k(t) \exp(ikx + i\beta_k) = \sum_k E_k \exp\{-i\Omega_k t + ikx + i\beta_k\} \tag{7.5}$$

where β_k's are the initial phases and the brackets $\langle ......\rangle$ indicate ensemble average over the initial phases. The turbulent wave particle interactions are represented by a diffusion process in the velocity space and the ensemble average distribution function evolve according to the equation,

$$\frac{\partial}{\partial t}\langle f_\alpha(v,t)\rangle = \frac{\partial}{\partial v} D(v) \frac{\partial}{\partial v}\langle f_\alpha(v,t)\rangle, \tag{7.6}$$

where the diffusion coefficient is given by

$$D(v) \equiv \frac{e^2}{m^2}\sum_k |E_k(t)|^2 \int_0^\infty d\tau \exp\left\{ i(\Omega_k - kv)\tau - \frac{1}{3}k^2 D(v)\tau^3 \right\}. \quad (7.7)$$

On using Eqs. (7.4) to (7.6) in Eq. (7.3), we obtain the dispersion relation

$$\epsilon(\Omega_k, k) \equiv 1 + \sum_\alpha \frac{\omega_\alpha^2}{ik}\int dv \frac{\partial \langle f_\alpha \rangle}{\partial v}\int_0^\infty d\tau \exp\left\{ i(\Omega_k - kv)\tau - \frac{1}{3}k^2 D\tau^3 \right\} = 0, \quad (7.8)$$

which describes the nonlinear behaviour of the system. On putting D = 0 in the exponent of Eq. (7.8) we get the linear dispersion relation. While deriving Eq. (7.8) the diffusion coefficient D has been assumed to be independent of v. Such a restriction, however, is not necessary in the formalism of Weinstock (1969).

Eqs. (7.6) to (7.8) completely describe the evolution of the system within the limitations of the formalism used, e.g. the mode coupling terms are neglected. In the initial stages the wave grows by extracting energy from the streaming motion of the beam. As the amplitude of the wave increases the particles start feeling its presence. This takes the system to the nonlinear regime and the orbits of the particles now get perturbed by the growing amplitude of the wave. Finally when the saturation takes place, the wave-particle interaction reaches an equilibrium state. This readjustment of the waves and the particles against each other is the physical origin of the diffusion process envisaged in Eqs. (7.6) and (7.7).

The unstable waves start growing from the initial thermal noise level of the plasma and hence the diffusion process, which is dependent on the largeness of the amplitude, may be neglected during the initial stage of the evolution. Thus putting $D = 0, \langle f_{\alpha}\rangle_0 = f_{o\alpha}(v)$ into (7.8) we obtain the linear dispersion relation

$$\epsilon_L(\Omega_k, k) \equiv 1 + \sum_{\alpha=b,p} \frac{\omega_\alpha^2}{ik} \int dv \frac{\partial}{\partial v} f_{o\alpha}(v) \int_0^\infty d\tau \exp\{i(\Omega_k - kv)\tau\}$$

$$\equiv 1 - \sum_{\alpha=b,p} \omega_\alpha^2 \int dv \frac{f_{o\alpha}(v)}{(\Omega_k - kv)^2} = 0, \qquad (7.9)$$

where $\omega_\alpha^2 = (4\pi n_{o\alpha} e^2/m)$ is the electron plasma frequency.

VII.3 Effects of Thermal Motion on the Linear Stability

Since we are dealing with low temperatures, we shall solve the dispersion relation (7.9) by iteration. Now for a cold beam-plasma, we have

$$f_{op}(v) = \delta(v), \quad f_{ob}(v) = \delta(v-u),$$

and the dispersion relation is

$$\epsilon_L^0 \equiv 1 - \frac{\omega_p^2}{\Omega_k^2} - \frac{\omega_b^2}{(\Omega_k - ku)^2} = 0. \qquad (7.10)$$

This dispersion relation has been solved, to the lowest order, by Briggs (1971). On defining

$$\epsilon_p = 1 - \frac{\omega_p^2}{\Omega_k^2} \quad \text{and} \quad \delta = \Omega_k - ku$$

and taking δ to be small compared to Ω_k, from Eq. (7.10) we get

$$\frac{\omega_b^2}{\delta^2} = \left.\epsilon_p\right)_{\Omega_k = ku} + \left.\frac{\partial \epsilon_p}{\partial \Omega_k}\right)_{\Omega_k = ku} \delta + \frac{1}{2} \left.\frac{\partial^2 \epsilon_p}{\partial \Omega_k^2}\right)_{\Omega_k = ku} \delta^2 + \cdots \qquad (7.11)$$

