STUDIES IN STRUCTURE AND PROPERTIES OF LIGHT NUCLEI

by

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PREFACE

A major problem in the theory of nuclei is to obtain an understanding of the nuclear forces operating within nuclei as distinct from the forces between two free nucleons. Despite the large amount of work by Brueckner and many others (which has helped to clarify and, to a large extent, justify the conceptual basis of nuclear shell model), calculations of nuclear properties, such as energy levels starting with free nucleon-nucleon interactions, present considerable mathematical complexity and are not in general easy to carry out without the aid of large computers. As a result, most shell model calculations are in practice done by assuming a relatively simple effective interaction. In this dissertation an attempt is made to study in a systematic way the nature of this effective interaction in several simple nuclei.

The work of Brueckner and others shows that the effective nuclear forces are non-local or momentum-dependent and also configuration-dependent. However, they are non-singular, i.e., will not have an infinite repulsive core, as is generally assumed in free nucleon forces. With these considerations in mind, we suggest in chapter II a method by which nuclear spectra may be calculated, and which may exhibit clearly the above properties of the effective forces. The idea is to

express the matrix elements of the nuclear Hamiltonian in terms of the matrix elements of the interaction in two-particle states of different spin, isotopic spin and relative orbital angular momentum. This procedure would be somewhat analogous to the analysis of the nucleon-nucleon scattering data in terms of phase shifts in different states of two nucleons. It is pointed out that, if enough experimental data on energy levels is available, it should be possible to study directly the nature of the effective interaction in different spin and isospin states and, in particular, a simple type of momentum-dependence, viz., angular-momentum-dependence of the forces, can be easily brought out. The formalism for such calculations is described in chapter II, and it is then applied to some simple nuclei such as Ti^{50}, O^{18} and T = 1 states in Li^{6}. Although the experimental data is not sufficient at present to make more definite analysis, it is pointed out that the even-state (singlet) interactions appear to be relatively configuration-independent, whereas the odd-state (triplet) forces appear to vary from nuclei to nuclei. More data will not only enable us to make these statements more precise, but will also enable us to extend the analysis to other states (such as T = 0 states), and other heavier nuclei. We would like to stress the importance of our method of analysing the effective interaction in states of relative orbital angular momentum which should be more powerful and useful to

elucidate the properties of the effective interaction rather than the conventional method of expressing the interaction in terms of various multipoles.

In chapter III some additional considerations are presented for the special case of isotopes O^{18} and O^{19}. In the first section, the results of our analysis of chapter II are compared with the results of other authors who made calculations of O^{18} spectrum. Next, the parameters of the effective interaction are tested by calculating the energy levels of O^{19}. It is found in chapter II that several different sets of parameters could give equally satisfactory results for O^{18}. Here we find that, if some of the excited states of O^{19} have their spins and parities measured, it may be possible to distinguish between some of these different sets of parameters. Finally, in section 3, the available data on the energy levels of O^{18} and O^{19} (eight levels) is reanalysed (by partial inversion of the energy matrices) to determine the matrix elements of the interaction in O^{18} states. This provides an additional check on the interaction parameters discussed in chapter II.

At the end of the thesis, reprints of our published work are included. These refer to the work included in the thesis as well as some work (nuclear interactions in $s_{1/2}$-doublets, etc.) not included in the thesis in the interest of homogeneity.

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CHAPTER I

INTRODUCTION

1. Nuclear Shell Model

Theoretical studies of the structure and properties of nuclei have been made for several years now, within the framework of the independent particle model or the nuclear shell model. The idea of the shell model with the introduction of a strong spin-orbit term was resurrected by Mayer and by Jensen et al, and it has had immense success in interpreting experimental results. The basic postulate of the shell model is that the complex interactions between the nucleons in a nucleus (about which we have very little precise information) may be largely replaced by an overall effective potential field V acting on each nucleon independently. There are already available several excellent reviews on the subject,[1,2] and we may briefly sketch some of the important aspects of the shell model which will be relevant for our subsequent discussions.

The solution of the Schrodinger equation for a nucleon moving in the potential V gives a complete set of orthonormal eigenfunctions, characterised by a set of quantum numbers, and the energy eigenvalues. We now consider the A (= N + Z) nucleons of the nucleus to occupy a set of these single particle eigenstates. Such a set of A occupied eigenstates will be

called a configuration. The total nuclear wavefunction in the simplest approximation is the properly anti-symmetrised product (Slater determinant) of the occupied eigenstates.

From an empirical analysis of the nuclear properties such as binding energies, ground state spins and parities, reduced widths of the low lying energy levels etc., we can obtain a rough picture of the single particle energy level scheme. We require that the potential V should, to a reasonable approximation, reproduce this empirical pattern. In all explicit shell model calculations, detailed properties of V do not enter at all. In such explicit calculations, the single particle energy levels, or rather the relative energies of the levels involved, are chosen as given by the experiments. Similarly the single particle wavefunctions are generally chosen to be those of a simple harmonic oscillator with a range parameter suitably related to the nuclear radius. This choice of the wavefunction is primarily dictated by ease in subsequent shell model calculations, and is partly justified by the argument that a harmonic oscillator potential with an appropriate spin-orbit term provides a fairly adequate explanation of the observed single particle level schemes. It should be possible to derive more detailed properties of V by empirical analysis of nuclear properties. Although this appears not to have been done systematically, we note that the inclusion of the spin-orbit term was dictated by the empirical pattern of the magic numbers; or the

observation of large quadrupole moments and transitions for many nuclei indicated that at least for some nuclei the single particle potential must deviate from spherical symmetry[3]. Other features of the potential V have become clear from recent theoretical analysis[4] of the foundations of the shell model. It has been shown from the equality of the binding energy per nucleon in the nucleus that the potential V must have a velocity dependence, so that it has a depth of about 70 MeV for the most tightly bound nucleons and about 35 MeV for the most weakly bound ones[5]. One may also surmise that the shape of the nuclear potential should roughly follow the nuclear density distribution, which can be well described in terms of a Woods-Saxon shape[6],

$$\left[1 + \exp(r-R)/b\right]^{-1}$$

Thus the harmonic oscillator is only an approximation, and it is well known that the use of the harmonic oscillator wavefunctions gives rise to incorrect binding energy results.

The other important concept underlying the shell model is that of the configuration space. We have defined the configuration as a set of particle states occupied by individual nucleons. A state of a given configuration is obtained by suitably coupling the angular momenta, spins and isotopic spins of individual nucleons to obtain the total angular momentum etc. Of course, the overall nuclear wavefunction must be made fully antisymmetric in all the

nucleons. In general, a given nuclear state may then be obtained by superposition of the suitable states of one or more configurations. In the lowest approximation the configuration space chosen is one dimensional i.e. only one configuration is assumed to describe the nuclear state satisfactorily. Various specialised versions of shell model are obtained depending upon the choice of the configuration space. For studying some detailed properties of the nuclear states it may be necessary to extend the configuration space to many dimensions, i.e. consider superposition or mixing of many configurations. In general the configuration space chosen depends upon the nature of the problem, the relative importance of the nearby configurations, and the ease of computation. It has been customary in shell model to include only the few energetically lowest configurations, and diagonalise the Hamiltonian matrix constructed in this limited space. The analysis of Brueckner and others[4] has provided an understanding of the validity of this procedure.

We may point out that in the model as described hitherto, there is considerable degeneracy in energy of nuclear states, i.e., there will exist several states with different spins J, but all degenerate in energy. For example, if we describe the nucleus O^{18} by the simple choice, viz., configuration $(d_{5/2})^2$, it will have three states with total angular momentum $J = 0, 2,$ and 4. Energetically, all the three states are degenerate, since they arise from the same configuration. Such degeneracies

are, however, not experimentally observed. To account for the observed splitting of these states, the shell model now introduces interactions between the nucleons; in other words correlations amongst the nucleons (which were till now considered independent), but only as small perturbations. It should be remarked that some correlations are already imposed by the action of the Pauli principle, and should be distinguished from these additional correlations which are now introduced empirically. The choice of these perturbing interactions has also been largely dictated by the ease of computation. Intuitively, one may characterise these inter-nucleon forces as residual effective interactions after the single particle potential V is extracted from the exact many body problem. In absence of apriori knowledge of these interactions, it has been customary to choose them in the variational spirit with a number of free parameters to be determined by comparison with the experimental results.

These nuclear interactions which we shall call effective interactions (they have also been referred to as pseudo potentials) are considered rather weak, so that the Hamiltonian matrix can be evaluated in Born approximation (first order perturbation theory) and the configuration mixing introduced (in particular high configurations) is small. They are generally chosen as central two-body forces of the form

$$V_{12} = V_0 \left[A_0 + A_1 M + A_2 B + A_3 MB \right] f(r_{12}) \qquad (1.1)$$

where M, B are the space and spin exchange operators (Majorana and Bartlett) and A_k are constants. The function $f(r_{12})$ of the distance between the two nucleons is chosen of the Gaussian or Yukawa shape, although a square-well or a δ-function are also occasionally used. Some calculations have also been made with inclusion of tensor or two-body spin-orbit forces. We should also add that, generally, more than two-body forces are neglected, and there are some phenomenological calculations which show that this seems to be a reasonably good approximation[7]. In general, then, we would have a very large number of arbitrary parameters in the interaction, and unless enough experimental data is available, or a systematic investigation can be made, or unless there is additional justification for choice of some of the parameters, it would be difficult to attach much physical significance to such empirical nuclear interactions. It does not appear to us that a satisfactory investigation on the nature of V_{12}, its uniqueness in a particular case, or for a group of nuclei, or a variation of its parameters as we go from one nucleus to another has been reported. Perhaps there has not been enough experimental data.

Even with the rather stringent limitation of only two-body central interactions, we find that most of the authors have chosen different interactions mostly as a matter of expediency. Table I lists some of the interactions reported in the literature; the list though not exhaustive, is representative. Listed are strengths of

Table 1

The relative strengths of the interactions, P_{ij}, in states with $i = 2T + 1$ and $j = 2S + 1$. P_{13} denotes the strength of interaction in $T = 0$, $S = 1$ state and is normalised to unity.

Author \ P_{ij}	P_{33}	P_{11}	P_{31}	P_{13}	Reference
Rosenfeld	-0.34	-1.78	0.60	1.0	8
Inglis	-0.60	-1.00	0.60	1.0	8
Soper	0.15	0.00	0.46	1.0	9
Meshkov & Ufford	0.20	-0.20	0.60	1.0	10
Talmi & Unna	-0.39	-0.37	0.51	1.0	11
Barker	0.38	0.81	0.50	1.0	12
Elliott & Flowers	-0.26	0.50	0.70	1.0	13
Peaslee	0.00	0.56	0.34	1.0	14
Pandya & Shah	1.10	0.78	0.43	1.0	15

the interaction in various spin and isotopic spin states of two nucleons, for example P_{31} denotes the strength of the interaction in $T = 1$, $S = 0$ state (singlet even). These are related to A_k of equation (1.1) by

$$\begin{aligned} P_{13} &= A_0 + A_1 + A_2 + A_3 = 1 \quad \text{(normalisation condition)} \\ P_{11} &= A_0 - A_1 - A_2 + A_3 \\ P_{31} &= A_0 + A_1 - A_2 - A_3 \\ P_{33} &= A_0 - A_1 + A_2 - A_3 \end{aligned} \tag{1.2}$$

It has been generally found that the effective two-body interaction appears to have a rather long range - of the order of the nuclear size - compared to the range of the free nucleon-nucleon interactions. Table 1 shows that there seems to be little agreement on the exchange nature of the interaction. These effective interactions do not appear to be of the Serber type. In particular, the forces in the odd states seem to be much less well determined, compared to the even states, for P_{31} appears to have a rather well defined value of ≈ 0.55.

A major aim of the work reported in this thesis is to attempt a systematic study as far as possible of the nature of the effective nuclear forces. The investigations of Brueckner et al previously mentioned have given considerable insight in the general qualitative nature of the effective interactions. In the next section we outline some of the ideas of the Brueckner theory, and then make

additional remarks on the problem of the effective nuclear interactions.

2. Brueckner Theory

The nuclear shell model as described in the previous section is an empirical model. It is based on an assumption that the nucleons in a nucleus move independently of each other in a common potential well; in other words, the Hartree-Fock procedure of atomic spectroscopy can also be applied to study nuclear spectra. The work initiated by Brueckner et al and carried on, extended and expanded by many investigators has now enabled us to understand why the shell model procedure has been so successful and has actually provided a more rigorous and somewhat modified Hartree-Fock procedure for application to nuclei.

2a. Realistic two-body potential

Any theoretical study of the nuclear structure starting from the first principles necessarily needs a complete knowledge of the nucleon-nucleon forces. A satisfactory procedure would be to obtain firstly from a theory of elementary particles the nature of the nucleon-nucleon forces, and then to proceed to solve the Schrödinger equation for the many-body problem with these interactions. However, at present, it is not even possible to derive the nucleon-nucleon forces rigorously from Meson theory. Our present knowledge of nuclear

forces is derived in a semi-phenomenological manner from a study of two nucleon systems such as the deuteron and the nucleon-nucleon scattering experiments. Two-nucleon potentials can be chosen to fit the experimental data and static as well as velocity-dependent or non-local forces have been used. Although non-local potentials have been explored by several authors, the static potentials, particularly those associated with the names of Gammel-Thaler and Signell-Harsmak appear to be more popular. It is now known that the interactions are strong, short ranged, have a hard repulsive core (which may be simulated by a non-local, non-singular potential) and are spin- and isotopic-spin-dependent. The potentials have central, tensor and spin-orbit terms and taking our cue from meson theory, they are given Yukawa shapes, i.e.

$$V = \infty \qquad r < r_c$$
$$= V_0 \left[\exp(-\mu r)\right]/\mu r \qquad r > r_c \text{ etc.}$$

The strength and range (V_0 and μ) in various states (18) can be specified, although not always unambiguously. The forces in T = 0 states are not precisely or unambiguously determined. We give in table 2 a set of parameters obtained by Gammel and Thaler[16]. These parameters have been used for nuclear spectroscopy calculations by Banerjee and Dutta Roy[17]. (For other sets of potentials also used in nuclear calculations see N. Moshkov[18] and Dawson, Talmi and Walecka[19].

Table 2

Parameters of the Gammel-Thaler potentials.
The potentials have the Yukawa shape $V_0[\exp(-\mu r)]/\mu r$ outside an infinite repulsive core of radius 0.4 fm [16].

State	Strength V_0 (MeV)	Inverse range μ $(fm)^{-1}$
Singlet even	-434	1.45
Triplet odd		
Central	-14	1.00
Tensor	22	0.80
Spin-orbit	-7315	3.70
Singlet odd	130	1.00
Triplet even		
Central	-877.4	2.091
Tensor	-159.4	1.045
Spin-orbit	-5000	3.70

Two points should be made. Firstly, although the role of non-local potentials is not yet clearly established, there seems to be an increasing awareness of their importance in view of the fact that they are non-singular and easy to handle in applying to calculations of nuclear structure. Secondly, one should remember that the analysis of two-nucleon systems only gives knowledge of nucleon-nucleon forces "on the energy shell". Matrix elements of the interaction "off the energy shell" can only be studied from more-than-two-body systems, such as H^3, He^3 or scattering of nucleons by deuterons etc. There is further the problem of possible existence of three-body or many-body forces[20,21] and the best that can be said is that at present there seems to be no definite phenomenological evidence in favour of their existence.

2b. Hamilton Matrix

We have now to consider the nucleus as an assembly of a large number of nucleons (A) interacting via the strong, short-ranged interactions described above. The total Hamiltonian is given by

$$H = \sum_{i=1}^{A} T_i + \sum_{i<j}^{A} V_{ij} \qquad (2.1)$$

where T_i = kinetic energy of the i^{th} nucleon and V_{ij} the two-body interaction potential. The energy of the system can be calculated from the Schrödinger equation

$$H\Psi = E\Psi \qquad (2.2)$$

The conventional Hartree-Fock procedure cannot be followed to obtain a solution of the above equation, when the Hamiltonian contains strong (and infinite in static case) forces. The achievement of the Brueckner theory is to provide a modified prescription for the evaluation of the energy of the state. This can be described as follows:

Introduce a single particle potential V_i operating on each nucleon i. Then a single particle wave equation can be written:

$$(T_\ell + V_\ell)\phi_\ell = \left(-\frac{\hbar^2}{2m}\nabla_\ell^2 + V_\ell\right)\phi_\ell = E_\ell\phi_\ell \ . \qquad (2.3)$$

With the solutions ϕ_ℓ and the single particle energies E_ℓ of the above equation, we construct a "reaction matrix" or a K - matrix,

$$(\ell m|K|rs) = (\ell m|V_{ij}|rs) + \sum_{pq} (\ell m|V_{ij}|pq)\frac{Q}{\Delta E_{pq}}(pq|K|rs) \qquad (2.4)$$

where the matrix elements in the space of the single particle states ϕ_ℓ etc. are denoted by the well known bracket notation, and ΔE_{pq} is the excitation energy of the two particle state (pq)

$$\Delta E_{pq} = E_\ell + E_m - E_p - E_q \ . \qquad (2.5)$$

The projection operator Q takes care of the exclusion principle, and is zero for all states for which $\Delta E_{pq}=0$ or approximately zero - as we discuss later, and Q = 1 otherwise. Thus K matrix is obtained from the solution of the above integral equation. Next, a self-consistency requirement is imposed by condition that the single particle potential should be given by

$$(\ell|V|m) = \sum_n \left\{ (\ell n|K|mn) - (\ell n|K|nm) \right\}. \qquad (2.6)$$

Thus the potential V obtained from the above equation should be identical with the potential with which we started in the equation (2.3), $V_\ell = (\ell|V|\ell)$. When the self-consistency requirement is satisfied, one may obtain the so-called energy shift,

$$\epsilon_{\ell m} = (\ell m|K|\ell m) - (\ell m|K|m\ell) \qquad (2.7)$$

and the total energy E of the system is then

$$E = \sum_\ell T_\ell + \sum_{\ell<m} \epsilon_{\ell m}$$
$$= \sum_\ell T_\ell + \tfrac{1}{2}\sum_\ell V_\ell \qquad (2.8)$$

We do not go into the details of the Brueckner theory, since these are adequately described in literature, and the present work does not deal with it. It is enough to point out that the determination of the

energy shift or the energy of the system is very similar to the usual Hartree-Fock method, except that here instead of the simple two-body interaction V_{ij}, one uses the reaction matrix K.

Let us summarise some of the results of the Brueckner formalism which are of interest to us.

(i) The single particle potential V occuring in the above formalism is a non-local potential, and therefore clearly the question arises as to how far it is proper to approximate it by the harmonic oscillator potential. We do not know about this, but most of the calculations on nuclear spectroscopy that have been recently performed following the ideas of Brueckner[17,18,19,22,23] appear to avoid this problem and use harmonic oscillator wavefunctions ϕ_ℓ to evaluate the reaction matrix. Banerjee and Dutta Roy remark that this procedure will not produce any large errors.

(ii) Although the two-body interaction V_{ij} is singular containing infinite repulsive core etc., it turns out that the K-matrix is much weaker, finite and analytically smooth and well-behaved. Thus it is possible to justify the use of first order

perturbation theory for the K-matrix (or its approximations i.e. the effective interactions), whereas for the V-matrix, this would lead to infinities. On the other hand even if the V_{ij} is local, the K-matrix is non-local and density dependent, as well as dependent on the configuration of the shell model states that is being considered. Thus in a strict sense the K-matrix is not a two-body operator, but for a given configuration space, it may be approximately considered as a two-body potential.

(iii) The effective nuclear interactions that have been used in shell model computations should not then be compared with the real nucleon-nucleon interaction V_{ij}, but should be regarded as approximate models of the K-matrix. Thus one should not expect them to have hard-cores or saturating properties etc.

(iv) One of the most important results that emerges is a consequence of the Pauli principle operating in nuclei. It is found that the form of the K-matrix in configuration space differs from V_{ij} only

at short distances $r \lesssim 1\ fm$ (called the healing distance). We shall have occasion to comment on this feature later in this thesis.

