STUDIES OF NONLINEAR WAVES IN PLASMAS

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TO

MY UNCLE

SARODE NARAYANA RAO

CERTIFICATE

I hereby declare that the work presented in this Thesis is original and has not formed the basis for the award of any degree or diploma by any University or Institution.

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ABSTRACT OF THE THESIS

We investigate, in this Thesis, some problems pertaining to the existence and propagation of nonlinear ion acoustic waves and nonlinear amplitude modulated, high frequency Langmuir waves and the associated low frequency ion waves. In the case of nonlinear ion acoustic waves, we first investigate the physical implications of the different sets of stretched co-ordinates employed in the Reductive Perturbation Method for obtaining the relevant evolution equation, namely, the Korteweg-de Vries equation. It is found that the two sets of co-ordinates yield different Korteweg-de Vries equations in the (x,t) co-ordinate system and that the initial value problems associated with these equations are different. The experimental implications of these differences have been discussed. We investigate, next, the effect of plasma inhomogeneities on the propagation characteristics of the nonlinear ion acoustic waves using the Reductive Perturbation Analysis. In the presence of spatial gradients in the ion density and ion temperature, these waves are found to be governed by a modified Korteweg-de Vries equation. Soliton solution of this equation shows that as the nonlinear ion acoustic waves propagate along the ion temperature (or density) gradient, their amplitudes are reduced. A linear analysis of the problem of ion acoustic wave propagation in inhomogeneous plasmas has also been carried out.

For the problem of nonlinear, amplitude modulated Langmuir waves we develop a theory valid in the entire range of the soliton Mach number, namely, $0 < M < 1$. A set of governing equations for the

stationary propagation of the high frequency Langmuir waves and the associated ion waves has been derived by taking into account the full ion nonlinearity and complete departures from the charge neutrality for the low frequency ion waves. A method is then developed to solve these coupled, nonlinear equations. This method is capable of taking into account any arbitrary degree of ion nonlinearity, consistent with the nonlinearity retained in the Langmuir field amplitude. A class of double-hump Langmuir solitons having non-zero Langmuir field intensity at the centre is found for intermediate values of the Mach number in the range $0 < M < 1$. These solutions are found to provide a smooth transition from single-hump Langmuir solitons to the double-hump Langmuir solitons having zero Langmuir field intensity at the centre. The regions of parameter values for the existence of different types of Langmuir soliton solutions are explicitly obtained. The existence and structure of these solutions have been studied by means of the Sagdeev Potential Analysis. The theory developed here yields, under suitable limiting conditions, various Langmuir soliton solutions discussed earlier by other authors. Finally, a conjecture is made about the existence of many-hump Langmuir solitons for higher order nonlinearities in the low frequency ion potential.

CONTENTS

CHAPTER I

INTRODUCTION

One of the most important and interesting characteristics of plasmas is their ability to sustain a rich variety of wave phenomena. This is due to the fact that the interactions between the plasma particles are long-range in nature and, macroscopically, plasmas behave like fluids. The presence of an external magnetic field further increases the number of waves which plasmas can support. The study of wave phenomena in plasmas is interesting as well as important in two ways: firstly, as a means of understanding the various complicated processes that take place in plasmas under different conditions and, secondly, as a diagnostic tool in experimental studies of plasma systems. Both these aspects are of great importance in the study of plasmas, one of the principal objectives of which is the controlled release of energy from thermonuclear reactions.

I.1 Role of Nonlinearities in Plasmas

In the last few decades, a great deal of theoretical and experimental work has been carried out on linear waves and instabilities in plasmas. A fairly complete account of these studies is available in (Stix, 1962; Akhiezer et al, 1975) and in the reviews (Bernstein & Trehan, 1960; Allen & Phelps, 1977). However, the usual linear approximation that the wave amplitudes are infinitesimally small has very limited applicability in many practical situations. For instance, when the waves are linearly unstable, the wave amplitudes grow exponentially in accordance with the predictions of linear theories. As the amplitudes grow, the linear theories are no longer applicable. Also, other processes like particle trapping, resonant wave-particle effects, wave decay and higher harmonic generation, etc., which are not considered in linear theories become quite important. Some of these processes actually account for the saturation of wave amplitudes observed in experiments on linearly unstable modes. Thus, the analysis of effects of nonlinearities on waves and instabilities forms an integral part of the study of plasma systems. While the study of weakly nonlinear systems is a first step in overcoming the limitations of linear theories, a thorough analysis of the effects of strong nonlinearities is very essential for a more complete understanding of the nonlinear plasma dynamics and the turbulence phenomena in plasmas. A detailed account of these studies can be found in Davidson (1972), Tsytovich (1970), Sagdeev & Galeev (1969), Kalman & Feix (1969), Whitham (1974), Leibovich & Seebass (1974), and in Kadomtsev & Karpman (1971), Franklin (1977).

I.2 Solitary Waves - Their Importance and Applications

Among the host of nonlinear phenomena that have been considered in plasmas as well as in other fields, the concept of Solitary waves or Solitons has come to occupy an important place in recent years. Solitons can be loosely defined as stationary, localized, finite energy wave packets which arise due to the balance between the effects of nonlinearity and dispersion; (for a more precise definition see, for instance, Scott et al (1973)). Dispersive effects are very important for the formation of solitary waves in any system. A nonlinear system without any dispersion injects the initial pulse energy into higher frequency modes through harmonic generation whereas, in dissipative systems, this leads to the formation of 'shocks'. On the other hand, in a linear system with dispersion alone, various Fourier components of a given initial pulse propagate with different phase velocities and, hence, separate out from each other in course of time. This eventually leads to the spreading of the initial pulse because of dispersion. However, under suitable circumstances, a proper balance between these two competing processes can be brought about and this results in the formation of solitary waves or solitons.

A broad classification of solitons into two groups is generally based on the considerations of the strength of dispersive effects. When the effect of dispersion is weak, the solitons are generally governed by the usual Korteweg-de Vries (K-dV) equation (or any of its generalizations) and are known as K-dV solitons. Such equations have been used to describe various phenomena like (i) self-trapping of heat pulses in solids (Tappert & Varma, 1970), (ii) shallow water wave propagation (Zabusky & Galvin, 1971), and (iii) propagation of waves in anharmonic lattices (Zabusky, 1973). On the other hand, when the dispersive effects are strong, solitary wave solutions can be obtained for the amplitude of the modulated waves, and these are called 'envelope solitons'. In many cases, envelope solitons are governed by a nonlinear Schrodinger-like equation. This equation has been used to describe (i) self-trapping phenomena in nonlinear optics (Karpman & Kruskal, 1969), (ii) filamentary structure of light beams in nonlinear liquids (Bespalov & Talanov, 1966), (iii) propagation of stationary optical pulses in dispersive dielectric fibres (Hasegawa & Tappert, 1973), and is (iv) related to the Ginzburg-Landau equation in superconductivity (de Gennes, 1966).

There are numerous other nonlinear equations which have soliton type of solutions. Among them, the following equations are more commonly used in different fields: the sine-Gordon equation for (i) the propagation of crystal dislocations (Frenkel & Kontorova, 1939), (ii) propagation of magnetic flux on a Josephson line (Scott, 1970); the Toda lattice equation for the motion on a one-dimensional lattice of mass points interacting through nonlinear potentials (Toda, 1970); the Boussinesq equation for shallow water waves (Hirota, 1973a); the Hirota equation (Hirota, 1973b) which is a simultaneous generalization of the K-dV equation and the nonlinear Schrodinger equation; the Born-Infeld equation which is a nonlinear modification of the Maxwell equations (Barbashov & Chernikov, 1967). These equations have found many applications in such diverse fields as Solid State Physics, Laser Physics, Astrophysics, Nuclear Physics and, recently, in Particle Physics. Excellent reviews on solitons in plasmas as well as in other fields can be found in Scott et al (1973), Lonngren & Scott (1978), Bishop & Schneider (1978), Bhatnagar (1979) and in Physica Scripta (1979, Vol.20, pp.291).

I.3 Solitons in Plasmas

Much of the current interest in solitary waves in plasmas started with the work of Zabusky & Kruskal (1965) who made a computer study of the K-dV equation through numerical integration. Some of the more important results of their investigation can be summarized as follows: (i) Under a wide variety of conditions, any given initial pulse breaks up into a number of solitons which move in the plasma with different, constant velocities; (ii) Solitons nonlinearly interact with each other and after the interaction, they emerge out without any changes in their shapes or in their velocities, thus retaining their identities; (iii) Periodically, they reconstruct the initial state with almost the same phase and thereby exhibit a recurrence phenomenon similar to the one first observed by Fermi et al (1965) in their numerical study of a discretized, weakly nonlinear string. It is the particle-like behaviour of these nonlinear entities in their interactions with each other that led Zabusky & Kruskal (1965) to coin the name "Soliton" for them.

With the work of Zabusky & Kruskal (1965), a many-faceted investigation about the formation and interaction of solitary waves in plasmas has been launched. It is now well established that many of the normal modes of plasma give rise to corresponding solitary waves in the weakly nonlinear regime (Jeffrey & Kakutani, 1972; Karpman, 1975a; Ichikawa & Watanabe, 1977). The first attempt in this direction was made by Gardner & Morikawa (1960) who derived the K-dV equation for weakly nonlinear hydromagnetic waves in a cold plasma. By taking finite electron temperature into account Kawahara (1969) showed that weakly nonlinear magneto-acoustic waves are governed by the usual K-dV equation whereas the Alfven solitary wave is described by a modified K-dV equation. The nonlinear evolution of small but finite amplitude ion acoustic waves in a collisionless plasma was later shown to be governed by the K-dV equation (Washimi & Taniuti, 1966). While nonlinear drift waves in a magnetized plasma show soliton type of behaviour (Petviashvili, 1967; 1977), a K-dV type of equation for these waves was explicitly derived by Orefice & Pizzoli (1970), Nozaki & Taniuti (1974) and Todoriki & Sanuki (1974).

While the above analyses are concerned with weakly dispersive systems, modulational instabilities play an important role in strongly dispersive media. Linear stability analyses show that a wide variety of finite amplitude waves become modulationally unstable when subjected to long-wavelength perturbations (Kakutani & Sugimoto, 1974; Hasegawa, 1975). Here again, the wave steepening due to the nonlinearity is balanced by the dispersive effects of the wave and the resulting state is that of the 'envelope soliton'. It is interesting to note here that this phenomenon is very much similar to self-focussing and self-contraction of wave packets, first discovered in nonlinear optics (Askaryan, 1962; Chiao et al, 1964; Akhmanov et al, 1968). In plasmas, many waves show this type of behaviour. Taniuti & Washimi (1968) were the first to show that the modulational instability and the self-trapping phenomena of hydromagnetic waves along the external magnetic field in a cold plasma could be described by a nonlinear Schrodinger equation. Similar analyses for electron- and ion-cyclotron waves have been carried out by Hasegawa (1970; 1971).

While the modulational instability of Langmuir waves was considered by Vedenov & Rudakov (1965), the nonlinear evolution as well as the so-called 'collapse phenomena' of these waves has been discussed by Zakharov (1972) in some detail. An important application of the envelope Langmuir solitons to the strong Langmuir turbulence in plasmas is due to Rudakov (1973) and Kingsep et al (1973) who treat the turbulent state as a collection of interacting, randomly distributed Langmuir solitons. For the one-dimensional case, Kingsep et al (1973) obtained the turbulence spectra $\langle |E_k|^2 \rangle \propto k^{-2}$, where k is the wave number of the Langmuir waves and $|E_k|$, the Langmuir field amplitude. It may be noted here that such turbulent spectra have been obtained in a number of one-dimensional numerical studies of strong Langmuir turbulence (see, for instance, Sudan, 1973; Kruer et al, 1973). As other examples in this type of analyses, amplitude modulated upper-hybrid waves (Kaufman & Stenflo, 1975) and ion acoustic waves (Tidman & Stainer, 1965; Kakutani & Sugimoto, 1974) are also found to be describable through nonlinear Schrodinger equation. Finally, the similarity between the long-time behaviour of the modulational instability and the Fermi-Pasta-Ulam recurrence phenomenon has been pointed out by Lake et al (1977), Yuen & Ferguson (1978) and Janssen (1981).

I.4 Motivation for the Present Work

Among the different types of solitary waves that have been discussed in the literature, those of ion acoustic waves and Langmuir waves have attracted much attention in the last decade. As pointed out earlier, these are respectively governed by the K-dV equation and the nonlinear Schrodinger equation (or the Zakharov equations). Both these are prototype equations for a large number of nonlinear phenomena where dispersive effects (weak or strong) are also important. Mathematically, the K-dV equation is the simplest of all nonlinear evolution equations of its kind, and the studies of this equation have led to much of the present understanding of some of the nonlinear effects. A significant outcome of these studies has been the discovery of Inverse Scattering Method (Gardner et al, 1967) which is an analytic method for solving a class of nonlinear equations as initial value problems. While further studies of this equation are continuing (Miura, 1976), a great deal of the current studies of solitary waves in plasma physics have centred around the analysis of the Zakharov equations (Zakharov, 1972) and their solutions. One of the motivations for these studies comes from their possible application to some problems in strong Langmuir turbulence (Zakharov, 1972; Rudakov, 1973).

As is well known, the existing approaches to the problem of strong Langmuir turbulence can be divided into two groups: (i) the statistical approach developed by Weinstock & Bezzerides (1973), Khakimov & Tsytovich (1973) and others, and (ii) the dynamical approach started by Zakharov (1972) and Rudakov (1973). The starting point in the statistical approach is to consider the developed modulational instability and describe the density depletions statistically through relevant correlation functions. Even though this approach offers a better description of the turbulent state, it suffers from loss of information about the detailed dynamical behaviour of modulational interactions.

The dynamical approach, on the other hand, consists of three steps: (i) to identify certain nonlinear dynamical entities induced by modulational interactions; (ii) to study interactions of these entities among themselves and with other plasma waves/particles and, finally, (iii) to obtain the turbulence spectra through an analysis of their statistical behaviour. With regard to the first step, it is now believed that the nonlinear entities like Langmuir solitons (in one-dimensional case), cavitons (the infinitely increasing local density depressions in higher dimensions; for the spherically symmetric case, ref. Zakharov (1972)), etc., indeed represent the type of dynamical entities one is looking for in the dynamical approach. Not much work has, however, been carried out so far on the interaction processes involving these solitons (for a review of the above two aspects, see, for instance, Thornhill & ter Haar, 1978; Rudakov & Tsytovich, 1978)).

For the amplitude modulated Langmuir waves, Zakharov (1972) derived a set of equations in an adiabatic approximation. In the quasi-static case $M \ll 1$ (M = Mach number), these equations lead to the non-linear Schrodinger equation from which Rudakov (1973) obtained an envelope soliton as a stationary solution. Kingsep et al (1973) have tried to model the one-dimensional strong Langmuir turbulence as a one-dimensional gas of these Rudakov solitons in a close packing.

The Rudakov soliton is a solution of the Zakharov equations in only one of the possible limits - the quasi-static limit. Karpman soliton (Karpman, 1975b) is another solution in the near-sonic limit $M \lesssim 1$. However, the Zakharov equations, themselves, are valid only for small amplitude for the ion wave; that is, these equations take into account only the linear ion response and assume charge neutrality for the low frequency ion motion. One could, therefore, consider a generalization of the Zakharov equations to include nonlinear ion dynamics and a departure from charge neutrality. A weakly nonlinear ion response of the K-dV type was considered by Nishikawa et al (1974) along with departures from charge neutrality. This leads to a different type of dynamical entity: a double-hump soliton with the Langmuir field amplitude vanishing at the centre of the soliton. The associated ion wave is given, in this case, by a plasma density depression with a maximum depth at the centre of the soliton.

One could, in principle, include any arbitrary degree of ion nonlinearity along with the complete Poisson equation for the low frequency waves, and investigate the various possible forms of Langmuir soliton solutions. (We have carried out such an investigation in Chapter IV, and have indeed found a class of double-hump solition solutions having non-zero Langmuir field intensity at the centre of the solitons; further, these solutions are found to provide a smooth transition from the Rudakov soliton at one end ($M << 1$) to the solitons of Nishikawa et al as well as Karpman at the other end ($M \lesssim 1$)).

It is clear that strongly nonlinear Langmuir turbulence will have ion wave associated with it in the form of density depletions brought about by the Langmuir wave induced ponderomotive force. It has, in fact, been found that collisions of the solitons of Nishikawa et al (1974) can lead to their complete destruction resulting in the emission of Langmuir and ion sound waves (Appert & Vaclavik, 1977).

The formulation of a theory of strong Langmuir turbulence, thus, requires that we must study, in detail, the various properties of these nonlinear entities and the dynamics of their interactions with other nonlinear entities, and other ion acoustic and Langmuir waves. It is with this view that we have undertaken the investigations carried out in this Thesis, which constitutes only a first step towards such a programme.

I.5 Scope of the Thesis

We thus investigate, in this thesis, some problems relating to the existence and propagation characteristics of nonlinear ion acoustic waves as well as nonlinear, amplitude modulated Langmuir waves and the associated ion waves. A brief summary of each one of these investigations is given in the following.

In Chapter II, we investigate, for weakly nonlinear ion acoustic waves, the physical implications of the use of two different sets of stretched co-ordinates in the framework of Reductive Perturbation Analysis which yield the K-dV equations of identical form in the two sets of the co-ordinates. This problem has significance when one compares the results of an experiment on K-dV solitons with the theoretical expectations. We find that even though the two sets of stretched co-ordinates have the same ordering with respect to the smallness parameter used in the Reductive Perturbation Analysis, the K-dV equations (and hence their solutions) obtained through them differ substantially. In particular, the soliton solutions corresponding to the two sets have different propagation characteristics like the amplitude and the width. Further, we discuss the differences between the two K-dV equations when transformed into the laboratory frame of reference and show that the two sets of stretched co-ordinates correspond to different ways of launching the solitons in experiments. We compare also the contributions coming from the second-order solutions with the first-order solutions for both the sets of stretched co-ordinates.

Chapter III deals with the analysis of the effects of plasma inhomogeneities on the propagation characteristics of linear and nonlinear ion acoustic waves. Most often, plasmas are inhomogeneous and the plasma inhomogeneities can appreciably modify the wave propagation characteristics. We consider, in this chapter, inhomogeneities which arise due to spatial gradients in the ion density and ion temperature assuming them to be slowly varying but having arbitrary distribution with respect to the space co-ordinate. First, we carry out the linear analysis of the basic fluid equations and show that as the linear ion acoustic wave propagates along the ion temperature (or ion density) gradient, its amplitude decreases. The nonlinear analysis of the basic equations is, then, carried out using the Reductive Perturbation Analysis. This yields a modified K-dV equation as the governing equation for the propagation of nonlinear ion acoustic waves in inhomogeneous plasmas. Soliton solutions of this equation indicate that as the ion acoustic solitary wave propagates along the ion temperature (or ion density) gradient, its amplitude decreases. Thus, the effects of spatial gradients in the ion temperature and ion density on the amplitudes of the linear as well as nonlinear ion acoustic waves are found to be similar. The changes in the wave number (for the linear wave) and the width (for the nonlinear wave) then follow as a consequence of the changes in the amplitudes. We also show that when the two gradients are in opposite directions with suitable scale lengths, the amplitudes of the waves remain constant whereas other characteristics like the width and the wavenumber change.

In Chapter IV, we make a systematic and self-consistent analysis of the problem of amplitude modulated Langmuir waves and the associated ion waves, and develop a theory valid in the entire range of the Mach number, namely, $0 < M < 1$. Starting with the basic fluid equations, we first obtain a set of governing equations for the amplitude of the high frequency Langmuir waves (E) and for the low frequency ion potential (Φ). While the equation for the Langmuir field amplitude has been derived from relevant fluid equations by averaging them over the 'fast time' ω_{pe}^{-1} (ω_{pe} is the usual Langmuir wave frequency corresponding to the unperturbed state), the equation for the low frequency potential has been derived from the ion fluid equations by taking into account full ion nonlinearity and complete departures from charge neutrality (for the low frequency waves) through the Poisson equation. We, next, develop a method for solving these equations wherein arbitrary degree of ion nonlinearity consistent with the nonlinearity retained in the Langmuir field amplitude can be taken into account. For small values of Mach number ($M << 1$), we obtain, from our solutions, the single-hump Rudakov solitons (Rudakov, 1973) where only the linear ion dynamics is considered. On the other hand, as the Mach number is increased the solution for E^2 becomes narrower and smaller, and for values of M beyond a certain critical Mach number M_{crit} the solution for E^2 develops a dip at the centre whose depth increases with further increase of the Mach number until E^2 vanishes at the centre for a certain limiting Mach number, the cut-off Mach number M_{cut}. We thus obtain a class of double-hump soliton solutions (for Mach numbers in the range $M_{crit} < M < M_{cut}$) which have non-zero Langmuir field intensity at the centre of the solitons. These solutions provide a smooth transition from single-hump Langmuir solitons (Rudakov, 1973; Karpman, 1975b) to the double-hump Langmuir solitons having zero Langmuir field intensity at the centre (Nishikawa et al, 1974). The existence of such a smooth transition is further confirmed by explicitly carrying out the Sagdeev potential analyses of the relevant equations for the Langmuir field amplitude and the low frequency ion potential. Finally, we make a conjecture about the existence of many-hump Langmuir soliton solutions for higher order nonlinearities in the low frequency ion potential.

Before considering these analyses in detail in the succeeding Chapters we discuss below, briefly, a few of the mathematical methods that are commonly used for treating nonlinear equations one usually encounters in Plasma Physics as well as in other fields, and some of which we have used in our investigations.

I.6 Mathematical Methods for Nonlinear Equations

It is a well known fact that the system of equations one usually encounters in the analysis of nonlinear problems in any field are not generally amenable to exact analytic solutions. Even if the exact solution of a problem can be found explicitly, it may be difficult for physical and mathematical interpretations. However, if one is looking for some specific nonlinear effects, special methods (both exact and perturbative) do exist. We discuss below a few of such methods which are widely used in the studies of solitary wave phenomena in plasmas. Needless to say that, quite often, depending on the need, new techniques have to be devised for solving nonlinear equations.