The growth rate is maximum when $\Omega_k \simeq \omega_p \simeq ku$. On neglecting terms $O(\delta^2)$, Eq. (7.11) immediately offers the solution

$$\Omega_0 = \omega_0 + i\gamma_0 \qquad (7.12)$$

with

$$\omega_0 = \omega_p - \frac{1}{\sqrt{3}}\gamma_0 \quad \text{and} \quad \gamma_0 = \frac{\sqrt{3}}{2}\left(\frac{\eta}{2}\right)^{1/3} \omega_p ,$$

where $$\eta = \frac{\omega_b^2}{\omega_p^2} = \frac{n_{ob}}{n_{op}} .$$

In the present analysis $\eta^{1/3}$ will be taken as the parameter of smallness and all the terms upto second order in this parameter will be retained throughout. With this Eq. (7.11) reduces to a quartic equation in δ, which by iterative procedure can be further reduced to a cubic equation in δ, namely

$$\delta_0^4 - \frac{2}{3}\omega_p \delta^3 + \frac{1}{3}\omega_b^2 \omega_p^2 = 0 , \qquad (7.13)$$

where
$$\delta_0 = -\frac{1}{\sqrt{3}}\gamma_0 + i\gamma_0 .$$

Eq. (7.13) can now be solved to give

$$\Omega_k = \Omega_{L0} = \omega_{L0} + i\gamma_{L0} , \qquad (7.14)$$

where

$$\omega_{L0} = \omega_p \left(1 - \frac{1}{\sqrt{3}} \frac{\gamma_0}{\omega_p} - \frac{1}{3} \frac{\gamma_0^2}{\omega_p^2} - \frac{2}{3\sqrt{3}} \frac{\gamma_0^3}{\omega_p^3}\right), \tag{7.15}$$

and

$$\gamma_{L0} = \gamma_0 \left(1 - \frac{1}{\sqrt{3}} \frac{\gamma_0}{\omega_p} - \frac{5}{9\sqrt{3}} \frac{\gamma_0^3}{\omega_p^3}\right). \tag{7.16}$$

In order to include the thermal motion in the system, we use the distributions (7.1) and (7.2) into (7.9) to obtain the dispersion relation

$$\epsilon_L(\Omega_k, k) \equiv 1 - \sum_\alpha \frac{\omega_\alpha^2}{2k^2 v_\alpha^2} Z'(\zeta_\alpha) = 0, \tag{7.17}$$

where $Z'(\zeta_\alpha)$ is the derivative of the dispersion function $Z(\zeta_\alpha)$ (Fried et al. 1961). The arguments of Z are defined by

$$\zeta_\alpha = \frac{\Omega_k - kU\delta_{\alpha b}}{\sqrt{2}\, k v_\alpha}, \quad \alpha = p, b. \tag{7.18}$$

$\delta_{\alpha b}$ in (7.18) is the kronecker delta. Since the thermal velocities involved are small, we can use the inequality

$$|\zeta_\alpha|^2 \ll 1, \text{ i.e., } \left(\frac{kv_p}{\omega_p}\right)^2 \ll 1 \quad \text{and} \quad \left(\frac{kv_b}{\gamma_0}\right)^2 \ll 1.$$

These inequalities hold good for the experiments of Mizuno and Tanaka (1972) and Carr et al. (1973). We take both these quantities to be of order $\epsilon = \eta^{1/3}$; this implies that $(v_b/v_p) \sim \epsilon$. On using the asymtotic expansion for $Z(\zeta_\alpha)$ and on retaining terms $O(\epsilon^2)$, Eq. (7.17) reduces to

$$\epsilon_L(\Omega_k,k) \equiv \epsilon_p - \chi_b = 0, \tag{7.19}$$

where

$$\epsilon_p = 1 - \frac{\omega_p^2}{\Omega_k^2} - \frac{3k^2 v_p^2 \omega_p^2}{\Omega_k^4}, \tag{7.20}$$

and

$$\chi_b = \frac{\omega_b^2}{(\Omega_k - ku)^2} + \frac{3k^2 v_b^2 \omega_b^2}{(\Omega_k - ku)^4}.$$

For low temperatures we can take $\Omega_k = \Omega_{LO} + \Delta$; thus Eq. (7.19) becomes

$$\epsilon_p(\Omega_{LO}+\Delta,k) = \chi_b(\Omega_{LO}+\Delta,k). \tag{7.21}$$

Taylor expanding both sides of this equation around $\Omega_k = \Omega_{LO}$ and keeping term $O(\epsilon^2)$, we get

$$a\Delta^2 - b\Delta + c = 0, \tag{7.22}$$

with

$$a = \frac{3\omega_b^2}{(\Omega_{LO} - ku)^4}, \quad b = \frac{2\omega_b^2}{(\Omega_{LO} - ku)^3} + \frac{2\omega_p^2}{\Omega_{LO}^3}$$

and

$$c = \frac{3k^2 v_b^2 \omega_b^2}{(\Omega_{LO} - ku)^4} + \frac{3k^2 v_p^2 \omega_p^2}{\Omega_{LO}^4}. \tag{7.23}$$