(v) In the shell model calculations, it was invariably assumed that for the configuration space one should include only the (energetically) lowest few configurations, and then the Hamiltonian matrix constructed in Born approximation in this space should be diagonalised. This approximation has received justification from Brueckner theory. As may be seen from equation (2.4) the higher two-particle excitations are all included already in the definition of the K-matrix. In practice one defines the operator Q so that in the sum over all excited states, all those states which are approximately degenerate with the ground state are excluded. The "approximately degenerate" has to be defined, but perhaps it is alright to define this to mean all configurations within about 2 MeV of the ground state. This should be adequate for considering the low lying states of nuclei. We shall adopt this definition[17] in our work. It is clear that the form of the K-matrix or the effective interaction will depend upon

the choice of the configuration space in which the Hamiltonian matrix is constructed. Although this was intuitively recognised, it is very well brought out from the Brueckner formalism, and this fact has to be borne in mind while comparing the different effective interactions.

We have summarised the results that are of interest to us, and hope to discuss the results of our calculations of effective interactions in various nuclei in the light of the above concepts.

3. Scope of the thesis

We have seen in the previous section that to evaluate the energy of a nuclear state, one must evaluate the corresponding K-matrix. The practical evaluation of the K-matrix for finite nuclei, is however quite difficult, particularly for potentials which are singular and contain infinite hard cores. As we mentioned above, the actual self-consistency problem is generally not attempted; even so the evaluation of the K-matrix involves a considerable amount of numerical computation. Since the work in this thesis was begun, several calculations[17,18,19] have been published which evaluate the K-matrix for various potentials and the energy levels of some nuclei. Although larger and larger number of such calculations will be performed in the near future, it would perhaps still be

desirable to have a simple model interaction, an approximation to the actual K-matrix which one can use in shell model calculations quickly, simply and effectively, to calculate the nuclear energy levels.

Usually, in the shell model calculations of nuclear spectra, one assumes some interaction as discussed in the previous section and calculates the resulting spectrum for comparison with experiment. In recent years an alternate procedure is also widely used. This is to take the observed spectrum and deduce from it some properties of the effective interaction. This process can be described as follows: For a given nucleus experiments may provide spins and parities and energies of a number of low levels. Other experiments such as stripping, or pick up of a nucleon or electromagnetic transitions might provide some information regarding the wavefunction of these states. If we have thus enough information on the eigenvalues (energies) and the eigenfunctions of different states, it may be possible to invert the Hamiltonian matrix (completely or partially) to obtain explicitely (all or some)the matrix elements of the interaction. We shall discuss it in detail later, but only mention it here that this entire procedure depends upon the choice of a suitable configuration space. Thus from a given nucleus or several nuclei of neighbouring mass values involving the same configurations, it may be possible to extract several matrix elements of the interaction. These matrix elements may then be analysed further

to obtain some properties of the effective interaction and its parameters. With some luck and enough experimental data, one may be thus able to obtain some information on the variation of the K-matrix with mass or configurations of nuclei etc. In chapter II we discuss in detail the methodology adopted by us. We would like to consider that the method of approach and analysis presented here is an important contribution, and a very useful one, whether or not the results obtained in this and subsequent chapters are completely satisfactory. This technique is then applied to simple nuclei such as Li^6, O^{18}, Ti^{50} etc. The energy levels are analysed and the results on effective interactions are presented.

In chapter III we present some additional calculations on the nuclei O^{18} and O^{19}. Combining the available information on these two nuclei, it is possible again to obtain more information on the nature of the effective interaction.

An additional investigation carried out during the course of this work is presented in the Appendix. This includes an analysis of effective nuclear interaction in $s_{1/2}$-$d_{3/2}$ subshell.

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CHAPTER II

EFFECTIVE INTERACTIONS IN NUCLEI

1. Introduction

A really interesting problem in shell model is to determine the nature of the effective two-body interaction in nuclei. As we mentioned in chapter I, it is possible to evaluate nuclear properties using a free nucleon-nucleon interaction such as Hamada-Thaler potential and the Brueckner-Bethe-Goldstone formalism; however, such calculations are rather complicated. Thus there exists a case for a serious attempt to obtain a relatively simple effective nuclear interaction for use in shell model calculations to predict nuclear properties with a fair degree of accuracy. This may be done by a detailed systematic study of the matrix elements of the effective interaction in states of two nucleons in various configurations. In this chapter we outline a method which we believe should be very useful for such a study and apply it to some simple calculations.

It is well known that the remarkable successes of the spherical shell model in predicting many of the nuclear properties and particularly the energy level schemes, have been obtained for nuclei near the closed shells, i.e. for nuclei with one, two or three nucleons (or holes) outside a closed shell. For simplicity, and to ensure that the results are dependable, we consider only nuclei with two nucleons outside a closed shell.

We further restrict ourselves to consideration of only $T = 1$ of two nucleons. The work described here can be extended also to $T = 0$ states in a similar way. However, experimental information on $T = 0$ states is not in an equally satisfactory state. The energy levels of three-nucleon systems (closed shell + 3) can easily be written down in terms of the matrix elements of the effective interaction in two nucleon states, with the use of fractional parentage coefficients. We discuss an example in the next chapter. A simultaneous analysis of related two- and three-nucleon systems would provide information on the role of three-body forces or change of effective interaction, if any, with change in the number of extra-core nucleons.

A standard technique for evaluating the matrix elements of a given two-body potential† is to expand it in Legendre polynomials[24]

$$V(\vec{r}_2 - \vec{r}_1) = \sum_k V_k(r_1, r_2) P_k(\cos\theta_{12}) \qquad (1.1)$$

and to express the matrix element in terms of Slater integrals,

$$F_k^{(abcd)} = \int V_k(r_1, r_2) R_a(r_1) R_b(r_2) R_c(r_1) R_d(r_2) \times d\vec{r}_1 d\vec{r}_2 . \qquad (1.2)$$

The effect of the potential is then entirely given in terms of the Slater integrals F_k. The spin-angle integrations

† We consider only local potentials, $V(\vec{r}_2 - \vec{r}_1)$.

are easily carried out in a general way using the tensor-algebraic techniques developed by Racah. Thus the energy levels of nuclei in p-shell can be expressed in terms of only two radial integrals F_0 and F_2, and in s-d shell by seven radial integrals[13]. Given an arbitrary local potential interaction, one may evaluate all the Slater integrals required in the calculation. However, for non-local potential this method becomes very cumbersome. There is however an alternate method which is much more flexible, more elegant and can reveal much better the detailed nature of the nuclear interaction. We shall show that it is also much easier to apply in actual calculations.

It may be remarked that in the analysis of the nucleon-nucleon scattering data, it is found convenient to analyse the data first in terms of the scattering phase shifts in states of different spin, isotopic-spin and relative orbital angular momentum ℓ. These phase shifts can then be discussed in terms of different potential models. Such an approach may well be useful in nuclear spectroscopy. In this case, one can analyse the available data on nuclear energy level spectra to obtain the matrix elements of the interaction in states of two nucleons with different spin, isotopic-spin and relative orbital angular momentum. One may then analyse these matrix elements of the potential in different states of two nucleons, and the possible configuration dependence

of the potential, locality or velocity dependence (i.e. dependence on ℓ) of the interaction etc. In the next section we describe in detail the procedure to be followed for such an analysis.

2. Method of Analysis

We shall consider only simple light nuclei such as Li^6, O^{18} and Ti^{50} (Fe^{54}). The single particle potential due to the core consisting of closed shells is taken to be

$$V = V_c + a(\vec{l}\cdot\vec{s}) = \tfrac{1}{2} m\omega^2 r^2 + a(\vec{l}\cdot\vec{s}) \qquad (2.1)$$
$$= \hbar\omega\nu r^2 + a(\vec{l}\cdot\vec{s})$$

and the value of the parameter a is suitably fixed. The single particle energy levels appropriate for the nucleus under consideration are taken from experimental data on closed shell plus one nucleon system. The single particle wavefunctions are chosen to be harmonic oscillator wavefunctions[24].

$$\Psi_{n\ell jm}(r,\theta,\phi) = \frac{R_{n\ell}(r)}{r} \sum_{m_\ell m_s} \langle \ell s m_\ell m_s | jm\rangle Y_\ell^{m_\ell}(\theta,\phi)\chi_s^{m_s} \qquad (2.2)$$

where $\langle \ell s m_\ell m_s | jm\rangle$ is the usual Clebsh-Gordan coefficient and $\chi_s^{m_s}$ is the spin wavefunction. The radial part $R_{n\ell}(r)$ is given as

$$R_{n\ell}(r) = \left[\frac{2^{\ell-n+2}\,(2\ell+2n+1)!!\,(2\nu)^{\ell+3/2}}{\sqrt{\pi}\, n!\,[(2\ell+1)!!]^2} \right]^{\frac{1}{2}} e^{-\nu r^2} r^{\ell+1} L^{(\ell+1/2)}_{n+\ell+1/2}(2\nu r^2) \quad (2.3)$$

where

$$L^{\ell+\frac{1}{2}}_{n+\ell+\frac{1}{2}}(2\nu r^2) = \sum_{k=0}^{n} (-1)^k 2^k \binom{n}{k} \frac{(2\ell+1)!!}{(2\ell+2k+1)!!} (2\nu r^2)^k \quad (2.4)$$

= Associated Laguerre polynomial.

The empirical parameter $r_\ell^2 = \frac{1}{\nu}$ appearing in the wavefunction is so adjusted that the r.m.s. radius calculated with these wavefunctions,

$$\langle r^2 \rangle = \tfrac{1}{2}(n+\tfrac{3}{2}) r_\ell^2 \quad (2.5)$$

is the same as that given by electron scattering experiments, or in absence of this data, by the simple expression $\sqrt{\frac{3}{5}} \times 1{\cdot}2$ fm.

The lowest few levels of a nucleus may be described in terms of an appropriate configuration space, in which, according to the usual practice, justified by the results of the Brueckner theory, only the lowest few energetically "almost degenerate" configurations are included. We can write the two-nucleon wavefunction

as†

$$|JM\rangle = |(n_1 l_1 s_1) j_1 (n_2 l_2 s_2) j_2 : JM\rangle$$

$$= \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | JM \rangle \Psi_{n_1 l_1 j_1 m_1}(r_1, \theta_1, \varphi_1) \Psi_{n_2 l_2 j_2 m_2}(r_2, \theta_2, \varphi_2) \qquad (2.6)$$

and the energy as

$$E_J = \epsilon(n_1 l_1 j_1) + \epsilon(n_2 l_2 j_2). \qquad (2.7)$$

The degeneracies in the above spectrum are removed by the introduction of the effective two-body interaction V_{12}. The Hamiltonian matrix in the assumed configuration space is constructed for each value of J,

$$\langle (n_1 l_1 s_1) j_1 (n_2 l_2 s_2) j_2 : JM | V_{12} | (n_1' l_1' s_1') j_1' (n_2' l_2' s_2') j_2' : JM \rangle \qquad (2.8)$$

and is diagonalised explicitely to obtain the eigenvalues and eigenfunctions of the lowest few states of interest.

In the above equation (2.8) the two-nucleon wavefunctions are written in j-j coupling formalism. The first step is then to convert them to L-S coupling, by a suitable

---

† We do not write down the isotopic-spin part explicitely since $T = 1$ is assumed in all cases. Again, although explicit antisymmetrisation of the above wavefunction is not displayed, it is taken account of in all calculations. See final equation (2.12).

transformation,

$$|(n_1 l_1 s_1) j_1 (n_2 l_2 s_2) j_2 : JM\rangle$$

$$= \sum_{LS} A \begin{pmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{pmatrix} |(n_1 l_1 n_2 l_2) L (s_1 s_2) S : JM\rangle \qquad (2.9)$$

where A - represents LS-jj transformation coefficient[25].

In the next step the space part of the wavefunction is expanded in terms of centre of mass and relative orbital angular momentum states of the two particles. This transformation has recently been widely used in nuclear spectroscopy calculations[26,27,28,29].

$$|(n_1 l_1 n_2 l_2) L\rangle = \sum_{N\Lambda n l} B^{n_1 l_1 n_2 l_2}_{N\Lambda n l}(L) \; |(N\Lambda n l) L\rangle \qquad (2.10)$$

where the coefficients $B^{n_1 l_1 n_2 l_2}_{N\Lambda n l}(L)$ are tabulated by Brody and Moshinsky[30].

Finally, we recouple the angular momenta, so that the two-nucleon spin state and the relative orbital angular momentum state are coupled together to give total angular momentum χ.

$$|(n_1 l_1 n_2 l_2) L (s_1 s_2) S : JM\rangle$$

$$= \sum_{N\Lambda n l} B^{n_1 l_1 n_2 l_2}_{N\Lambda n l}(L) \; |(N\Lambda n l) L (s_1 s_2) S : JM\rangle \qquad (2.11)$$

$$= \sum_{\substack{N\Lambda n l \\ \chi}} B^{n_1 l_1 n_2 l_2}_{N\Lambda n l}(L) \; U(\Lambda l J S; L\chi) \; |N\Lambda (n l, S)\chi : JM\rangle .$$

Here $U(abcd;ef)$ is the normalised Racah coefficient.

Now combining above equations, it is possible to write the matrix element of the Hamiltonian in equation (2.8), in terms of the matrix elements of V_{12} in two nucleon states of total spin S, and relative orbital angular momentum l.

$$\langle (n l, S) x \,|\, V_{12} \,|\, (n' l', S') x \rangle .$$

Thus

$$\langle (n_1 l_1 s_1) j_1 (n_2 l_2 s_2) j_2 : JM \,|\, V_{12} \,|\, (n_1' l_1' s_1') j_1' (n_2' l_2' s_2') j_2' : JM \rangle$$

$$= \sum_{\substack{LSL'S' \\ N\Lambda n l n' l' \chi}} a a' \left[1 + (-1)^{S+l}\right] \left[1 + (-1)^{S'+l'}\right] \times$$

$$\times A \begin{pmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{pmatrix} A \begin{pmatrix} l_1' & s_1' & j_1' \\ l_2' & s_2' & j_2' \\ L' & S' & J \end{pmatrix} B^{n_1 l_1 n_2 l_2}_{N\Lambda n l}(L) \, B^{n_1' l_1' n_2' l_2'}_{N\Lambda n' l'}(L') \times \qquad (2.12)$$

$$\times U(\Lambda l J S'; L \chi) U(\Lambda l' J S'; L' \chi) \times$$

$$\times \langle (n l, S) \chi \,|\, V_{12} \,|\, (n' l', s') \chi \rangle$$

where a and $a' = \frac{1}{2}$ for equivalent particles

$= \frac{1}{\sqrt{2}}$ for inequivalent particles.

The factors in the first line of the right side of the above equation take care of explicit antisymmetrisation of the wavefunctions.

It is possible to simplify the above expression in some approximation. For example, if one considers only central interactions V^{c}_{12}, we obtain $\ell = \ell'$, $s = s'$ and the matrix element is independent of the value of x. In that case the sum over x can be carried out directly and one obtains,

$$\left\langle (n,\ell,s_1) j_1 (n_2 \ell_2 s_2) j_2 : JM \,\middle|\, V^{c}_{12} \,\middle|\, (n'\ell' s_1') j_1' (n_2' \ell_2' s_2') j_2' : JM \right\rangle$$

$$= \sum_{\substack{LL'S \\ N\Lambda n\ell n'}} a a' \left[1 + (-1)^{\ell+S} \right]^2 A \begin{pmatrix} \ell_1 & s_1 & j_1 \\ \ell_2 & s_2 & j_2 \\ L & S & J \end{pmatrix} A \begin{pmatrix} \ell_1' & s_1' & j_1' \\ \ell_2' & s_2' & j_2' \\ L' & S & J \end{pmatrix} \times$$

$$B^{n,\ell_1,n_2\ell_2}_{N\Lambda n\ell}(L)\, B^{n'\ell_1'n_2'\ell_2'}_{N\Lambda n'\ell}(L')\, \delta_{LL'} \left\langle n\ell, S \,\middle|\, V^{c}_{12} \,\middle|\, n'\ell, S \right\rangle . \qquad (2.13)$$

Now if we study only two-nucleon states with isotopic-spin $T = 1$, the matrix elements of the interaction need be considered only in singlet even ($S = 0$, ℓ = even) and triplet odd ($S = 1$, ℓ = odd) states. Further, if we make an additional approximation of considering only singlet even interactions (Serber-type forces), we obtain

$$\left\langle (n,\ell,s_1) j_1 (n_2 \ell_2 s_2) j_2 : JM \,\middle|\, V^{cs}_{12} \,\middle|\, (n'\ell' s_1') j_1' (n_2' \ell_2' s_2') j_2' : JM \right\rangle$$

$$= \sum_{\substack{N\Lambda \\ n n'\ell}} 4\, a a'\, A \begin{pmatrix} \ell_1 & s_1 & j_1 \\ \ell_2 & s_2 & j_2 \\ L & 0 & J \end{pmatrix} A \begin{pmatrix} \ell_1' & s_1' & j_1' \\ \ell_2' & s_2' & j_2' \\ L & 0 & J \end{pmatrix} \times$$

$$B^{n,\ell_1,n_2\ell_2}_{N\Lambda n\ell}(L=J)\, B^{n'\ell_1'n_2'\ell_2'}_{N\Lambda n'\ell}(L=J) \left\langle n\ell \,\middle|\, V^{cs}_{12} \,\middle|\, n'\ell \right\rangle . \qquad (2.14)$$

If we further consider only interactions in singlet s-states ($\ell = 0$), the above expression becomes even more simple.

We note that tables of A- and B-coefficients are available[25,30], and the radial matrix elements can be written down once and for all as a function of $\lambda = r_0/r_\ell$ for a given potential shape, so that the evaluation of any matrix element is quite simple. The expression for the radial matrix elements $\langle n\ell | V_{12} | n\ell \rangle$ in terms of I_ℓ are given in table 1. Table 2 gives the values of these matrix elements that we shall need in subsequent calculations. For this evaluation we assume the potential to have Gaussian shape $\exp[-(r/r_0)^2]$, and the matrix elements are listed as functions of λ , in units of the depth of the potential, V_0. We note that $I_\ell = (\lambda^2/1+\lambda^2)^{\ell+3/2}$. Figure 1 also shows a plot of the matrix elements $\langle n\ell | V_{12} | n\ell \rangle$ in units of I_{0s}, as a function of λ . Here it may be remarked that when we take the limiting case, $\lambda \to 0$, I_{0s} tends to some finite value, so that $I_{0s} \neq 0$, but some finite constant and all other $I_{n\ell}\,(\ell \neq 0)$ are zero. It can be seen from table 1 that in such a limiting case $I_{1s} = 1.5\, I_{0s}$, $I_{2s} = 1.875\, I_{0s}$, $I_{3s} = 2.187\, I_{0s}$ etc. Similarly, for $\lambda \to \infty$, we get in the limit, $I_{n\ell} = I_{0s}$ for all $n\ell$. A further advantage of this formalism is that we can consider the matrix elements of V_{12} in states of different ℓ and s as arbitrary parameters, and provided sufficient experimental data is available, we can determine them empirically,

Table 1

The expressions for the matrix elements $\langle n\ell | V_{12} | n\ell \rangle$ in terms of $I_\ell = (\lambda^2/1+\lambda^2)^{\ell+3/2}$, where $\lambda = r_0/r_\ell$

$\langle n\ell \mid V_{12} \mid n\ell \rangle$	The expressions in terms of I_ℓ.
$\langle 0s \mid V_{12} \mid 0s \rangle$	I_0
$\langle 0p \mid V_{12} \mid 0p \rangle$	I_1
$\langle 0d \mid V_{12} \mid 0d \rangle$	I_2
$\langle 0f \mid V_{12} \mid 0f \rangle$	I_3
$\langle 0g \mid V_{12} \mid 0g \rangle$	I_4
$\langle 0h \mid V_{12} \mid 0h \rangle$	I_5
$\langle 0i \mid V_{12} \mid 0i \rangle$	I_6
$\langle 1s \mid V_{12} \mid 1s \rangle$	$1.5I_0-3I_1+2.5I_2$
$\langle 2s \mid V_{12} \mid 2s \rangle$	$1.875I_0-7.5I_1+16.25I_2-17.5I_3+7.87I_4$
$\langle 3s \mid V_{12} \mid 3s \rangle$	$2.1875I_0-13.1250I_1+45.9375I_2-96.25I_3$ $+123.0625I_4-86.6230I_5+26.8125I_6$
$\langle 1p \mid V_{12} \mid 1p \rangle$	$2.5I_1-5I_2+3.5I_3$
$\langle 2p \mid V_{12} \mid 2p \rangle$	$4.375I_1-17.5I_2+33.25I_3-31.50I_4+12.375I_5$
$\langle 1d \mid V_{12} \mid 1d \rangle$	$3.50I_2-7I_3+4.5I_4$
$\langle 2d \mid V_{12} \mid 2d \rangle$	$7.875I_2-31.50I_3+56.25I_4-49.50I_5+17.875I_6$
$\langle 1f \mid V_{12} \mid 1f \rangle$	$4.5I_3-9.0I_4+5.5I_5$
$\langle 1g \mid V_{12} \mid 1g \rangle$	$5.5I_4-11.0I_5+6.5I_6$

Table 2

The matrix elements $I_{nl} = \langle nl | V_{12} | nl \rangle$ for the Gaussian potential $\exp(-r^2/r_l^2)$ for different values of $\lambda = r_0/r_l$.