I.6.1 Reductive Perturbation Method

Perturbation methods offer an easy way of handling nonlinear equations when the nonlinearities involved are weak. Among the numerous perturbation methods that are available now (Nayfeh, 1973), the so-called Reductive Perturbation Method (Taniuti & Wei, 1968) has been extensively developed and applied to a wide variety of nonlinear phenomena (for a thorough review of this method, ref. Suppl. Progr. Theor. Phys., No.55, 1974). The basic aim of this method is to 'reduce' a class of systems of nonlinear equations to a more tractable single nonlinear equation which, for specific values of the parameters involved, yields the Burgers equation or the K-dV equation or the nonlinear Schrodinger equation or a variety of modifications thereof. While originally the method was developed for homogeneous systems, it was later extended by Asano & Ono (1971) for inhomogeneous systems as well. In the following, we give a brief outline of the reductive perturbation method as applied to homogeneous, weakly nonlinear and weakly dispersive (or dissipative) systems.

A weakly nonlinear system can generally be represented by the system of equations,

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + \left[\sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}\left(H_{\alpha}^{\beta}\frac{\partial}{\partial t} + K_{\alpha}^{\beta}\frac{\partial}{\partial x}\right)\right]U = 0, \tag{1.1}$$

where $p \geq 2$; x and t are respectively the space and time coordinates, and U is a column vector with n components $u_1, \ldots\ldots, u_n$ ($n \geq 2$) representing various physical quantities; A, H_{α}^{β} and K_{α}^{β} are $n \times n$ matrices whose elements are functions U. Linearizing eqs. (1.1) and using Fourier transforms in space and time, the dispersion relation can be obtained for the frequency ω and the wavenumber k. If we now restrict only to long wavelength perturbations, then, the dispersion relation takes the form

$$\frac{\omega}{k} = \lambda_0 + \mu k^{p-1} + O\left(k^{2(p-1)}\right), \tag{1.2}$$

where λ_0 is the linear phase velocity of the waves in the long wavelength limit and μ is a constant. In plasmas, p equals 3 for ion acoustic and magnetoacoustic waves, and for Alfven waves propagating in a direction oblique to the magnetic field direction; p equals 2 for Alfven waves propagating along the magnetic field direction.

The characteristic curves of the reduced equation obtained by neglecting the last term in eqs.(1.1) can be written in the following form:

$$\frac{dx}{dt} = \lambda_0 + \epsilon \lambda_1 + O(\epsilon^2), \tag{1.3}$$

where ϵ is a small, positive parameter specifying the magnitude of the nonlinearity. Comparing eq.(1.2) with eq.(1.3), we note that the coupling of the nonlinear effect with dispersion (or dissipation) is possible in the order of ϵ if $k \approx \epsilon^a$ where $a = 1/(p - 1)$. On the other hand, since in the limit $k \longrightarrow 0$ the phase velocity ω/k of the wave is constant, it is more convenient to go over to a frame of reference moving with a velocity λ_0. Thus, one introduces a set of 'stretched co-ordinates' defined by (Taniuti & Wei, 1968)

$$\xi = \epsilon^a (x - \lambda_0 t),$$
$$\tau = \epsilon^{a+1} t. \tag{1.4}$$

We may point out here that this transformation is not unique and one can as well introduce the following set of stretched co-ordinates (Taniuti & Wei, 1968):

$$\xi = \epsilon^{a}(x - \lambda_0 t),$$

$$\tau = \epsilon^{a+1} \cdot x. \tag{1.4a}$$

Obviously, the sets of stretched co-ordinates (1.4) and (1.4a) have the same ordering with respect to the smallness parameter ϵ. However, the evolution equations obtained through them in the x - t co-ordinates are different and hence give rise to different time evolutions of a given initial pulse (some of these differences for the case of nonlinear ion acoustic waves have been considered in Chapter II).

Assuming now that there exists a constant solution U_0 such that U, A, H_{α}^{β} and K_{α}^{β} can be developed as a power series in ϵ, we write

$$U = U_0 + \epsilon U_1 + \epsilon^2 U_2 + \cdots,$$

$$A = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \cdots; \text{ etc.} \tag{1.5}$$

Substituting eqs. (1.4) (or eqs. (1.4a)) and (1.5) into eqs. (1.1) the coefficients of various powers of ϵ are equated to zero to yield sets of equations corresponding to different orders in ϵ. The equations corresponding to the zeroth and first order in ϵ relate, respectively, the different unperturbed quantities and the first order perturbed quantities. Using these equations, the equations corresponding to second order in ϵ can be reduced to an equation of the form

$$\frac{\partial u^{(1)}}{\partial \tau} + C_1 u^{(1)} \frac{\partial u^{(1)}}{\partial \xi} + C_2 \frac{\partial^p u^{(1)}}{\partial \xi^p} = 0, \tag{1.6}$$

where $u^{(1)}$ is one of the components of the column vector U_1 and C_1, C_2 are constants which are known functions of the unperturbed quantities. Note that eq.(1.6) becomes, with appropriate values of C_1 and C_2, the Burgers equation for $p = 2$ and K-dV equation for $p = 3$. The above procedure can easily be extended for higher powers of ϵ and, correspondingly, the governing equations for the higher order perturbed quantities can be obtained.

The Reductive Perturbation Method has been extended so as to be applicable to strongly dispersive nonlinear systems also (Taniuti, 1974). The reduced equation for such systems is the nonlinear Schrodinger equation, namely,

$$i \frac{\partial u^{(1)}}{\partial \tau} + P \frac{\partial^2 u^{(1)}}{\partial \xi^2} + Q |u^{(1)}|^2 u^{(1)} = 0, \tag{1.7}$$

where $u^{(1)}$ now represents the complex amplitude of the modulated waves and P, Q are constants. Obviously, the reduced equations (1.6) and (1.7) are much simpler than the original set of equations and can readily be solved using the Inverse Scattering Method discussed below.

I.6.2 Inverse Scattering Method

One of the most significant and far-reaching results obtained in recent years through the studies of K-dV equation has been the discovery of Inverse Scattering Method whereby one can solve the initial value problem for a class of nonlinear partial differential equations. The method was, originally, developed by Gardner et al (1967) for the K-dV equation and it was, later, extended and applied to other equations like the nonlinear Schrodinger equation (Zakharov & Shabat, 1972), the Modified K-dV equation (Wadati, 1972), etc. A notable contribution in this direction is by Ablowitz et al (1973) who set up a general inverse scattering framework for solving a class of nonlinear evolution equations of physical significance encompassing the K-dV equation, the sine-Gordon equation, the sinh-Gordon equation, the Benny-Newell equation and their various generalizations. The original inverse scattering method of Gardner et al (1967) has been expressed in an elegant and general form by Lax (1968).

The procedure in the inverse scattering method or in any of its generalizations consists of four steps: (i) constructing an appropriate linear scattering (eigenvalue) problem in the 'space' variable where the solution to the nonlinear evolution equation plays the role of the potential; (ii) choosing the 'time' dependence of the eigen functions in such a way that the eigenvalues remain invariant with respect to time as the potential evolves according to the given evolution equation; (iii) solving the direct scattering problem at the initial time and determining the time dependence of the scattering data and, finally, (iv) carrying out the inverse scattering problem at later times, that is, to construct the potential knowing the (discrete) eigenvalues corresponding to the bound states and the time dependence of other scattering data. The last step can be written in terms of a linear integral equation (or a coupled set of linear integral equations) from which one can compute the solution to the evolution equation for all times. Obviously, the advantage of such a procedure is that instead of solving a nonlinear equation one needs only to solve now a coupled set of linear equations. In the following, we describe briefly the above method for solving the K-dV equation. Relevant details can be found in, for example, Davidson (1972).

Consider, then, the K-dV equation for the variable $v(\xi, \tau)$ obtained from eq.(1.6) for p = 3 and for appropriate values of C_1 and C_2, in the form

$$\frac{\partial v}{\partial \tau} - 6v\frac{\partial v}{\partial \xi} + \frac{\partial^3 v}{\partial \xi^3} = 0, \tag{1.8}$$

where ξ and τ are respectively the space and time variables. It is required to solve eq.(1.8) as an initial value problem for a given $v(\xi, \tau = 0)$ and obtain $v(\xi, \tau)$ for all τ. For this, let us assume, for the time being, that $v(\xi, \tau)$ is known and treat it as the potential in the time-independent Schrodinger equation,

$$\frac{d^2\Psi}{d\xi^2} + \left[E - V(\xi, \tau)\right]\Psi = 0, \tag{1.9}$$

where τ is treated as a parameter. The quantities E and Ψ, then, depend on τ parametrically. If at some fixed value of τ, say, $\tau = 0$, $v(\xi, \tau)$ is sufficiently well behaved, then, the direct scattering problem associated with eq.(1.9) can be solved. Equation (1.9) admits, in general, a finite number of bound states with energies $E_n = -k_n^2$, $n = 1,2......, N$ and a continuum of states with energies $E = k^2$; and one can obtain the corresponding scattering data. The solution of the inverse problem requires a knowledge of the discrete eigenvalues E_n, corresponding eigen functions Ψ_n and the **reflection** coefficient $R(k, \tau)$; the transmission coefficient $T(k, \tau)$ is not needed for obtaining the solution of the inverse problem.

When the potential in eq.(1.9) satisfies the K-dV equation (1.8), the scattering parameters are given by (Davidson, 1972)

$$k_n(\tau) = k_n(0),$$

$$C_n(\tau) = C_n(0)\cdot \exp(4k_n^3\tau),$$

$$R(k,\tau) = R(k,0)\cdot \exp(i8k^3\tau), \qquad (1.10)$$

where C_n is the amplitude of the wave function,

$$\Psi_n = C_n \cdot \exp(-k_n\xi). \qquad (1.11)$$

Then, the required solution $v(\xi, \tau)$ of the K-dV equation (1.8) is given by

$$V(\xi,\tau) = -2\frac{d}{d\xi}K(\xi,\xi,\tau) \tag{1.12}$$

where $K(\xi, \xi, \tau)$ is the solution to the Gelfand-Levitan integral equation

$$K(\xi,\eta,\tau) + B(\xi+\eta,\tau) + \int_{\xi}^{\infty} d\eta' \cdot B(\eta+\eta',\tau)\cdot K(\xi,\eta',\tau) = 0 \tag{1.13}$$

and $B(\xi, \tau)$ is defined by

$$B(\xi,\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \cdot R(k,\tau)\cdot \exp(ik\xi) + \sum_{n=1}^{N} C_n^2(\tau)\cdot \exp(-k_n(\tau)\,\xi) \tag{1.14}$$

Thus, the problem of solving the nonlinear equation (1.8) has been reduced to solving two coupled linear equations, namely, the time-independent Schrodinger equation (1.9) and the Gelfand-Levitan integral equation (1.13). In many cases, this is a big gain.

I.6.3 Sagdeev Potential Analysis

Even though the methods outlined above are frequently used in the discussion of nonlinear wave phenomena, they have certain limitations. The reduction procedure of the first method is applicable when the linear dispersion relation of the system of equations under consideration is of a particular form and when the nonlinearities to be considered are weak. On the other hand, the inverse scattering procedure for obtaining the exact analytic solutions is limited, in its applications, to a particular class of nonlinear evolution equations. However, even when the system of equations are more general and are not amenable to explicit solutions, it is possible to analyse the nature of its solutions under different conditions by using a procedure first employed by Sagdeev (1966). This procedure is particularly suited for finding out the existence of soliton type of stationary, localized solutions for a given set of nonlinear equations.

Let the given set of equations be represented by the system

$$L_i(u_1, u_2, \ldots, u_N) = 0, \quad i = 1, 2, \ldots, N \tag{1.15}$$

where the variables u_i represent various physical quantities and L_i are functions of differential operators with respect to the space and time co-ordinates x and t respectively. For stationary solutions, it is more convenient to go over to a frame of reference moving with the wave and, accordingly, define the variable,

$$\xi = x - at, \tag{1.16}$$

where a is the velocity of the stationary wave. Using eq.(1.16), the variables x and t in eqs.(1.15) can be transformed away in favour of ξ. The resulting set of equations are then suitable combined and integrated with respect to ξ, if necessary, to yield a single nonlinear equation of the form

$$\frac{1}{2}\left(\frac{du}{d\xi}\right)^2 + V(u) = 0, \tag{1.17}$$

where u represents one of the variables u_i, i = 1,2,....,N. If we now consider the variable u as analogous to the space co-ordinate and ξ as analogous to the time co-ordinate, then, eq.(1.17) is equivalent to the equation of motion for a particle of unit mass in a potential V(u). In eq.(1.17), the first term represents the 'kinetic energy' of the particle and the second term the 'potential energy'. The quasi-potential V(u) is called the Sagdeev potential.

One can now analyse the potential V(u) for the existence of soliton type of localized solutions for u(ξ). For simplicity, let u be positive in $-\infty < \xi < +\infty$ and V(u) be single valued with respect to u (this corresponds to single-hump soliton solutions for u(ξ) with the maximum at $u = u_m$). Integrating eq.(1.17) with respect to ξ once, we obtain,

$$\xi = \int_{u_0}^{0} \left[-2V(u)\right]^{-1/2} \cdot du. \tag{1.18}$$

Clearly, for real values of ξ, V(u) must be negative in the range, $0 < u < u_m$, where u_m is the maximum of the solution $u(\xi)$; then, the lower limit of the integration in eq.(1.18) lies in the range $0 < u_0 < u_m$. The value of ξ given by eq.(1.18) can be thought of as the 'time' taken by the quasi-particle to move in the quasi-potential V(u) from $u = u_0$ to $u = 0$. Consider now the behaviour of V(u) in the neighbourhood of $u = 0$. Expanding V(u) in Taylor series around $u = 0$, we obtain,

$$V(u) \simeq V(0) + au + bu^2, \tag{1.19}$$

where

$$V(0) = V(u=0),$$

$$a = \left.\frac{\partial V}{\partial u}\right|_{u=0}, \quad b = \frac{1}{2} \left.\frac{\partial^2 V}{\partial u^2}\right|_{u=0}. \tag{1.20}$$

Given $V(0) = 0$, $a = 0$ and b is finite, it is easy to see from eqs. (1.18) and (1.19) that the point (0,0) in the u-V plane is mapped to the points $(\pm\infty, 0)$ in the ξ-u plane. Similarly, one can analyse the behaviour of the potential V(u) in the neighbourhood of $u = u_m$. It, then, follows that if

$$V(u=u_m) = 0, \quad \left.\frac{\partial V}{\partial u}\right|_{u=u_m} \neq 0, \tag{1.20a}$$

and $\partial^2 V/\partial u^2$ is finite at $u = u_m$, then, the point $(u_m, 0)$ in the u-V plane is mapped to finite ξ points in the ξ-u plane.

We thus obtain the following conditions for the existence of single-hump, localized solutions for $u(\xi)$:

$$V(u=0) = 0 = V(u=u_m),$$

$$\left.\frac{\partial V}{\partial u}\right|_{u=0} = 0,$$

$$\left.\frac{\partial V}{\partial u}\right|_{u=u_m} \neq 0, \tag{1.21}$$

together with the requirements that $V(u) < 0$ in the range $0 < u < u_m$ and that $\partial^2 V/\partial u^2$ be finite at $u = 0$ as well as at $u = u_m$. The quasi-particle motion in potential V(u), then, consists of a single-transit of the particle from $u = 0$ to $u = u_m$, and back.

The above analysis can be extended to the case when V(u) is a multi-valued function of u, in which case the solution for $u(\xi)$ will consist of many-humps (an analysis of this type for the amplitude modulated Langmuir waves has been carried out in Chapter IV).

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CHAPTER II

EFFECTS OF STRETCHED CO-ORDINATES AND HIGHER ORDER CONTRIBUTIONS IN THE REDUCTIVE PERTURBATION METHOD FOR NONLINEAR ION ACOUSTIC WAVES

II.1 Introduction

Among the various normal modes of plasmas ion acoustic waves were one of the first to be studied both theoretically and experimentally. It is well known that in the linear approximation the ion acoustic waves, in a homogeneous collisionless plasma, are non-dispersive in the long wavelength limit with the dispersion relation $\omega = C_s k$ (ω is the wave frequency, k the wavenumber). They, thus propagate with the speed C_s, the ion acoustic speed, which is determined essentially by the electron temperature T_e and the ion mass m_i, $C_s^2 = T_e/m_i$. This is because for these low frequency oscillations the inertia is provided by the ions while the restoring force is provided by the hot electrons. However, if contributions due to finite wavenumbers are taken into account, then, the waves become dispersive and this leads to the spreading of any given initial pulse. On the other hand, when the waves are of finite amplitudes, the effects of nonlinearity become important and this leads to a steepening of the wavefronts. Under suitable conditions, a balance between these two competing processes, namely, the wave spreading due to the dispersion and the wave-steepening due to the nonlinearity can be brought out, resulting in the formation of ion acoustic solitons.

The effects of nonlinearity and dispersion of ion acoustic waves were first considered by Sagdeev (1966) who predicted the existence of ion acoustic solitons. Later, Washimi & Taniuti (1966) showed that small but finite amplitude ion acoustic waves are governed by the usual Korteweg-de Vries (K-dV) equation which had been numerically studied earlier by Zabusky & Kruskal (1965). An analytic method, namely, the Inverse Scattering Method for solving the K-dV equation as an initial value problem was developed by Gardner et al (1967). Experimental studies of ion acoustic solitons have been carried out by Ikezi et al (1970), Ikezi (1973) and others. While the above analyses are restricted to the propagation of ion acoustic waves in one-dimension, extensions to two and three dimensions in space have been carried out by many authors (Maxon & Viecelli, 1974; Ogino & Takeda, 1976; Chen & Schott, 1976; Nishida et al, 1978; Nishida et al, 1979).

We consider, in this Chapter, the differences in the description of weakly nonlinear ion acoustic waves arising due to the use of different sets of 'stretched co-ordinates' in the reductive perturbation method. As is well known, the K-dV equation which governs the evolution of weakly nonlinear ion acoustic waves in a homogeneous plasma can be derived by the reductive perturbation method which makes use of a set of slowly varying co-ordinates called 'stretched co-ordinates' (ref. § I.6.1). As pointed out in Chapter I, this set of stretched co-ordinates is not unique and there exist in the literature two different sets of stretched co-ordinates which have same ordering with respect to the relevant smallness parameter, giving rise to similar K-dV equations for the two sets. One may then ask whether the solitary waves governed by the K-dV equations obtained through these different stretchings have the same or different propagation characteristics in the (x, t) space. This problem is important when one compares the result of experiments on soliton propagation with the theoretical predictions. In particular, one would like to ascertain the proper equation whose predictions to compare the experimental results with.

Thus, for one-dimensional propagation of ion acoustic solitons in a homogeneous plasma, we show here that the K-dV equations obtained through these stretched co-ordinates have identical forms in the two sets of the co-ordinates. When we transform these K-dV equations to the (x, t) co-ordinates, we find them to be completely different (Rao & Varma, 1977). While the highest derivative (third order) in one is with respect to the time co-ordinate, it is with respect to the space co-ordinate (x) in the other. If we seek soliton solutions of these equations such that they correspond to the same propagation velocity u in the x-t space, we find these solitons to have different dependence of their amplitude and width on the velocity of propagation. We further discuss the implications of these equations and their solutions from the experimental view point and suggest that these may describe solitons corresponding to two different ways of launching them in the experiments.

We also discuss here the contributions of the second order perturbed quantities in the reductive perturbation analysis for ion acoustic waves. It turns out that for both the sets of stretched co-ordinates, the second order perturbed quantities are governed by linear, inhomogeneous equations driven by source terms which are functions of the first order perturbed quantities. The solutions of these equations are compared with the solutions for the first order perturbed quantities. For both the sets of stretched co-ordinates the effect of the second order perturbed quantities is to reduce the amplitudes of the first order solutions.

II.2 Derivation of the K-dV Equations

We consider a homogeneous, collisionless plasma in one-dimension with cold ions and hot isothermal electrons. The basic set of fluid equations required to derive the K-dV equation for weakly non-linear and weakly dispersive ion acoustic waves in such a system are the continuity and momentum conservation equations for the ions and the Poisson equation. These are, respectively, given by

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0, \tag{2.1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial x} = 0, \tag{2.2}$$

$$\frac{\partial^2 \phi}{\partial x^2} - \exp(\phi) + n = 0, \tag{2.3}$$

where in eq.(2.3) we have used the Boltzmann distribution for the electron number density obtained from the momentum conservation equation for electrons by neglecting the electron inertia. The notations used in eqs.(2.1) - (2.3) are as follows: n is the ion number density, v the ion fluid velocity, ϕ represents the electrostatic potential and, x and t are the space and time co-ordinates respectively. All these quantities are normalized with respect to the equilibrium parameters, namely, plasma density (n_0), ion acoustic velocity (C_s), a characteristic potential (T_e/e), electron Debye length ($\lambda_{De}^2 = T_e/4\pi n_0 e^2$) and ion plasma period ($\tau_{pi}^2 = m_i/4\pi n_0 e^2$).

The linear dispersion relation for the ion acoustic waves can be obtained by linearizing the set of eqs.(2.1) - (2.3) and by assuming the various perturbed quantities to vary as $\sim \exp\left[i(kx - \omega t)\right]$. This yields the dispersion relation

$$\omega^2 = k^2 (1 + k^2)^{-1}. \tag{2.4}$$

If solved for k in terms of ω, eq.(2.4) yields,

$$k^2 = \omega^2 (1 - \omega^2)^{-1}. \tag{2.5}$$

Since we are considering long wavelength ($k \ll 1$) and low frequency ($\omega \ll 1$) oscillations, the dispersion relations (2.4) and (2.5) can be approximated to give, respectively,

$$\omega \approx k - \frac{1}{2} k^3, \tag{2.6}$$

and

$$k \approx \omega + \frac{1}{2} \omega^3. \tag{2.7}$$

By making use of eqs.(2.6) and (2.7), one can now analyse the phase factor $(kx-\omega t)$ and this suggests the following sets of stretched co-ordinates (Washimi & Taniuti, 1966):

$$\xi = \epsilon^{1/2} (x - t),$$

$$\tau = \epsilon^{3/2} \cdot t, \tag{2.8}$$

and

$$\xi = \epsilon^{1/2} (x - t)$$

$$\eta = \epsilon^{3/2} \cdot x \tag{2.9}$$

where ϵ is the smallness parameter characterizing the strength of the nonlinearity, and is related to the amplitude of the wave.