In deriving Eq. (7.22), we have made use of Eq. (7.10). On substituting Eq. (7.14) into Eq. (7.22), we obtain $\Omega_L = \omega_L + i\gamma_L$, where

$$\omega_L = \omega_p\left(1 - \frac{1}{\sqrt{3}}\frac{\gamma_o}{\omega_p} - \frac{1}{3}\frac{\gamma_o^2}{\omega_p^2} - \frac{2}{3\sqrt{3}}\frac{\gamma_o^3}{\omega_p^3} + \frac{1}{\sqrt{3}}\frac{\gamma_o}{\omega_p}\frac{k^2 v_p^2}{\omega_p^2} + \frac{1}{2}\frac{k^2 v_p^2}{\omega_p^2} - \frac{\sqrt{3}}{16}\frac{k^4 v_p^4}{\gamma_o \omega_p^3} - \frac{\sqrt{3}}{4}\frac{k^2 v_b^2}{\gamma_o \omega_p}\right) \tag{7.24}$$

and

$$\gamma_L = \gamma_0\left(1 - \frac{1}{\sqrt{3}}\frac{\gamma_0}{\omega_p} - \frac{k^2 v_p^2}{\omega_p^2} - \frac{3}{16}\frac{k^4 v_p^4}{\gamma_0^2\omega_p^2} - \frac{3}{4}\frac{k^2 v_b^2}{\gamma_0^2}\right). \quad (7.25)$$

In the above expressions for ω_L the last two terms are of order ϵ^3; in order to be consistent with our approximations, we must neglect these terms. From Eq. (7.24), it is apparent that thermal motions increase the oscillation frequency. Because of the finite thermal velocity, the plasma electrons become more mobile and respond to the oscillations more easily thereby increasing its frequency. The effect on the growth of the amplitude of these oscillations is, however, just the opposite. The growth corresponds to increasing the space density of charge or leading to stronger charge bunching. When the electrons posses random thermal velocities they are less easily localized in space than when they are cold. Consequently the growth rate decreases with increasing thermal motion, as depicted by Eq. (7.25)

VII.4 Diffusion Coefficient and Nonlinear Dispersion Relation

As the beam-plasma system evolves, the nonlinear effects become more and more important. The feedback action of the growing wave on the particles may be considered, as indicated before, to be a diffusion process in the velocity space. This diffusion brings about the saturation of the growth.

The growth rate of the instability peaks around $\omega_p = ku$. So from the initial thermal noise present in the plasma, a number of waves centred around the above start growing. However, this fastest growing wave begins to suppress the growth of the neighbouring waves and thus generate a narrow spectrum. The random thermal motion of the particles coupled with this narrow but finite spectrum of waves provide the stochasticity required for describing the system by a diffusive process which is fully described by Eq. (7.6).

The orbit integral of Eq. (7.8), namely

$$I = \int_0^\infty d\tau \exp\{i(\omega_k - kv + i\gamma_k)\tau - \tfrac{1}{3}k^2 D\tau^3\}$$

can be rewritten as

$$I(y) = (k^2 D)^{-1/3}\int_0^\infty d\tau \exp(-y\tau - \tfrac{1}{3}\tau^3), \qquad (7.26)$$

with $y = \{\gamma_k + i(kv - \omega_k)\}(k^2 D)^{-1/3}$. If we consider the number of trapped particles to be small, then the orbit integral of (7.26) can be expanded (cf. Gupta, 1972) as

$$(k^2 D)^{1/3} I(y) = \frac{1}{y}\left(1 - \frac{8}{9\xi^2} + \cdots\right), \text{ for } |\xi| = \left|\frac{2}{3}y^{3/2}\right| \gg 1.$$

Consequently, we obtain

$$I(y) \simeq \frac{1}{\gamma_k + i(kv - \omega_k)} - \frac{2k^2 D}{\{\gamma_k + i(kv - \omega_k)\}^4}. \qquad (7.27)$$

Putting this into Eq. (7.7), we get the diffusion coeff- icient for the untrapped particles

$$D(v) = \frac{e^2}{m^2}\sum_k \left|E_k(t)\right|^2 \frac{\gamma_k}{\gamma_k^2 + (kv - \omega_k^2)} . \tag{7.28}$$

In the limit of small γ_k this expression reduces to that of quasilinear diffusion coefficient (Sagdeev and Galeev 1969), thus indicating that the resonance broadening in (7.28) is due to finite γ_k which corresponding to strong turbulence.

On using Eq. (7.27) for the orbit integral and writing

$$\langle f_\alpha(v,t)\rangle \simeq f_{o\alpha}(v) + \langle f_\alpha(v,t)\rangle_1 ,$$

the nonlinear dispersion relation of (7.8) simplifies to

$$\epsilon_{NL}(\Omega_k, k) \equiv \epsilon_L(\Omega_k, k) + \sum_\alpha \chi_\alpha(\omega_k, \gamma_k) = 0, \tag{7.29}$$

where $\epsilon_L(\Omega_k, k)$ is given by Eq. (7.17) and

$$\chi_\alpha(\omega_k,\gamma_k) = \frac{\omega_\alpha^2}{ik}\int dv \left\{ \frac{\frac{\partial}{\partial v}\langle f_\alpha\rangle_1}{\gamma_k + i(kv-\omega_k)} - \frac{2k^2 D \frac{\partial}{\partial v} f_{o\alpha}(v)}{\{\gamma_k + i(kv-\omega_k)\}^4}\right\}, \tag{7.30}$$

with

$$\langle f_\alpha\rangle_1 = \frac{1}{2\gamma_k}\frac{\partial}{\partial v} D(v) \frac{\partial}{\partial v} f_{o\alpha}(v). \tag{7.31}$$