I_{nl}	0.5	0.8	1.0	1.2
I_{0s}	0.0695	0.2437	0.3536	0.4536
I_{1s}	0.0696	0.1730	0.2310	0.2726
I_{2s}	0.0809	0.1377	0.1739	0.2127
I_{3s}	0.0736	0.1239	0.1518	0.1803
I_{0p}	0.0179	0.0951	0.1768	0.2676
I_{1p}	0.0282	0.1030	0.1547	0.2052
I_{2p}	0.0342	0.0966	0.1361	0.1743
I_{0d}	0.0036	0.0371	0.0924	0.1590
I_{1d}	0.0083	0.0540	0.0994	0.1481
I_{2d}	0.0128	0.0533	0.1008	0.1367
I_{0f}	0.0007	0.0145	0.0442	0.0932
I_{1f}	0.0020	0.0261	0.0606	0.1038
I_{0g}	0.0001	0.0057	0.0221	0.0550
I_{1g}	0.0005	0.0130	0.0363	0.0692
I_{0h}	0.0000	0.0023	0.0110	0.0325
I_{0i}	0.0000	0.0009	0.0055	0.0192

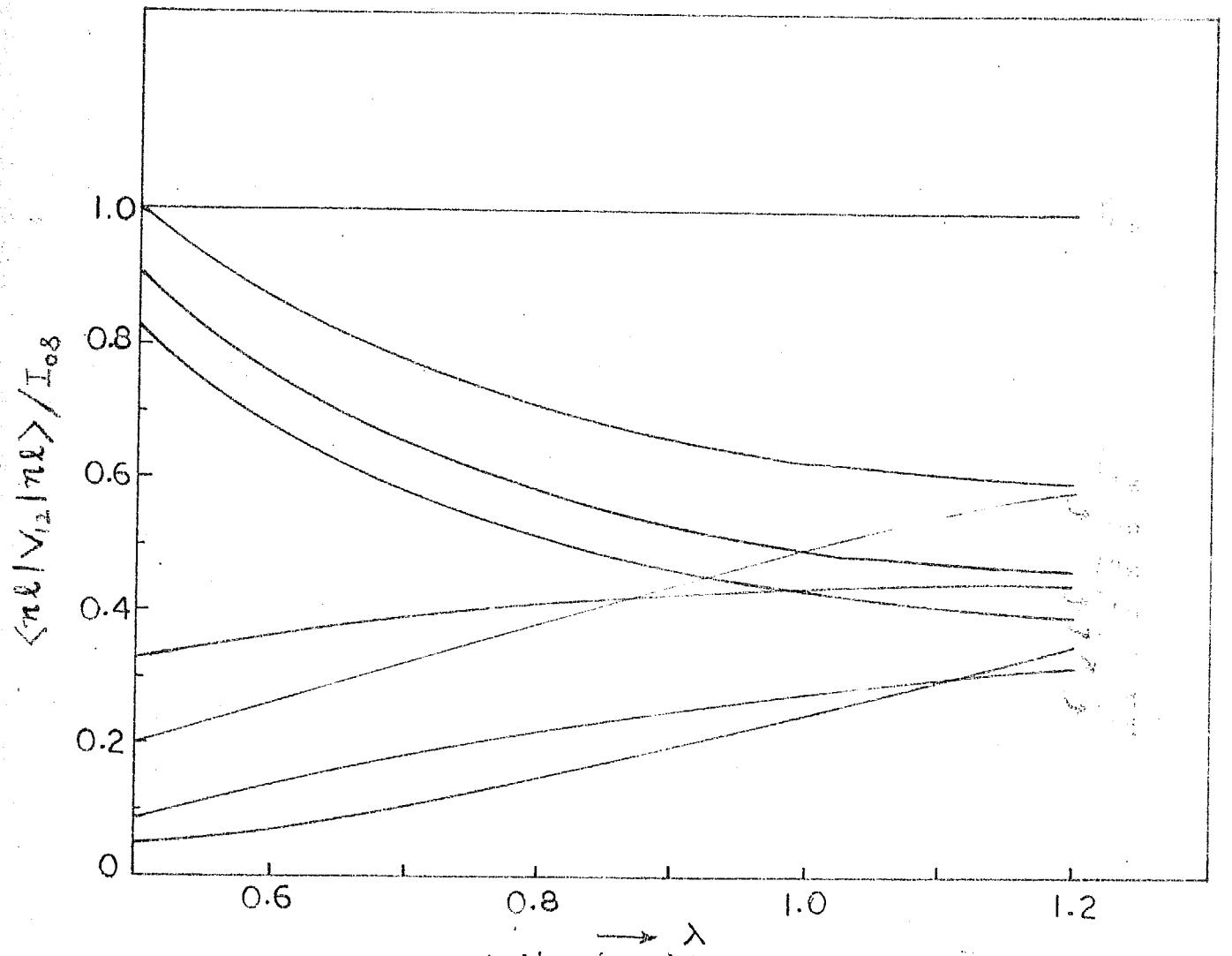

Fig. 1. The matrix $\langle n\ell | V_{12} | n\ell \rangle$ [illegible] for [illegible] $\lambda = \kappa_0/\kappa_{1s}$.

without having to assume anything about the nature of the potential. Unfortunately, in general, not enough experimental data is available, and one has to make some assumptions on the nature of the potential. In the next section we shall treat in some detail the extraction of the matrix elements from available data on various nuclei.

3. Pairing Energies

We would now describe some results on the nature of the energy levels to be expected on the basis of a very simple assumption viz., consider singlet s-state interactions only.

We consider two equivalent nucleons $(nlj)^2$. The energy of a state of spin J of this configuration is then, for singlet s-state ($S = 0$, $l = 0$) interactions only,

$$E\left[(nlj)^2 : J\right] = \sum_{Nn} A^2 \begin{pmatrix} l & \frac{1}{2} & j \\ l & \frac{1}{2} & j \\ J & 0 & J \end{pmatrix} B^2_{NJn0}(J) \langle ns | V_{12} | ns \rangle . \quad (3.1)$$

Now if we consider the states with $J = 0$, then we can study the behaviour of pairing energies $E\left[(j)^2_0\right]$ for s-interactions only in different configurations. For such a case the expression for the energy of the state is particularly simple,

$$E\left[(j)^2_0\right] = \frac{2j+1}{2(2l+1)} \sum_{Nn} B^2_{N0n0}(0) \langle ns | V_{12} | ns \rangle . \quad (3.2)$$

Thus for a given ℓ, the dependence on j is only given by the factor $(2j+1)$. The values of $E^2/2\ell+1$ required in the above formula are given in table 3. The final results would depend upon the behaviour of the matrix elements with n, and this in turn would depend upon the range of the potential or the parameter $\lambda = r_0/r_\ell$. Let $I_0 = \langle os|V_{12}|os\rangle$, $I_1 = \langle op|V_{12}|op\rangle$ etc. Then we put $X_1 = I_1/I_0$, $X_2 = I_2/I_0$ etc. and write the expressions for $2E_0/(2j+1)I_0$ knowing the values of $E^2/2\ell+1$ from table 3, as

$$\begin{aligned}
\frac{2E_0}{(2j+1)I_0} &= 1 \quad \text{for } (os)^2 \\
&= 0.166(1+X_1) \quad \text{for } (op)^2 \\
&= 0.033(1+X_2) + 0.111X_1 \quad \text{for } (od)^2 \qquad (3.3) \\
&= 0.007(1+X_3) + 0.050(X_1+X_2) \quad \text{for } (of)^2 \\
&= 0.0015(1+X_4) + 0.016(X_1+X_3) \\
&\quad + 0.040X_2 \quad \text{for } (og)^2
\end{aligned}$$

We also give a plot of $2E_0/(2j+1)I_0$ for different oscillator orbits ℓ, and different choice of behaviour of I_n with n in figure 2. The results are given in table 4. The general trend is quite clear, viz., E_0 increases with j for the given ℓ, and $E_0/2j+1$ decreases with ℓ.

The energy levels (with $T = 1$) of some simple configuration $(p_{3/2})^2$, $(d_{3/2})^2$, $(d_{5/2})^2$, $(f_{5/2})^2$ and $(f_{7/2})^2$ are evaluated for singlet s-state interactions only using equation (2.16). Figure 3 shows the separation of levels $J = 0$ and $J = 2$ as a function of λ, and figure 4 shows

Table 3

The table gives the values of $B^2/2\ell+1$ for different configuration showing the ℓ dependence of E_0 in equation (3.2).

Configuration	N	n	B	B^2	$\frac{B^2}{2\ell+1}$
$(os)^2$	0	0	1.0	1.0	1.0
$(op)^2$	1	0	0.707	0.5	0.166
	0	1	-0.707	0.5	0.166
$(od)^2$	0	2	0.408	0.166	0.033
	2	0	0.408	0.166	0.033
	1	1	-0.745	0.555	0.111
$(of)^2$	0	3	0.294	0.050	0.007
	3	0	-0.294	0.050	0.007
	1	2	-0.592	0.350	0.050
	2	1	0.592	0.350	0.050
$(og)^2$	0	4	0.119	0.014	0.0015
	4	0	-0.119	0.014	0.0015
	1	3	0.414	0.171	0.019
	3	1	-0.414	0.171	0.019
	2	2	0.400	0.360	0.040

Table 4

The values of $2B_0/(2j+1)I_0$ for different configurations corresponding to different nature of interactions.

No.	Parameters	$(\sigma_s)^2$	$(\sigma_p)^2$	$(\sigma_d)^2$	$(\sigma_f)^2$	$(\sigma_g)^2$
a	Long range force $X_n=1$ for all n	1	0.338	0.177	0.114	0.121
b	Hypothetical $X_n=0$ for all n	1	0.166	0.033	0.007	0.0015
c	δ-fn. $X_1 = 1.80$ $X_2 = 1.875$ $X_3 = 2.00$ $X_4 = 2.10$	1	0.415	0.261	0.190	0.146
d	$\lambda = 0.8$ $X_1 = 0.71$ $X_2 = 0.67$ $X_3 = 0.60$ $X_4 = 0.40$	1	0.384	0.131	0.075	0.048
e	$\lambda = 1.0$ $X_1 = 0.53$ $X_2 = 0.49$ $X_3 = 0.43$ $X_4 = 0.40$	1	0.371	0.119	0.066	0.042

$\frac{2E_0}{(2j+1)I_0}$

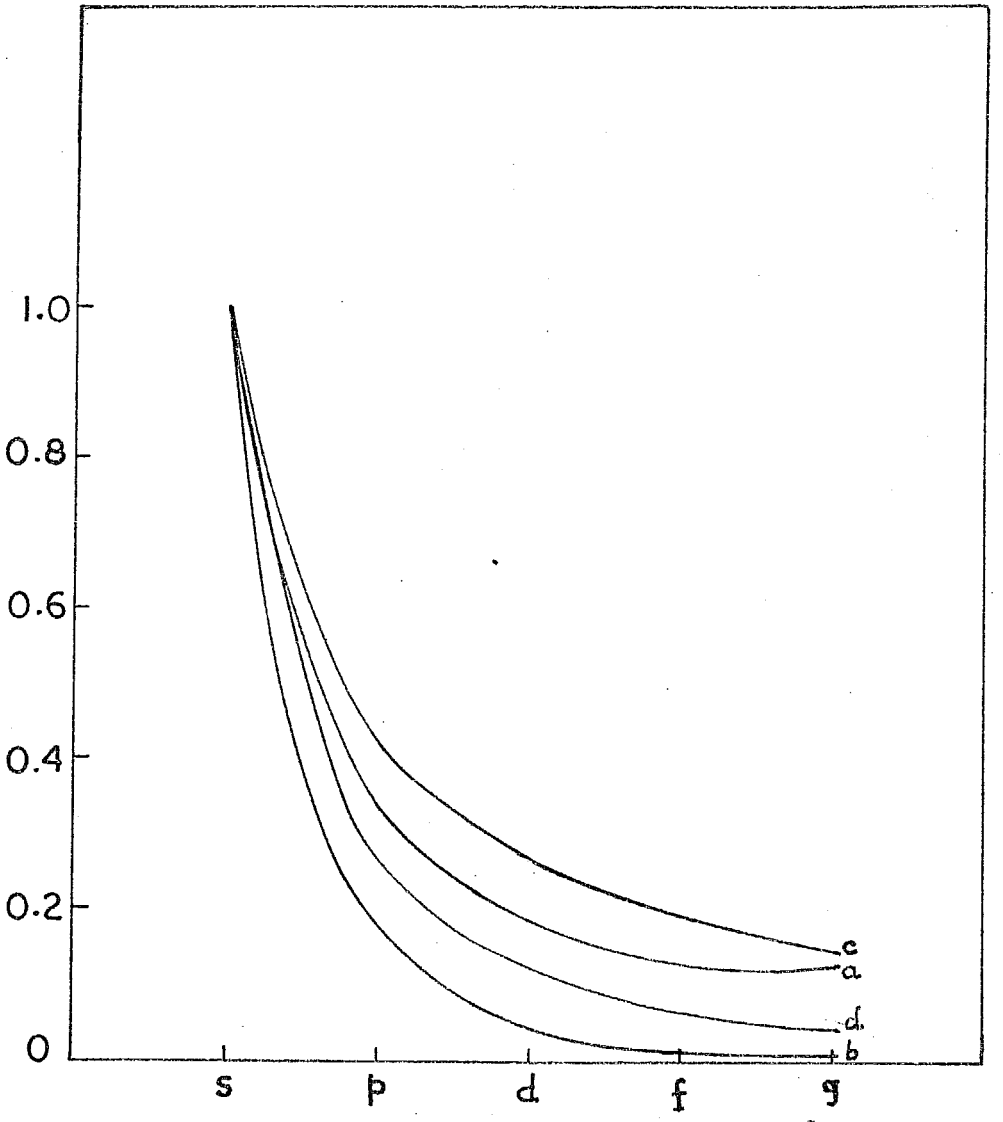

Fig. 7. $2E_0/(2j+1)I_0$ for different interactions against different oscillator orbitals ℓ.

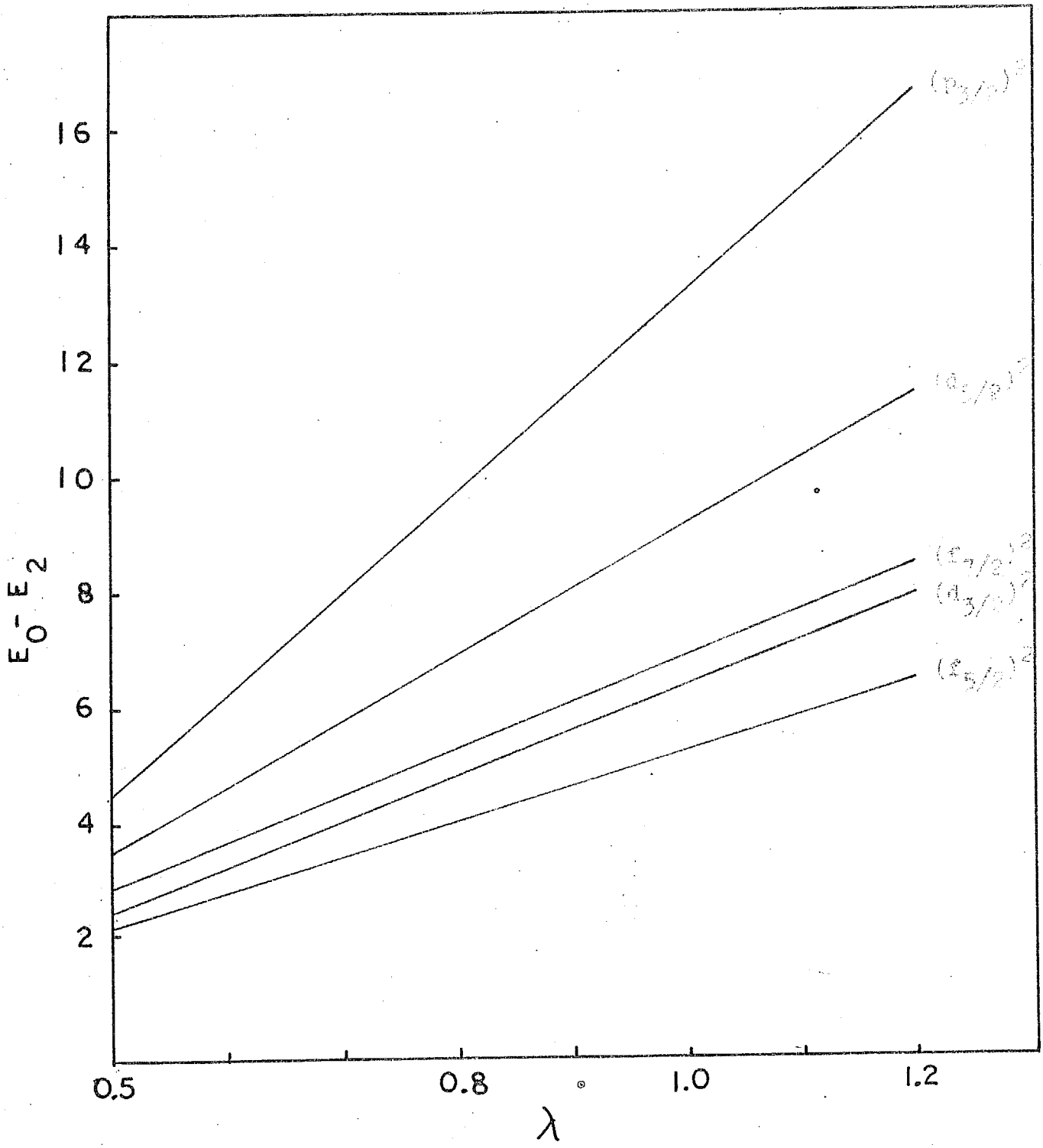

Fig. 1. $E_0 - E_2$ (in arbitrary units) as a function of λ for different two nucleon configurations.