To obtain the K-dV equations corresponding to the sets of stretched co-ordinates (2.8) and (2.9), we now carry out the reduction procedure outlined in § I.6.1. Accordingly, we expand the quantities n, v and ϕ as

$$n = 1 + \epsilon n_1 + \epsilon^2 n_2 + \epsilon^3 n_3 + \cdots,$$
$$V = \epsilon V_1 + \epsilon^2 V_2 + \epsilon^3 V_3 + \cdots,$$
$$\phi = \epsilon \phi_1 + \epsilon^2 \phi_2 + \epsilon^3 \phi_3 + \cdots, \tag{2.10}$$

and substitute these expansions along with the stretchings (2.8) or (2.9) into eqs.(2.1) - (2.3). This yields the K-dV equations,

$$\frac{\partial \phi_1}{\partial \tau} + \phi_1 \frac{\partial \phi_1}{\partial \xi} + \frac{1}{2}\frac{\partial^3 \phi_1}{\partial \xi^3} = 0, \tag{2.11}$$

and

$$\frac{\partial \phi_1}{\partial \eta} + \phi_1 \frac{\partial \phi_1}{\partial \xi} + \frac{1}{2}\frac{\partial^3 \phi_1}{\partial \xi^3} = 0, \tag{2.12}$$

corresponding to the two stretchings given by eqs.(2.8) and (2.9) respectively. The two equations (2.11) and (2.12) are identical in form and differ in one of their independent variables, namely, τ and η being different.

II.3 Solutions of the K-dV Equations

The K-dV equations (2.11) and (2.12) admit soliton solutions with a constant velocity u in the (x, t) space. In order to facilitate a proper comparison of the propagation characteristics of the solitons described by the eqs. (2.11) and (2.12), we must obtain solutions having the same propagation velocity u in the (x, t) space in both cases. Thus, we look for solutions of Φ_1 which depend on the variables x and t through a variable $z = \epsilon^{\frac{1}{2}}(x-ut)$ where u is normalized with respect to the ion acoustic velocity, C_s. The factor $\epsilon^{\frac{1}{2}}$ ensures the large width of the solitons.

Using the definitions of (ξ , τ) and (ξ , η) we find

$$Z = \xi - a\tau \quad ; \quad a = (u-1)/\epsilon \tag{2.13}$$

corresponding to the stretchings (2.8) and eq. (2.11), and

$$Z = u\xi - a\eta , \tag{2.14}$$

corresponding to the stretchings (2.9) and eq. (2.12). Then eq. (2.11) can be easily integrated for stationary solutions of the form $\Phi_1 = \Phi_1(z)$, subject to the boundary conditions,

$$\Phi_1 , \frac{d\Phi_1}{dZ} , \frac{d^2\phi}{dZ^2} \longrightarrow 0 \quad \text{as} \quad |Z| \rightarrow \infty, \tag{2.15}$$

to obtain the solution

$$\Phi_1 = 3(u-1)\cdot \mathrm{sech}^2\left[\left\{\frac{u-1}{2}\right\}^{1/2}(x-ut)\right], \tag{2.16}$$

where $\Phi_1 = \epsilon\,\phi_1$. Similarly, the soliton solution of the eq.(2.12) subjected to the same boundary conditions (2.15) is given by

$$\Phi_1 = \frac{3(u-1)}{u}\cdot \mathrm{sech}^2\left[\left\{\frac{u-1}{2u^3}\right\}^{1/2}(x-ut)\right]. \tag{2.17}$$

Obviously, the solitons represented by the solutions (2.16) and (2.17) have different propagation characteristics. These differences, together with the implications of the differences in the two K-dV equations (2.11) and (2.12) are discussed in the next section.

II.4 Comparison of the Soliton Solutions

Comparing the two solutions (2.16) and (2.17), we note that while the two solitons propagate with the same velocity u in the (x, t) space, the dependence on u of their amplitudes and widths is quite different. Figures (1) and (2) show the plots of amplitudes and widths obtained from eqs.(2.16) and (2.17) as function of the soliton velocity u for both the cases. The differences are small for velocities $u \gtrsim 1$ whereas for higher velocities, they become quite appreciable. However, for sufficiently large values of u the contribution from the higher order perturbed quantities become important. This will be considered in the next section.

There is another important difference between eqs.(2.11) and (2.12), and their solutions which does not show up in the stationary

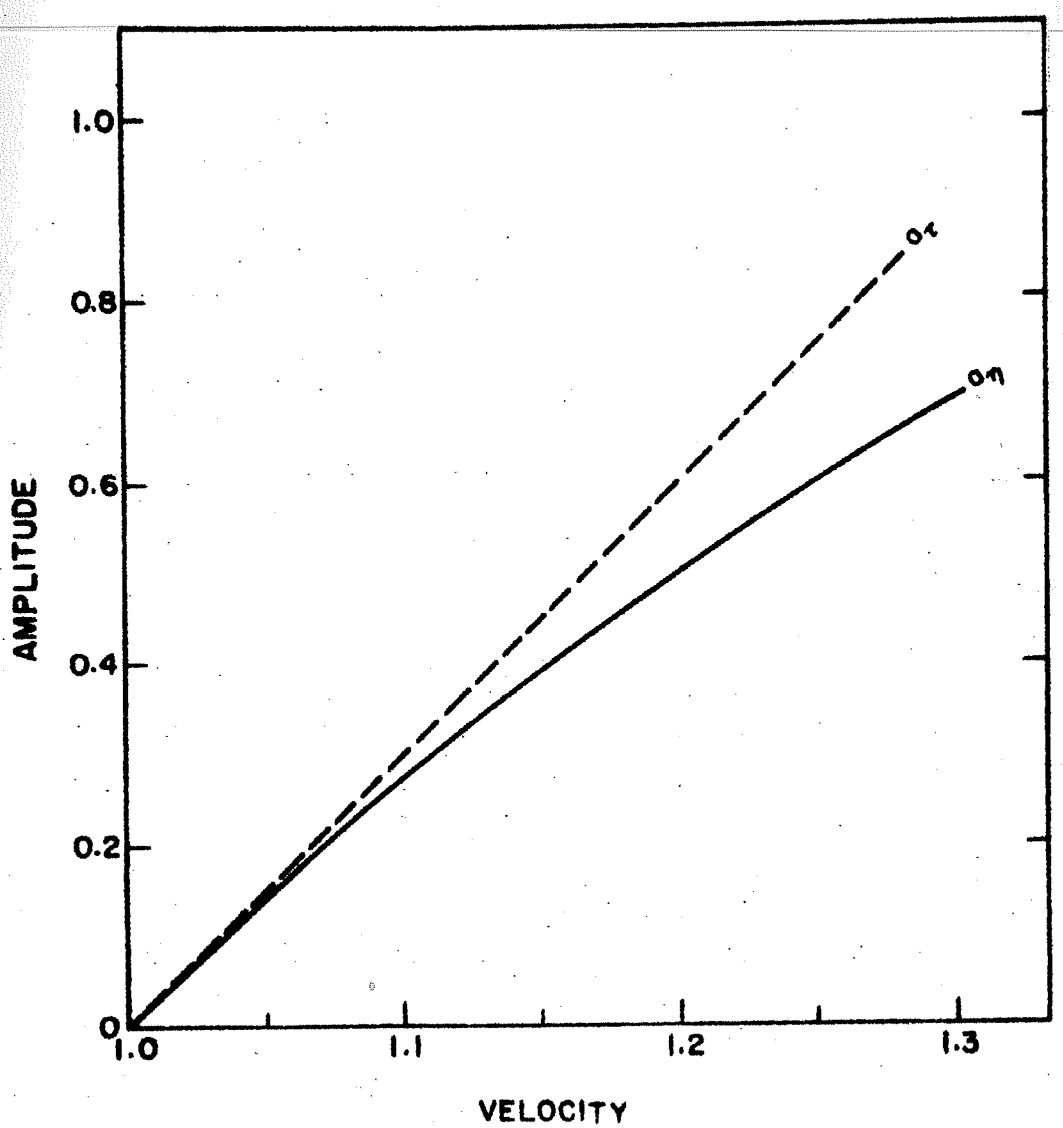

FIGURE (1). Plot of the amplitudes a_{τ} and a_{η} of the ion acoustic K-dV solitons as functions of the soliton Mach number. The subscripts τ and η denote the type of stretched co-ordinate used in obtaining the solutions.

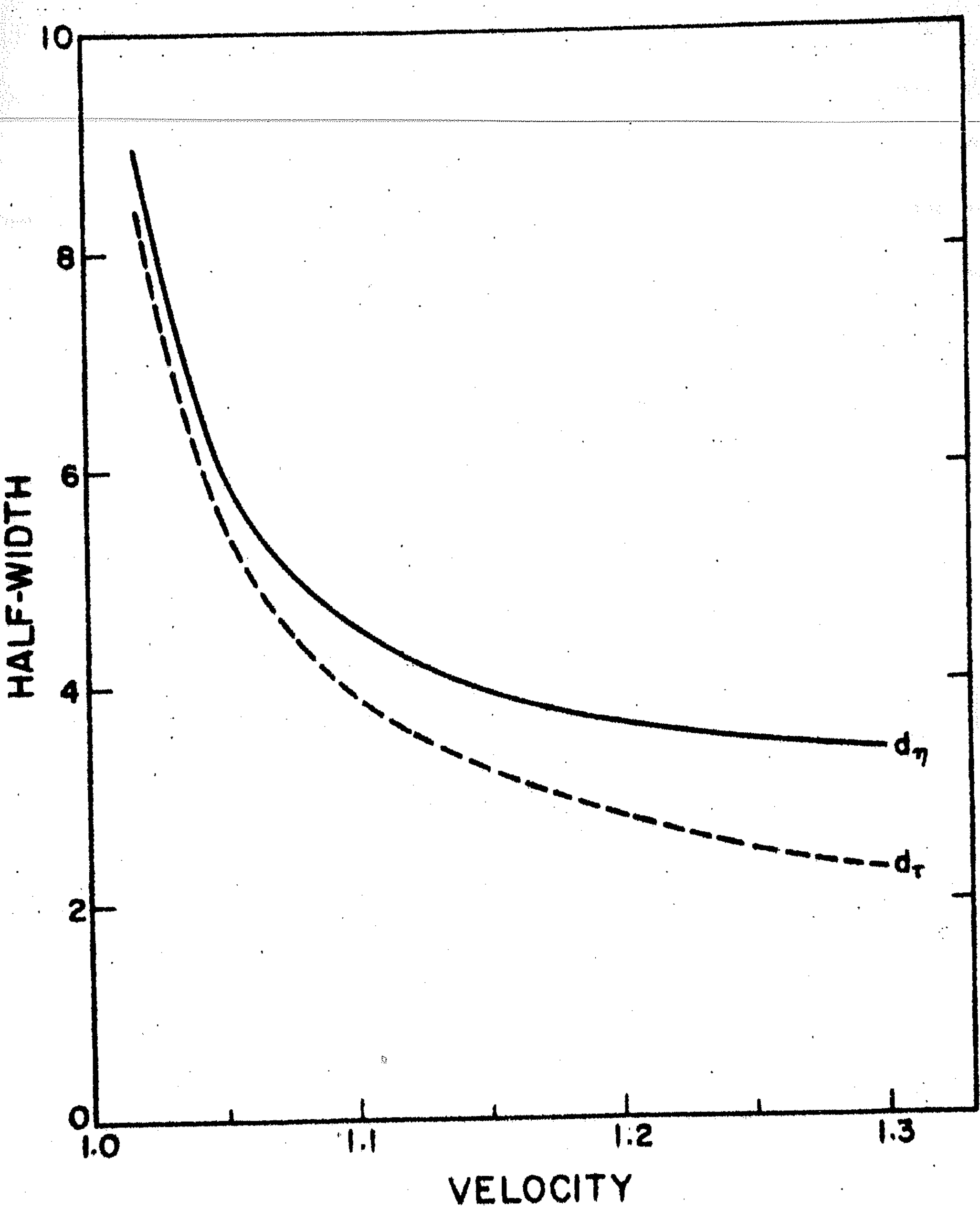

FIGURE (2). Plot of the half-widths d_τ and d_η of the ion acoustic K-dV solitons as functions of the soliton Mach number. The subscripts τ and η denote the type of stretched co-ordinate used in obtaining the solutions.

solutions (2.16) and (2.17). Here, the identity of the x and t variables is somewhat lost because of the solitons being stationary in the frame moving with the soliton velocity. The real difference between the eqs.(2.11) and (2.12) appears when we transform them into the (x, t) variables. From eqs.(2.8) and (2.9), the inverse transformations are given, respectively, by

$$x = \epsilon^{-1/2}\xi + \epsilon^{-3/2}\tau,$$
$$t = \epsilon^{-3/2}\tau, \tag{2.18}$$

and,

$$x = \epsilon^{-3/2}\eta,$$
$$t = \epsilon^{-3/2}\eta - \epsilon^{-1/2}\xi. \tag{2.19}$$

Using eqs.(2.18) and (2.19), we transform the K-dV equations (2.11) and (2.12) into the ones in the x-t variables and obtain respectively

$$\frac{\partial \phi_1}{\partial t} + \frac{\partial \phi_1}{\partial x} + \epsilon\,\phi_1 \frac{\partial \phi_1}{\partial x} + \frac{1}{2}\frac{\partial^3 \phi_1}{\partial x^3} = 0, \tag{2.20}$$

and

$$\frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_1}{\partial t} - \epsilon\,\phi_1 \frac{\partial \phi_1}{\partial t} - \frac{1}{2}\frac{\partial^3 \phi_1}{\partial t^3} = 0, \tag{2.21}$$

A comparison between eqs.(2.20) and (2.21) shows, except for changes of sign, the interchange of the roles of x and t variables between the two equations. Obviously, these two equations will give different time

Equation (2.20) involves a term with a first order time derivative of Φ_1. As an initial value problem, then, one would need to specify a function $\Phi_0(x)$, that is, the values of Φ_1 at all points of space at some initial time. If we, therefore, specify a spatial distribution for Φ_1 at an initial time, the resulting soliton should correspond to eqs.(2.8), (2.11) and (2.16). On the other hand, eq.(2.21) involves a third order derivative of Φ_1 with respect to time and only a first order space derivative of Φ_1. Therefore, to solve this equation as an initial value problem, one would need to specify at all space points the value of the function Φ_1 along with its first and second time derivatives at the initial time. If the spatial distribution of the initial disturbance be taken to be a δ-function at certain point and a time varying disturbance be applied to it (with its first and second time derivatives along with the value being specified), then this corresponds to the standard experimental technique of using a potential pulse on a grid for launching the solitons (John & Saxena, 1976; Ikezi et al, 1970). The resulting soliton would correspond to eqs.(2.9), (2.12) and (2.17).

On the other hand, to obtain a soliton corresponding to the set of eqs.(2.8), (2.11) and (2.16), one would have to specify a potential distribution initially all over space. This will have to be done by distributing a number of grids in the plasma volume and applying appropriate potentials to them at only the initial time (that is, a δ-function in time). This seems to be a more difficult proposition from an experimental view point than the other one. Thus, clearly, the solitons observed in the experiments so far (for instance, John & Saxena

1976; Ikezi et al, 1970) should be compared for their propagation characteristics with the solitons that are derived from eqs.(2.9) and (2.17).

II.5 Higher Order Contributions in the Reductive Perturbation Method

In the above discussion, we have considered the K-dV equations (2.11) and (2.12) which govern the first order perturbed quantities in the expansions (2.10). Notwithstanding the fact that the K-dV equation takes into account the basic nonlinear and dispersive effects, from the point of view of perturbation approaches it is important to evaluate the contributions coming from higher order terms and compare them with the contributions due to the first order perturbed quantities. Such an analysis was first carried out by Ichikawa et al (1976) who made a quantitative analysis of the higher order contributions in the reductive perturbation framework for the weakly nonlinear ion acoustic waves by considering the stretchings given by eqs.(2.8). Further developments in the evaluation of higher order contributions are due to Sugimoto & Kakutani (1977), Watanabe (1978), Kodama & Taniuti (1978), Kodama (1978) and others. In the following, we extend the analysis of Ichikawa et al (1976) for the other set of stretched co-ordinates given by eqs.(2.9) and compare the contributions coming from the second order perturbed quantities from those of the first order perturbed quantities.

The evolution equations for the higher order perturbed quantities can be obtained by substituting the stretchings (2.8) (or (2.9)) and the expansions (2.10) into the basic equations (2.1) - (2.3), and equating the coefficients of like powers of ϵ to zero. In particular, the equation for the second order perturbed quantities can be obtained by collecting the terms corresponding to third order in ϵ and eliminating the third order quantities n_3, v_3 and Φ_3 from the resulting set of equations. This gives the equation,

$$\frac{\partial \phi_2}{\partial \tau} + \frac{\partial}{\partial \xi}(\phi_1 \phi_2) + \frac{1}{2}\frac{\partial^3 \phi_2}{\partial \xi^3} = S_1(\phi_1), \tag{2.22}$$

corresponding to the stretched co-ordinates (2.8), and the equation

$$\frac{\partial \phi_2}{\partial \eta} + \frac{\partial}{\partial \xi}(\phi_1 \phi_2) + \frac{1}{2}\frac{\partial \phi_2}{\partial \xi^3} = S_2(\phi_1), \tag{2.23}$$

for the other set of stretched co-ordinates given by eqs.(2.9). The source terms S_1 and S_2 in eqs.(2.22) and (2.23) are given by

$$S_1(\phi_1) = \frac{1}{2}\phi_1 \frac{\partial^3 \phi_1}{\partial \xi^3} - \frac{5}{4}\frac{\partial \phi_1}{\partial \xi}\cdot\frac{\partial^2 \phi_1}{\partial \xi^2} - \frac{3}{8}\frac{\partial^5 \phi_1}{\partial \xi^5}, \tag{2.24}$$

$$S_2(\phi_1) = \phi_1 \frac{\partial^3 \phi_1}{\partial \xi^3} + \frac{\partial \phi_1}{\partial \xi}\cdot\frac{\partial^2 \phi_1}{\partial \xi^2} - \frac{1}{2}\phi_1^2 \frac{\partial \phi_1}{\partial \xi}$$
$$- \frac{3}{2}\phi_1 \frac{\partial \phi_1}{\partial \eta} - \frac{3}{4}\frac{\partial^3 \phi_1}{\partial \eta\, \partial \xi^2} \tag{2.25}$$

where the first order solutions Φ_1 are respectively given by eqs. (2.16) and (2.17). Equations (2.22) and (2.23) are linear, inhomogeneous equations for Φ_2 driven by the source terms which are functions of the first order perturbed quantity only. This implies that while the fundamental nonlinear effect is fully accounted for in the lowest order evolution equation, namely, the K-dV equation, the equations for higher order quantities describe the interaction between the higher order dispersion and the fundamental nonlinearity.

The stationary solutions of the evolution equations (2.22) and (2.23) can be obtained by the method of variation of parameters (Ichikawa et al, 1976). This yields the solution,

$$\Phi_2 = 9A^4 \cdot \operatorname{sech}^2(AZ) \cdot \left[2AZ \cdot \tanh(AZ) + 7\operatorname{sech}^2(AZ) - 8\right] \quad (2.26)$$

for the equation (2.22), and,

$$\Phi_2 = 9u^4B^4 \cdot \operatorname{sech}^2(BZ) \cdot \left[-2BZ \cdot \tanh(BZ) + 4\operatorname{sech}^2(BZ) - \frac{14}{3}\right] \quad (2.27)$$

for the equation (2.23). The notations used in eqs.(2.26) and (2.27) are:

$$\Phi_2 = \epsilon^2 \phi_2 \, , \quad Z = x - ut;$$

$$A = \left(\frac{u-1}{2}\right)^{1/2}, \quad B = \left(\frac{u-1}{2u^3}\right)^{1/2}$$

As expected, the second order contributions are different for the two sets of stretched co-ordinates. The total perturbed potential ϕ is given by the expansion in eqs.(2.10). Keeping terms up to ϵ^2, we have for ϕ,

$$\Phi = \epsilon \phi_1 + \epsilon^2 \phi_2 = \Phi_1 + \Phi_2 . \tag{2.28}$$

Substituting for Φ_1 and Φ_2 from eqs.(2.16), (2.26) and (2.17), (2.27) corresponding to the two sets of stretched co-ordinates (2.8) and (2.9), we finally obtain the total perturbed potential as,

$$\begin{aligned} \phi = {} & 6A^2 \cdot \operatorname{sech}^2(AZ) \\ & + 9A^4 \cdot \operatorname{sech}^2(AZ) \\ & \cdot \left[2AZ \cdot \tanh(AZ) + 7\operatorname{sech}^2(AZ) - 8 \right] \end{aligned} \tag{2.29}$$

for the stretched co-ordinates (ξ , η) defined by eqs.(2.8), and

$$\begin{aligned} \phi = {} & 6B^2u^2 \cdot \operatorname{sech}^2(BZ) \\ & + 9B^4u^4 \cdot \operatorname{sech}^2(BZ) \\ & \cdot \left[-2BZ \cdot \tanh(BZ) + 4\operatorname{sech}^2(BZ) - \frac{14}{3} \right] \end{aligned} \tag{2.30}$$

for the stretched co-ordinates (ξ , η) defined by eqs.(2.9)

The amplitudes of the solitons given by eqs.(2.29) and (2.30) are plotted as functions of the soliton velocity in Figure (3). It is clear from this figure that for both sets of the stretched co-ordinates the effect of second order perturbed quantities is such as to reduce the amplitudes of the first order solutions. In Figure (4), we plot explicitly the actual profiles of the solutions for a particular value of the soliton velocity, u = 1.2. We thus find that the widths of the solutions are also reduced due to the contributions coming from second order perturbed quantities.