Eq. (7.30), with the help of Eq. (7.28), can be written in terms of the dispersion function $Z(\zeta_\alpha)$. Once again for

$|\zeta_\alpha|^2 \gg 1$, we can use the asymtotic expansion for $Z(\zeta_\alpha)$. Then after some straightforward but cumbersome algebra, we get

$$\chi_\alpha(\omega_k,\gamma_k) = \frac{\omega_\alpha^2\omega_B^4}{2^4\gamma_k^4 k^2 v_\alpha^2}\Big\{\frac{1}{\zeta_\alpha^2} + 2\sqrt{2}i\frac{\beta}{\zeta_\alpha^3} + \frac{(-6\beta^2+\frac{3}{2})}{\zeta_\alpha^4}$$

$$+(-8\sqrt{2}i\beta^3 + 6\sqrt{2}i\beta)\frac{1}{\zeta_\alpha^5} + (10\beta^4 - 30\beta^2 + \frac{15}{4})\frac{1}{\zeta_\alpha^6}$$

$$-\frac{1}{\zeta_\alpha^{*2}} - \frac{3}{2\zeta_\alpha^{*4}} - \frac{15}{4\zeta_\alpha^{*6}} + 4i\sqrt{\pi}\,\zeta_\alpha^* e^{-\zeta_\alpha^{*2}}\Big\}, \quad (7.32)$$

where $\omega_B^4 = \left|\frac{ekE_k(t)}{m}\right|^2$, $\beta = \frac{\gamma_k}{kv_\alpha}$ and ζ_α^* is the complex conjugate of ζ_α. ω_B defines the bounce frequency of a particle of mass m and charge e bouncing in the potential of the electric field $E_k(t)$. Now for $kv_p \ll \omega_p$ and $kv_b \ll \gamma_o$, $\zeta_\alpha^* e^{-\zeta_\alpha^{*2}} \to 0$ and hence Eq. (7.32) reduces to

$$\chi_\alpha(\omega_k,\gamma_k) = \frac{\omega_\alpha^2\omega_B^4}{8\gamma_k^4}\Big\{\frac{1}{(\Omega_k - ku\delta_{\alpha b})^2} + \frac{4i\gamma_k}{(\Omega_k - ku\delta_{\alpha b})^3}$$

$$+(-12\gamma_k^2 + 3k^2v_\alpha^2)\frac{1}{(\Omega_k - ku\delta_{\alpha b})^4} + \frac{-32i\gamma_k^3 + 24i\gamma_k k^2 v_\alpha^2}{(\Omega_k - ku\delta_{\alpha b})^5}$$

$$+\frac{40\gamma_k^4 - 120\gamma_k^2 k^2 v_\alpha^2 + 15k^4v_\alpha^4}{(\Omega_k - ku\delta_{\alpha b})^6} - \frac{1}{(\Omega_k^* - ku\delta_{\alpha b})^2}$$

$$-(3k^2v_\alpha^2)\frac{1}{(\Omega_k^* - ku\delta_{\alpha b})^4} - 15k^4v_\alpha^4\frac{1}{(\Omega_k^* - ku\delta_{\alpha b})^6}\Big\} \quad (7.33)$$

Separating the real and the imaginary parts of Eq. (7.33), we obtain

$$\mathrm{Re}\,\chi_\alpha = \frac{\omega_\alpha^2\omega_B^4}{\{(\omega_k - ku\delta_{\alpha b})^2+\gamma_k^2\}^6}\Big\{-5(\omega_k - ku\delta_{\alpha b})^6$$

$$-37(\omega_k - ku\delta_{\alpha b})^4\gamma_k^2 + 117(\omega_k - ku\delta_{\alpha b})^2\gamma_k^4 - 11\gamma_k^6$$

$$+2k^2v_\alpha^2\big[105(\omega_k - ku\delta_{\alpha b})^4$$

$$-126(\omega_k - ku\delta_{\alpha b})^2\gamma_k^2 + 9\gamma_k^4\big]\Big\}, \tag{7.34}$$

and

$$\mathrm{Im}\,\chi_\alpha = \frac{2\omega_\alpha^2\omega_B^4(\omega_k - ku\delta_{\alpha b})\gamma_k}{\{(\omega_k - ku\delta_{\alpha b})^2+\gamma_k^2\}^6}\Big\{3(\omega_k - ku\delta_{\alpha b})^4$$

$$+54(\omega_k - ku\delta_{\alpha b})^2\gamma_k^2 - 29\gamma_k^4$$

$$+\frac{6k^2v_\alpha^2}{\gamma_k^2}\Big[5(\omega_k - ku\delta_{\alpha b})^4 - 26(\omega_k - ku\delta_{\alpha b})^2\gamma_k^2$$

$$+9\gamma_k^4 + \frac{5}{8}\frac{k^2v_\alpha^2}{\gamma_k^2}\big(-3(\omega_k - ku\delta_{\alpha b})^4$$

$$+10(\omega_k - ku\delta_{\alpha b})^2\gamma_k^2 - 3\gamma_k^4\big)\Big]\Big\} \tag{7.35}$$

In the limit of vanishing thermal velocity of the particles, expressions (7.34) and (7.35) reduce to those of Gupta (1972) for the cold case.