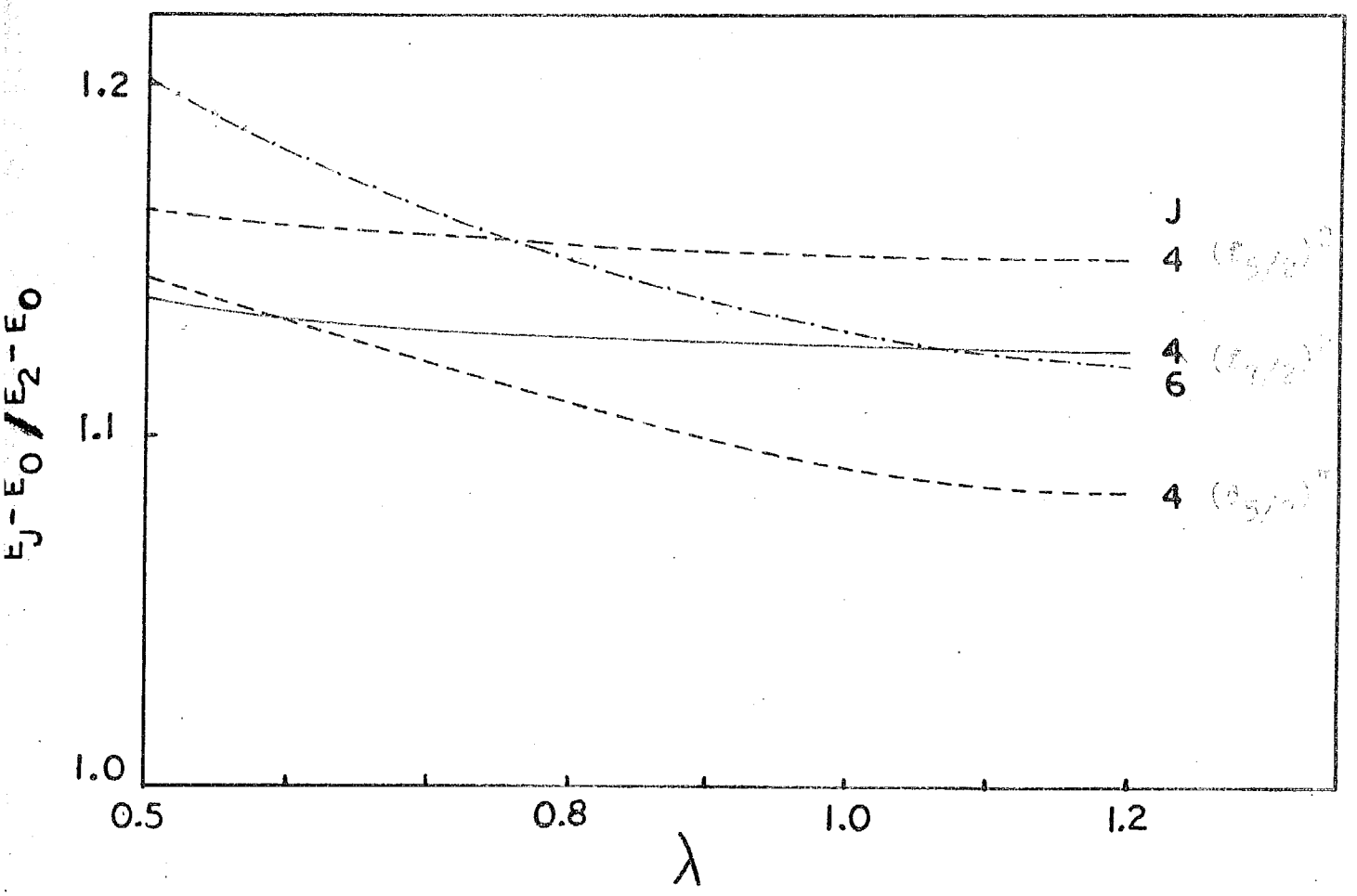

Fig.4. The splittings of the seniority 2 levels relative to $E_2 - E_0$ as a function of λ for different configurations.

the splitting of the seniority 2 levels $J \neq 0$ relative to the $J = 0$, 2 separation varying with λ.

4. Analysis of closed shell+2 nucleon configurations

In this section we use the formalism outlined in section 2 to study the nature of the effective interaction which would correctly predict the energy levels of some simple configurations in nuclei Li^6, O^{18} and Ti^{50}. Our purpose is to see if a single set of interaction parameters can explain energy levels in all these nuclei, or if there is any evidence for configuration-dependence of the interaction. This analysis can be further extended to cover heavier nuclei such as Zr^{90} or two-nucleon configuration in the vicinity of A = 208. Since the present investigation reported here is tentative and designed primarily to see if the methodology of section 2 can be successfully applied, our programme for the present has not been ambitious. It is however hoped to undertake a thorough systematic analysis of a large number of two nucleon configurations in near future.

Although in principle we can consider all the matrix elements $\langle n\ell, S:\alpha | V_{12} | n'\ell', S:\alpha \rangle$ in a Hamiltonian matrix as independent variables to be determined empirically from an analysis of the experimental data, we shall find that in general there is not enough data to do this. We therefore make some simple assumptions.

Firstly, we consider only central two-body interactions. Secondly, we neglect the contributions to the energy of the matrix elements in states with $\ell \geq 3$. This seems reasonable in view of the combined effects of the Pauli principle (which gives rise to a finite healing distance) and the centrifugal barrier. One can also see from figure 1 that at a reasonable value of range parameter λ, the matrix elements of higher states are quite small. We note that if the non-central forces are included in the analysis, the coupling of the $\ell = 1$ and $\ell = 3$ states may play an important role.

We may then consider the strength and range of the potential in each of the states $\ell = 0$, 2 ($S = 0$) and $\ell = 1$ ($S = 1$) as independent parameters. (Note that for a definite range of the interaction the matrix elements $\langle n\ell | V_{12} | n\ell \rangle$ for different n but same ℓ are all related). Even this gives too many arbitrary parameters for the data available at present, as we shall see. Therefore we shall consider two types of potentials: (a) we assume the even state interaction to be operative only in $\ell = 0$ state, and no interaction in $\ell = 2$ state. This makes the interaction rather strongly non-local. (b) More conventionally, we shall consider the same potential to be operative in $\ell = 0$ and $\ell = 2$ states. We shall allow the odd and even state interaction to have different range and strengths. In most of the usual shell model calculations the same range is assumed for interactions in all states.

The observed level schemes[31] of Li^6, O^{18} and Ti^{50} are shown in the figure 6. Spins and parities of the lowest six states of Li^6 are well known. These states are attributed to the $(p)^2$ configuration of the two extra-core nucleons, and have been analysed by a number of authors[9,10,32,33] to derive information on the nature of the nuclear forces. Our discussion shall be mainly in the framework of the analysis of Pinkston and Brennan[33]. The energy levels of the $(p)^2$ configuration in the intermediate coupling shell model scheme are described by the spin-orbit interaction parameter $\underline{a}$, and the matrix elements of V_{12} in the six states ^1S, ^1D, $^3P(T=1)$ and ^3S, ^3D, $^1P(T=0)$ of the two nucleons; here S, P, D refer to $L = 0, 1, 2$ in our notation. The Hamiltonian matrices can be atleast partially inverted to obtain some of the parameters directly in terms of the observed energies. The value of the parameter $\underline{a}$ is in this way determined to be $\underline{a} = -1.55$ MeV. It is also found that the major components of the lowest six states are singlet and triplet even states, and their energies are relatively insensitive to the matrix elements in the P-states. With reasonable assumptions, Pinkston and Brennan find

$$\begin{aligned} \langle {}^1S | V_{12} | {}^1S \rangle &= -3.85 \text{ MeV} \\ \langle {}^1D | V_{12} | {}^1D \rangle &= -2.12 \text{ MeV} \\ \langle {}^3P | V_{12} | {}^3P \rangle &\approx 0 \quad \text{MeV}. \end{aligned} \tag{4.1}$$

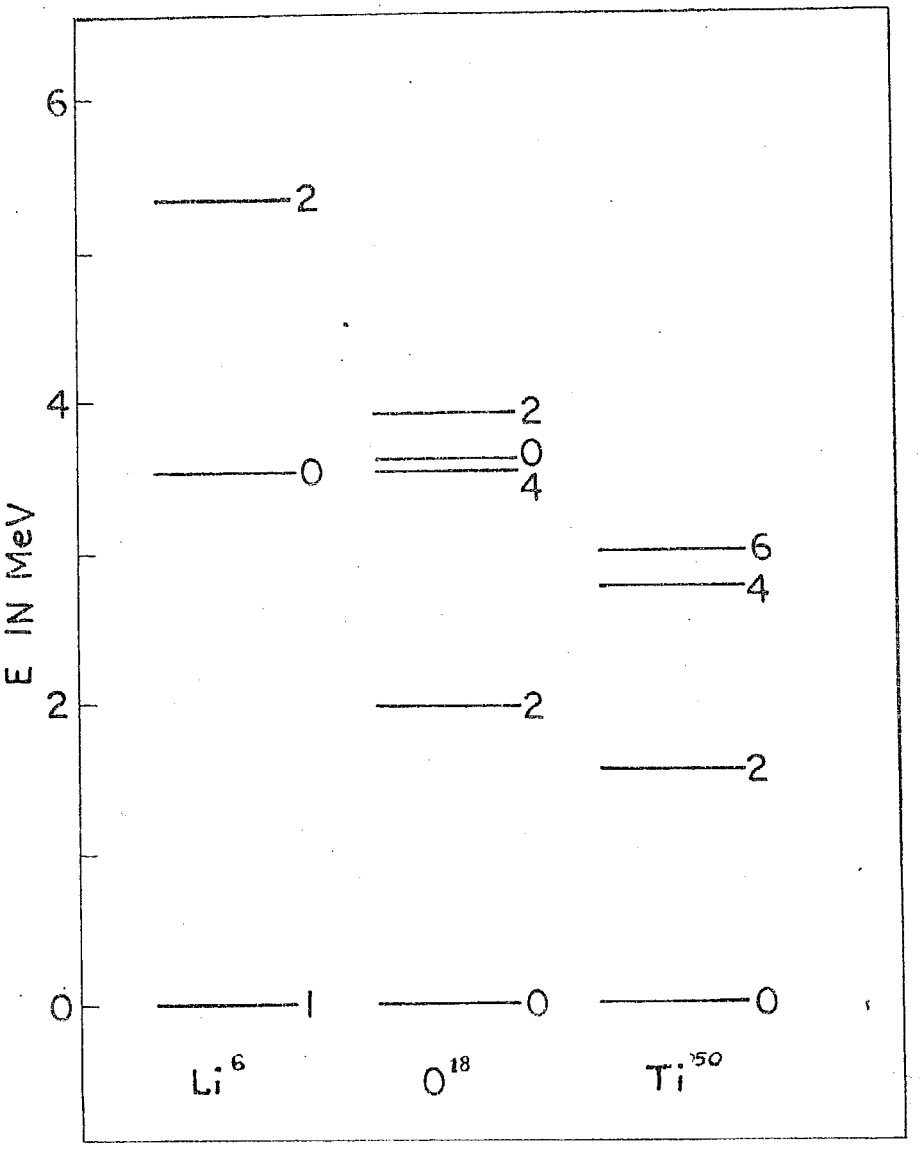

Fig. 7. The energy levels observed in nuclei Li^6, O^{18} and Ti^{50}.

These values can be made more precise, when the P-states which are expected to occur at $\sim$ 8-10 MeV are identified. In terms of states of relative orbital angular momenta† matrix elements in ^1S and ^1D states can immediately be written down

$$\langle {}^1S | V_{12} | {}^1S \rangle = \frac{1}{2}\left[\langle 0s | V_{12} | 0s \rangle + \langle 1s | V_{12} | 1s \rangle\right]$$

$$\langle {}^1D | V_{12} | {}^1D \rangle = \frac{1}{2}\left[\langle 0s | V_{12} | 0s \rangle + \langle 0d | V_{12} | 0d \rangle\right] \qquad (4.2)$$

From the known values of these two matrix elements on the left hand side, and with the assumption of V_{12} to be a Gaussian shape potential it is possible to determine the parameters λ and the depth V_0 of this potential in singlet even states. If we assume the effective interaction to be operative only in s-states ($l = 0$) we obtain

$$\langle 0s | V_{12} | 0s \rangle = -4.24 \text{ MeV}$$

$$\langle 1s | V_{12} | 1s \rangle = -3.46 \text{ MeV} \qquad (4.3)$$

and

$$V_0 = -24 \text{ MeV}$$

$$\lambda = 0.68 .$$

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† The analysis of Pinkston and Brennan has already yielded matrix elements in L-S coupling representation, hence we do not need the A-coefficient in this case.

On the otherhand, if we assume that the same interaction operates in $\ell = 0, 2$ states, we find

$$\langle 0S | V_{12} | 0S \rangle = -4.0 \text{ MeV}$$
$$\langle 0d | V_{12} | 0d \rangle = -0.20 \text{ MeV}$$
$$\langle 1S | V_{12} | 1S \rangle = -3.70 \text{ MeV} \qquad (4.4)$$

and

$$V_0 = -34 \text{ MeV}$$
$$\lambda = 0.57.$$

The range of the potential can be obtained by calculating the value of r_ℓ. For harmonic oscillator wavefunction, the root mean square radius is given by

$$\langle r^2 \rangle = \frac{5}{4} r_p^2 = (2.7)^2 fm^2 \qquad (4.5)$$

where the last step is obtained from the electron scattering data[34]. Thus $r_p = 2.5$ fm, and, for $\lambda = 0.62$ and 0.57 the range r_0 has the values 1.7 fm and 1.4 fm respectively.

In the above analysis the p-state matrix element was assumed to be zero following Pinkston and Brennan. We wish to emphasise that the knowledge of the odd-state interactions in light nuclei is very very inadequate, particularly since the energies of the lowest few levels (against which our nuclear force parameters can be tested) are not affected to any large extent by these odd-state interactions. We have already listed in table 1 of chapter I many different exchange interactions used in

light nuclei. We note that the range of the interaction in all cases is assumed to be same for odd- and even-state interactions.

One of the aims of our analysis is to obtain a better knowledge of p-state interactions. It is clear empirically that at least for Li^6 the p-state matrix element is small. Taking the potential strength of Boyer[9] or Meshkov and Ufford[10], and the same range for odd and even forces, we find

$$\langle 0p|V_{12}|0p\rangle \simeq -0.5 \text{ MeV}.$$

This is consistent with the zero value of Pinkston and Brennan, and inserting this value in the Hamiltonian matrices, we can easily show that the singlet even energies are not essentially affected.

Now let us consider the nuclei O^{18} and Ti^{50}. The lowest few levels will be attributed to the configurations of the last two nucleons in the unfilled subshells. In the spirit of the Brueckner theory we include for configuration mixing only approximately degenerate orbits. We therefore restrict ourselves to configurations which lie within 3-4 MeV of the ground state configuration. Thus for O^{18}, we only consider the states

$$(d_{5/2})^2 \; J = 0, 2, 4 \quad (d_{5/2}s_{1/2}) \; J = 2,3 \text{ and}$$

$$(s_{1/2})^2 \; J = 0,$$

and for Ti^{50}

$(f_{7/2})^2$ J = 0,2,4,6 $(f_{7/2}p_{3/2})$ J = 2, 4 and

$(p_{3/2})^2$ J = 0,2.

The energy separation of the single particle levels $d_{5/2}$ and $s_{1/2}$ is taken as 0.88 MeV and that of $f_{7/2}$ and $p_{3/2}$ to be 1.95 MeV. Hamiltonian matrices are now constructed using equation (2.13).

Let us first consider effective interactions to be operative in s-state only. Hamiltonian matrices are diagonalised for V_0 = -25, -30, -35 and -40 MeV and ranging from 0.5 to 1.2. In table 5 we give the matrix elements in units of V_0 for O^{18} and in table 6, those of Ti^{50}. One can take a fixed value of V_0 and multiply the matrix elements given in tables 5 and 6 and add the appropriate single particle energy and carry out the explicit diagonalisation for each value of λ. We give the exact calculations for two values of V_0 = -35 and -40 MeV as shown in figures 6, 7, 8 and 9.

We see that for O^{18}, the qualitative features of the level spectrum can be easily obtained. For each value of V_0 and a corresponding value of λ (λ = 0.85 for V_0 = -35 MeV, and λ = 0.65 for V_0 = -40 MeV), we can obtain a first excited state 2 at about 2 MeV, and a group of states† with J = 4,0* and 2* at 2.8-3.5 MeV.

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† The notation here is that the unstarred and starred values refer to the lowest and the first excited states of given J.

Table 5

The matrix elements in units of V_0 for 0^{12} for different values of $\lambda = r_0/r_l$ (singlet s-state interactions only).

Configuration.	J \ λ	0.5	0.8	1.0	1.2
$\langle d^2_{5/2} \mid d^2_{5/2} \rangle$	0	0.0469	0.0958	0.1304	0.1575
	2	0.0113	0.0269	0.0348	0.0438
	4	0.0057	0.0183	0.0255	0.0340
$\langle s^2_{1/2} \mid s^2_{1/2} \rangle$	0	0.0380	0.0843	0.1160	0.1483
$\langle d_{5/2} s_{1/2} \mid d_{5/2} s_{1/2} \rangle$	2	0.0164	0.0440	0.0638	0.0816
$\langle d^2_{5/2} \mid d_{5/2} s_{1/2} \rangle^2$		0.0100	0.0352	0.0358	0.0456
$\langle d^2_{5/2} \mid s^2_{1/2} \rangle$	0	0.0160	0.0304	0.0548	0.0699

Table 6

The matrix elements in units of V_0 for Ti^{50} for different values of λ (singlet s-state interactions only).

Configuration.	J \ λ	0.6	0.8	1.0	1.2
$\langle f^2_{7/2} \mid f^2_{7/2} \rangle$	0	0.0333	0.0726	0.0934	0.1151
	2	0.0093	0.0190	0.0246	0.0305
	4	0.0053	0.0121	0.0160	0.0200
	6	0.0040	0.0109	0.0158	0.0203
$\langle p^2_{3/2} \mid p^2_{3/2} \rangle$	0	0.0290	0.0650	0.0895	0.1123
	2	0.0074	0.0176	0.0248	0.0311
$\langle f_{7/2}p_{3/2} \mid f_{7/2}p_{3/2} \rangle$	2	0.0154	0.0332	0.0433	0.0503
	4	0.0060	0.0162	0.0233	0.0302
$\langle f^2_{7/2} \mid p^2_{3/2} \rangle$	0	0.0158	0.0333	0.0451	0.0563
	2	0.0035	0.0078	0.0109	0.0139
$\langle f^2_{7/2} \mid f_{7/2}p_{3/2} \rangle$	2	0.0082	0.0190	0.0262	0.0303
	4	0.0045	0.0099	0.0132	0.0165
$\langle p^2_{3/2} \mid f_{7/2}p_{3/2} \rangle$	2	0.0031	0.0107	0.0165	0.0214

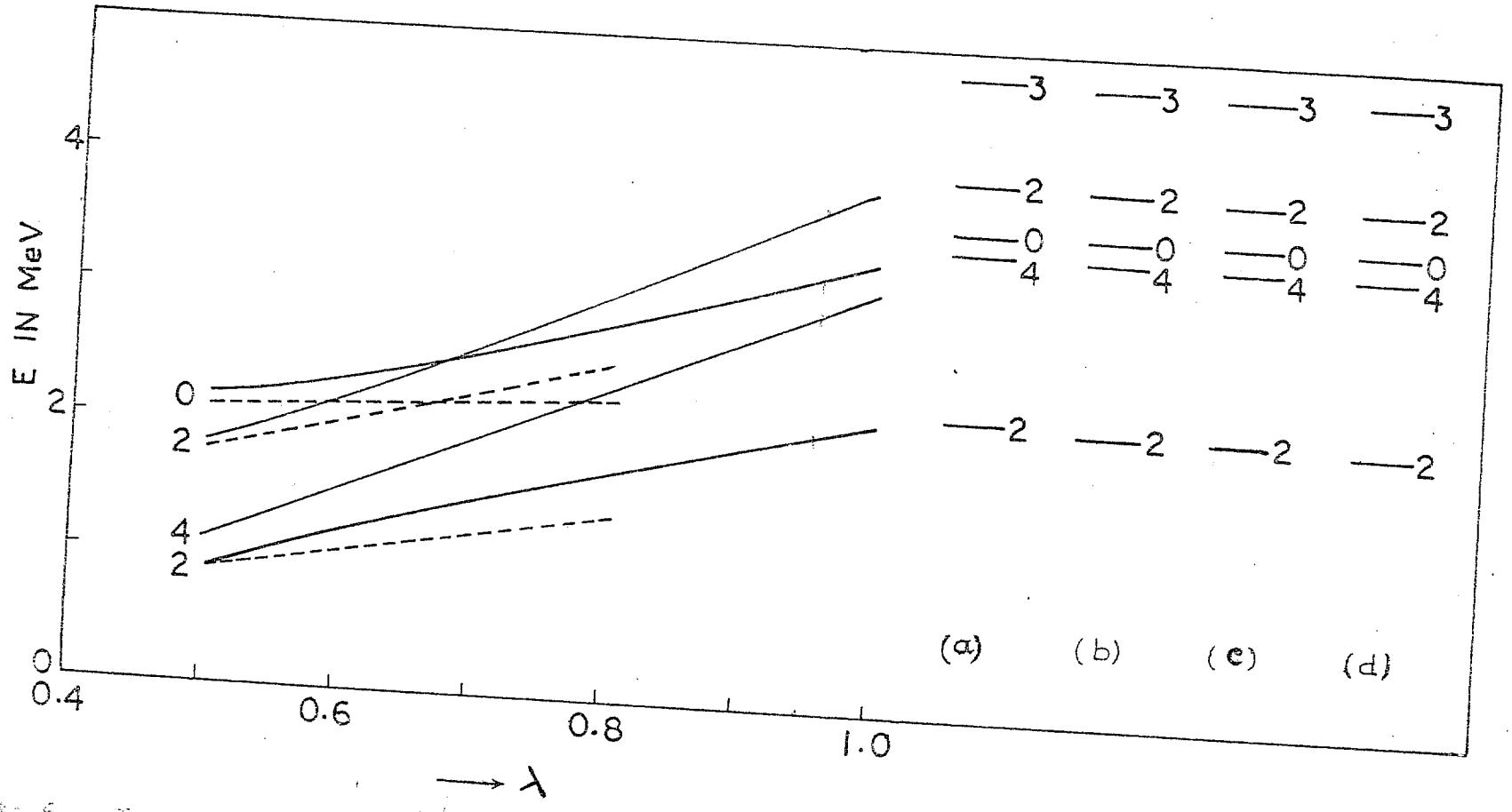

Fig. 5. Theoretical level ordering in O^{18} as a function of range parameter λ for $V_0 = -25$ MeV. The full curve represent the levels with singlet s- state interactions only. The dotted curves represent the levels with singlet s- and d- state interactions. The right hand side of the figure gives the level schemes obtained with parameters $V_0 = -25$ MeV, $\lambda = 1.0$ (i.e. s-state interactions only) and triplet p-state interactions with parameters (a) $V_t = -27.28$ MeV, $\lambda = 0.5$ (b) $V_t = -5.29$ MeV, $\lambda = 0.8$ (c) $V_t = -5.28$ MeV, $\lambda = 1.0$ and (d) $V_t = -4.10$ MeV, $\lambda = 1.0$. The state $J = 0$ is normalised to the value $E = 0$.