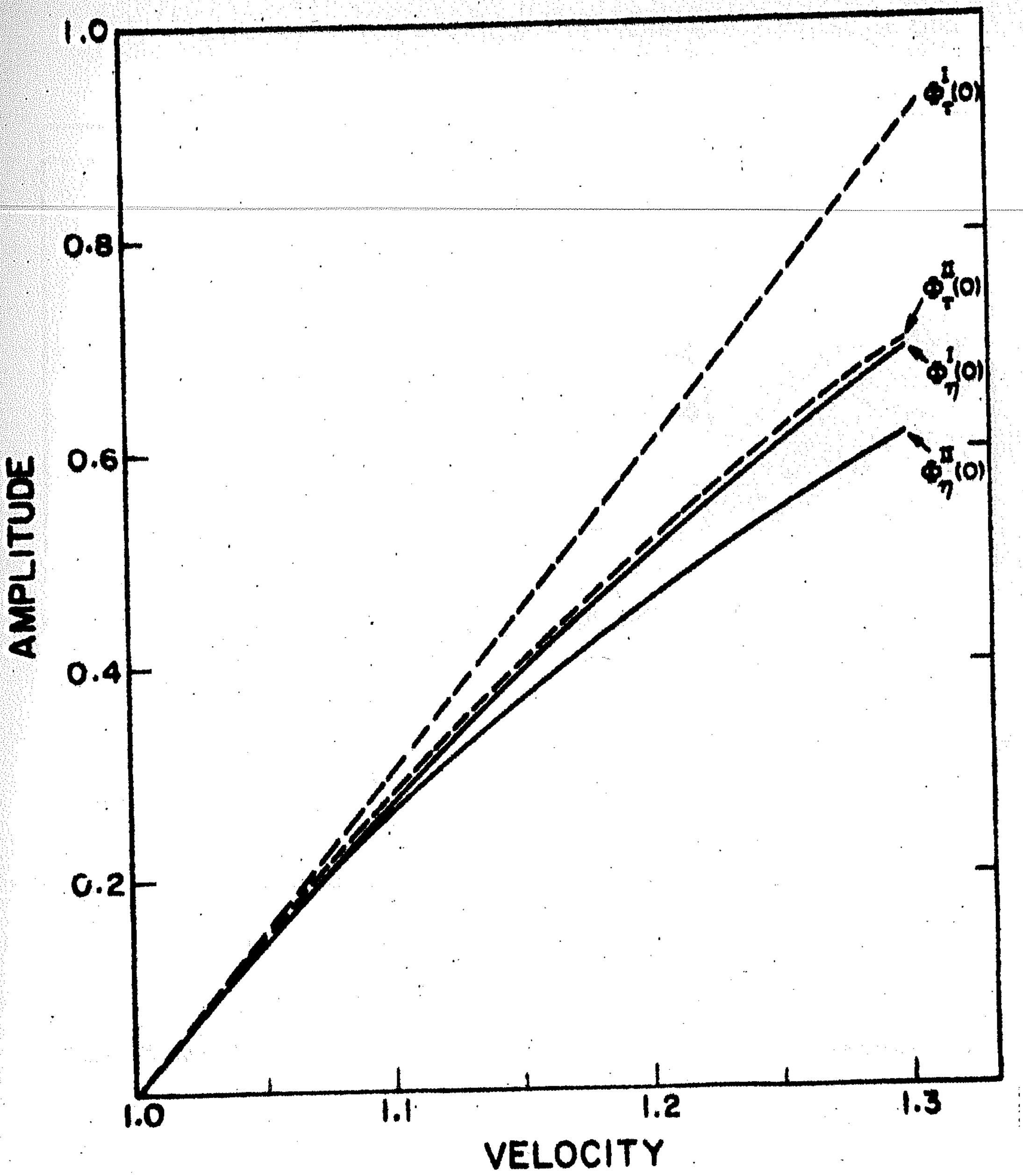

FIGURE (3). Plot of the amplitudes of the first order and total (up to second order) solutions for the ion acoustic solitons as functions of the soliton Mach number. Here, $\Phi^{I}(0) = \Phi_1(0)$ and $\Phi^{II}(0) = \Phi_1(0) + \Phi_2(0)$. The subscripts τ and η denote the type of stretched co-ordinate used in obtaining the solutions.

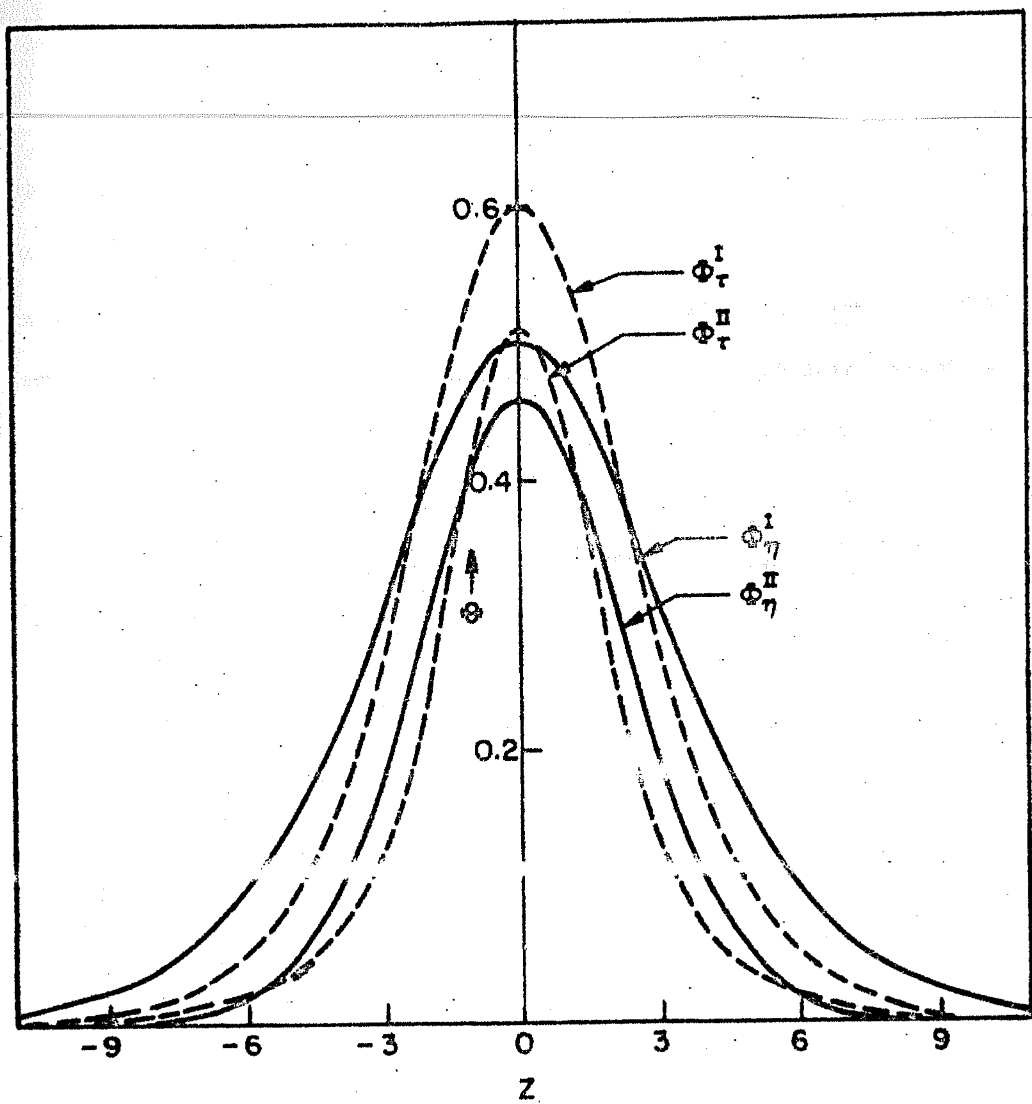

FIGURE (4). Plot of the solutions $\Phi^{I}(Z) = \Phi_1(Z)$ and $\Phi^{II}(Z) = \Phi_1(Z) + \Phi_2(Z)$ for a value of the soliton Mach number, u=1.2 . The subscripts τ and η denote the type of stretched co-ordinate used in obtaining the solutions (2.29) & (2.30). Note that for both the sets of stretched co-ordinates, the amplitudes and the widths of the first order solutions are reduced due to the contributions coming from the second order quantities.

II.6 Conclusions

We have considered, in this Chapter, the differences that arise in the description of the weakly nonlinear ion acoustic waves in the reductive perturbation method where different sets of stretched co-ordinates are used. By carrying out the usual reductive perturbation analysis on the basic set of fluid equations we have shown that even though the two sets of stretched co-ordinates have the same ordering with respect to the smallness parameter, the K-dV equations obtained through them differ substantially. The soliton solutions of the two K-dV equations are found to have different velocity dependences of their amplitudes and widths. By analysing the two K-dV equations in the (x, t) co-ordinates, we deduce that the initial value problems associated with them are different and hence give rise to different time evolutions for some given initial pulse. From an experimental view point, the two sets of stretched co-ordinates are shown to correspond to different ways of launching the solitons in experiments.

We have also analysed the contributions coming from higher order perturbed quantities in the reductive perturbation analysis. We find that for both the sets of stretched co-ordinates, the second order perturbed quantities are governed by linear, inhomogeneous equations driven by source terms which are functions of only the first order perturbed quantities. This indicates that while the K-dV equation accounts for the basic nonlinearity present in the system, the interaction between the fundamental nonlinearity and the higher order dispersion is described by the evolution equations for the higher order perturbed quantities.

A comparison of the second order solutions with the first order solutions is also made. It is found that for higher soliton velocities, the higher order contributions are considerably important.

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CHAPTER III

LINEAR AND NONLINEAR ION ACOUSTIC WAVES IN INHOMOGENEOUS PLASMAS

III.1 Introduction

Various aspects of linear and nonlinear wave propagation in homogeneous plasmas have been extensively studied over the last few decades. From the theoretical point of view, it is much simpler to consider homogeneous systems where the unperturbed quantities do not vary with respect to the space or time co-ordinates. However, many physical systems of practical interest are inhomogeneous in nature. When a system is not homogeneous, the interactions of the waves (linear or nonlinear) with the inhomogeneities become important and these can appreciably modify the wave propagation characteristics. Thus, the study of the effects of inhomogeneities on the wave propagation in plasmas is important both theoretically and experimentally.

Plasma inhomogeneities can be broadly classified into two groups: (i) spatial inhomogeneities where the unperturbed quantities are functions of the space co-ordinates and are time independent, and (ii) temporal inhomogeneities where the unpertuebd quantities are time-varying but spatially homogeneous. Typical examples of the interactions of the linear and nonlinear waves with such inhomogeneities are the damping-growth transition of ion acoustic waves propagating in a spatially inhomogeneous plasma, steepening of sound waves propagating upwards in the atmosphere, decay of the solitary water waves in the interaction with the bottom irregularities, growing or damping of waves in a homogeneously expanding or contracting gas and so on.

The propagation of linear ion acoustic waves in a spatially inhomogeneous plasma with density gradients has been considered by many authors. D'Angelo (1968) suggested a model for the heating of the solar corona based on the Landau damping of ion acoustic waves in the coronal regions. This suggestion was later verified theoretically by Parkinson & Schindler (1969) and by Liu (1970) who considered the initial value problem associated with the propagation of ion acoustic waves in a collisionless, gravity-supported plasma. For laboratory plasmas, the effect of density gradients on the linear acoustic waves has been considered, theoretically as well as experimentally, by D'Angelo et al (1975), D'Angelo et al (1976), Gunzburger (1978), Mateev & Zhelyazkov (1978) and others. Doucet et al (1974) have carried out a theoretical and experimental analysis of the boundary-value problem associated with the linear ion acoustic waves propagating in a collisionless, finite-size plasma having an arbitrary equilibrium density profile.

In case of the nonlinear waves, Asano & Ono (1971) have extended the usual Reductive Perturbation Method so as to be applicable to inhomogeneous systems as well. These authors show that the basic set of equations for a class of nonlinear dispersive (or dissipative) waves in an inhomogeneous medium can be reduced to a single nonlinear equation, namely, a modified Korteweg-de Vries (K-dV) equation (or a modified Burgers equation). Theoretical analyses of modified K-dV equations for a wide variety of inhomogeneities have been carried out by Tappert & Zabusky (1971), Johnson (1973), Hirota & Satsuma (1976), Ko & Kuehl (1978), Kaup & Newell (1978). In particular, the effect of density gradient on the propagation of ion acoustic solitary waves has been considered by Nishikawa & Kaw (1975) and Gell & Gomberoff (1977).

In this Chapter, we consider the problem of the propagation of both the linear and nonlinear ion acoustic waves in a plasma in the presence of spatial gradients in the ion density as well as in the ion temperature. As pointed out earlier, the propagation of linear ion acoustic waves in the presence of a density gradient alone has been considered by Doucet et al (1974). We extend their treatment of the linear waves to include also the spatial inhomogeneity in the ion temperature.

The analysis of the problem of nonlinear ion acoustic waves in the plasma having spatial gradients in the ion density and ion temperature has been carried out within the framework of the Reductive Perturbation Method (Asano & Ono, 1971). For these nonlinear waves, Nishikawa & Kaw (1975) and Gell & Gomberoff (1977) have analysed the effect of plasma density gradient alone. The treatments given by these authors, however, seem to be inadequate or inconsistent in certain respects. A steady state inhomogeneity that is assumed in these treatments has, clearly, to be maintained by an appropriate distribution of sources and sinks in the plasma. However, because of widely different inertia of the electrons and ions, it is obvious that such an inhomogeneity induces an ambipolar electric field and, in turn, a zeroth order plasma fluid velocity. Further, as discussed by Asano & Ono (1971), the co-ordinate stretching appropriate for a spatially inhomogeneous plasma is different, in form, from the one for a homogeneous plasma.

Both Nishikawa & Kaw (1975) and Gell & Gomberoff (1977) have used the co-ordinate stretchings appropriate for a homogenous plasma in their treatments and have, moreover, not included the zeroth order plasma fluid velocity induced by the inhomogeneity. Also, while Nishikawa & Kaw (1975) consider only a linear density gradient, it is easy to generalize the treatment to an arbitrary but weak density (and temperature) inhomogeneities. On the other hand, the predictions of the treatment given by Gell & Gomberoff (1977) regarding the density dependence of the amplitude and the width of the ion acoustic solitary waves do not agree with the results of the experiment on the K-dV soliton propagation in density gradients carried out by John & Saxena (1976).

In our analysis of the problem of the propagation of ion acoustic solitary waves in inhomogeneous plasmas, we include also the ion temperature gradient and attempt to remove the above shortcomings of the earlier treatments. Thus, we start with the basic fluid equations of continuity, momentum balance and the equation of state for the ions. Using the set of stretched co-ordinates appropriate for an inhomogeneous system, we carry out the reductive perturbation analysis of these equations taking, now, into account all the zeroth order unperturbed quantities in the density, the velocity and the potential. The equation, thus obtained, for the first order perturbed quantities turns to be a modified K-dV equation with space dependent coefficients and having now additional terms arising because of the inhomogeneities. It has been possible, however, through a series of transformations to again obtain a K-dV equation with constant coefficients for the new dependent and independent variables. This enables us to obtain an exact analytic solution of the original equation.

Our results show that the effects of ion temperature gradient on the propagation characteristics of the linear and nonlinear ion acoustic waves are similar to those due to the ion density gradient (ref. Doucet et al, 1974; Nishikawa & Kaw, 1975). In particular, we show that as the waves (linear and nonlinear) travel in the direction of increasing ion temperature (or density), their amplitudes decrease. The modifications in the other propagation characteristics like the wave-number (for the linear waves) and the width (for the nonlinear waves) then follow as a consequence of the changes in the amplitudes. When the two graidnets are in opposite directions with appropriate scale lengths, we find that the wave amplitudes remain constant while the wave number and the width change as the waves propagate in the gradient regions.

III.2 Basic Equations

We consider a one-dimensional, collisionless, inhomogeneous plasma having spatial gradients in the ion density as well as in the ion temperature. The electrons are assumed to be hot and isothermal (because of their large thermal conductivity at high temperatures), and the ions to be warm and adiabatic. Under this assumption, the electron temperature T_e is much larger than the ion temperature $T_i(x)$ and, hence, the Landau damping of the waves can be neglected.

Since we are considering low frequency, long wavelength oscillations the system is well described by means of the fluid equations. The ions are thus governed by the equations of continuity and momentum balance, and the adiabatic law with $\gamma = 3$ (because of one-dimensionality) as the equation of state. On the other hand, for electrons, the following simplification can be made: Since $m_e << m_i$, the ion acoustic wave phase velocity is much smaller than the electron thermal velocity and hence the inertia of the electrons can be neglected. This, coupled with the momentum conservation equation for electrons, leads to the Boltzmann law for the electron density distribution. The system of equations is then closed by the Poisson equation which relates the difference in the ion and electron charge densities to the electric field of the wave. Neglecting transport processes such as heat conduction, viscosity, etc., the relevant equations can be written in the form

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0, \tag{3.1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial x} + \frac{1}{n}\frac{\partial p}{\partial x} = 0, \tag{3.2}$$

$$\frac{\partial^2 \phi}{\partial x^2} - g(x_0) \cdot \exp(\phi) + n = 0, \qquad (3.3)$$

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} + 3p\frac{\partial v}{\partial x} = 0, \qquad (3.4)$$

where the last equation is obtained from the adiabatic law for ions. In eqs. (3.1) - (3.4), the following notations are used: n is the ion density, v the ion fluid velocity, p the ion pressure, ϕ the electrostatic potential and, x and t are the space and time co-ordinates respectively. All these quantities are normalized respectively with respect to the standard plasma parameters, namely, the plasma density N, ion acoustic velocity C_s, electron pressure (NT_e), a characteristic potential (T_e/e), electron Debye length $((T_e/4\pi Ne^2)^{\frac{1}{2}})$ and the ion plasma period $((m_i/4\pi Ne^2)^{\frac{1}{2}})$, all these quantities being defined at some particular value of x, say, $x = x_0$. The quantity $g(x_0)$ in eq. (3.3) is a known function of x_0.

III.3 Linear Analysis

We now carry out a linear analysis of the problem by linearizing the set of equations (3.1) - (3.4). For mathematical simplicity we omit, in the linear analysis, the zeroth order fields and velocities. This can be physically justified if one is avoiding regions of large density gradients (or discontinuities), as is the case in many experiments on linear ion acoustic waves (see, for instance, Doucet et al, 1974). We, then, expand the various quantities as

$$n = n_0(x) + \tilde{n}_1(x,t), \tag{3.5}$$

$$V = \tilde{V}_1(x,t), \tag{3.6}$$

$$\phi = \tilde{\phi}_1(x,t), \tag{3.7}$$

$$p = \tilde{P}_1(x,t), \tag{3.8}$$

where the perturbed quantities are denoted by tilde. The set of eqs. (3.1) - (3.4) can now be linearized in the perturbed quantities using eqs.(3.5) - (3.8). Further, without loss of generality, we can assume the time dependence of all the perturbed quantities to be given as follows (such an assumption actually corresponds, in experimental situations, to the excitation of the waves by a grid on which a continuous-wave RF signal of angular frequency ω is applied):

$$\tilde{A}_1(x,t) = A_1(x)\cdot \exp(-i\omega t), \tag{3.9}$$

where $A \equiv (n, v, \phi, p)$ and ω is the wave frequency which is a slowly varying function of x. Using eqs.(3.9) and the eqs.(3.5) - (3.8) in eqs.(3.1) - (3.4), we obtain the following set of equations for the space dependent amplitudes of the perturbed quantities.

$$i\omega n_1 = n_0 \frac{\partial V_1}{\partial x} + V_1 \frac{\partial n_0}{\partial x}, \tag{3.10}$$

$$-i\omega n_0 v_1 = n_0 \frac{\partial \phi_1}{\partial x} + \frac{\partial p_1}{\partial x} \tag{3.11}$$

$$n_1 = n_0 \phi_1 \tag{3.12}$$

$$i\omega p_1 = v_1 \frac{\partial p_0}{\partial x} + 3 p_0 \frac{\partial v_1}{\partial x} \tag{3.13}$$

where we have negelected the derivatives of ω with respect to x and have assumed charge neutrality in obtaining eq. (3.12). Eliminating the quantities n_1, ϕ_1 and p_1 in eqs. (3.10) - (3.13), we finally obtain an equation for v_1 in the form,

$$(n_0 + 3p_0)\frac{d^2 v_1}{dx^2} + \left(\frac{dn_0}{dx} + 4\frac{dp_0}{dx}\right)\frac{dv_1}{dx} + k_0^2 n_0 v_1 = 0, \tag{3.14}$$

where k_0 is the wavenumber of the ion acoustic waves in a uniform plasma. Equation (3.14) is the governing equation for the linear ion acoustic waves propagating in an inhomogeneous plasma with ion density and ion temperature gradients. One can immediately see that for the case of cold ions, $p_0 = 0$ and the above equation reduces to the equation

$$\frac{d^2 v_1}{dx^2} + \left(\frac{1}{n_0}\frac{dn_0}{dx}\right)\frac{dv_1}{dx} + k_0^2 v_1 = 0, \tag{3.15}$$

which was obtained earlier by Doucet et al (1974) for an inhomogeneous plasma with only a density gradient.