VII.5 Nonlinear Saturation

The dispersion relation (7.29) may be solved to obtain the nonlinear effects on the growth and the oscillation characteristics of the system under consideration. Let us write

$$\Omega_k = \omega_k + i\gamma_k = \Omega_L + \delta\Omega_k, \tag{7.36}$$

where Ω_L is defined in Eq. (7.24) and $\delta\Omega_k = \delta\omega_k + i\delta\gamma_k$ with $|\delta\Omega_k| \ll |\Omega_L|$.

Now Eq. (7.29) may be Taylor expanded to give

$$-\delta\Omega_k\left(\frac{\partial\epsilon_L}{\partial\Omega}\right)_{\Omega=\Omega_L} \simeq \chi_p(\omega_L,\gamma_L) + \chi_b(\omega_L,\gamma_L). \tag{7.37}$$

On separating the real and imaginary parts of Eq. (7.37) we get two simultaneous equations whose solutions are:

$$\delta\omega_k = -\frac{b_1}{a_1}\left(1 + \frac{a_2}{a_1}\frac{c_1}{b_1} - \frac{a_2^2}{a_1^2}\right) \tag{7.38}$$

and

$$\delta\gamma_k = -\frac{c_1}{a_1}\left(1 - \frac{a_2}{a_1}\frac{b_1}{c_1} - \frac{a_2^2}{a_1^2}\right), \tag{7.39}$$

where

$$\begin{aligned} a_1 &= -\operatorname{Re}\left(\frac{\partial\epsilon_L}{\partial\Omega}\right)_{\Omega=\Omega_L} \\ &= -\frac{6}{\omega_p}\left(1 + \frac{2}{\sqrt{3}}\frac{\gamma_0}{\omega_p} - 2\frac{\gamma_0^2}{\omega_p^2} + \frac{\sqrt{3}}{4}\frac{k^2 v_p^2}{\gamma_0\omega_p}\right. \\ &\qquad \left. + \frac{33}{2}\frac{k^2 v_p^2}{\omega_p^2} - \frac{3}{16}\frac{k^4 v_p^4}{\gamma_0^2\omega_p^2} + \frac{3}{4}\frac{k^2 v_b^2}{\gamma_0^2}\right), \end{aligned} \tag{7.40}$$

$$a_2 = -\mathrm{Im}\left(\frac{\partial E_L}{\partial \Omega}\right)_{\Omega=\Omega_L}$$

$$= \frac{12}{\omega_p}\left(\frac{\gamma_0}{\omega_p} + \sqrt{3}\,\frac{\gamma_0^2}{\omega_p^2} - \frac{3}{8}\,\frac{k^2 v_p^2}{\gamma_0 \omega_p} + \frac{3\sqrt{3}}{16}\,\frac{k^4 v_p^4}{\gamma_0^2 \omega_p^2} - \frac{3\sqrt{3}}{8}\,\frac{k^2 v_b^2}{\gamma_0^2} - \frac{3\sqrt{3}}{8}\,\frac{k^2 v_b^2}{\omega_p^2}\right), \qquad (7.41)$$

$$b_1 = -\mathrm{Re}\,\chi_b(\omega_L, \gamma_L) = -\frac{5\times 3^3}{2^5}\,\frac{\omega_b^2 \omega_B^4}{\gamma_0^6}\left(1 + \frac{17\sqrt{3}}{10}\,\frac{\gamma_0}{\omega_p} + \frac{9\sqrt{3}}{40}\,\frac{k^2 v_p^2}{\gamma_0 \omega_p}\right), \qquad (7.42)$$

$$b_2 = -\mathrm{Re}\,\chi_p(\omega_L, \gamma_L) \qquad (7.43)$$

$$= -5\,\frac{\omega_B^4}{\omega_p^4}$$

$$c_1 = -\mathrm{Im}\,\chi_b(\omega_L, \gamma_L)$$

$$= -\frac{3^4\times\sqrt{3}}{2^6}\,\frac{\omega_b^2 \omega_B^4}{\gamma_0^6}\left(1 - \frac{5}{\sqrt{3}}\,\frac{\gamma_0}{\omega_p} + \frac{11\sqrt{3}}{4}\,\frac{k^2 v_p^2}{\gamma_0 \omega_p} - \frac{29}{2}\,\frac{\gamma_0^2}{\omega_p^2} + \frac{27}{2}\,\frac{k^2 v_p^2}{\omega_p^2} + \frac{111}{32}\,\frac{k^4 v_p^4}{\gamma_0^2 \omega_p^2} - \frac{17}{4}\,\frac{k^2 v_b^2}{\gamma_0^2}\right) \qquad (7.44)$$

and

$$c_2 = -\mathrm{Im}\,\chi_p(\omega_L, \gamma_L) = -\frac{6\gamma_0 \omega_B^4}{\omega_p^5}\left(1 + \frac{10 k^2 v_p^2}{\gamma_0^2}\left(1 - \frac{5 k^2 v_p^2}{8\gamma_0^2}\right)\right). \qquad (7.45)$$