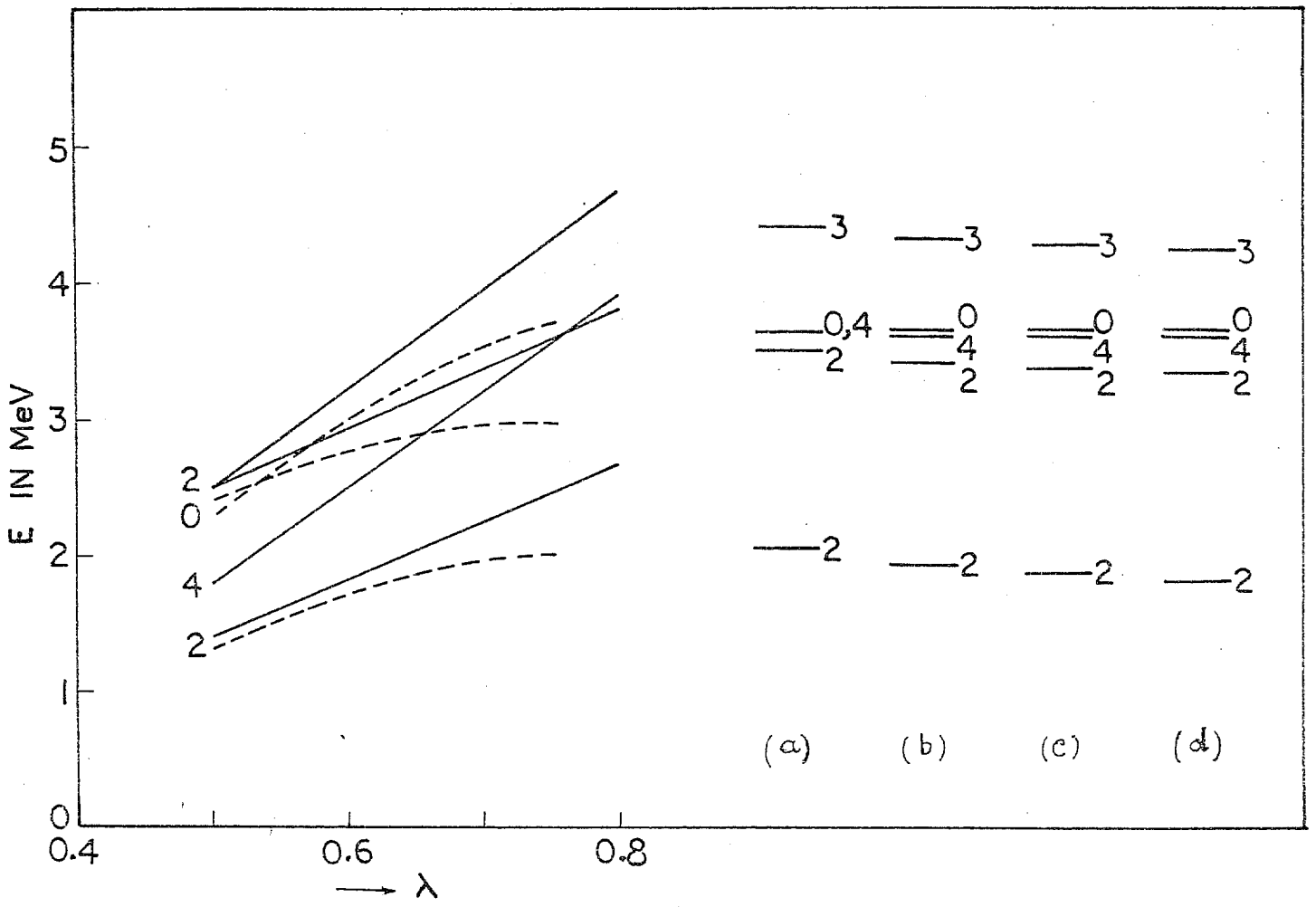

Fig. ?. Theoretical level ordering in O^{16} as a function of range parameter λ for V_0 = -[illegible] MeV. The full curves represent the levels with singlet s-states interactions only. The dotted curves represent the levels with singlet s- and d-state interactions. The right-hand side of the figure gives the level schemes obtained with parameters V_0 = -30 MeV, λ = 0.65 (singlet s- and d- state interactions) and triplet s-state interactions with parameters (a) V_0 = -196.4 MeV, λ = [illegible]; (b) V_0 = -25.4 MeV, λ = [illegible]; (c) V_0 = -[illegible] MeV, λ = [illegible] and (d) V_0 = -16.4 MeV, λ = [illegible]. The state [illegible] is [illegible] as [illegible].

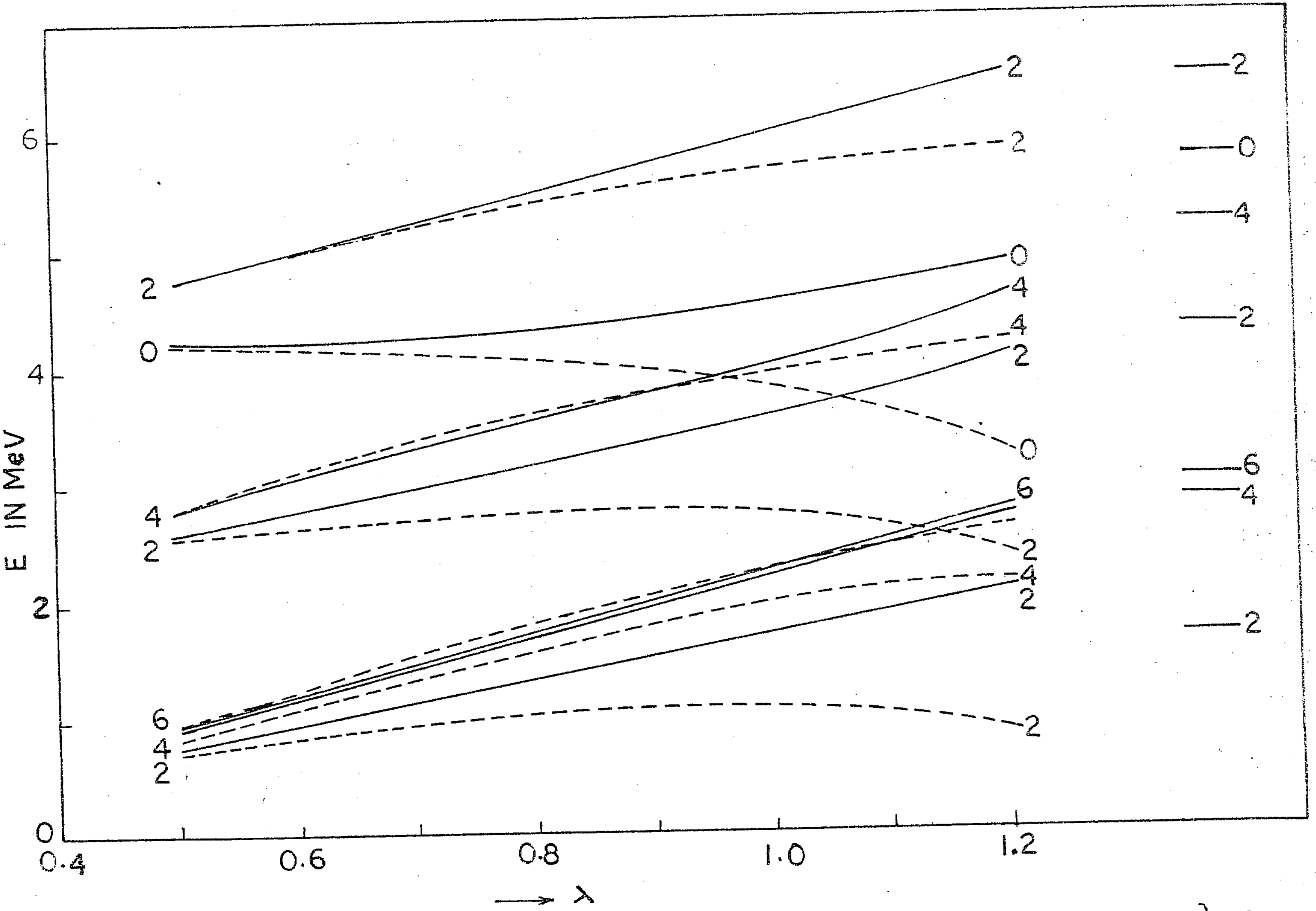

Fig.8. Theoretical level ordering in Ti^{50} as a function of the range parameter λ for $V_0 = -27$ MeV. The full curves represent the levels with singlet s-state interactions only. The broken curves represent the levels with singlet s- and d- state interactions. The right hand side of the figure gives the level scheme obtained with parameters $V_0 = -25$ MeV and $\lambda = 0.8$ (for s-state interactions) and triplet s-state interactions with parameters $V_0 =$ [illegible] MeV and $\lambda = 0.5$. The state J = 0 is normalised to E = 0.

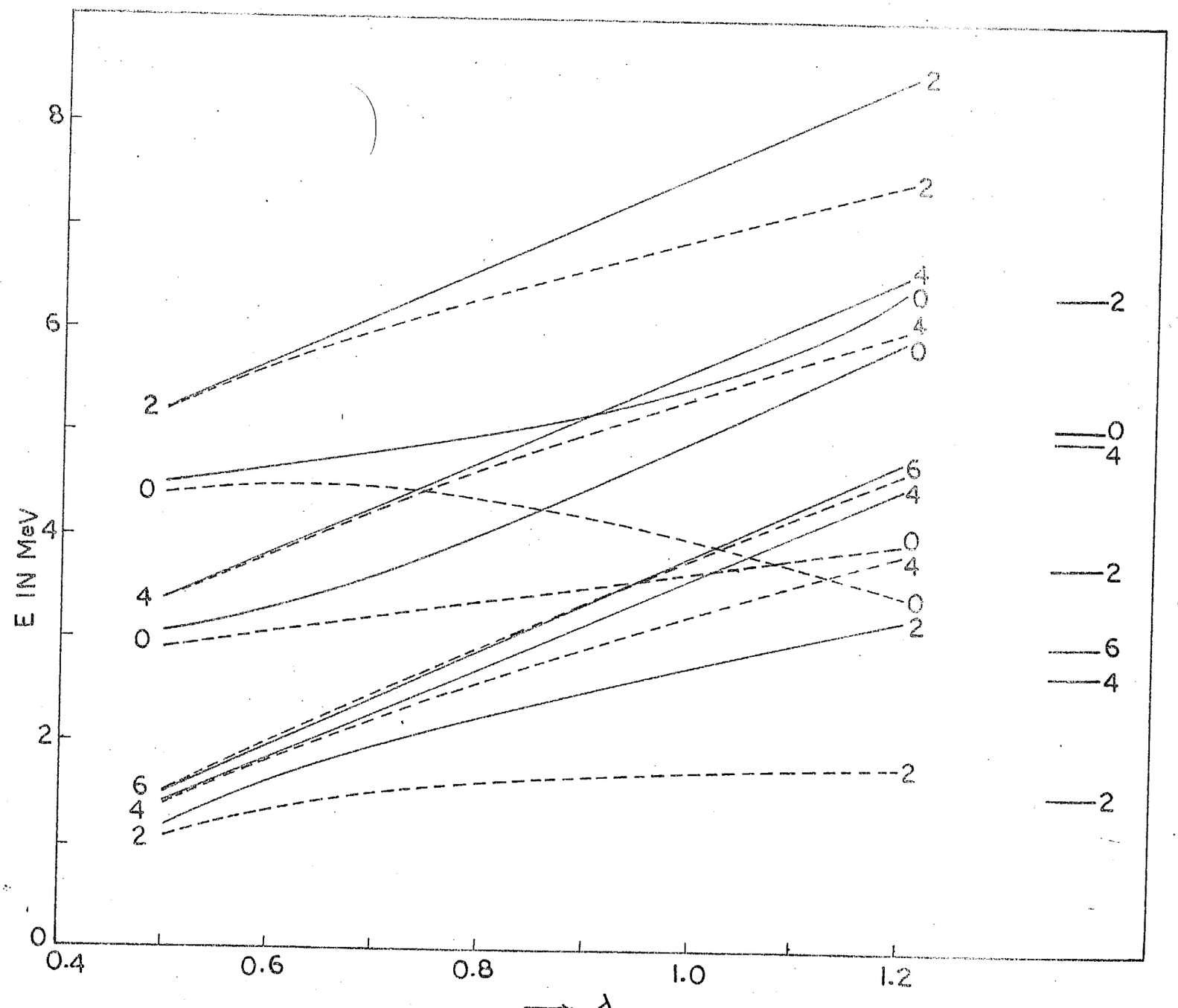

Fig.3. Theoretical level ordering in Ti^{50} as a function of range parameter λ for $V_0 = -40$ MeV. The full curves represent the levels with singlet s-state interactions only. The dotted curves represent the levels with singlet s- and d-state interactions. The [illegible] side of the figure gives the level scheme obtained with parameters $V_0 = -40$ MeV, λ [illegible] singlet s- and d- state interactions) and triplet p-state interactions with parameters V_1 [illegible] and $\lambda = 0.5$. The state J = 0 is normalised to E = 0.

However this triad of states is somewhat low compared to the experimental results. On the other hand for Ti^{50}, the agreement between theory and experiment is quite poor. If the first excited state is correctly fitted (at V_o = -25 MeV, λ = 0.33 or V_o = -40 MeV, λ = 0.57) the states 4, 6 occur about 1 MeV too low (at 1.75 MeV). We next note that for V_o = -25 MeV, the values of λ obtained above λ = 0.35 (O^{18}) and 0.33 (Ti^{50}) give for the range of the potential ℓ_o = 1.7 and 2.0 fm respectively, values which are in fair agreement with the range obtained for Li^6, see equation (4.3). Thus out of many possible sets of parameters which give almost similar results for singlet s-state interactions we choose for further investigation parameters corresponding to V_o = -25 MeV.

It is now of interest to investigate the effect of triplet p-state interactions. We give in tables 7 and 8, the matrix elements in triplet p-state in units of V_1, the strength of the potential, for O^{18} and Ti^{50} respectively. The results of the calculations show that the excitation energy of the states J = 2 and 2' for O^{18} and the state J = 2 for Ti^{50} are not significantly affected by the p-state forces. However, for O^{18} the states J = 4 and 0' are pushed up by attractive interactions and are depressed by repulsive interactions. This clearly argues in favour of attractive p-state forces, and show that by inclusion of suitable interactions one can obtain agreement with the experiments. In figure 6, we also show the energy levels

Table 7

The matrix elements in units of V_1 for O^{18} for different values of λ (triplet p-state interactions only)

Configuration.	J \ λ	0.5	0.8	1.0	1.2
$\langle d^2_{5/2} \mid d^2_{5/2} \rangle$	0	0.0094	0.0306	0.0663	0.0946
	2	0.0071	0.0367	0.0674	0.1012
	4	0.0009	0.0043	0.0083	0.0134
$\langle s^2_{1/2} \mid s^2_{1/2} \rangle$	0	-	-	-	-
$\langle s_{1/2} d_{5/2} \mid s_{1/2} d_{5/2} \rangle$	2	0.0054	0.0252	0.0446	0.0653
	3	0.0067	0.0315	0.0563	0.0833
$\langle d^2_{5/2} \mid s_{1/2} d_{5/2} \rangle$	2	-	-	-	-
$\langle d^2_{5/2} \mid s^2_{1/2} \rangle$	0	-	-	-	-

Table 8

The matrix elements, in units of V_1 for Ti^{50} for different values of λ (triplet p-states interactions only).

Configuration.	J \ λ	0.6	0.8	1.0	1.2
$\langle f^2_{7/2} \mid f^2_{7/2} \rangle$	0	0.0129	0.0411	0.0639	0.0871
	2	0.0123	0.0406	0.0632	0.0864
	4	0.0065	0.0264	0.0432	0.0606
	6	0.0057	0.0306	0.0563	0.0861
$\langle f^2_{5/2} \mid p^2_{3/2} \rangle$	0	0.0039	0.0143	0.0329	0.0319
	2	0.0079	0.0285	0.0458	0.0639
$\langle f_{7/2}p_{3/2} \mid f_{7/2}p_{3/2} \rangle$	2	0.0060	0.0230	0.0424	0.0610
	4	0.0034	0.0182	0.0330	0.0498
$\langle f^2_{7/2} \mid p^2_{3/2} \rangle$	0	-	-	-	-
	2	-	-	-	-
$\langle f^2_{7/2} \mid f_{7/2}p_{3/2} \rangle$	2	-	-	-	-
	4	-	-	-	-
$\langle p^2_{3/2} \mid p_{3/2}f_{7/2} \rangle$	2	-	-	-	-

calculated for O^{18} with p-state parameters adjusted to obtain the best fit. Again it is found that these parameters cannot be uniquely fixed with the given data, and several different values of V_1 and ρ_1 are possible (such as V_1 = -37 MeV, ρ_1 = 0.9 fm and V_1 = -3.7 MeV, ρ_1 = 2.2 fm). Here, the s-state parameters are so chosen as to give the best fit for the J = 2 and 2+ states which are not affected much by the p-state forces (V_0 = -25 MeV, ρ_0 = 1.8 fm) and the p-state parameters are so chosen as to keep J = 0+ state always at 3.63 MeV.

In case of Ti^{50} one finds that with suitable attractive p-state forces, one can raise the J = 4, 6 states and also increase their separation. In this case we also find that if the range of the triplet p-interaction is large (corresponding to $\lambda > 0.3$) the J = 6 state is depressed below the J = 4 state. This result leads us to effectively shorter range for p-state forces. In figure 8 we show one set of parameters (V_1 = -155 MeV, λ = 0.5) for p-states which give a good fit to the experimental data.

We can now summarise the results. It is possible to choose a singlet s-state interaction which has the same parameters in Li^6, O^{18} and Ti^{50} nuclei. Its parameters seem to be reasonable. On the other hand p-state interaction parameters appear to change systematically as we go from A = 6 to A = 50. For Ti^{50} the interaction appears to be quite strong with a rather short range (V_1 = -155 MeV, ρ_1 = 1.2 fm).

For O^{18}, the parameters are not uniquely fixed, but if we take the same range viz., $\xi_1 = 1.2$ fm the strength V_1 definitely appears to be weaker, perhaps by a factor of 3. For comparison, the matrix element $\langle op | V_{12} | op \rangle$ in O^{18} would have a value of ≈ -0.7 MeV, whereas in Ti^{50} it has the value ≈ -2.3 MeV. It was concluded that in Li^6, although the p-states are not yet observed, $\langle op | V_{12} | op \rangle \lesssim 0.5$ MeV.