Equation (3.14) can be solved for any arbitrary but slowly varying functions $n_0(x)$ and $p_0(x)$. However, solutions of eq. (3.14) have

$$V_1(x) = u(x) \cdot \cos(kx), \tag{3.16}$$

where k is the wavelength and $u(x)$ represents the slow spatial variation of the wave amplitude as the wave propagates into the inhomogeneous regions. Substituting eq. (3.16) into eq. (3.14) and assuming k to be slowly varying function of x, we obtain the following equation:

$$\left[2k(n_o + 3P_o)\frac{du}{dx} + \left(\frac{dn_o}{dx} + 4\frac{dP_o}{dx}\right)ku\right] \cdot \sin(kx)$$

$$-\left[(n_o+3P_o)\left(\frac{d^2u}{dx^2} - k^2 u\right)\right.$$

$$\left. + \left(\frac{dn_o}{dx} + 4\frac{dP_o}{dx}\right)\frac{du}{dx} + k_o^2 n_o u\right] \cdot \cos(kx) = 0. \tag{3.17}$$

Sufficient conditions for eq. (3.17) to be satisfied for all values of x are that the coefficients of the sine and cosine terms must be equal to zero separately. This yields the equations,

$$2(n_o+3P_o)\frac{du}{dx} + \left(\frac{dn_o}{dx} + 4\frac{dP_o}{dx}\right)u = 0, \tag{3.18}$$

$$(n_o+3P_o)\frac{d^2u}{dx^2} + \left(\frac{dn_o}{dx} + 4\frac{dP_o}{dx}\right)\frac{du}{dx}$$

$$\left[(k_o^2 - k^2)n_o - 3P_o k^2\right]u = 0, \tag{3.19}$$

which must be satisfied simultaneously. These equations, then, determine the slow variations in the amplitude and the wavenumber as the wave propagates in the gradient regions. The general solution of eq. (3.18) can easily be obtained as

$$u(x) = C \cdot \exp\left[-\int \frac{\left(\frac{dn_0}{dx} + 4\frac{dp_0}{dx} \right)}{2(n_0 + 3p_0)} dx \right], \tag{3.20}$$

where C is the constant of integration. Given $n_0(x)$, the amplitude u(x) can be obtained by performing the integration in eq.(3.20). Substituting the resulting solution into eq.(3.19), we obtain an equation for the variation of the wavenumber.

It is of interest to note here that if the gradients in $n_0(x)$ and $P_0(x)$ are in opposite directions satisfying the relation

$$\frac{dn_0}{dx} = -4\frac{dp_0}{dx}, \tag{3.20a}$$

then, from eq.(3.18) it follows that the amplitude of the ion acoustic wave remains constant as the wave propagates in the inhomogeneous regions. However, as is clear from eq.(3.19), the wavenumber no longer remains constant. Similar results are obtained in the next section for the nonlinear ion acoustic waves also.

Various special cases of the solution (3.20) can now be considered. For this, we assume the ions to be governed by the ideal gas law, namely, p = nT. Thus, $p_0 = n_0T_0$ where $T_0(x)$ is the given ion temperature distribution with respect to the space variable x and is normalized with respect to the constant electron temperature T_e. Substituting the above expression for p_0 in eq.(3.20), we obtain,

$$u(x) = C \cdot \exp\left[-\frac{1}{2}\int \frac{g(x)}{1+3T_0}\right] \cdot \exp\left[-2\int T_0 \frac{g(x)+h(x)}{1+3T_0}\right], \tag{3.21}$$

where

$$g(x) = \frac{1}{n_0}\frac{dn_0}{dx}, \quad h(x) = \frac{1}{T_0}\frac{dT_0}{dx},$$

and these are related to the characteristic lengths of variations of the two gradients.

Case (i): $T_0 = 0$

When the ion temperature is zero, the solution (3.21) reduces to

$$u(x) = C \cdot n_0^{-1/2}, \tag{3.22}$$

which is the result obtained by Doucet et al (1974) for an inhomogeneous plasma with only a density gradient.

Case (ii): T_0 = a constant

In this case, $h_0(x) = 0$ and from the solution (3.21), we obtain

$$u(x) = n_0^{-\frac{1}{2}\left(\frac{1+4T_0}{1+3T_0}\right)} \tag{3.23}$$

Thus, it follows that the presence of a constant ion temperature further increases the effect of the density gradients on the amplitude of the ion acoustic wave.

Case (iii): $T_0 = T_0(x)$

When both the gradients are present, the exact behaviour of the wave amplitude is given by the solution (3.21). Given $n_0(x)$ and $T_0(x)$, one can do the integrations on the right hand side of eq.(3.21) and study the effect of inhomogeneities on the propagation characteristics of the linear ion acoustic wave.

Thus, we see that the presence of finite ion temperature which could be a constant or varying as a function of x indeed modifies the propagation characteristics of the linear ion acoustic waves. In the next section, we discuss these modifications for the weakly nonlinear ion acoustic wave.

III.4 Nonlinear Analysis

The evolution of weakly nonlinear ion acoustic waves in a inhomogeneous plasma can be studied by means of the Reductive Perturbation Method developed by Asano & Ono (1971). This method was used in our earlier analysis (Rao & Varma, 1979) for the weakly nonlinear ion acoustic waves in a inhomogeneous plasma having only the ion density gradient. In this analysis, the zeroth order fields and the fluid velocities that exist in the system due to the presence of the inhomogeneities were taken into account and a set of stretched co-ordinates appropriate for the inhomogeneous media was used. In this section, we extend the above analysis for the case when ion temperature gradient is also present.

III.4.1 Reductive Perturbation Analysis

The basic equations required to derive the governing equations for the nonlinear ion acoustic waves in the inhomogeneous plasma are given by eqs.(3.1) - (3.4). To carry out the reductive perturbation analysis of these equations, we define the following set of stretched co-ordinates which is appropriate for the spatially inhomogeneous plasmas (Asano & Ono, 1971):

$$\xi = \epsilon^{1/2} \left(\int^{x} \frac{dx'}{\lambda_0(x')} - t \right),$$

$$\eta = \epsilon^{3/2} x, \tag{3.24a}$$

where ϵ is a small, positive parameter characterizing the strength of the nonlinearity. $\lambda_0(x)$ is the phase velocity of the frame moving with the soliton and will be determined later self-consistently. From eqs.(3.24a) we obtain following inverse transformations,

$$x = \epsilon^{-3/2} \eta,$$

$$t = \epsilon^{-3/2} \int^{\eta} \frac{d\eta'}{\lambda_0(\eta')} - \epsilon^{-1/2} \xi, \tag{3.24b}$$

Using eqs.(3.24a) and (3.24b), one can easily obtain the following transformations for the space and time derivatives:

$$\frac{\partial}{\partial x} \equiv \frac{\epsilon^{1/2}}{\lambda_0(\eta)} \cdot \frac{\partial}{\partial \xi} + \epsilon^{3/2} \frac{\partial}{\partial \eta},$$

$$\frac{\partial}{\partial t} \equiv -\epsilon^{1/2} \frac{\partial}{\partial \xi}, \tag{3.25a}$$

and

$$\frac{\partial}{\partial \xi} \equiv -\epsilon^{-1/2} \frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial \eta} \equiv \epsilon^{-3/2} \frac{\partial}{\partial x} + \frac{\epsilon^{-3/2}}{\lambda_0(x)} \frac{\partial}{\partial t}. \tag{3.25b}$$

Since we are considering only spatial gradients, $n_0(x)$ and $p_0(x)$ are functions of x only. Hence, they satisfy the conditions,

$$\frac{\partial n_0}{\partial \xi} = 0 \; , \; \frac{\partial p_0}{\partial \xi} = 0 \; ; \tag{3.26a}$$

also, as λ_0 is a function of x only, we have,

$$\frac{\partial \lambda_0}{\partial \xi} = 0. \tag{3.26b}$$

Using eqs.(3.25a), we write the eqs.(3.1) - (3.4) in terms of (ξ, η) co-ordinates as,

$$\frac{\partial n}{\partial \xi}+\frac{1}{\lambda_0}\frac{\partial}{\partial \xi}(nv)+\epsilon\frac{\partial}{\partial \eta}(nv)=0, \tag{3.27}$$

$$-n\frac{\partial v}{\partial \xi}+\frac{nv}{\lambda_0}\frac{\partial v}{\partial \xi}+\epsilon nv\frac{\partial v}{\partial \eta}$$

$$+n_0\frac{1}{\lambda_0}\frac{\partial \phi}{\partial \xi}+\epsilon n\frac{\partial \phi}{\partial \eta}+\frac{1}{\lambda_0}\frac{\partial p}{\partial \xi}+\epsilon\frac{\partial p}{\partial \eta}=0, \tag{3.28}$$

$$\frac{\epsilon}{\lambda_0^2}\frac{\partial^2 \phi}{\partial \xi^2}+\frac{2\epsilon^2}{\lambda_0}\frac{\partial^2 \phi}{\partial \xi \partial \eta}-\frac{\epsilon^2}{\lambda_0^2}\frac{\partial \lambda_0}{\partial \eta}\frac{\partial \phi}{\partial \xi}$$

$$+\epsilon^3\frac{\partial^2 \phi}{\partial \eta^2}-g(n_0)\cdot \exp(\phi)+n=0, \tag{3.29}$$

$$-\frac{\partial p}{\partial \xi}+\frac{v}{\lambda_0}\frac{\partial p}{\partial \xi}+\epsilon v\frac{\partial p}{\partial \eta}+\frac{3p}{\lambda_0}\frac{\partial v}{\partial \xi}$$

$$+3\epsilon p\frac{\partial v}{\partial \eta}=0. \tag{3.30}$$

We now expand the various quantities n, v, ϕ, and p in terms of the smallness parameter ϵ as

$$\psi=\psi_0+\epsilon\psi_1+\epsilon^2\psi_2+\cdots, \tag{3.31}$$

where $\psi \equiv (n, v, \phi, p)$ and the subscript '0' denotes the corresponding unperturbed quantities which are slowly varying functions of the space co-ordinate x . Substituting the expansions (3.31) into the eqs.(3.27) - (3.30) and equating the coefficients of same powers of ϵ to zero, we get sets of equations corresponding to different orders in

ϵ . The zeroth order equations, thus obtained, coupled with eqs. (3.26a) and (3.26b) yield

$$\frac{\partial V_0}{\partial \xi} = 0 \, , \, \frac{\partial \phi}{\partial \xi} = 0. \tag{3.32}$$

The first order equations in ϵ obtained from eqs. (3.27) - (3.30) are respectively given by

$$-\frac{\partial n_1}{\partial \xi} + \frac{n_0}{\lambda_0}\frac{\partial V_1}{\partial \xi} + \frac{V_0}{\lambda_0}\frac{\partial n_1}{\partial \xi} + \frac{\partial}{\partial \eta}(n_0 V_0) = 0, \tag{3.33}$$

$$-n_0 \frac{\partial V_1}{\partial \xi} + \frac{n_0 V_0}{\lambda_0}\frac{\partial V_1}{\partial \xi} + n_0 V_0 \frac{\partial V_0}{\partial \eta}$$

$$+ n_0 \cdot \frac{1}{\lambda_0} \cdot \frac{\partial \phi_1}{\partial \xi} + n_0 \frac{\partial \phi_0}{\partial \eta} + \frac{1}{\lambda_0}\frac{\partial P_1}{\partial \xi} + \frac{\partial P_0}{\partial \eta} = 0, \tag{3.34}$$

$$-n_0 \phi_1 + n_1 = 0, \tag{3.35}$$

$$-\frac{\partial P_1}{\partial \xi} + \frac{V_0}{\lambda_0}\frac{\partial P_1}{\partial \xi} + V_0 \frac{\partial P_0}{\partial \eta} + \frac{3 P_0}{\lambda_0}\frac{\partial V_1}{\partial \xi}$$

$$+ 3 P_0 \frac{\partial V_0}{\partial \eta} = 0, \tag{3.36}$$

where we have used eqs. (3.26a), (3.26b) and (3.32). We now impose the boundary condition that the plasma is homogeneous at the boundary, that is,

$$\left.\begin{array}{l} n_1 = V_1 = \Phi_1 , \\ V_0 , \Phi_0 , P_0 \rightarrow 0 , \\ \quad n_0 , \lambda_0 \rightarrow 1 , \end{array}\right\} \quad \text{as} \quad |\xi| \longrightarrow \infty , \tag{3.37}$$

and integrate the eqs.(3.33) - (3.36) once with respect to ξ to obtain respectively,

$$V_1 = R\, n_1 - Q\,\xi , \tag{3.38}$$

$$\Phi_1 = (\lambda_0 - V_0)\, V_1 - \frac{1}{n_0} P_1 - \frac{\lambda_0}{n_0}\left(n_0 V_0 \frac{\partial V_0}{\partial \eta} + n_0 \frac{\partial \Phi_0}{\partial \eta} + \frac{\partial P_0}{\partial \eta} \right) \xi , \tag{3.39}$$

$$n_1 = n_0 \Phi_1 , \tag{3.40}$$

$$P_1 = \frac{1}{(\lambda_0 - V_0)} \left[3P_0 (R\, n_0 \Phi_1 - \xi Q) + \lambda_0 \left(V_0 \frac{\partial P_0}{\partial \eta} + 3 P_0 \frac{\partial V_0}{\partial \eta} \right) \xi \right] \tag{3.41}$$

where

$$R = \frac{1}{n_0} (\lambda_0 - V_0) \;, \; Q = \frac{\lambda_0}{n_0} \frac{\partial}{\partial \eta} (n_0 V_0) .$$

Equations (3.38) - (3.41) constitute a set of inhomogeneous equations for the perturbed quantities n_1, v_1, Φ_1, and p_1. Using eqs.(3.38), (3.40) and (3.41), we eliminate v_1 and p_1 from eq.(3.39) in terms of Φ_1 to get,

$$\Phi_1 = \xi \Big[-n_0 Q(\lambda_0 - V_0)^2 + 3p_0 Q - \lambda_0 \Big(V_0 \frac{\partial p_0}{\partial \eta} + 3p_0 \frac{\partial V_0}{\partial \eta} \Big) - \lambda_0(\lambda_0 - V_0)\Big(n_0 V_0 \frac{\partial V_0}{\partial \eta} + n_0 \frac{\partial \phi_0}{\partial \eta} + \frac{\partial p_0}{\partial \eta} \Big) \Big] \cdot \Big[n_0(\lambda_0 - V_0) - R n_0^2 (\lambda_0 - V_0)^2 + 3R n_0 p_0 \Big]^{-1} . \tag{3.42}$$

In eq.(3.42), the right hand side depends on the zeroth order quantities whereas Φ is a first order quantity. Since the first order quantities cannot be determined in terms of zeroth order quantities alone, we make the expression on the right hand side of eq.(3.42) an indeterminate quantity with respect to the zeroth order quantities; that is, we put both the numerator and the denominator of the eq.(3.42) equal to zero separately.

Putting the denominator of the expression on the right hand side of eq.(3.42) equal to zero, we obtain the equation,

$$\lambda_0 = V_0 + (1 + 3p_0/n_0)^{1/2} , \tag{3.43}$$

which determines λ_0 self-consistently in terms of the unperturbed quantities. Similarly setting the numerator equal to zero, we get, after using the eq.(3.43),

$$n_0 \lambda_0 (\lambda_0 - V_0) \frac{\partial V_0}{\partial \eta} + V_0 \frac{\partial n_0}{\partial \eta} + \lambda_0 \frac{\partial p_0}{\partial \eta} + n_0(\lambda_0 - V_0) \frac{\partial \phi_0}{\partial \eta} = 0. \tag{3.44}$$

This equation then gives a relationship between the unperturbed quantities. Given n_0, the unperturbed potential Φ_0 can be calculated from the zeroth order equation obtained from eq.(3.29). Knowing n_0, ϕ_0 and p_0, the zeroth order fluid velocity induced by the inhomogeneities can be calculated from eq.(3.44).

To derive the equation describing the evolution of the ion acoustic solitary waves in the presence of inhomogeneities, we look for a nonlinear equation for the first order perturbed quantities. To this end, we consider the equations in the second order in ϵ arising from eq.(3.27) - (3.30), these are:

$$-\frac{\partial n_2}{\partial \xi} + \frac{1}{\lambda_0}\frac{\partial}{\partial \xi}\left(n_0 V_2 + n_1 V_1 + n_2 V_0\right) + \frac{\partial}{\partial \eta}\left(n_0 V_1 + n_1 V_0\right) = 0, \tag{3.45}$$

$$-n_0\frac{\partial V_2}{\partial \xi} - n_1\frac{\partial V_1}{\partial \xi} + \frac{n_0 V_0}{\lambda_0}\frac{\partial V_2}{\partial \xi} + n_0 V_0\frac{\partial V_1}{\partial \eta} + \frac{1}{\lambda_0}\left(n_0 V_1 + n_1 V_0\right)\frac{\partial V_1}{\partial \xi} + \left(n_0 V_1 + n_1 V_0\right)\frac{\partial V_0}{\partial \eta}$$
$$+ \frac{1}{\lambda_0}\left(n_0\frac{\partial \phi_2}{\partial \xi} + n_1\frac{\partial \phi_1}{\partial \xi}\right) + \left(n_0\frac{\partial \phi_1}{\partial \eta} + n_1\frac{\partial \phi_0}{\partial \eta}\right) + \frac{1}{\lambda_0}\frac{\partial P_2}{\partial \xi} + \frac{\partial P_1}{\partial \eta} = 0, \tag{3.46}$$

$$\frac{1}{\lambda_0^2}\frac{\partial^2 \phi_1}{\partial \xi^2} - n_0\phi_2 - \frac{1}{2}n_0\phi_1^2 + n_2 = 0, \tag{3.47}$$

$$-\frac{\partial P_2}{\partial \xi} + \frac{1}{\lambda_0}\left(V_0\frac{\partial P_2}{\partial \xi} + V_1\frac{\partial P_1}{\partial \xi}\right) + \left(V_0\frac{\partial P_1}{\partial \eta} + V_1\frac{\partial P_0}{\partial \eta}\right)$$
$$+ \frac{3}{\lambda_0}\left(P_0\frac{\partial V_2}{\partial \xi} + P_1\frac{\partial V_1}{\partial \xi}\right) + 3\left(P_0\frac{\partial V_1}{\partial \eta} + P_1\frac{\partial V_0}{\partial \eta}\right) = 0. \tag{3.48}$$

Equations (3.45) - (3.48) can be considered as a set of four equations for the four second order quantities n_2, v_2, Φ_2 and p_2 from which any three of the quantities can be eliminated to obtain an equation for the fourth one. Thus, when n_2, v_2 and p_2 are eliminated, we get the following equation for Φ_2:

$$\Phi_2 = F \cdot \left[n_0^2(\lambda_0 - V_0) - n_0^2(\lambda_0 - V_0)^3 + 3P_0 n_0(\lambda_0 - V_0) \right]^{-1} \qquad (3.49)$$

where F is a known function of the unperturbed and first order perturbed quantities, and their derivatives. It is easy to see that the denominator in eq.(3.49) is zero by virtue of eq.(3.43). In order, however, that Φ_2 be finite it follows that the numerator should also be equal to zero simultaneously. Putting, therefore, the function F to be equal to zero and eliminating the quantities n_1, v_1 and p_1 in favour of Φ_1 using eqs.(3.38) - (3.41), we obtain, after some simplifications, the following modified K-dV equations as the governing equation for the first order perturbed potential Φ_1 (Rao & Varma, 1978):

$$\frac{\partial \Phi_1}{\partial \eta} + \frac{1}{\mu_0 \lambda_0^2} \Phi_1 \frac{\partial \Phi_1}{\partial \xi} + \frac{1}{2 n_0 \mu_0 \lambda_0^4} \frac{\partial^3 \Phi_1}{\partial \xi^3}$$

$$- \left(\frac{1}{\lambda_0} \frac{\partial \lambda_0}{\partial \eta} - \frac{1}{\mu_0} \frac{\partial \mu_0}{\partial \eta} \right) \xi \frac{\partial \Phi_1}{\partial \xi} + \frac{1}{2} \left(\frac{1}{n_0} \frac{\partial n_0}{\partial \eta} + \frac{3}{\mu_0} \frac{\partial \mu_0}{\partial \eta} \right) \Phi_1 = 0, \qquad (3.50)$$

where

$$\mu_0 = \left(1 + \frac{3P_0}{n_0} \right)^{1/2}. \qquad (3.51)$$

This equation, then, governs the propagation of weakly nonlinear ion acoustic waves in an inhomogeneous plasma with spatial gradients in the ion density and ion temperature.

Before obtaining the stationary solution of the above equation, we discuss, briefly, the various limiting cases. First of all, for a homogeneous plasma with zero ion temperature, we have $n_0 = 1$, $p_0 = 0$ and $v_0 = 0$. Equations (3.43) and (3.51) then give respectively $\lambda_0 = 1$ and $\mu_0 = 1$. For these values of λ_0 and μ_0, eq.(3.50) reduces to the usual K-dV equation for a homogeneous, cold ion plasma, first derived by Washimi & Taniuti (1966). On the other hand, for a cold ion plasma with only the ion density gradient present, $p_0 = 0$ and hence, $\mu_0 = 1$. Equation (3.50) reduces, in this case, to the modified K-dV equation obtained earlier by Rao & Varma (1979) and, further, for the approximation $\lambda_0 \simeq 1$, it reduces to the equation obtained by Nishikawa & Kaw (1975). In the next section, we obtain the stationary solution of eq.(3.50) by means of certain transformations.