On substituting Eqs. (7.40) - (7.45) and retaining ϵ^2 terms, we immediately get the nonlinear contribution to the oscillation frequency and growth rates which are given by

$$\delta\omega_k = -\frac{5\sqrt{3}}{4}\,\frac{\omega_B^4}{\gamma_0^3}\left(1 + \frac{13\sqrt{3}}{30}\,\frac{\gamma_0}{\omega_p} + \frac{\sqrt{3}}{5}\,\frac{k^2 v_p^2}{\gamma_0 \omega_p}\right) \qquad (7.46)$$

and

$$\delta\gamma_k = -\frac{9}{8}\frac{\omega_B^4}{\gamma_0^3}\left(1 - \frac{1}{3\sqrt{3}}\frac{\gamma_0}{\omega_p} + \frac{5}{\sqrt{3}}\frac{k^2 v_p^2}{\gamma_0 \omega_p} - \frac{49}{18}\frac{\gamma_0^2}{\omega_p^2} - \frac{37}{3}\frac{k^2 v_p^2}{\omega_p^2} + \frac{101}{32}\frac{k^4 v_p^4}{\gamma_0^2 \omega_p^2} - \frac{15}{2}\frac{k^2 v_b^2}{\gamma_0^2}\right). \quad (7.47)$$

The nonlinear growth rate is thus given by

$$\begin{aligned}\gamma_k &= \gamma_L + \delta\gamma_k \\ &= \gamma_L\left(1 - \frac{9}{8}\frac{\omega_B^4}{\gamma_0^4}\mu\right),\end{aligned} \quad (7.48)$$

where

$$\mu = 1 + \frac{2}{3\sqrt{3}}\frac{\gamma_0}{\omega_p} + \frac{5}{\sqrt{3}}\frac{k^2 v_p^2}{\gamma_0 \omega_p} - \frac{5}{2}\frac{\gamma_0^2}{\omega_p^2} - \frac{29}{3}\frac{k^2 v_p^2}{\omega_p^2} + \frac{107}{32}\frac{k^4 v_p^4}{\gamma_0^2 \omega_p^2} - \frac{27}{4}\frac{k^2 v_b^2}{\gamma_0^2}.$$

The growth of the instability saturates when $\gamma_k = 0$ which happens only if

$$1 = \frac{9}{8}\frac{\omega_B^4}{\gamma_0^4}\mu . \quad (7.49)$$

In the limit of vanishing thermal velocities of the particles and considering only the zero order term, Eq. (7.49) reduces to the condition obtained by Gupta (1972) for the cold beam-plasma case.

The saturation level of the electric field fluctuations can now be obtained from Eq. (7.49) and is given by

$$|E_k(t)|^2 = \frac{8}{9}\gamma_0^4\left|\frac{m}{ek}\right|^2\frac{1}{\mu} = |E_k(t)|_0^2\,\mu^{-1};$$

where $|E_k(t)|^2$ is the saturation level for the cold case. The above expression gives a saturation spectrum $|E_k(t)|^2 \sim k^{-2}$. This agrees with the Langmuir turbulence spectrum obtained by Kingsep et al. (1973). However for longwave region the spectrum is $|E_k|^2 \sim k^{-2.84}$ (Tsytovich 1972). The effect of the thermal motions on the saturation level is seen from the above relation which may be rewritten as

$$|E_k(t)|^2 = |E_k(t)|_0^2 \Big(1 - \frac{2}{3\sqrt{3}}\frac{\gamma_0}{\omega_p} - \frac{143}{54}\frac{\gamma_0^2}{\omega_p^2} - \frac{5}{\sqrt{3}}\frac{k^2 v_p^2}{\gamma_0 \omega_p} + \frac{107}{9}\frac{k^2 v_p^2}{\omega_p^2} + \frac{479}{96}\frac{k^4 v_p^4}{\gamma_0^2 \omega_p^2} + \frac{27}{4}\frac{k^2 v_b^2}{\gamma_0^2}\Big). \tag{7.50}$$

The dominant term arising from the thermal motion is the term $(-5k^2v_p^2/\sqrt{3}\ \gamma_o\ \omega_p)$ and consequently the saturation level is lowered with respect to the level for the zero thermal velocity case. The finite thermal velocity of the particles enhances the diffusive interaction between the waves and the particles.