Next, we consider the results of the alternative assumption according to which the same interaction is operating in both the even states $\ell = 0$ and $\ell = 2$. In the same way as above, the Hamiltonian matrices were diagonalised for $V_0 = -25$, -30, -35 and -40 MeV and λ ranging from 0.5 to 1.2. In table 9, we give the matrix elements in units of V_0 for O^{18} and those for Ti^{50} in table 10. Here also we give the exact calculations for only two values of V_0 viz., -25 and -40 MeV. The energy levels calculated on this basis for even state interactions only are shown by dotted lines in the figures 6 - 9. It will be noticed the changes in the excitation energies are most marked for the states with $J = 0^+$, 2 and 5^+, which are now depressed. As we have previously remarked, the p-state interaction do not affect the $J = 2$ state much, and hence we choose the even state parameters to obtain a good fit for this state. (Here also, it may be noted that in O^{18} the p-state parameters are so chosen as to give $J = 0^+$ state at 3.62 MeV). This leads us to choose $V_0 = -40$ MeV,

Table 9

The matrix elements, in units of V_0 for O^{19} for different values of λ (singlet s- and d-state interactions)

Configuration.	λ / J	0.5	0.3	1.0	1.2
$\langle d^2_{5/2} \mid d^2_{5/2} \rangle$	0	0.0471	0.0933	0.1323	0.1630
	2	0.0132	0.0394	0.0659	0.0930
	4	0.0069	0.0202	0.0309	0.0419
$\langle s^2_{1/2} \mid s^2_{1/2} \rangle$	0	0.0400	0.1049	0.1651	0.2341
$\langle d_{5/2} s_{1/2} \mid d_{5/2} s_{1/2} \rangle$	2	0.0136	0.0624	0.1026	0.1458
$\langle d^2_{5/2} \mid s^2_{1/2} \rangle$	0	0.0153	0.0313	0.0373	0.0393
$\langle d^2_{5/2} \mid s_{1/2} d_{5/2} \rangle$	2	0.0100	0.0225	0.0269	0.0270

Table 10

The matrix elements in units of V_0 for Ti^{50} for different values of λ (singlet s- and d-states interactions)

Configuration.	J \ λ	0.5	0.8	1.0	1.2
$\langle f^2_{7/2} \mid f^2_{7/2} \rangle$	0	0.0395	0.0778	0.1041	0.1326
	2	0.0122	0.0373	0.0618	0.0907
	4	0.0064	0.0173	0.0280	0.0410
	6	0.0041	0.0119	0.0132	0.0245
$\langle p^2_{3/2} \mid p^2_{3/2} \rangle$	0	0.0309	0.0792	0.1187	0.1599
	2	0.0036	0.0256	0.0403	0.0576
$\langle f_{7/2}p_{3/2} \mid f_{7/2}p_{3/2} \rangle$	2	0.0132	0.0530	0.0930	0.1132
	4	0.0064	0.0192	0.0302	0.0414
$\langle f^2_{7/2} \mid p^2_{3/2} \rangle$	0	0.0145	0.0347	0.0274	0.0275
	2	0.0033	0.0062	0.0075	0.0032
$\langle f^2_{7/2} \mid f_{7/2}p_{3/2} \rangle$	2	0.0031	0.0147	0.0159	0.0147
	4	0.0045	0.0093	0.0114	0.0127
$\langle p^2_{3/2} \mid f_{7/2}p_{3/2} \rangle$	2	0.0033	0.0034	0.0195	0.0239

λ = 0.70 for Ti^{50} and V_0 = -40 MeV, λ = 0.65 for O^{18}. (We remark that for V_0 = -25 MeV and $\lambda > 0.8$ in O^{18}, the wavefunction of the J = 2 state gives a dominant component of $(s_{1/2}d_{5/2})$ configuration). The corresponding ranges in the two nuclei would be 1.5 fm and 1.2 fm. It may be remembered that for Li^6, we had very similar parameters, equation (4.4). The above choice of the parameters gives rather low values for the excitation energies of the higher states. This can be remedied to a large extent as in the previous case by inclusion of p-state forces. Figures 7 and 8 also give the results obtained for the level spectra with the inclusion of the p-state forces. Again a reasonably good agreement between theory and experiment is obtained. For O^{18} the J = 2$^+$ state is predicted a little lower than its observed position. For Ti^{50} the p-state interaction required has again a short range R_1 = 1.2 fm and depth -35 MeV, whereas for O^{18} one can fit the data with several different p-state interactions, one of which would be V_1 = -45 MeV, R_1 = 1.2 fm. Thus again the p-state force appears to grow stronger as we go to heavier nuclei.

It should be kept in mind that there is yet some uncertainty about the energy level scheme of Ti^{50}; the 6 level is yet to be identified, although we have assumed it to be at 3.0 MeV. The identification of this level as well as of the spins and parities of other excited states would certainly enable us to refine and improve the present analysis.

Now we add one remark on yet unobserved $J = 3^+$ state which is predicted from $(s_{1/2}^2 d_{5/2})$ configuration in O^{18}. This state is solely due to triplet p-state interactions and the matrix elements in units of V_1 are given in table 7. Since it has not been possible to choose a unique set of parameters for triplet p-state interactions, we give the predicted position of this level for all sets of parameters which give equally good fit for the remaining known energy levels of O^{18}. The position of this level corresponding to parameters given in figure 6 is at $\sim$ 4.7 MeV and corresponding to those in figure 7 is at $\sim$ 4.3 MeV. The experimental evidence for the existence of this state will be able to throw some light on the nature of triplet odd-state interactions. It may be noted that $J = 3^-$ level has been detected at $\sim$ 4.5 MeV. The existence of this level should be a point of interest.

8. Conclusions

In the previous sections we have made an attempt to study systematically the parameters of the effective two-body nuclear interaction for a few configurations in terms of some simple models. It is hardly possible with the available data to decide in favour of a unique model for this effective "residual" interaction. We have considered only central forces, and neglected their contribution to the energy from states of relative orbital angular momenta $\ell \geq 3$. This seems to us a good assumption. We find that the available energy level

data on the configurations chosen for study here, can be explained in terms of either

(i) a non-local even interaction which operates only in $\ell = 0$ states with $V_0 = -25$ MeV and range $r_0 = 1.8 - 2.0$ fm and an odd-state interaction in $\ell = 1$ state which appears to change as we go to heavier nuclei or higher configurations, growing stronger and perhaps shorter ranged, or

(ii) an even state interaction with $V_0 = -40$ MeV, $r_0 = 1.2 - 1.4$ fm operating in both s- and d-states, and an interaction in p-state which again appears to increase in strength as one goes to higher shell model levels. We also note that the interaction in the p-state is definitely found to be attractive.

Much additional work remains to be done. The effect of including tensor forces in the odd-states has to be investigated. It is of interest to consider configurations with three identical nucleons outside a closed shell and compare the effective interactions in such configurations with our results above, e.g. in the pairs of nuclei O^{18}, O^{19} and Ti^{50}, V^{51}. Of these pairs, we make study of O^{18}, O^{19} pair in the chapter III. Similar calculations may also be carried out for heavier nuclei such as Ni^{60}, Zr^{90}, Zr^{92} and even Pb^{206} etc.

One may expect that in relative orbital angular momentum states with $\ell \geq 1$, the effect of hard cores or singularities in the realistic nucleon-nucleon interactions would not be of any critical importance for shell model wavefunctions, and in such states the reaction matrix and the potential matrix should give rather similar results. Thus to a reasonable approximation the contribution to the energy of a state from such states with $\ell \geq 1$ in equation (2.13), may be calculated by using a simple potential which fits well the low energy nucleon-nucleon data. Then the matrix elements in the s-state may alone be considered as empirical parameters to be determined so as to give a good fit to the observed level spectrum. These matrix elements would then be the reaction matrix elements in singlet s-states for finite nuclei, and may be compared with similar matrix elements calculated from realistic potentials (local or non-local) that have been proposed by many authors. It would not be too difficult to evaluate the reaction matrix for singlet s-states even for finite nuclei.

We wish to emphasise the approach and methodology of this chapter, which we hope will enable a more useful and flexible analysis to be made of the effective interaction when enough experimental data is available on both $T = 1$ and $T = 0$ states of two nucleon configurations.

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CHAPTER III

AN ANALYSIS OF THE ENERGY LEVELS OF O^{18} AND O^{19}.

1. Introduction

In this chapter we consider somewhat more carefully, and also with a slightly different approach the energy levels of O^{18} and O^{19} and their relationship. It is obvious that if the same configuration space is assumed to describe the low-lying energy levels of O^{18} and O^{19}, the matrix elements of the Hamiltonian for different states of O^{19} can be directly expressed in terms of the matrix elements of O^{18}. Such an analysis - described in section 4 - is independent of any assumptions on the nature of the two-body interactions. We use such an analysis to obtain some sort of a check on the results obtained in the previous chapter. In section 3, a more direct approach is used, i.e. assuming the Hamiltonian matrix elements for O^{18} given by different types of interactions considered in the previous chapter. We evaluate the corresponding energy level spectrum of O^{19}, and compare with experiment to see if it is possible to discriminate between the several interactions which all give the same results for O^{18} level scheme. In section 2, we briefly discuss our results for O^{18} energies and wavefunctions comparing them with the results obtained by other authors who have previously made similar calculations.

2. Comparison with results of other authors

In view of the simplicity of structure, O^{18} and O^{19} have attracted considerable attention from several authors over many years. We mention the early work of Elliott and Flowers[13], and Redlich[34]. More recently Dawson, Talmi and Walecka[19] have used the Brueckner-Bethe-Goldstone[35] formalism to calculate the energy levels of O^{18}, using a reaction matrix calculated from the free nucleon-nucleon potential of Brueckner-Gammel-Thaler[36]. Some information on the wavefunctions of the various states involved is also derived from an analysis of the deuteron stripping reactions by McFarlane and French[37].

In the analysis of Elliott and Flowers, and Redlich, the configuration space included in addition to the $d_{5/2}$ and $s_{1/2}$ orbitals, also the $d_{3/2}$ orbital. As discussed in chapter II, we feel that in the spirit of the Brueckner theory, it appears unjustified to include the $d_{3/2}$ state, since it lies quite well above the $d_{5/2}$ state ($\sim$ 5 MeV). The calculations of Dawson, Talmi and Walecka also in the first approximation neglect the $d_{3/2}$ state, and their results as well as those of the earlier authors on the wavefunctions of the O^{18} - O^{19} states show quite clearly that even when the $d_{3/2}$ state is included in the configuration space, the components of the wavefunctions containing this state are rather small, and their effect on

the energy of the state would be even smaller. Therefore we can compare our results with those of the others neglecting the $d_{3/2}$ state, without introducing any important inconsistency.

Our calculated energy levels agree much better with experimental results than those of the other authors as shown in figure 1. This is inevitable since our potential is constructed in such a way as to give the best fit with the experimental results. A check of our calculations would be provided by the prediction of the $J = 3^+$ level $(s_{1/2}d_{5/2})$ in O^{18}. This we predict at 4.8 to 4.7 MeV (see figures 6 and 7 of chapter II). Dawson, Talmi and Walecka's calculation also predicts 3^+ level in this range. It is very desirable that this be checked experimentally. Location of this state is very important for determining the p-state forces, since it depends only on the p-state interaction. If the energy of this state is $E_3 < 4.8$ MeV, the matrix element of the interaction $\langle s_{1/2}d_{5/2}; 3|V_{12}|s_{1/2}d_{5/2}; 3\rangle$ will be < 0, whereas if $E_3 > 4.8$ MeV, the matrix element is > 0.† Thus in former case the p-state interaction would be attractive, whereas in the latter case it would be repulsive. Of course, in either case, it is certainly small in magnitude, but if, experimentally, it turns out to be repulsive, our analysis of the chapter II would have to be slightly revised.

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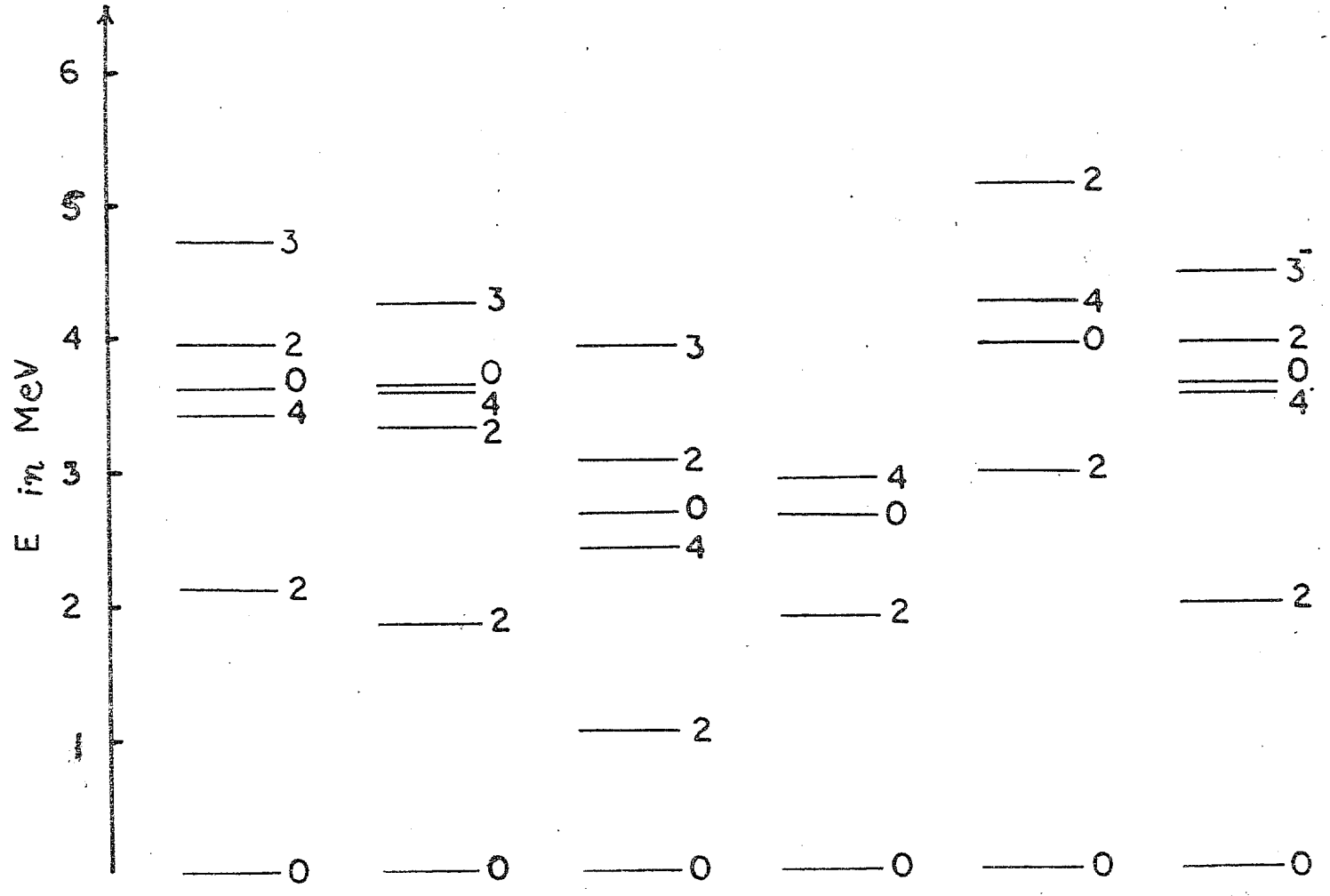
† This follows because $E_3 - E_0 = \langle s_{1/2}d_{5/2}|V_{12}|s_{1/2}d_{5/2}\rangle_3 + (\Delta = 0.85 \text{ MeV}) + 3.02 \text{ MeV}$.


E in MeV
6
5
4
3
2
3
2
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2
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3
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4
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2
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4
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Since our interest is in devising a potential which would be a good model for the actual reaction matrix, it might be of interest to compare in detail the matrix elements calculated by our model and by the more involved reaction matrix calculation of Dawson, Talmi and Walecka[19]. Their calculations give for the Talmi integral $I_{os} = \langle os \,|\, V_{12} \,|\, os \rangle$ the best value of -3.3 MeV, which also explains fairly satisfactorily the observed spectrum. The values of I_{os} obtained by us using the parameters $V_o = -25$ MeV (singlet s-interactions only) and $\lambda = 1.0$ is -3.84 MeV and using parameters $V_o = -40$ MeV (singlet s- and d-interactions) and $\lambda = 0.70$ is -7.80 MeV. This compares very well with the value obtained by Dawson, Talmi and Walecka. The detailed comparison of the values of different $I_{n\ell}$ obtained by us with those of Dawson, Talmi and Walecka is given in table 1 (singlet even) and table 2 (triplet odd) for one set of parameters (a) only (i.e. $V_o = -25$ MeV, $\lambda = 1.0$). The results for other set of parameters (b) are rather similar. We also compare the relevant matrix elements for various states with those obtained by the above authors in table 3 for set of parameters (a) and in table 4 for set of parameters (b).

We further note that Dawson, Talmi and Walecka give also results for O^{18} spectrum calculated by using a simple non-singular central potential which has been used by Multhan and Sugawara[38] to fit the low energy

Table 1

The values of $I_{n\ell} = \langle n\ell | V_{12} | n\ell \rangle$ for different n and ℓ for singlet potential.

$I_{n\ell}$	Present calculations with parameters $V_0 = -35$ MeV $\lambda = 1.0$	Dawson, Talmi and Walecka
I_{0s}	-3.040	-3.323
I_{1s}	-3.525	-3.988
I_{2s}	-4.353	-3.296
I_{0p}	-4.420	-2.313
I_{0d}	-2.210	-0.777
I_{1d}	-2.459	-1.329
I_{0f}	-1.105	-0.257
I_{0g}	-0.553	-0.106

Table 2

The values of $I_{n\ell} = \langle n\ell | V_{12} | n\ell \rangle$ for different n and ℓ for triplet odd potential.

I_n \ λ	0.5	0.8	1.0	1.2	Dawson, Talmi and Walecka.
I_{0p}	-0.868	-0.864	-0.837	-0.800	-0.321
I_{1p}	-1.036	-0.963	-0.817	-0.758	-0.402
I_{0d}	-0.134	-0.330	-0.463	-0.586	-0.118
I_{0f}	-0.026	-0.120	-0.234	-0.345	-0.054

Table 1

The numerical values in MeV of the matrix elements of various states in O^{18} using our parameters (V_0 = -23 MeV, λ = 1.0) for singlet even and those corresponding to figure 6 of chapter II for triplet odd, and those obtained by Dawson, Talmi and Walecka.

(a) $\langle d^2_{5/2} | V_{12} | d^2_{5/2} \rangle$

J	Present calculations singlet even	Dawson et al singlet even	Present calculations triplet odd.	Dawson et al. triplet odd.
0	-3.160	-3.209	λ =0.6, -0.360 λ =0.8, -0.352 λ =1.0, -0.351 λ =1.2, -0.350	+1.245
2	-0.870	-1.133	λ =0.6, -0.364 λ =0.8, -0.327 λ =1.0, -0.357 λ =1.2, -0.374	-0.419
4	-0.663	-0.671	λ =0.6, -0.034 λ =0.8, -0.043 λ =1.0, -0.047 λ =1.2, -0.050	-0.093

(b) $\langle d^2_{5/2} | V_{12} | s^2_{1/2} \rangle$

J	Present calculation singlet even	Dawson et al singlet even
0	-1.370	-0.982

Table 3 (Continued)

(c) $\langle s^2_{1/2}|V_{12}|s^2_{1/2}\rangle$

J	Present calculation singlet even	Dawson et al singlet even
0	-2.900	-3.019

(d) $\langle d^2_{5/2}|V_{12}|d_{5/2}s_{1/2}\rangle$

J	Present calculation singlet even.	Dawson et al singlet even	Present calculation Triplet odd.	Dawson et al Triplet odd
2	-0.895	-0.824	-	+0.044

(e) $\langle d_{5/2}s_{1/2}|V_{12}|d_{5/2}s_{1/2}\rangle$

J	Present calculation singlet even.	Dawson et al singlet even	Present calculation Triplet odd.	Dawson et al Triplet odd.
2	-1.695	-1.928	λ =0.6,-0.201 λ =0.8,-0.224 λ =1.0,-0.238 λ =1.2,-0.244	+0.237
3	-	-	λ =0.6,-0.249 λ =0.8,-0.280 λ =1.0,-0.296 λ =1.2,-0.308	-0.640

Table 4

The numerical values in MeV of the matrix elements of various states in O^{18} using parameters $V_o = -40$ MeV, $\lambda = 0.65$ for singlet even and those corresponding to figure 7 of chapter II for triplet odd.