III.4.2 Solution of the Modified K-dV Equation

The exact soliton solution of the modified K-dV equation (3.50) can be obtained by making use of certain dependent and independent variable transformations similar to the ones given, for example, by Asano & Ono (1971) and Nishikawa & Kaw (1975). Defining the following new dependent and independent variables,

$$\tilde{\phi}_1 = (n_0 \mu_0^3)^{1/2} \phi_1 ,$$

$$\tilde{\xi} = \left(1 + \lambda n \frac{\lambda_0}{\mu_0}\right) \xi, \qquad \tilde{\eta} = \eta, \tag{3.52}$$

eq. (3.50) can be reduced to the form,

$$\frac{\partial \tilde{\Phi}_1}{\partial \tilde{\eta}} + \frac{1}{N_0^{1/2}} \tilde{\Phi}_1 \frac{\partial \tilde{\Phi}_1}{\partial \tilde{\xi}} + \frac{1}{2N_0} \frac{\partial^3 \tilde{\Phi}_1}{\partial \tilde{\xi}^3} = 0, \tag{3.53}$$

where $N_0 = n_0 \mu_0 \lambda_0^4$. In order to reduce the eq. (3.53) to the usual K-dV equation with constant coefficients, we introduce the independent variables ζ and τ as

$$\zeta = \left[N_0(\tilde{\eta})\right]^{1/4} \cdot \tilde{\xi}, \qquad \tau = \int^{\tilde{\eta}} \left[N_0(\tilde{\eta}')\right]^{-1/4} \cdot d\tilde{\eta}', \tag{3.54}$$

in terms of which eq. (3.53) takes the form

$$\frac{\partial \tilde{\Phi}_1}{\partial \tau} + \tilde{\Phi}_1 \frac{\partial \tilde{\Phi}_1}{\partial \zeta} + \frac{1}{2} \frac{\partial^3 \tilde{\Phi}_1}{\partial \zeta^3} + \beta \zeta \frac{\partial \tilde{\Phi}_1}{\partial \zeta} = 0, \tag{3.55}$$

where $\beta = (\partial N_0 / \partial \eta) / 4 N_0^{3/4}$.

The last term on the left hand side of eq. (3.55) can be eliminated by defining another set of independent variables,

$$\tilde{\xi} = 1 - \int \beta(\tau)\, d\tau$$
$$\tilde{\tau} = \tau . \tag{3.56}$$

These transformations, then, finally enable us to reduce eq.(3.55) to the usual K-dV equation for a homogeneous plasma, namely,

$$\frac{\partial \tilde{\phi}_1}{\partial \tilde{\tau}} + \tilde{\phi}_1 \frac{\partial \tilde{\phi}_1}{\partial \tilde{\xi}} + \frac{1}{2} \frac{\partial^3 \tilde{\phi}_1}{\partial \tilde{\xi}^3} = 0 . \tag{3.57}$$

The soliton solution of this equation is well known and is given by

$$\tilde{\phi}_1 = 3a \cdot \operatorname{sech}^2 \left[\left(\frac{a}{2}\right)^{1/2} (\tilde{\xi} - a\tilde{\tau}) \right] , \tag{3.58}$$

where 'a' is a constant. In terms of (ξ , η) variables, the above solution becomes

$$\phi_1 = \frac{3a}{(n_0 \mu_0^3)^{1/2}} \cdot \operatorname{sech}^2 \left[\left(\frac{a}{2}\right)^{1/2} \left\{ N_0^{1/4} \left(1 - \ln N_0^{1/4}\right) \right.\right.$$
$$\left.\left. \left(1 + \ln \frac{\lambda_0}{\mu_0}\right) \xi - a \int \frac{d\eta}{N_0^{1/4}} \right\} \right] . \tag{3.59}$$

From this solution, it is obvious that the propagation characteristics of the ion acoustic solitons are modified by the presence of ion temperature gradient.

III.4.3 Results and Discussions

Assuming now the ideal gas law for the ions, we have

$p_0 = n_0 T_0$ where $T_0(x)$ is the spatially varying ion temperature distribution. The expression for μ_0 then becomes,

$$\mu_0 = \left(1 + 3\,T_0(x)\right), \tag{3.60}$$

where $T_0(x)$ is normalized with respect to the constant electron temperature T_e. Combining eq.(3.60) with the solution (3.59), we obtain the following result: as the soliton propagates in the direction of the ion temperature gradient, the amplitude decreases. The modifications in the propagation velocity and the width of the soliton then follow as a consequence of this change in the soliton amplitude. These results can be readily understood as follows: as the soliton propagates into higher ion density (or temperature) regions, the dispersive term in eq.(3.50) reduces. Thus, a smaller amplitude is required to balance this term with the nonlinear term. Hence, as long as the soliton solution is maintained, the soliton amplitude keeps reducing as it travels along the gradient regions.

The above analysis shows that the effect of ion temperature gradient on the soliton amplitude is similar to that of the ion density gradient. This can also be seen directly from the governing eq.(3.50) : the last term on the left hand side of eq.(3.50) gives rise to the modifications in the soliton amplitude and, its dependence on the inhomogeneous quantities $n_0(x)$ and $\mu_0(x)$ is the same except for a numerical factor. This similarity brings out the following interesting result: When the two gradients in $n_0(x)$ and $\mu_0(x)$ are in opposite directions satisfying the relation,

$$\frac{1}{n_o}\frac{dn_o}{dx} = -\frac{1}{\mu_o}\frac{d\mu_o}{dx}, \tag{3.61}$$

then, the last term in eq.(3.50) becomes zero and hence the soliton amplitude remains constant. However, the width and the propagation velocity of the soliton change as it propagates in the gradient regions. It is interesting to note here that a similar result was obtained for the linear ion acoustic waves in § III.3.

III.5 Summary

We have considered in this Chapter the modifications in the propagation characteristics of linear and nonlinear waves as they propagate in an inhomogeneous plasma having spatial gradients in the ion density as well as the ion temperature. The treatment employed here for the linear waves is similar to the one given by Doucet et al (1974) where only the density gradient was considered. On the other hand, for the weakly nonlinear waves we carry out the Reductive Perturbation Analysis of the basic equations and thereby derive a more general modified K-dV equation for the inhomogeneous plasma. The soliton solution of this equation has been explicitly obtained by making use of certain dependent and independent variable transformations. Our results show that the effect of the ion temperature gradient on the propagation characteristics of the ion acoustic waves (linear or nonlinear) is similar to that of the ion temperature gradients. In particular, we show that for both the linear as well as nonlinear ion acoustic waves the wave amplitude is reduced as the wave propagates into regions of higher ion density or ion temperature regions. The fact that the effects of density and temperature gradients on linear or nonlinear ion acoustic waves are similar in nature leads to an interesting result : When the two gradients are in opposite directions with appropriate scale lengths, then, the wave amplitudes remain constant whereas the wavenumber (for the linear waves) and the width (for the nonlinear waves) change as the waves propagate in the gradient regions. These results are explained on simple physical grounds.

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CHAPTER IV

A THEORY FOR LANGMUIR SOLITONS

IV.1 Introduction

The problem of amplitude modulated Langmuir waves has been extensively considered over the last several years (Rudakov & Tsytovich, 1978; Thornhill & ter Haar, 1978). As is well known, a large amplitude Langmuir wave becomes modulationally unstable when subjected to long wavelength perturbations, thus leading to a concentration of plasmons in some regions of space and depletion in others. The ponderomotive force due to the high frequency Langmuir field acting on the electrons, then, expels the latter from regions where Langmuir field intensity is concentrated more. The ions follow the electrons and this results in the reduction of plasma number density in regions of higher plasmon density which, then, leads to further trapping of the plasmons. However, this self-consistent phenomenon of Langmuir wave trapping attains, in the time domain, an equilibrium state with the two competing forces, namely, the ponderomotive force due to the high frequency field and the force due to particle pressure balancing each other. One obtains, in this manner, localized stationary entities known as Langmuir solitons.

IV.2.1 Brief Review of Earlier Analyses

Many of the analyses of Langmuir solitons (Abdulleov et al, 1975; Degtyarev et al, 1975; Schmidt, 1975; Lebedev & Tsytovich, 1975; Gibbons et al, 1977; Pereira, 1977; Pereira & Sudan, 1977; D'Evelyn & Morales, 1977; Wang, 1978; Wardrop & ter Haar, 1978; Goldman & Nichonson, 1978; Nicholson & Goldman, 1978; Wang, 1980) are based on the Zakharov equations (Zakharov, 1972) which are a set of coupled equations: (i) a Schrodinger-like equation for the modulated amplitude of the high frequency Langmuir waves and (ii) the wave equation for the low frequency ion waves which is now driven by the ponderomotive force of the Langmuir waves. The Schrodinger-like equation is obtained from the electron fluid equations through an averaging over the fast-time scale ω_{pe}^{-1}, where ω_{pe} is the usual Langmuir wave frequency corresponding to the equilibrium density. The wave equation, on the other hand, is derived from the fluid equations for the low frequency ion waves by taking into account only the linear response of the low frequency motion. The charge-neutrality is, therefore, assumed for the low frequency part of the ion motion.

Rudakov (1973) obtained a quasi-static ($M \ll 1$, where M is the Mach number of the soliton normalized to the ion acoustic velocity), single-hump Langmuir soliton as a stationary solution of the Zakharov equations while Karpman (1975) obtained a near-sonic ($M \lesssim 1$) soliton solution, also a single-hump solution. However, when the Mach number takes near-sonic values, neither the linear response for the low frequency motion nor the charge-neutrality (for the low frequency part) is, in general, a good approximation. Makhankov (1974a, b) considered a Boussinesq type of equation for the ion waves and obtained a correction to the single-hump soliton solutions for near-sonic velocities. Nishikawa et al (1974), on the other hand, considered a K-dV type of weakly nonlinear wave equation for the low frequency motion of the ions; they, thereby, obtained a near-sonic soliton which has a node at the centre for the Langmuir field amplitude and, hence, a double-hump structure for the plasmon number density.

Schamel et al (1977) have taken complete ion nonlinearity into account, but have assumed charge-neutrality for the low frequency motion to close their system of equations. For small amplitude waves, these authors obtain explicit solutions while for finite amplitudes, they obtain, instead, a set of 'existence relations'. However, the assumption of charge neutrality is not quite consistent with taking full ion nonlinearity into account. Recently, Laedke & Spatschek (1979) have considered the effects of charge non-neutrality and ion nonlinearity in a perturbative manner.

While the above analyses are theoretical in nature, experimental investigations on Langmuir solitons have been carried out in recent years. The double-hump solitons of Nishikawa et al (1974) have been observed experimentally by Ikezi et al (1974) while structures similar to single-hump Rudakov solitons have been observed by Wong & Quon (1975), by Ikezi et al (1976) in the interaction of electron beams with plasmas, and by Kim et al (1974) for a R.F. electric field in a plasma.

The single-hump, quasi-static Rudakov soliton and the double-hump, near-sonic soliton of Nishikawa et al (1974) obviously have different forms and, further, they exist for widely separated regions of Mach number values in the range $0 < M < 1$. It is, therefore, desirable to investigate the existence of soliton solutions for intermediate values of the Mach number in the range $0 < M < 1$ and further, to understand if it is possible to go over smoothly from single-hump solutions to the double-hump solutions as one increases the Mach number. In order to be able to do so, a more general treatment of the problem retaining full ion nonlinearity and taking complete departures from charge-neutrality (through Poisson's equation) for the low frequency waves is needed.

IV.2.2 Abstract of the Present Analysis

We present, in this Chapter, a systematic and self-consistent analysis of the above problem, valid in the entire range of the Mach number, namely, $0 < M < 1$. We first derive the equations of evolution for the system by taking into account full ion nonlinearity and complete departures from charge-neutrality for the low frequency motion. A method is then developed to solve these coupled, nonlinear equations whereby any degree of ion nonlinearity consistent with the nonlinearities retained in the Langmuir field amplitude can be taken into account (Varma & Rao, 1980). We, thus, find that the single-hump solutions corresponding to the Rudakov solitons exist for small values of the Mach number. As the Mach number is increased, the solution for the plasmon number density (E^2) becomes narrower with a corresponding increase in the amplitude of the low frequency ion potential (Φ). For values of the Mach number beyond a critical value M_{crit}, the solution for E^2 develops a dip at the centre of the soliton, which becomes deeper with the increase of the Mach number until E^2 vanishes at the centre for a limiting Mach number M_{cut}. The solutions obtained for values of the Mach number $M = M_{cut}$ are then identified with the solutions of Nishikawa et al (1974).

We, thus, obtain a class of double-hump Langmuir soliton solutions with $E^2 \neq 0$ at the centre of the solitons for the Mach numbers in the range $M_{crit} < M < M_{cut}$. These solutions provide a smooth transition with respect to the Mach number from single-hump soliton solutions to the double-hump soliton solutions for E^2 obtained by Nishikawa et al (1974). We also obtain explicitly the parameter regions, in the (M, Δ) parameter space, for the existence of different types of Langmuir solitons where Δ is the normalized nonlinear frequency shift in the Langmuir wave frequency. The Sagdeev potential analyses of the relevent equations for E^2 and Φ confirm the above features of the Langmuir solitons.

IV.3 Equations of Evolution for the System

We consider one-dimensional Langmuir waves in a collisionless, homogeneous plasma with no external magnetic field. The ponderomotive force (or the Miller force) due to the high frequency Langmuir field acting on the electrons tends to expel them from regions where the Langmuir field is concentrated more. As ions follow the electrons, this leads to a depletion in the plasma density in these regions. The formation of such a 'density-well' leads to further trapping of the Langmuir field. There are thus two types of oscillations involved here: (i) the high frequency electron oscillations corresponding to the Langmuir waves and, (ii) the low frequency ion and electron oscillations characteristic of the ion acoustic waves. We consider the nonlinear coupling of these two widely separated frequencies of oscillations following the Zakharov adiabatic approximation (Zakharov, 1972), but retaining now the Poisson equation and complete ion nonlinearity for the low frequency waves.

IV.3.1 The Electron Response

Consider first the high and low frequency responses of the electrons in the plasma, which are governed by the usual particle and momentum conservation equations, and the Poisson equation. These are respectively given by

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}\left(n_e V_e\right) = 0, \tag{4.1}$$

$$\frac{\partial v_e}{\partial t} + v_e \frac{\partial v_e}{\partial x} = \frac{e}{m_e}\frac{\partial \phi}{\partial x} - \frac{\gamma T_e}{m_e n_e}\frac{\partial}{\partial x}(n_e), \qquad (4.2)$$

$$\frac{\partial^2 \phi}{\partial x^2} = -4\pi e (n_i - n_e), \qquad (4.3)$$

where, in the standard notation, n_e, T_e, v_e and m_e denote the electron number density, electron temperature, electron fluid velocity and the electron mass respectively, while ϕ and n_i denote respectively the electrostatic potential and the ion number density. The quantity γ is the usual adiabatic exponent which takes the value 3 for adiabatic electrons in one-dimension(as in the high frequency case) and equals unity for the low frequency response (when the electrons are isothermal).

Let δn denote the high frequency part of the electron density perturbation corresponding to the Langmuir oscillations and $\delta n_e'$, the low frequency part corresponding to the ion acoustic oscillations. Similarly, let $\delta n_i'$ denote the ion density perturbation which is entirely of low frequency. The electron and ion number densities, can, then, be split as

$$n_e = n_0 + \delta n_e' + \delta n,$$
$$n_i = n_0 + \delta n_i', \qquad (4.4)$$

where n_0 is the common ion (or electron) number density. Likewise, if $\tilde{\phi}$ and $\tilde{\Phi}$ denote, respectively, the high and low frequency parts of perturbed electrostatic potential ϕ, then,

$$\phi = \tilde{\phi} + \tilde{\Phi}. \qquad (4.5)$$

Substituting eq.(4.5) into eq.(4.3) and separating the high and low frequency parts, we obtain the following Poisson equations which relate the two potentials $\tilde{\phi}$ and $\tilde{\Phi}$ to their corresponding charge density perturbations.

$$\frac{\partial^2 \tilde{\phi}}{\partial x^2} = 4\pi e \cdot \delta n \tag{4.6}$$

$$\frac{\partial^2 \Phi}{\partial x^2} = -4\pi e \left(\delta n_i' - \delta n_e'\right). \tag{4.7}$$

It may be noted here that the usual assumption of charge neutrality for the low frequency oscillations implies, in the notation of eq.(4.7), that $\delta n_i' = \delta n_e'$, and, hence, amounts to neglecting the Poisson equation (4.7). We, however, allow arbitrary departures from charge neutrality by making use of eq.(4.7). Equations (4.1) and (4.2) are now linearized in the high frequency perturbed quantities using eqs.(4.4). This yields the equations,

$$\frac{\partial}{\partial t}(\delta n) + n_0 \frac{\partial V_e}{\partial x} + \frac{\partial}{\partial x}\left(\delta n_e' \cdot V_e\right) = 0, \tag{4.8}$$

$$\frac{\partial V_e}{\partial t} - \frac{e}{m_e}\frac{\partial \tilde{\phi}}{\partial x} + \frac{3T_e}{m_e n_0}\frac{\partial}{\partial x}(\delta n) = 0. \tag{4.9}$$

Equations (4.8) and (4.9) together with the Poisson equation (4.6) then constitute the basic set of equations governing the high frequency response of electrons.

The low frequency response, on the other hand, is obtained by averaging the electron equation of motion over the fast time scale. This gives rise to the usual ponderomotive force $\sim \partial/\partial x(|\tilde{E}|^2)$ (the Miller force) where $\tilde{E}$ is the complex amplitude of the high frequency Langmuir field oscillations. Using this additional nonlinear force on the right hand side of the electron momentum equation (4.2) (with $\gamma = 1$, for the one-dimensional isothermal motion of the electrons corresponding to the low frequency oscillations) and neglecting the electron inertia for the low frequency waves, we obtain the following expression for the electron density low frequency perturbation.

$$\delta n_e' = n_o\left[\exp\left(\frac{e\Phi}{T_e} - \frac{|\tilde{E}|^2}{16\pi n_o T_e}\right) - 1\right]. \tag{4.10}$$

Following Zakharov (1972), we write the Langmuir field $\tilde{E}$ as a high frequency oscillating exponential modulated by a slowly varying low frequency amplitude E(x,t). Thus,

$$\tilde{E} = -\frac{\partial\tilde{\Phi}}{\partial x} = E(x,t)\cdot\exp(-i\omega_{pe}t), \tag{4.11}$$

where ω_{pe} is the electron plasma frequency corresponding to the unperturbed density. Using eq.(4.11) in eqs.(4.6), (4.8) and (4.9), and averaging the resulting equations over the fast time scale (ω_{pe}^{-1}), we obtain (Zakharov, 1972),

$$i\epsilon\frac{\partial E}{\partial t} + \frac{3}{2}\frac{\partial^2 E}{\partial x^2} = \frac{1}{2}(\delta n_e)E, \tag{4.12}$$

where, in deriving eq. (4.12), the second derivative of E with respect to t has been neglected. In eq. (4.12), $\epsilon = (m_e / m_i)^{1/2}$, $\delta n_e = \delta n_e' / n_0$ and all other quantities are normalized as follows: x and t with respect to the electron Debye length $\lambda_{De} = (T_e / 4\pi n_0 e^2)^{1/2}$ and the ion plasma period $\tau_{pi} = 2\pi/\omega_{pi}$, respectively, and $|E|$ with respect to $(4\pi n_0 e^2)^{1/2}$.

We now look for stationary solutions for E(x,t) in the form

$$E(x,t) = E_a(x - Mt) \cdot \exp\left[i\left\{X(x) + T(t)\right\}\right], \tag{4.13}$$

where M is the Mach number of the stationary solution (normalized with respect to the ion acoustic velocity, $C_s = (T_e / m_i)^{1/2}$). To determine the unknown functions X(x) and T(t), we substitute eq. (4.13) in eq. (4.12) and separate, in the resulting equation, the real and imaginary parts. The imaginary part of the equation gives, after integration, the equation

$$X(x) = \epsilon M x/3, \tag{4.14}$$

while the real part of the equation gives the following equation for the amplitude E_a:

$$3\frac{d^2 E_a}{d\xi^2} = (\lambda + \delta n_e) E_a, \tag{4.15}$$

where

$$\xi = x - Mt, \quad \lambda = 2\Delta + \epsilon^2 M^2/3, \tag{4.15a}$$

and $\Delta = \epsilon \cdot dT/dt$ is the normalized frequency shift in the Langmuir wave frequency.

IV.3.2 The Ion Response

The response of the ions in the system is, of necessity, of low frequency and is given by the equation of particle and momentum conservation, and the Poisson equation (4.7). Using the standard notations for the relevant variables, these can be written in the form

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v_i) = 0, \tag{4.16}$$

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} + \frac{\partial \Phi}{\partial x} = 0, \tag{4.17}$$

$$\frac{\partial^2 \Phi}{\partial x^2} + n_i - \exp\left(\Phi - \frac{E_a^2}{4}\right) = 0, \tag{4.18}$$

where in eq. (4.18), we have made use of the expressions for $\delta n_i'$ and $\delta n_e'$ given, respectively, by eqs. (4.4) and (4.10). Further, in eqs. (4.16) - (4.18), $\Phi = e\tilde{\Phi}/T_e$ and the variables n_i and v_i are normalized, respectively, with respect to n_0 and C_s; the variables x, t and E_a are non-dimensionalized as in § IV.3.1. In eq. (4.17) the ponderomotive force on the ions has been neglected since it is smaller than that on the electrons in proportion to the electron-to-ion mass ratio.

Assuming now that n_i, v_i and Φ are functions of the variable $\xi = (x - Mt)$ only, and using boundary conditions that

$$n_i \to 1, \quad v_i \to 0, \quad \Phi \to 0 \quad \text{as} \quad |\xi| \to \infty, \tag{4.19}$$

one can integrate eqs.(4.16) and (4.17) once with respect to ξ to give an expression for n_i in terms of Φ alone. Using this expression for n_i in eq.(4.18), we obtain the following equation for the low frequency ion wave potential:

$$\frac{d^2\Phi}{d\xi^2} = -M\left(M^2-2\Phi\right)^{-1/2} + \exp\left(\Phi - E^2/4\right). \tag{4.20}$$

Substituting for $\delta n_e = \delta n_e' / n_0$ from eq.(4.10) in eq.(4.15), the equation for E_a becomes,

$$3\frac{d^2E}{d\xi^2} = (\lambda-1)E + E\cdot\exp\left(\Phi - E^2/4\right), \tag{4.21}$$

where in eqs.(4.20) and (4.21), the subscript 'a' on E_a has been dropped for convenience.