On using Eq. (7.49), Eq. (7.46) becomes

$$\delta\omega_k = -\frac{10}{3\sqrt{3}}\gamma_0\Big(1 + \frac{19}{30\sqrt{3}}\frac{\gamma_0}{\omega_p} - \frac{22}{5\sqrt{3}}\frac{k^2 v_p^2}{\gamma_0 \omega_p}\Big),$$

and thus the oscillation frequency at saturation is given by

$$\omega_{ks} = \omega_p\Big(1 - \frac{13}{3\sqrt{3}}\frac{\gamma_0}{\omega_p} - \frac{46}{27}\frac{\gamma_0^2}{\omega_p^2} + \frac{97}{18}\frac{k^2 v_p^2}{\omega_p^2}\Big). \tag{7.51}$$

VII. 6 Energy Balance

In the present problem of beam-plasma interaction the initial energy comprises of the streaming energy of the beam electrons ($T_b^{st} = \frac{1}{2}\, n_{ob}\, mu^2$), the thermal energy of the plasma electrons ($T_p^{th} = \frac{1}{2}\, n_{op}\, mv_p^2$), the thermal energy of the beam electrons ($T_b^{th} = \frac{1}{2}\, n_{ob}\, mv_b^2$) and the initial fluctuation energy of the waves ($\mathcal{E}_i = \sum_k |E_k(0)|^2/8\pi$).

Initially the system is taken to be in thermal equilibrium so that $\mathcal{E}_i$ is the thermal noise. Usually this energy is very small compared to the mean particle kinetic energy, i.e., $\mathcal{E}_i \ll T^{th} = T_p^{th} + T_b^{th}$

The wave grows by extracting energy from the streaming motion of the beam and as the amplitude becomes appreciable, it influences the orbits of the particles -- thus changing their distribution and energy content. These physical processes are represented by the diffusion equation (7.6) with the diffusion coefficient given by Eq. (7.28).

The streaming velocity of the beam after the non-linear interaction is

$$u_s = \int dv \langle f_\alpha(v,t)\rangle v = u + \frac{1}{2\gamma_k}\int dv\, f_{ob}(v)\frac{\partial}{\partial v}D(v)$$

After substituting for D(v) from Eq. (7.28), the integral may be expressed in terms of the dispersion function $Z(\zeta)$ and its derivatives. Once again if we use the asymtotic expansion for $Z(\zeta)$, we obtain

$$\mathcal{U}_s = \mathcal{U} + \frac{\omega_B^4}{k} \frac{(\omega_k - k\mathcal{U})}{\{(\omega_k - k\mathcal{U})^2 + \gamma_k^2\}^2} \times \left\{ 1 + 6k^2 v_b^2 \frac{(\omega_k - k\mathcal{U})^2 - \gamma_k^2}{\{(\omega_k - k\mathcal{U})^2 + \gamma_k^2\}^2} \right\}. \quad (7.52)$$

On using Eqs. (7.24), (7.25) and (7.49) this gives

$$\mathcal{U}_s = \mathcal{U} - \frac{3\sqrt{3}}{16} \frac{\omega_B^4}{k\gamma_0^3} \left(1 + \sqrt{3}\frac{\gamma_0}{\omega_p} + \frac{\gamma_0^2}{\omega_p^2} + \frac{9}{2} \frac{k^2 v_p^2}{\omega_p^2} \right), \quad (7.53)$$

the energy being

$$\widetilde{T}_b^{st} = T_b^{st} - 2\mathcal{E} \left(1 + \frac{3\sqrt{3}}{4} \frac{\gamma_0}{\omega_p} - \frac{1}{3} \frac{\gamma_0^2}{\omega_p^2} + \frac{23}{4} \frac{k^2 v_p^2}{\omega_p^2} \right), \quad (7.54)$$

where $\mathcal{E} = \sum |E_k|^2 / 8\pi$ is the fluctuation energy after the nonlinear process. The changes in the energy of the different components shall be expressed in terms of $\mathcal{E}$.

The thermal energies of the plasma and the beam particles can be obtained in the same way. They are given by

$$\widetilde{T}_p^{th} = \left\langle \frac{1}{2} n_{op} m v^2 \right\rangle = T_p^{th} + \mathcal{E} \left(1 + \frac{2}{\sqrt{3}} \frac{\gamma_0}{\omega_p} + \frac{2}{3} \frac{\gamma_0^2}{\omega_p^2} + 8 \frac{k^2 v_p^2}{\omega_p^2} \right) \quad (7.55)$$

and

$$\widetilde{T}_b^{th} = \left\langle \frac{1}{2} n_{ob} m v^2 \right\rangle = T_b^{th} + \frac{7}{2\sqrt{3}} \frac{\gamma_0}{\omega_p} \mathcal{E} \left(1 + \frac{8}{21\sqrt{3}} \frac{\gamma_0}{\omega_p} + \frac{11}{7\sqrt{3}} \frac{k^2 v_p^2}{\gamma_0 \omega_p} \right). \quad (7.56)$$

From Eq. (7.54) it is obvious that the energy lost by the streaming motion of the beam increases because of the thermal motion. Also there is an increase in the thermal energies of the plasma and the beam, as is evident from Eqs. (7.55) and (7.56). On the other hand Eq. (7.50) shows a decrease in fluctuation energy with thermal motion. Thus the finite temperature of the particles result in an enhanced transfer of energy from the streaming motion into the thermal motion. The quasilinear theory predicts an equal sharing of the energy lost by the resonant particles among the non-resonant particles and the fluctuations (Davidson 1972). Equations (7.50), (7.54), (7.55) and (7.56) suggest a modification of this result; the particles pick up more energy than the waves during the nonlinear interaction.