(a) $\langle d^2_{5/2} | V_{12} | d^2_{5/2} \rangle$

J	Singlet even	Triplet odd	
0	-3.91	λ =0.5, -1.00 λ =1.0, -1.00	λ =0.8, -1.00 λ =1.2, -1.00
2	-0.96	λ =0.5, -0.78 λ =1.0, -1.02	λ =0.8, -0.93 λ =1.2, -1.07
4	-0.50	λ =0.5, -0.10 λ =1.0, -0.13	λ =0.8, -0.12 λ =1.2, -0.14

(b) $\langle d^2_{5/2} | V_{12} | s^2_{1/2} \rangle$

J	Singlet even
0	-1.01

(c) $\langle s^2_{1/2} | V_{12} | s^2_{1/2} \rangle$

J	Singlet even
0	-2.63

Table 4 (continued)

(d) $\langle d^2_{5/2}|V_{12}|d_{5/2}s_{1/2}\rangle$

J	Singlet even
2	-0.68

(e) $\langle d_{5/2}s_{1/2}|V_{12}|d_{5/2}s_{1/2}\rangle$

J	Singlet even	Triplet odd	
2	-1.48	λ=0.6, -0.57 λ=1.0, -0.69	λ=0.8, -0.64 λ=1.2, -0.70
3	-	λ=0.6, -0.71 λ=1.0, -0.84	λ=0.8, -0.80 λ=1.2, -0.88

nucleon-nucleon scattering data. For a Yukawa shape, their parameters are

$$V_{os} = -47 \text{ MeV} \qquad V_{ot} = -32 \text{ MeV}$$
$$\rho_{os} = 1.3 \text{ fm} \qquad \rho_{ot} = 1.4 \text{ fm} \tag{2.1}$$

It is somewhat remarkable that for our assumption of same interaction in all even states (i.e. $\ell = 0, 2$ here) we also obtain very similar parameters viz. $V_{os} = -40$ MeV, $\rho_{os} = 1.2$ fm and although triplet parameters are not uniquely specified, the above values are not in disagreement (see chapter II, figure 7). The agreement is not very surprising, since for only two extra-core nucleons, and with such a simple non-singular potential the reaction matrix may not be drastically different from the potential.

Finally, we remark on the wavefunctions obtained by us. In table 5 we give the wavefunctions for the $J = 0$ ground state obtained by Elliott and Flowers, Redlich, Dawson, Talmi and Walecka, and by us. In table 6 are listed the wavefunctions for $J = 2$ state, neglecting however the states involving $d_{3/2}$ orbital. It may be noted that a considerable amount of configuration mixing is predicted by all theoretical calculations. On the other hand, McFarlane and French find from an analysis of stripping reaction data (O^{17} (d,p) O^{18}) a much smaller amount of mixing i.e. the amplitude of the two states are

$$A\,(d_{5/2}s_{1/2}) \cong 0.36\text{-}0.43$$
$$A\,(d_{5/2})^2 \cong 0.89\text{-}0.93 \tag{2.2}$$

Table 3

The amplitudes of wavefunctions for $J = 0$, the ground state of O^{18}.

Configuration.	Reilich[34]	Elliott[13] & Flowers	Dawson[19] et al.	Present calculations parameters (a)	(b)
$(d_{5/2})^2$	0.66	0.89	0.89	0.91	0.89
$(s_{1/2})^2$	0.31	0.24	0.39	0.42	0.45
$(d_{3/2})^2$	0.40	0.30	0.23	-	-

Table 4

The amplitudes of wavefunctions for $J = 2$ state of O^{18}.

Configuration.	Reilich[34]	Dawson[19] et al	Present calculations parameters (a)	(b)
$(d_{5/2})^2$	0.71	0.79	0.74	0.75
$(d_{5/2}s_{1/2})$	0.64	0.54	0.63	0.65
$(d_{5/2}d_{3/2})$	-0.20	0.16	-	-
$(d_{3/2})^2$	0.14	0.10	-	-
$(d_{3/2}s_{1/2})$	0.20	-0.21	-	-

We do not discuss this discrepancy any further.

3. Energy levels of O^{19} (I)

In the previous chapter, we have discussed the nature of the effective nuclear interaction in T = 1 states of nuclei O^{18}, Ti^{50} etc. The parameters of the interaction in singlet even and triplet odd states of the two nucleons outside the closed shells were determined under two assumptions:

(a) the even state interactions are non-local and are effective in $\ell = 0$ (ℓ refers to relative orbital angular momentum), i.e. s-state only, and

(b) the even state interactions are the same in all states $\ell = 0, 2$ etc.

For O^{18}, the parameters of the even interactions (assumed to have Gaussian radial shape $V_0 \exp(-r^2/r_0^2)$ were determined to be (a) $V_0 = -25$ MeV, $\lambda = r_0/r_\ell = 1.0$ and (b) $V_0 = -40$ MeV, $\lambda = 0.65$, and several sets of corresponding odd-state interactions which give a good fit to the O^{18} spectrum were also given. In this section we apply these different sets of parameters to calculate the energy level spectrum of O^{19}, to see if these additional data can help to distinguish between the different sets.

The configurations that we consider for the T = 3/2 states of O^{19} are

(a) $(d_{5/2})^3$, (b) $(d_{5/2})^2 s_{1/2}$ and (c) $d_{5/2}(s_{1/2})^2$

The wavefunction corresponding to the configuration (a) is just the one hole wavefunction. For the configurations (b) and (c) the antisymmetrised wavefunction for $T = 3/2$ state is given by[39]

$$|(j)^2 J_0, j' : J\rangle$$
$$= \frac{1}{\sqrt{N}} \left\{ |j^2 : J_0\rangle \times |j'\rangle \right.$$
$$\left. + a^{-1} (-1)^{j+J} \sum_{J_1} (-1)^{1+J_1} U(jjJj'; J_0 J_1) |jj' : J_1\rangle \times |j\rangle \right\} \qquad (3.1)$$

where N is the normalisation coefficient given as

$$N = 3 \left\{ 1 - 2(-1)^{j+J} U(jjJj'; J_0 J_0) \delta_{nn'} \delta_{ll'} \delta_{jj'} \right\} \qquad (3.2)$$

and $a = a_{jj'} = 1/2$ if j and j' are equivalent

$= 1/\sqrt{2}$ if j and j' are nonequivalent (3.3)

The matrix elements for the configuration (a) for various allowed states can be written with the help of coefficients of fractional parentage as

$$\langle (d_{5/2})^3 J | V_{12} | (d_{5/2})^3 J \rangle$$
$$= 3 \sum_{J_1} \langle (d_{5/2})^3 J | \} (d_{5/2})^2 J_1 \rangle^2 \langle (d_{5/2})^2 J_1 | V_{12} | (d_{5/2})^2 J_1 \rangle \qquad (3.4)$$

where $\langle (d_{5/2})^3 J | \} (d_{5/2})^2 J_1 \rangle$ are the coefficients of

fractional parentage[40]. For the configurations (b) and (c) the matrix elements can be written as

$$\begin{aligned}
&\langle (j)^2 J_0, j' : J \mid V_{12} \mid (j_1)^2 J_0', j_1' : J \rangle \\
&= \frac{3}{\sqrt{N_1 N_2}} \Big\{ \delta_{J_0 J_0'} \delta_{j' j_1'} \langle j^2 J_0 \mid V_{12} \mid j_1^2 J_0 \rangle \\
&- \delta_{j' j_1} (-1)^{j+J} a_1^{-1} U(j j_1 J j_1'; J_0' J_0) \langle j^2 J_0 \mid V_{12} \mid j j_1' J_0 \rangle \\
&- \delta_{j_1' j} (-1)^{j+J} a^{-1} U(j j J j'; J_0 J_0') \langle j j' J_0' \mid V_{12} \mid j_1^2 J_0 \rangle \\
&+ \delta_{j j_1} (a a_1)^{-1} \sum_{J_1} U(j j J j'; J_0 J_1) U(j j J j_1'; J_0' J_1) \times \\
&\qquad \langle j j' J_1 \mid V_{12} \mid j j' J_1 \rangle \Big\}.
\end{aligned} \tag{3.6}$$

Thus we see that the matrix elements of states in O^{19} could be written in terms of the matrix elements of two-body states in O^{18} and one can employ the parameters of O^{18} to calculate the energy levels of O^{19}. The matrix elements for parameters (a) and (b) are given in tables 7 and 8. The matrix elements for triplet odd state interactions (triplet p- only) are also given in these tables for different ranges and strengths. The results of all calculations are given in figure 2. We can see that in case (a) the odd-state interactions appear to have only a small effect on the energies of the different levels and all the different sets of odd interactions of chapter II, give essentially the same results.

Table 7

The matrix elements of G^{19} for parameters (a) plus the triplet p-state interactions. Triplet p-interactions have different ranges and strengths. Single particle energy Δ is already included. The off-diagonal matrix elements do not have any contribution from odd-state interactions.

Configurations.	λ / V_1 / J	0.6 -37.23 MeV	0.8 -8.90 MeV	1.0 -5.30 MeV	1.2 -3.70 MeV
$\langle d^3_{5/2} \vert d^3_{5/2} \rangle$	3/2	-3.0282	-3.1694	-3.2381	-3.2772
	5/2	-4.3276	-4.3954	-4.4271	-4.4448
	9/2	-2.3934	-2.4306	-2.4507	-2.4774
$\langle d^2_{5/2} s^2_{1/2} \vert d^2_{5/2} s^2_{1/2} \rangle$	1/2	-3.6873	-3.7175	-3.7400	-3.7557
	3/2	-1.6830	-1.7172	-1.7663	-1.7969
	5/2	-1.0788	-1.1928	-1.2497	-1.2873
	7/2	-1.3929	-1.4313	-1.4503	-1.4634
	9/2	-0.4151	-0.4857	-0.5183	-0.5448
$\langle s^2_{1/2} d^4_{5/2} \vert s^2_{1/2} d^4_{5/2} \rangle$	5/2	-2.1708	-2.2275	-2.2511	-2.2682
$\langle d^3_{5/2} \vert d^2_{5/2} s^2_{1/2} \rangle$	3/2	1.5650	1.5650	1.5650	1.5650
$\langle d^3_{5/2} \vert d^2_{5/2} s^2_{1/2} \rangle$	9/2	-0.0600	-0.0600	-0.0600	-0.0600
$\langle d^3_{5/2} \vert d^2_{5/2} s^2_{1/2} \rangle$	5/2	-0.2975	-0.2975	-0.2975	-0.2975
$\langle d^3_{5/2} \vert s^2_{1/2} d^4_{5/2} \rangle$	5/2	-0.9125	-0.9125	-0.9125	-0.9125
$\langle d^2_{5/2} s^2_{1/2} \vert s^2_{1/2} d^4_{5/2} \rangle$	5/2	-0.5775	0.5775	0.5775	0.5775

Table 8

The matrix elements of O^{19} for parameters (b) plus the triplet p-state interactions. Triplet p-interactions have different ranges and strengths. Single particle energy Δ is already included. The off-diagonal matrix elements do not have any contribution from odd-state interactions.

Configurations.	λ / V_1 / J	0.5 -108.4 MeV	0.8 -85.30 MeV	1.0 -15.10 MeV	1.2 -10.60 MeV
$\langle d^3_{5/2} \mid d^3_{5/2} \rangle$	3/2	-4.1704	-4.5628	-4.7632	-4.8880
	5/2	-4.9204	-5.1083	-5.1934	-5.2395
	9/2	-2.5049	-2.6750	-2.7599	-2.8150
$\langle d^2_{5/2} s^1_{1/2} \mid d^2_{5/2} s^1_{1/2} \rangle$	1/2	-4.7122	-4.8440	-4.9130	-4.9705
	3/2	-2.6773	-2.9423	-3.0879	-3.1853
	5/2	-2.4084	-2.7321	-2.8592	-3.0105
	7/2	-1.8606	-1.7653	-1.8214	-1.8643
	9/2	-1.1564	-1.3493	-1.4459	-1.5315
$\langle s^2_{1/2} d^4_{5/2} \mid s^2_{1/2} d^4_{5/2} \rangle$	5/2	-2.5880	-2.7122	-2.7879	-2.8437
$\langle d^3_{5/2} \mid d^2_{5/2} s^1_{1/2} \rangle$	3/2	1.1880	1.1880	1.1880	1.1880
	9/2	-0.0440	-0.0440	-0.0440	-0.0440
	5/2	-0.2240	-0.2240	-0.2240	-0.2240
$\langle d^3_{5/2} \mid s^2_{1/2} d^4_{5/2} \rangle$	5/2	-0.6760	-0.6760	-0.6760	-0.6760
$\langle d^2_{5/2} s^1_{1/2} \mid s^2_{1/2} d^4_{5/2} \rangle$	5/2	0.4400	0.4400	0.4400	0.4400

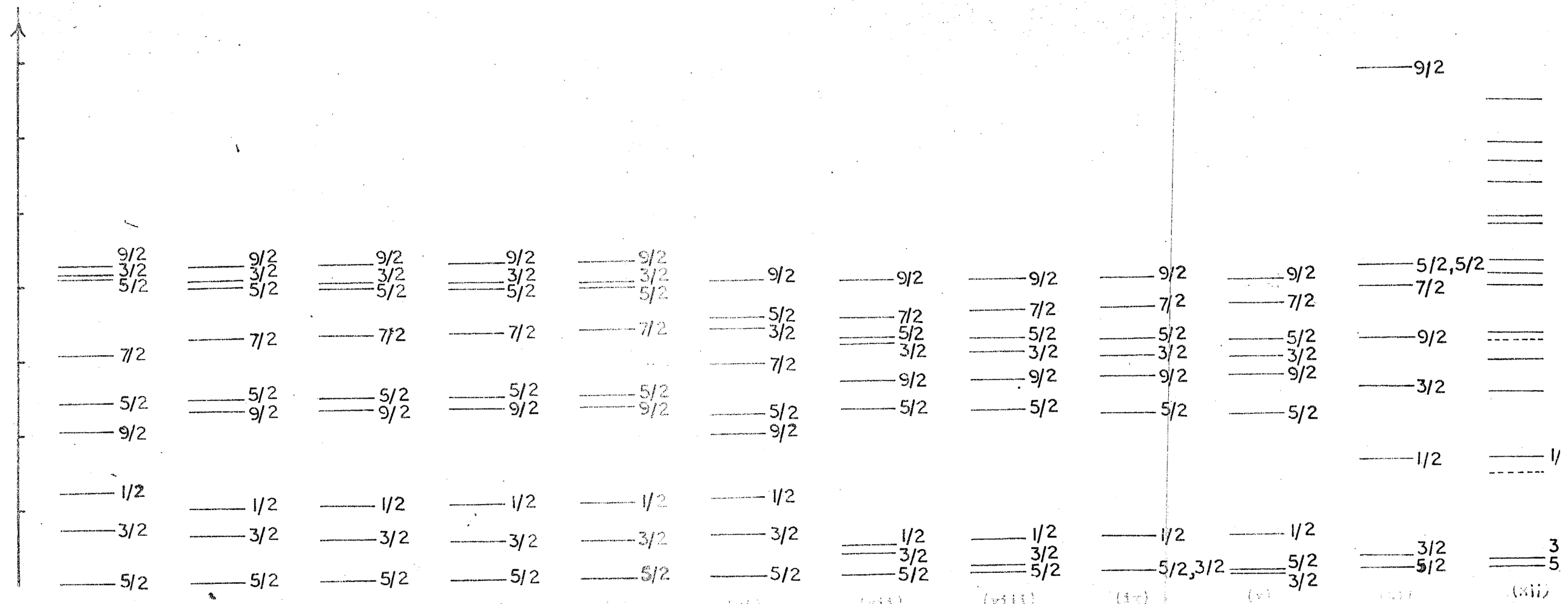

Fig. 2. Theoretical energy level scheme for ^{19}F. (i) (iv) (v) Parameters V_0 = -25 MeV, λ = 1.0 (singlet even-state interactions only); (ii), (iii), (vi) and (x) with parameters of (i) plus triplet even-state interactions of the same strength; (vii) V_0 = -17 MeV, λ = 0.5; V_1 = -7.0 MeV, λ = 0.5; and V_t = -3.7 MeV, λ = 1.0 respectively. (viii) (ix) Parameters V_0 = -40 MeV, λ = 0.65 (singlet even and triplet odd interactions); (xi), (xii) (ix) and (x) with parameters of (viii) plus triplet even-state interactions of V_t = -100 MeV, λ = 0.5; V_t = -20.40 MeV, λ = 0.8; V_t = -16.80 MeV, λ = 1.0; and V_t = -10.60 MeV, λ = 1.2 respectively. (x) Kurath [Phys. Rev. 101, 216 (1956)]. (xi) Experiment [42].

On the other hand for case (b) the odd-state interactions change considerably the energies of the states and in particular the energy of the lowest J = 3/2 state is quite different for different odd-state forces which give equally good fit for O^{18}. One can easily see that the set (iii) gives the best fit for the lowest J = 3/2 state. We also give for comparison the results recently reported by Talmi and Unna[41].

On the experimental side the situation is not very happy. Indeed, the energies for a number of levels have been measured[42] but there is no information available on spin-parity assignments to these levels.

Looking at the results it seems that the spins and parities of the levels between 2 and 3.5 MeV would be very important for distinguishing between the various schemes. In particular, it would be possible to distinguish between the schemes (a) and (b) if the spin of the 3.16 MeV level is determined and the existence of the level at 3.05 MeV is confirmed.

The experiments at present report three states between 2 and 3.5 MeV excitation above the ground state. The choice (a) of interaction parameters predicts three states in this region, 9/2, 5/2 (close to each other) and 7/2 - in that order. The choice (b) predicts two additional states in this region viz. 5/2 and 3/2. Talmi and Unna's results show only two states in this region,

3/2 and 5/2. Our feeling is that an overall best fit is given by choice (a) of the parameters, with old-state parameters remaining relatively indeterminate. Further experimental data would enable us to make more positive pronouncements.

4. Energy levels of O^{19} (II)

It is easy to see that if the low lying levels of O^{18} and O^{19} that we consider here are described only in terms of the $d_{5/2}$ and $s_{1/2}$ orbits, the nuclear spectroscopy involves only eight matrix elements of the effective interaction.

$$V_0 = \left\langle d_{5/2}^2;0 \middle| V_{12} \middle| d_{5/2}^2;0 \right\rangle \qquad V_0' = \left\langle (s_{1/2})^2;0 \middle| V_{12} \middle| (s_{1/2})^2;0 \right\rangle$$

$$V_2 = \left\langle d_{5/2}^2;2 \middle| V_{12} \middle| d_{5/2}^2;2 \right\rangle \qquad V_2' = \left\langle s_{1/2}d_{5/2};2 \middle| V_{12} \middle| s_{1/2}d_{5/2};2 \right\rangle$$

$$V_4 = \left\langle d_{5/2}^2;4 \middle| V_{12} \middle| d_{5/2}^2;4 \right\rangle \qquad V_3 = \left\langle s_{1/2}d_{5/2};3 \middle| V_{12} \middle| s_{1/2}d_{5/2};3 \right\rangle$$

$$V_{00} = \left\langle (s_{1/2})^2;0 \middle| V_{12} \middle| (d_{5/2})^2;0 \right\rangle \qquad (4.1)$$

$$V_{22} = \left\langle d_{5/2}^2;2 \middle| V_{12} \middle| s_{1/2}d_{5/2};2 \right\rangle .$$

In principle it should be possible to determine these matrix elements from the available information on the energies of lowest 6 states of O^{18} and the lowest 3 states of O^{19}. In practice, of course this would involve quite a bit of computation. In this section we make this calculation

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In reference (41) Talmi and Unna assume identically $V_{00} = V_{22} = 0$ for simplicity in computation.

to obtain the two-body matrix elements and compare them with the matrix elements calculated in chapter II with our various potential models.