Equations (4.20) and (4.21) are the required equations of evolution for the system under consideration (Varma & Rao, 1980). They constitute a coupled set of nonlinear equations for the propagation of the stationary, coupled solutions for the Langmuir field amplitude E and the associated ion wave potential Φ. Further, they involve two free parameters: the Mach number M and the nonlinear frequency shift Δ. Since complete ion nonlinearity along with Poisson's equation have been taken into account, these equations describe, in principle, the Langmuir solitons in the entire range of the Mach number, namely, $0 < M < 1$. In the next section, we develop a technique for solving these equations up to any arbitrary degree of ion nonlinearity, consistent with the nonlinearity retained in E .

IV.4 Method of Solution for the Equations of Evolution

We develop, in this section, a procedure for obtaining the solutions of the coupled equations (4.20) and (4.21) valid in the entire range of the Mach number, $0 < M < 1$. Our aim here is to eliminate the independent variable ξ between eqs. (4.20) and (4.21). To this end, we first note that these equations can be derived from a Lagrangian $L\ (E, \Phi, E_\xi, \Phi_\xi)$ given by

$$L = \frac{1}{2}\left(\frac{d\Phi}{d\xi}\right)^2 - \frac{3}{4}\left(\frac{dE}{d\xi}\right)^2 + M\left(M^2 - 2\Phi\right)^{1/2} - \frac{1}{4}(\lambda-1)E^2 + \exp\left(\Phi - E^2/4\right). \tag{4.22}$$

Since this Lagrangian L does not depend explicitly on the independent variable ξ, the corresponding Hamiltonian is a constant of motion with respect to ξ. Using the boundary conditions for localized solutions for E and Φ, namely,

$$E, \Phi, \frac{dE}{d\xi}, \frac{d\Phi}{d\xi} \longrightarrow 0 \qquad |\xi| \longrightarrow \infty, \tag{4.23}$$

we obtain the 'energy integral' in the form

$$1 + M^2 = \frac{3}{4}\left(\frac{dE}{d\xi}\right)^2 - \frac{1}{2}\left(\frac{d\Phi}{d\xi}\right)^2 + M\left(M^2 - 2\Phi\right)^{1/2} - \frac{1}{4}(\lambda-1)E^2 + \exp\left(\Phi - E^2/4\right). \tag{4.24}$$

Making use of the eq. (4.24), we can now eliminate the independent variable ξ between eqs. (4.20) and (4.21). This yields the following differential equation for $\Psi = E^2/4$ in terms of Φ alone (Varma & Rao, 1980):

$$
\begin{aligned}
&12\Psi\Big[M(M^2-2\Phi)^{1/2}-(1+M^2)-(\lambda-1)\Psi \\
&\qquad\qquad +\exp(\Phi-\Psi)\Big]\frac{d^2\Psi}{d\Phi^2} \\
&+9\Big[M(M^2-2\Phi)^{-1/2}-\exp(\Phi-\Psi)\Big]\left(\frac{d\Psi}{d\Phi}\right)^3 \\
&-6\Big[M(M^2-2\Phi)^{1/2}-(1+M^2)-2(\lambda-1)\Psi \\
&\qquad\qquad +(1-\Psi)\cdot\exp(\Phi-\Psi)\Big]\left(\frac{d\Psi}{d\Phi}\right)^2 \\
&-6\Psi\Big[M(M^2-2\Phi)^{-1/2}-\exp(\Phi-\Psi)\Big]\left(\frac{d\Psi}{d\Phi}\right) \\
&-4\Psi^2\Big[(\lambda-1)+\exp(\Phi-\Psi)\Big] = 0. \qquad (4.25)
\end{aligned}
$$

We may now attempt a power series solution for Ψ in terms of Φ in the form

$$\Psi = \sum a_n \Phi^n, \qquad (4.26)$$

where the coefficients a_n are to be determined so that eq. (4.26) is a solution of the eq. (4.25). In order, however, that the coefficients a_n

be finite in the limit $M \rightarrow 0$, it is more appropriate to introduce, in place of Φ, the variable Θ defined by

$$\Theta = \Phi / M^2 \tag{4.27}$$

Substituting for Φ from eq.(4.27) into eq.(4.26), the series for Ψ becomes

$$\Psi = \sum b_n \Theta^n, \tag{4.28}$$

where $b_n = M^{2n} a_n$. Using this expansion for Ψ in eq.(4.25) and expanding the various quantities in powers of Θ, we explicitly obtain expressions for b_n by equating like powers of Θ to zero. The coefficient b_0 is zero by virtue of the boundary conditions (4.23) while the coefficient b_1 is determined entirely in terms of the free parameters M and Δ, that is,

$$b_0 = 0,$$
$$b_1 = M^2 - 1 - 4M^2(2\Delta + \epsilon^2 M^2/3)/3 . \tag{4.29}$$

All the other higher coefficients are uniquely determined in terms of the lower coefficients. Some of them are listed in the Appendix.

One can, thus, solve for Ψ in terms of Θ up to any number of terms in the expansion (4.28). This solution, which is now consistent with the eqs.(4.20) and (4.21), can be substituted in eq.(4.20) giving an equation for Θ (or Φ) alone. The resulting equation can, in principle, be solved for $\Theta(\xi)$ with the boundary conditions (4.23). It is, however, possible to obtain analytic solution for

$\theta(\xi)$ if terms up to third order in θ are retained. In that case, the equation for θ is given by

$$M^2 \frac{d^2\theta}{d\xi^2} = \alpha_1 \theta + \alpha_2 \theta^2 + \alpha_3 \theta^3, \tag{4.30}$$

where,

$$\alpha_1 = -(1+\alpha),$$

$$\alpha_2 = -\frac{3}{2} - b_2 + \frac{1}{2}\alpha^2,$$

$$\alpha_3 = -\frac{5}{2} - b_3 + \alpha b_2 - \frac{1}{6}\alpha^3, \tag{4.31}$$

with

$$\alpha = -(1 + 4\lambda M^2/3). \tag{4.31a}$$

The solution of eq. (4.30) subject to the boundary conditions (4.23) is, then, obtained in the form (Varma & Rao, 1980)

$$\theta(\xi) = \frac{\beta_1 \beta_2 \cdot \operatorname{sech}^2[k(\xi-\xi_0)]}{\beta_1 - \beta_2 \tanh^2[k(\xi-\xi_0)]}, \tag{4.32}$$

where

$$\beta_{1,2} = \frac{1}{3\alpha_3}\left[-2\alpha_2 \mp (4\alpha_2^2 - 18\alpha_1\alpha_3)^{1/2}\right],$$

$$k = (\lambda/3)^{1/2}, \tag{4.33}$$

and $k\xi_0$ is the initial phase of the soliton which may be taken, without any loss of generality, to be equal to zero. The solution for

$E^2(\xi)$ is then obtained by substituting the solution (4.32) for $\Theta(\xi)$ in eq.(4.28); this yields,

$$E^2(\xi) = 4\sum_n b_n \Theta^n(\xi), \tag{4.34}$$

retaining now only appropriate number of terms.

The solutions (4.32) and (4.34) for Θ and E^2 involve the free parameters M and Δ, which appeared naturally in the equations of evolution, namely, eqs.(4.20) and (4.21). In place of Δ, we may, sometime, take E_0, the Langmuir field amplitude at the centre $\xi = 0$, as a free parameter. Then, Δ and E_0 are related through the relation (4.34) at $\xi = 0$.

IV.5 Discussion of the Solutions

The solutions (4.32) and (4.34) for the low frequency ion potential $\Theta(\xi)$ and the high frequency Langmuir field amplitude $E(\xi)$ are valid in the entire range of the Mach number, namely, $0 < M < 1$. We consider, in this section, the structure of these solutions for different values of the parameters M and Δ. In particular, we show here the existence of a class of double-hump Langmuir solitons having non-zero Langmuir field intensity at the centre $\xi = 0$. These solutions, then, provide a smooth transition from the sub-sonic and near-sonic single-hump Langmuir solitons to the near-sonic double-hump solitons of Nishikawa et al (1974) which have a zero Langmuir field intensity at the centre of the soliton.

In order to simplify the discussion, we consider here a specific case where the expansion in eq.(4.34) contains terms up to Θ^2 only, that is,

$$E^2(\xi) = 4\left[b_1\theta(\xi) + b_2\theta^2(\xi)\right], \tag{4.35}$$

with $\Theta(\xi)$ still being given by eq.(4.32):

$$\theta(\xi) = \frac{\beta_1\beta_2 \cdot \mathrm{sech}^2(k\xi)}{\beta_1 - \beta_2 \tanh(k\xi)}, \tag{4.36}$$

where the initial phase of the soliton is taken to be equal to zero. For the case considered here, the definitions (4.31) and (4.33) remain exactly the same except that in the expression for α_3 in eqs.(4.31), the coefficient b_3 is zero (ref. eq.(4.35)). For the sake of mathematical convenience, we first take M and Δ as the free parameters.

IV.5.1 Free Parameters: M and Δ

Typical plots of the solutions (4.35) and (4.36) for a value of Δ and for different values of M are shown respectively in Figures (1) and (2). For sub-sonic velocities, we obtain solutions for the Langmuir field intensity E^2 and the ion wave potential Φ which have single-hump and one bottom-well structure respectively. Such solutions are the ones obtained with the linear ion response by Rudakov (1973) using Zakharov equations. As the Mach number increases, the solution for $E^2(\xi)$ becomes narrower with a decrease in its amplitude.

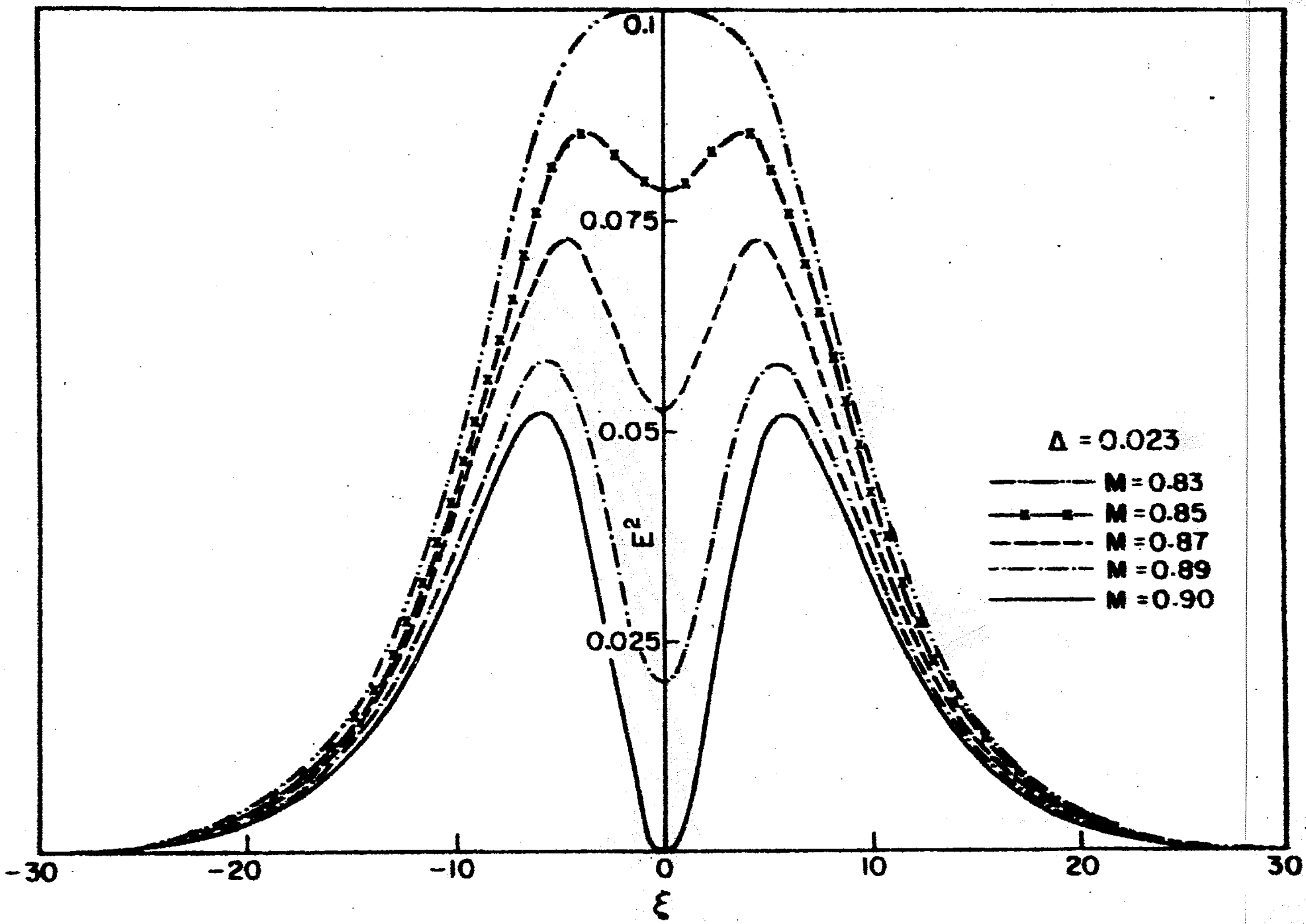

FIGURE (1). High frequency Langmuir field intensity $E^2(\xi)$ (eq.(4.35)) for a given value of the nonlinear shift in the Langmuir wave frequency Δ and for different values of the soliton Mach number M .

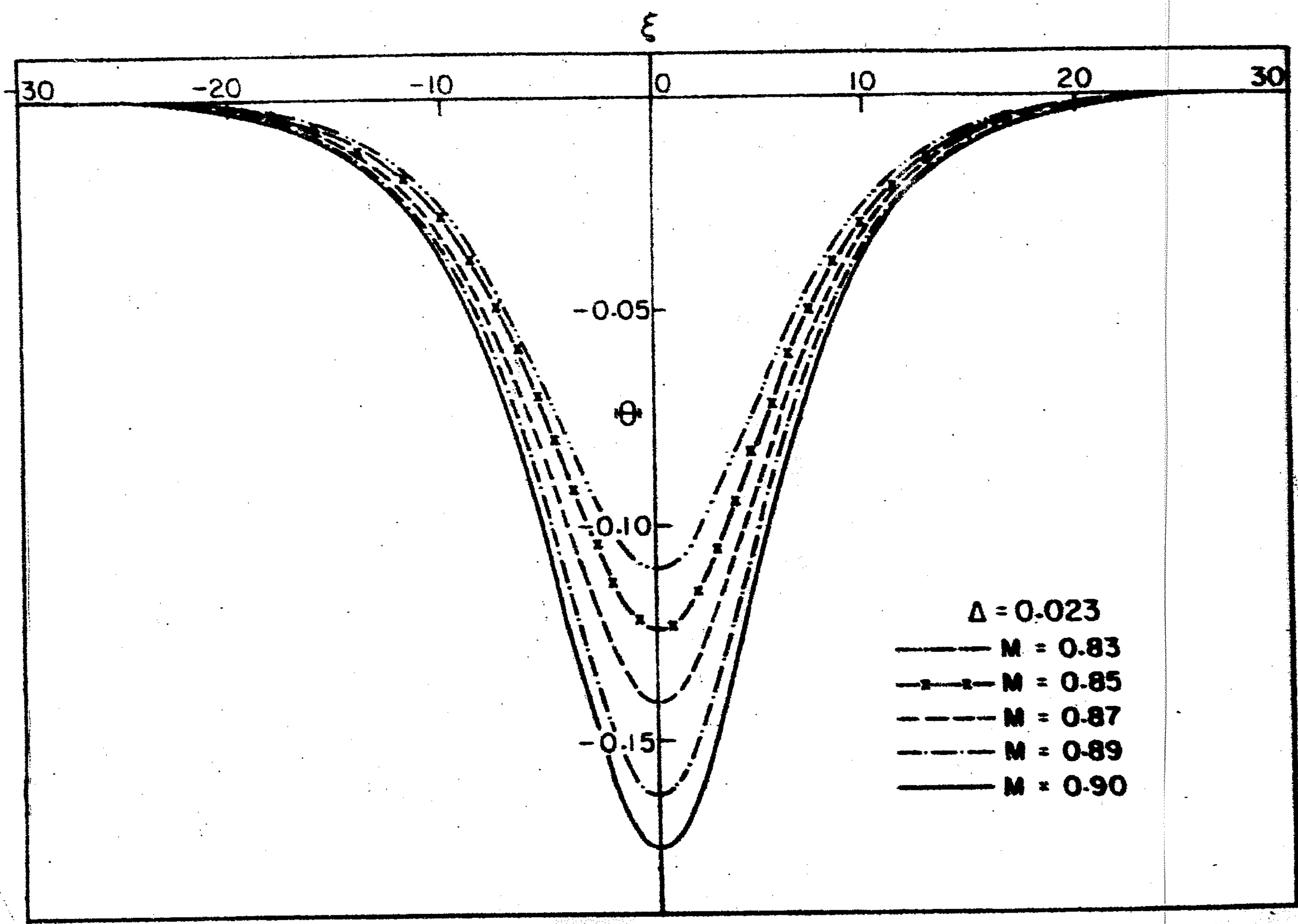

FIGURE (2). Low frequency ion wave potential $\Phi(\xi)$ (eq.(4.36)) for values of the parameters M and Δ corresponding to the solutions plotted in Figure (1).

Beyond a certain critical value of the Mach number, the solutions for $E^2(\xi)$ develops a dip at $\xi = 0$ whose depth increases with further increase of the Mach number. The critical Mach number (M_{crit}) beyond which the solutions for E^2, therefore, have double-hump structures is a function of Δ, and is calculated from the equation

$$b_1 + 2b_2\beta_2 = 0, \tag{4.37}$$

which is obtained from eqs.(4.35) and (4.36) as a condition that $d^2(E^2)/d\xi^2$ be zero at $\xi = 0$.

Since $E^2(\xi)$ is positive, its minimum value at $\xi = 0$ can only be zero. This corresponds to the soliton solutions with the maximum dip, which are then identified with the solutions obtained by Nishikawa et al (1974). For a given value of Δ, the maximum dip occurs for a limiting Mach number beyond which double-hump solutions for $E^2(\xi)$ with E^2 positive everywhere do not exist. Such a cut-off Mach number M_{cut} can be evaluated from the equation

$$b_1 + b_2\beta_2 = 0, \tag{4.38}$$

which is obtained from eq.(4.35) as a condition that $E^2 = 0$ at $\xi = 0$. Clearly, for a given value of $\Delta \neq 0$, the cut-off Mach number is always greater than the critical Mach number. Figure (3) shows the plot of M_{cut} and M_{crit} as functions of Δ in the (M, Δ) parameter space. Here, for values of M and Δ corresponding to the region below the shaded region, the solutions for $E^2(\xi)$ have single-hump structures, similar to the solutions obtained with the linear ion response (Rudakov,

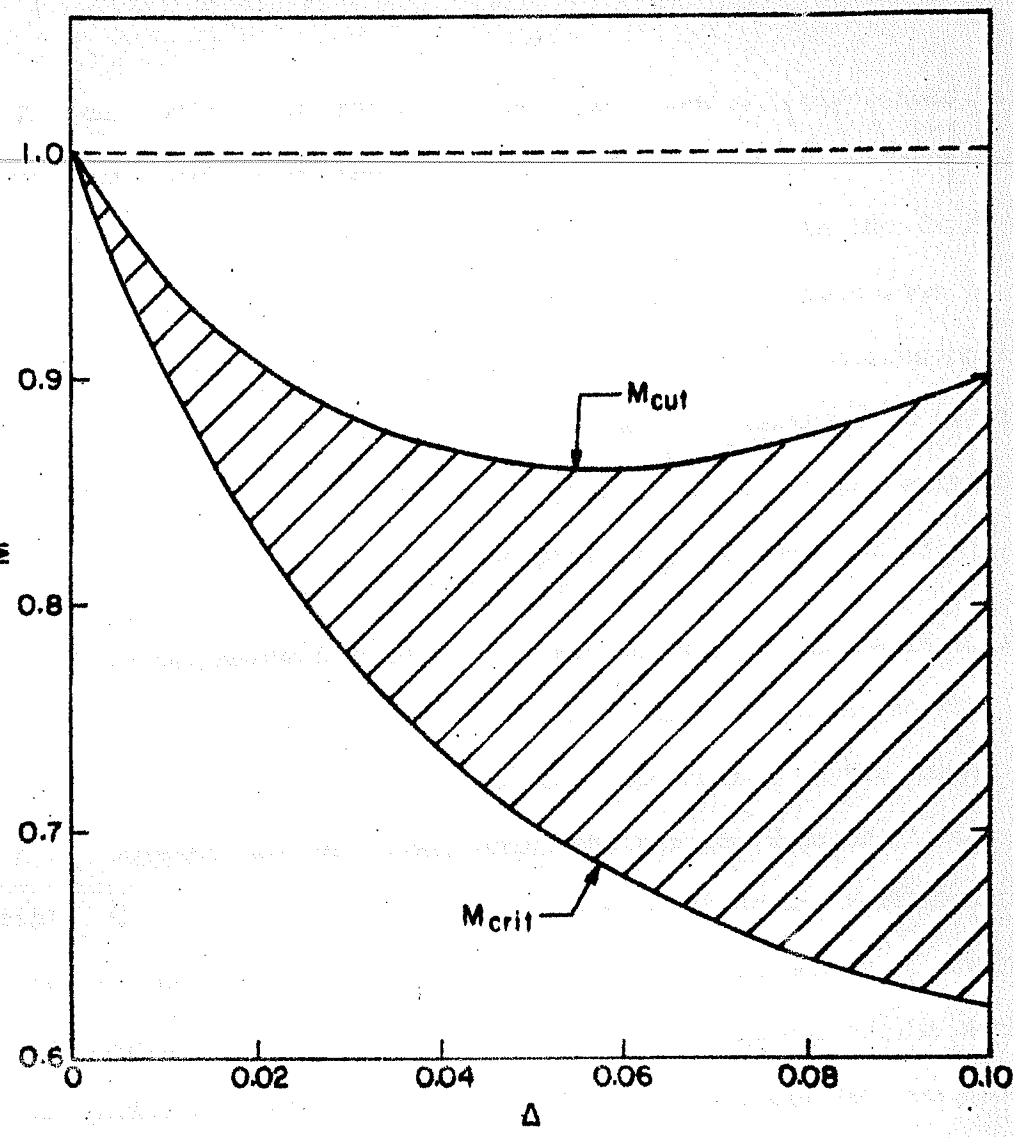

FIGURE (3). Plot of the Critical Mach number M_{crit} and the Cut-off Mach number M_{cut} as functions of the nonlinear frequency shift Δ in the (M, Δ) parameter space. For values of the parameters M & Δ corresponding to the shaded region, the solutions for $E^2(\xi)$ have double-hump structures with non-zero Langmuir field intensity at the centre of the solitons.