VII.7 Discussion and Conclusions

The nonlinear interaction of the growing waves with the nontrapped particles is the cause of the nonlinear processes studied here. The saturation of the beam-plasma instability due to trapping as discussed by Drummond et al. (1970) requires a single wave with large enough amplitude to trap the beam particles. Here we have discussed the diffusive interaction between the waves and particles. This could be considered as an alternative way of saturating the beam-plasma instability.

The basis for using the diffusion type interaction rather than the trapping process are:

i) The waves we discuss here are the ones that grow from the thermal noise and thus have a very small amplitude to begin with. In the experiments where the saturation due to particle trapping are observed (Mizuno et al. (1972) and Carr et al. (1973)), the wave is a large amplitude launched single wave. Also in these experiments the particles are found to be detrapped when other waves appear in the system. According to theoretical model of O'Neil et al. (1972), similar detrapping of particles would take place due to nonlocal interactions.

ii) When the waves grow up from the thermal noise, a number of waves around the, $\omega_p = ku$, mode grow and thus generate a spectrum of waves. The presence of a spectrum of waves, though narrow, is a favourable condition for the diffusion process to occur. Moreover diffusion can take place at not-so-large amplitude of the wave because unlike the trapping process it does not need a large critical amplitude.

iii) The random thermal motion of the particles also contribute to the stochasticity, which is the essential requirement for diffusion. The phenomenon in this case is analogous to the linear Landau damping, which is absent for particle distributions represented by delta functions.

Biskamp and Welter (1972) had shown that the diffusion model be valid if $k\,\Delta(\omega/k)_s > \omega_B$, $\Delta(\omega/k)_s$ being the spread in the phase velocity at saturation. In our case, from Eq. (7.51), we find that

$$k\,\Delta\left(\frac{\omega}{k}\right)_s \approx \frac{\Delta k}{k}\,\omega_p \approx \frac{\omega_B}{\epsilon}\,\frac{\Delta k}{k}$$

In beam-plasma case, it was pointed out by Drummond et al. (1970) that $(\Delta k/k) \sim \epsilon$ and hence it is a reasonable model to be used.

We have considered the beam particles to be nontrapped. As discussed by Gupta (1972), the required condition for nontrapping is

$$(kv-\omega_k)^2 > (k^2D)^{2/3} - \gamma_k^2,$$

which can be rewritten as $2\gamma_0^2 > (k^2D)^{2/3}$. And for nonresonant particles Eq. (7.28), gives

$$(k^2D)^{2/3} \simeq \left(\frac{e^2k^2|E_k(t)|^2}{m^2}\,\frac{\gamma_0}{\omega_p}\right)^{2/3} = \omega_B^2\left(\frac{\omega_b\gamma_0}{\omega_p^2}\right)^{2/3}.$$

Since at saturation $\omega_B \sim \gamma_0$, the condition for nontrapping is easily satisfied.

The beam-plasma instability discussed here results in the turbulence of the electron plasma or Langmuir waves. As shown in Chapter IV, the electric potential φ of these turbulent waves are governed by the nonlinear Schrödinger equation

$$i\frac{\partial\varphi}{\partial t}+\frac{1}{2}\frac{\partial^2\omega}{\partial k^2}\frac{\partial^2\varphi}{\partial\xi^2}-\frac{\partial\omega}{\partial|\varphi|^2}|\varphi|^2\varphi=0.$$

When $(\partial^2\omega/\partial k^2)(\partial\omega/\partial|\varphi|^2) < 0$, the system is modulationally unstable. From Eqs. (7.24) and (7.46) it is seen that the Langmuir turbulence is modulationally stable; this is in agreement with the results of Asano and Taniuti (1969).

The thermal motions of the particles, which are inevitably present in a plasma have an appreciable effect on the evolution of the system. The presence of the thermal motion results in the reduction of the saturation level of the electric field. This is because of the fact that the diffusion in velocity space, which brings about the saturation, is strengthened by the randomness in the particle motion. The streaming motion of the beam loses more energy in the presence of thermal motion. This energy is shared by the fluctuations and the particles (thermal motion), the latter taking the bigger fraction due to the finite temperature. It is known that the nonlinear saturation of the beam-plasma instability results in a heating of the plasma electrons. The particle thermal motions produce a more efficient heating by slowing down the beam further and by bringing the level of saturation down. As is seen in the various results, the thermal velocity of the plasma particles have a more prominent effect than that of the beam particles.

Initially the system under consideration consists of a plasma, a beam traversing through it, and background fluctuations. The beam drives the resonant mode unstable and this instability is stabilized by the feedback effect of the growing waves on the particles. The end product of the analysis is a plasma with a larger spread in the thermal velocities, a beam with a reduced streaming velocity but larger thermal velocity traversing through the system, and a large-amplitude-stable wave (the saturated wave). The final configuration is stable as far as the saturated wave is concerned but not necessarily against other modes. In fact the resonant mode ($\omega_p \simeq ku_s$) may now be expected to grow. The nonlinear behaviour of this later time instability may be studied approximately by carrying out the complete analysis all over again. However, there are other factors like the presence of a large amplitude wave, broader spectrum of waves, which should also be taken into consideration.

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