The binding energy of the two extra-core neutrons in O^{18} ground state can be obtained from the binding energy data as[19]

$$B.E.\ (O^{18}) - B.E.\ (O^{16}) - 2\left[B.E.(O^{17}) - B.E.(O^{16})\right]$$

$$= -12.215 + 2\ (4.146)\ MeV \qquad (4.2)$$

$$= -2.923\ MeV$$

and similarly the binding energy of the three neutrons in the ground state of O^{19} is given as

$$B.E.(O^{19}) - B.E.(O^{16}) - 3\left[B.E\ (O^{17}) - B.E.(O^{16})\right] \qquad (4.3)$$

$$= -3.72\ MeV.$$

Thus the absolute energies of states given experimentally in O^{18} and O^{19} are given in table 9.

Now obviously we obtain $V_4 = E_4 = -0.37$ MeV. Next, since the $J = 3^+$ state in O^{18} is not observed, there is no direct determination of V_3. Finally we note that E_0, E_0^* and E_2, E_2^* are the eigenvalues of the matrices

$$\begin{pmatrix} V_0 & V_{00} \\ & V_0' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} V_2 & V_{22} \\ & V_2' \end{pmatrix} \quad \text{respectively.}$$

Table 9

The absolute energies E_J of the states in O^{18} and O^{19}

O^{18}		O^{19}	
J	E_J (MeV)	J	E_J (MeV)
0	$E_0 = -3.92$	5/2	$E_{5/2} = -3.73$
2	$E_2 = -1.94$	3/2	$E_{3/2} = -3.62$
4	$E_4 = -0.37$	1/2	$E_{1/2} = -2.20$
0^*	$E_0^* = -0.39$		
2^*	$E_2^* = 0.0$		

Therefore for each assumed value of V_{00} and V_{22} one can calculate V_0, V_0' and V_2, V_2' by inversing the matrix. In table 10 we list the values of V_{00}, V_{22} and corresponding values of V_0, V_0', V_2 and V_2'. Negative sign for these off-diagonal matrix elements is chosen to give phase of the wavefunctions in agreement with the results of other calculations[13,19]. We discuss the explicit wavefunction a little later. For $V_{00} < -1.82$ MeV and $V_{22} < -0.37$ MeV, we would find in the wavefunction for the lowest $J = 0, 2$ states major components of $(s_{1/2})^2_0$ and $(s_{1/2}d_{5/2})_2$ states. Therefore we do not consider these values.

Table 10.

The values of V_o, V_o', V_2 and V_2' (in MeV) obtained by inverting the matrix for different values of V_{oo} and V_{22}.

V_{oo}	0	-0.5	-1.0	-1.5	-1.83
V_o	-3.92	-3.85	-3.68	-3.13	-2.11
V_o'	-2.03	-2.10	-2.33	-2.82	-3.85
V_{22}	0	-0.5	-0.75	-0.90	-0.97
V_2	-1.94	-1.80	-1.59	-1.33	-0.98
V_2'	-0.88	-1.02	-1.23	-1.49	-1.84

Thus we have obtained from the five known states of O^{18}, five matrix elements V_o, V_2, V_4, V_o' and V_2' in terms of three unknown matrices V_{oo}, V_{22} and V_3. Our next step is to construct Hamiltonian matrices for the states of O^{19} $J = 5/2$ (3 x 3), $J = 3/2$ (2 x 2) and $J = 1/2$ (1 x 1) for various sets of values of V_{oo}, V_{22} and V_3 and find the set which will best fit the energies of these states. The equations involved are already given in section 3. The results are summarised in figures 3, 4 and 5.

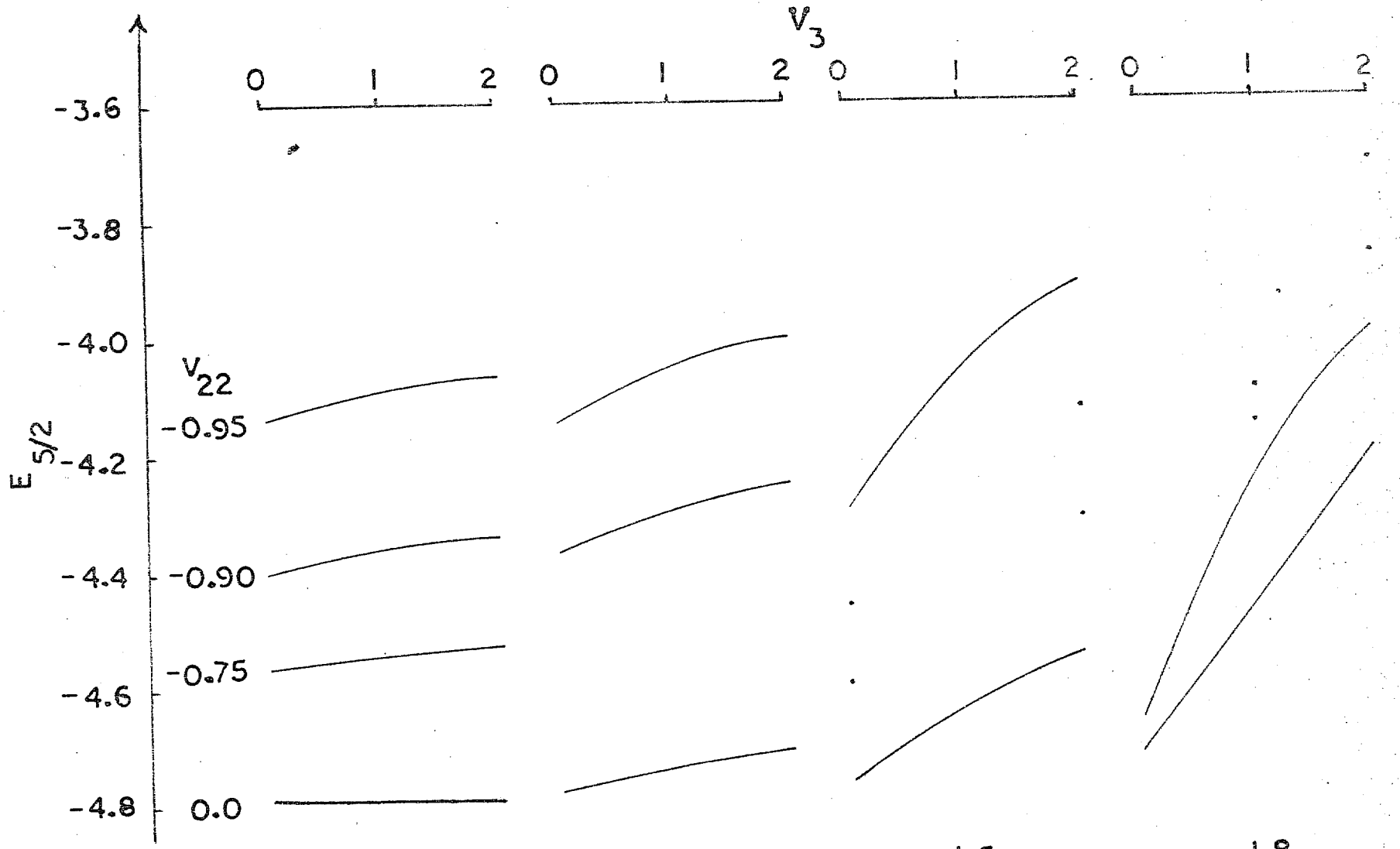

Fig.1. The state J= 5/2 as a function of V_{00}, V_{22} and V_3. $E_{5/2}$ was not calculated for the cases (i) V_{00}= -1.0 and V_{22}= -0.75 and (ii) V_{00}= -1.5, V_{22}= -0.75 and -0.90, and V_3= 1. For the cases V_{00} = -1.1, V_{22} = -0.90 and -0.95 the dominant component was found to be $(s_{1/2})^2 d_{5/2}$. Hence not to be considered.

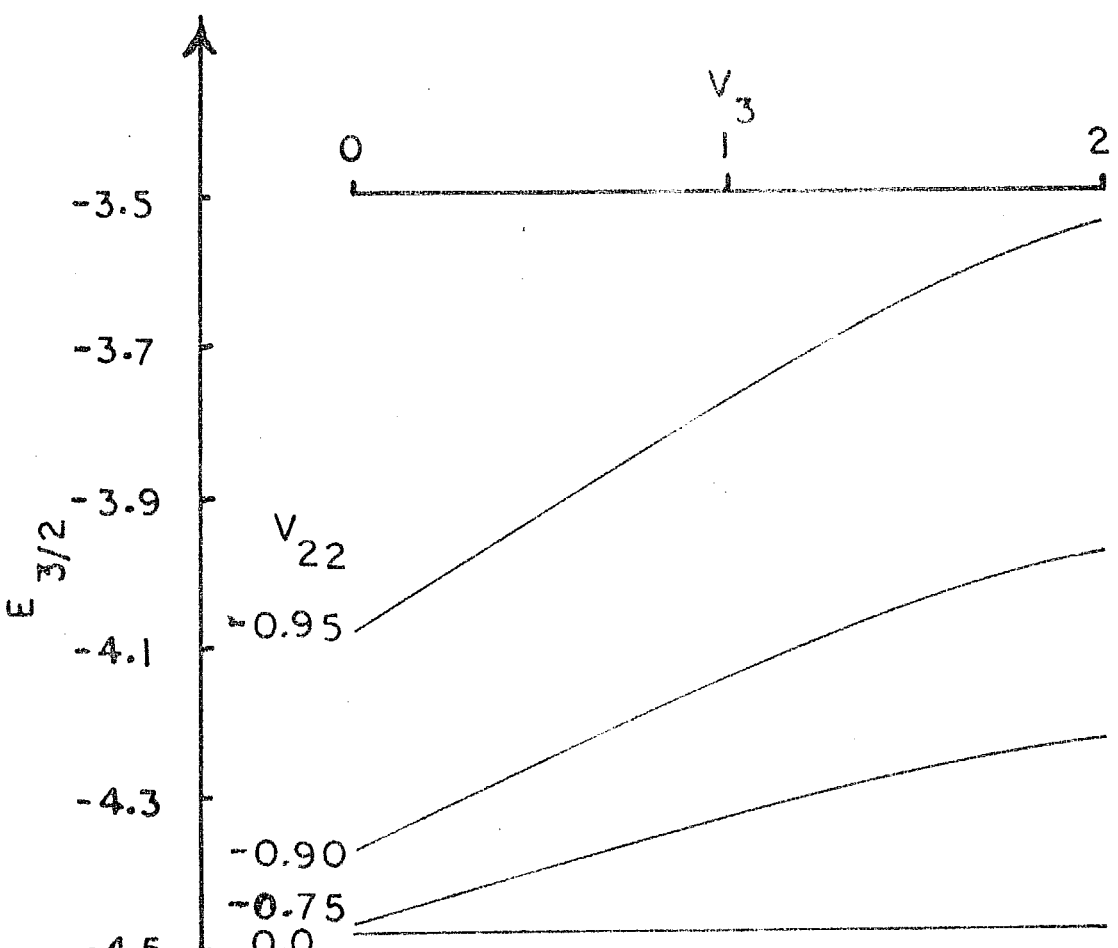

Fig. 4. The state $J = 3/2$ as a function of $V_{00} = V_{02}$ and V_3 for all values of V_{22}.

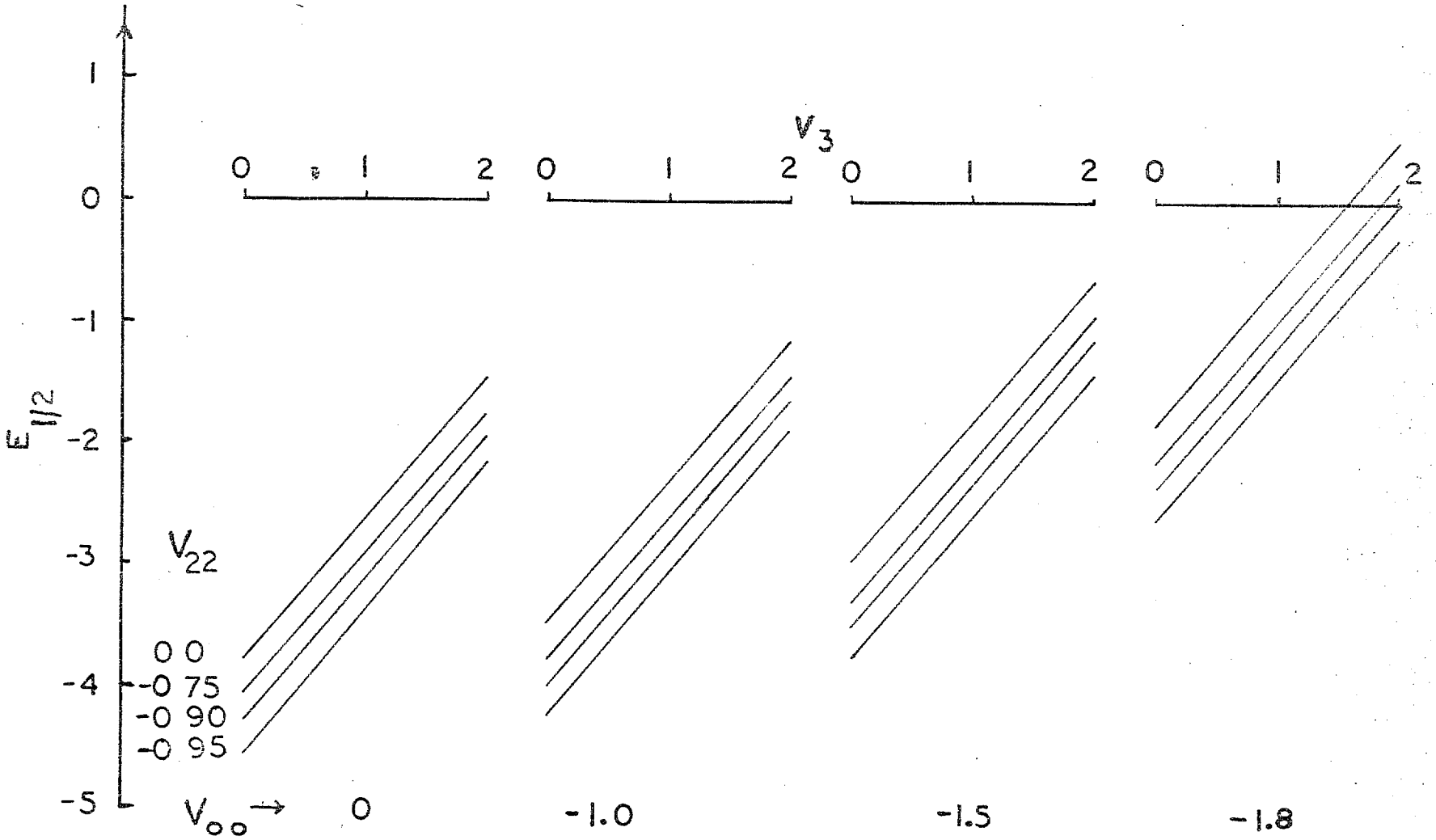

Fig.1. The state $J = 1/2$ as a function of V_{10}, V_{22} and V_{00}.

It seems to us that the best fit to O^{19} results is obtained for $V_{oo} = -1.5$ MeV, $V_{22} = -0.33$ MeV and $V_3 = +1.5$ MeV. This gives $E_{3/2} = -4.0$ MeV, $E_{3/2} = -3.83$ MeV, $E_{1/2} = -1.9$ MeV. We consider this agreement to be reasonably good since the calculation is heavily biased towards exact fitting of the five O^{18} states. A least square fit with equal bias to each of the 8 states would considerably improve the agreement.

To conclude we find that the following set of values provides a satisfactory fit to the known states of O^{18} and O^{19}.

$$\begin{aligned} V_0 &= -3.0 \text{ MeV} & V_0 &= -2.9 \text{ MeV} \\ V_2 &= -1.2 \text{ MeV} & V_2 &= -1.6 \text{ MeV} \\ V_4 &= -0.4 \text{ MeV} & V_3 &= +1.5 \text{ MeV} \\ V_{oo} &= -1.6 \text{ MeV} & V_{22} &= -0.33 \text{ MeV} \end{aligned} \qquad (4.5)$$

We remark that this predicts $J = 3^+$ state in O^{18} rather high i.e. at ~ 6 MeV. This we discuss below. Further $J = 9/2$ $(d_{5/2})^3$ state is also predicted by these parameters at about 2.1 MeV above the ground state 5/2.

Next we comment about the wavefunctions of the $J = 0, 2$ states of O^{18} given by these parameters. For $J = 0$ state the resulting wavefunction is reasonable, in the sense that it gives about 80 % $(d_{5/2})^2$ + 20% $(s_{1/2})^2$ mixing. This is in good agreement with other results.

On the other hand for $J = 2$ we find a rather large degree of mixing, about 55% $(d_{5/2})^2$ + 45% $(d_{5/2}p_{1/2})$. This appears to be rather large, especially in view of the results of McFarlane and French. Perhaps, if as mentioned above, a least square fitting is done, the values of the parameters may change a little, and may improve the result. But it is almost certain that a considerable mixing (greater than the values of McFarlane and French) will be predicted. We do not think this discrepancy is very serious.

The matrix elements obtained above agree well with the matrix elements obtained from our potential model and constitute another check on those results. (see tables 3 and 4 of section 1).

Two comments may be made. The matrix element V_0 obtained from the potential model appears to be somewhat larger than that found in this section. (However, owing to large repulsive contribution of tensor and spin-orbit forces, Dawson, Talmi and Walecka find this matrix element much smaller i.e. -2.0 MeV). Secondly, the matrix element V_3 has changed sign. The value +1.5 MeV obtained here does appear to be somewhat large. In view of the lack of experimental knowledge of the position of the $J = 3^+$ state, we do not explore this point further. If future experiments show the state to have indeed a positive matrix element $V_3 > 0$, our potential model will have to be somewhat revised.

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CHAPTER IV

CONCLUDING REMARKS

In this dissertation our main preoccupation has been the study of effective nucleon-nucleon forces in nuclei. The conceptual framework of the shell model supplemented by the work of Brueckner and colleagues has been delineated. We have pointed out that there is unsatisfactory and insufficient knowledge of the effective forces within this framework. It is very necessary, not only for pure shell model calculations of nuclear spectra, but also for a detailed understanding of the origin of "collective motions" within the framework of the shell model, that a reasonably reliable model effective interaction be available.

In chapter II, we have described a simple method by which the nuclear spectra can be calculated in terms of effective interactions in various states of relative orbital angular momentum. This technique we believe is more flexible and gives results of more general interest than the usual method of expanding the effective interaction in multipoles. We have then applied this method to learn something about the nature of the effective interaction which will give rise to some simple observed nuclear spectra, viz. energy levels

of (closed shell + two nucleons) $T = 1$ states. Unfortunately even in these simple nuclei, not enough experimental information is available to provide any clear cut answers. In particular, the knowledge of odd-state forces obtained is not satisfactory. However, the calculations reveal the limitations of experimental data and serve to point up the data required for complete satisfactory analysis. For example, the observation of $J = 3^+$ state in O^{18} and the spins and parities of the three states between 3.0 and 3.5 MeV excitation in O^{19} would help to determine the effective forces in these nuclei. The results presented here will be refined when such additional data becomes available. We believe that with availability of additional data, this method of analysis will enable us to study questions such as presence of non-central forces, or non-locality of forces etc. For example, the eight matrix elements involved in analysis of O^{18} and O^{19} (see equation (4.1) chapter III) can be further analysed in terms of the Talmi integrals $I_{n\ell} = \langle n\ell | V_{12} | n\ell \rangle$, once they are empirically determined as in section 4 of chapter III. It is easy to see that only seven Talmi integrals are involved, viz. I_{0s}, I_{1s}, I_{2s}, I_{0d}, I_{1d}, I_{0p} and I_{1p}.† Thus a determination of these will immediately tell us if the forces in the $\ell = 0$ and $\ell = 2$ states are the same or otherwise. We do not carry out this detailed analysis at present since the empirical information is not yet quite clear cut, as mentioned in the previous chapters.

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† With our restriction that contribution of f- and g-states ($\ell = 3, 4$) to interaction energies can be neglected.

It will also be desirable and instructive to extend the analysis to T = 0 states in odd-odd nuclei such as F^{18} when enough states are known in these. For example in F^{18}, the T = 0 states would be $(d_{5/2})^2$ J = 1, 3, 5, $(s_{1/2})^2$ J = 1, $(s_{1/2}d_{5/2})$ J = 2, 3. Experimentally, the two J = 1 states, one J = 3, and the J = 5 state are now located; if one or two additional states are determined, we can apply the above method of analysis to this nucleus also.

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