1973; Karpman, 1975). On the other hand, the shaded region gives the parameter values for which the Langmuir field intensity has a double-hump structure with $E^2 \neq 0$ at $\xi = 0$. The curve $M = M_{crit}$ in the (M, Δ) parameter space defines the transition from single-hump solutions to the double-hump solutions, whereas for values of M and Δ corresponding to the curve $M = M_{cut}$, we recover the solutions of Nishikawa et al (1974). Further, one can easily identify the parameter region for the existence of near-sonic, single-hump solutions discussed by Karpman (1975).

From the above discussion, it follows that there exists, for a given Δ, a smooth transition from single-hump solutions to the double-hump solutions as one increases the Mach number in the range $0 < M < 1$. However, for all these solutions the low frequency ion wave potential $\Phi(\xi)$ exhibits a one bottom-well structure throughout. In Figures (4) and (5), we plot the Langmuir field intensity E_0^2 and the ion wave potential Φ_0 at the centre $\xi = 0$ as functions of the Mach number for different values of Δ, whereas, in Figure (6), we plot Φ_0 as a function of Δ for the corresponding values of the critical and cut-off Mach numbers. In the following section, we carry out the Sagdeev potential analyses of the equations (4.30) and (4.35) and confirm the above mentioned features of the Langmuir solitons.

IV.5.2 Sagdeev Potential Analyses

We first carry out the Sagdeev potential analysis of the eq.(4.30). Integrating this equation with respect to ξ once and using the boundary conditions (4.23), we obtain

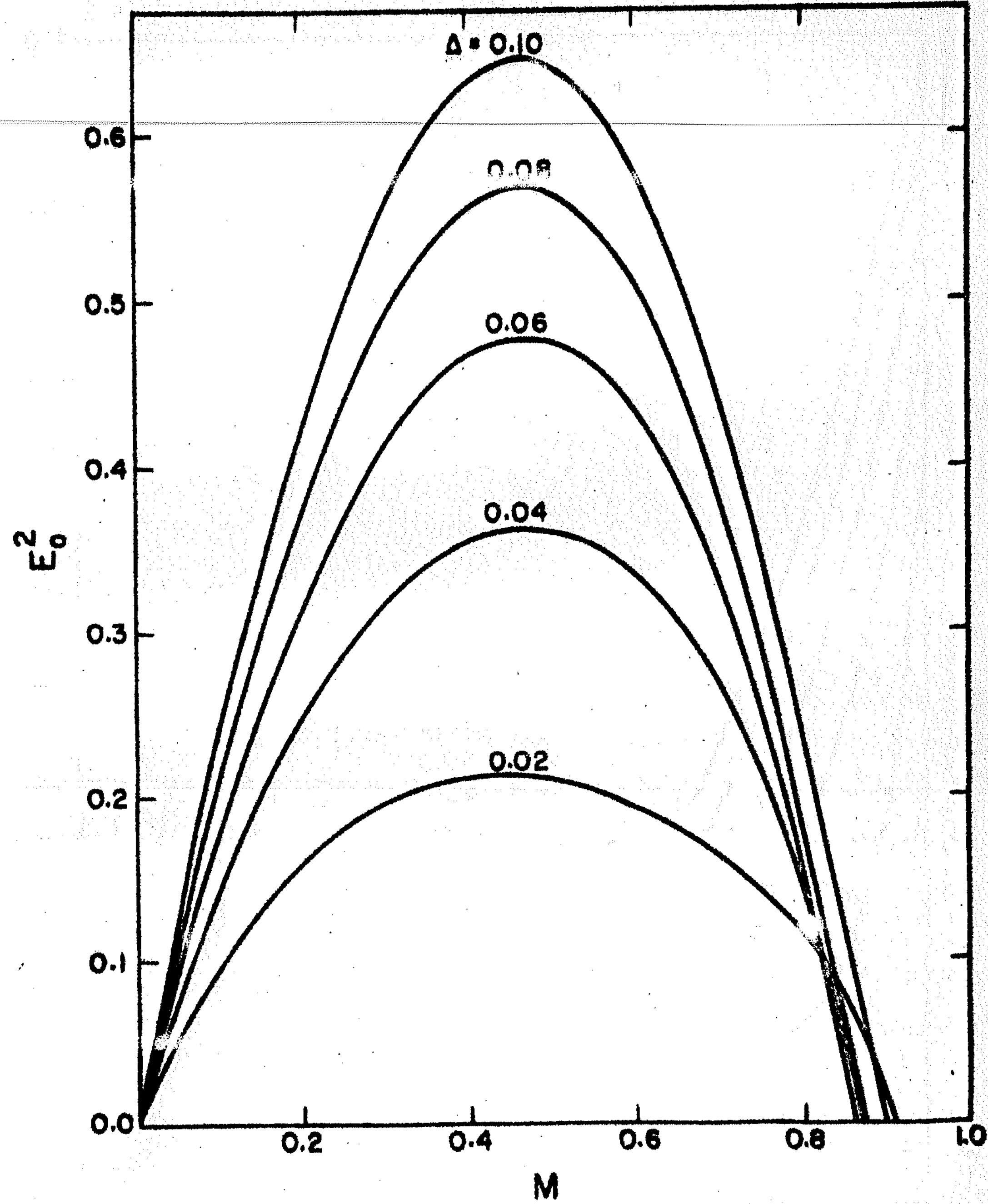

FIGURE (4). Langmuir field intensity E_0^2 at the centre of the soliton $\xi = 0$ (eq.(4.35)) as a function of the soliton Mach number M for different values of the nonlinear frequency shift Δ.

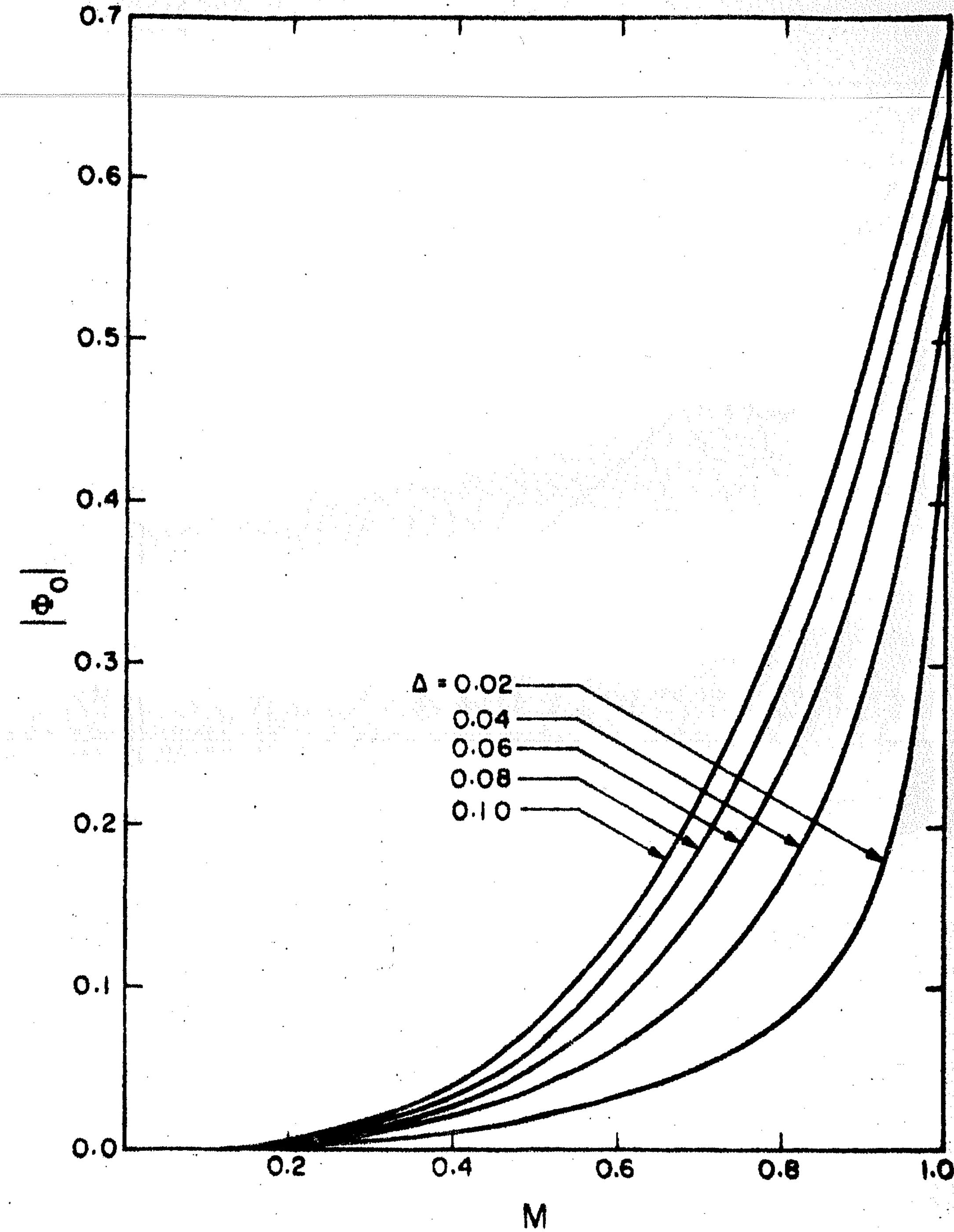

FIGURE (5). Plot of the amplitude of the ion wave potential Φ_0 (eq.(4.36)) for values of the parameters M and Δ as in Figure (4).

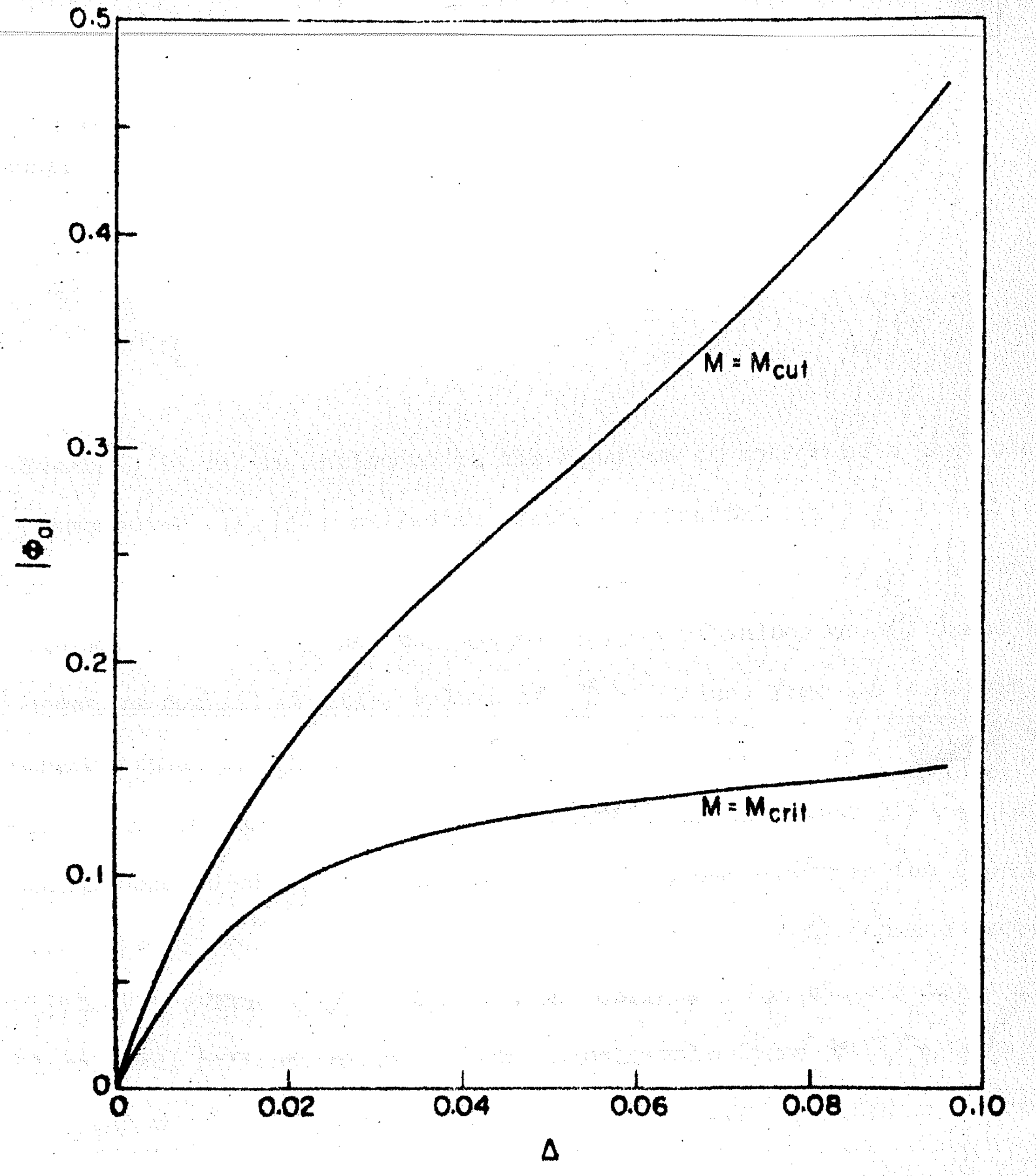

FIGURE (6). Plot of the amplitude of the ion wave potential Φ_0 (eq.(4.36)) as a function of the nonlinear frequency shift Δ for corresponding values of the Critical Mach number M_{crit} and the Cut-off Mach number M_{cut}.

$$\frac{1}{2}\left(\frac{d\Phi}{d\xi}\right)^2 + V_\Phi(\Phi) = 0, \tag{4.39}$$

where,

$$V_\Phi(\Phi) = -\frac{2M^4k^2}{\beta_1\beta_2}\cdot(\theta-\beta_1)(\theta-\beta_2)\theta^2. \tag{4.40}$$

Equation (4.39) is analogous to the equation of motion of a particle in a potential $V_\Phi(\Phi)$ called the Sagdeev potential (ref. § I.6.3). Consider the behaviour of V_Φ with respect to Φ : V_Φ is zero at minimum ($\theta = \beta_2$) and the maximum ($\theta = 0$) values of Φ , and is negative for all negative values of Φ . Also, from eq.(4.40) it follows that (i) $dV/d\Phi$ is zero at $\Phi = 0$ and (ii) $d^2V/d\Phi^2$ is finite at $\Phi = 0$. One can now analyse the behaviour of V_Φ in the neighbourhood of $\Phi = 0$ and show that the point (0,0) in the $\Phi - V_\Phi$ plane is mapped to the points ($\pm\infty$, 0) in the $\xi - \Phi$ plane (see, § I.6.3). Thus, V_Φ behaves appropriately for the existence of localized, soliton solutions. The single-valuedness of V_Φ with respect to Φ implies that the solution for $\Phi(\xi)$ has one bottom-well structure always. This is consistent with the solution (4.36) plotted in Figure (2) for different values of the Mach number. Typical plots of $V_\Phi(\Phi)$ are shown in Figure (7).

We next carry out the Sagdeev potential analysis for $E^2(\xi)$ (Rao & Varma, 1981). Differentiating eq.(4.35) with respect to ξ once and substituting for $d\Phi/d\xi$ from eqs.(4.39) and (4.40), we obtain

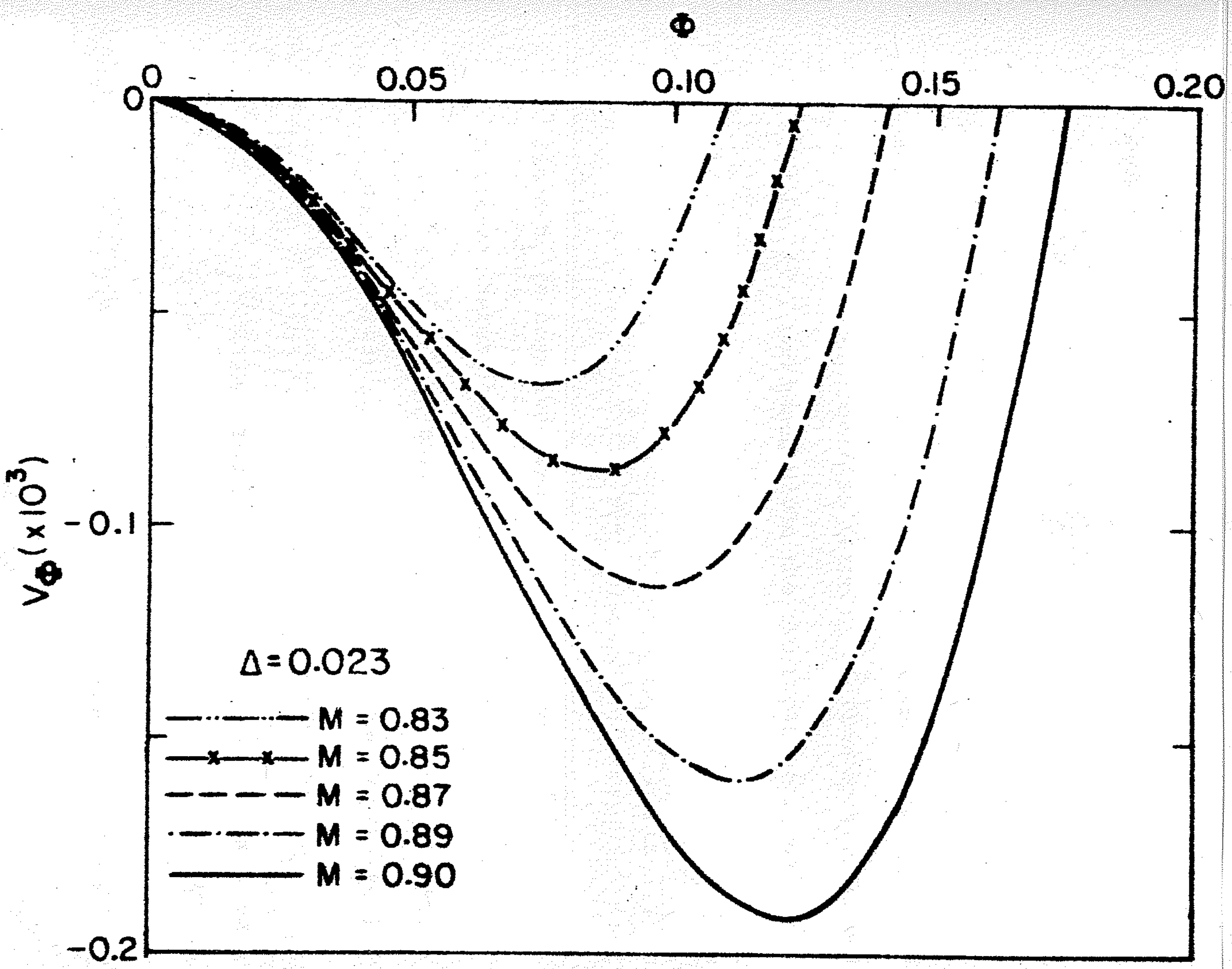

FIGURE (7). Sagdeev potential $V_\Phi(\Phi)$ as a function of Φ (eq.(4.40)) for values of the parameters M & Δ as in Figure (2). Note that $V_\Phi(\Phi)$ is always a single-valued function of Φ.

$$\frac{1}{2}\left(\frac{dE^2}{d\xi}\right)^2 + V_E(E^2) = 0, \tag{4.41}$$

where,

$$V_E(E^2) = -\frac{32k^2}{\beta_1\beta_2}(b_1 + 2b_2\theta)^2(\theta - \beta_1) \cdot (\theta - \beta_2)\,\theta^2, \tag{4.42}$$

and is the Sagdeev potential for solutions for $E^2(\xi)$. In eq.(4.42), θ is to be determined in terms of E^2 from eq.(4.35), which is a quadratic in θ, and hence gives the two roots θ_1 and θ_2 as,

$$\theta_{1,2} = \frac{1}{2b_2}\left[-b_1 \mp (b_1^2 + b_2E^2)^{1/2}\right]. \tag{4.43}$$

Since b_1 is negative always, both θ_1 and θ_2 are negative when b_2 is negative whereas only θ_1 is negative when b_2 is positive. The condition that $E^2 \leq b_1^2/|b_2|$ ensures the reality of $\theta_{1,2}$ when b_2 is negative. Using eqs.(4.43) in eq.(4.42), we note that V_E is, in general, a double-valued function of E^2, say V_1 and V_2; that is,

$$V_{1,2} = -\frac{32k^2}{\beta_1\beta_2}(b_1 + 2b_2\theta_{1,2})^2(\theta_{1,2} - \beta_1)(\theta_{1,2} - \beta_2)\,\theta_{1,2}^2. \tag{4.44}$$

Using eqs.(4.44), one can now examine $V_{1,2}(E^2)$ for the existence of soliton solutions for $E^2(\xi)$. Let E_m^2 be the maximum of E^2, that is, $E_m^2 = b_1^2/|b_2|$. From eqs.(4.44), it is easy to verify that

$$V_1(E^2)\Big|_{E^2=0} = 0,$$

$$V_2(E^2)\Big|_{E^2=0} = 0, \quad \text{if} \quad b_1 + b_2\beta_2 = 0,$$

$$V_{1,2}(E^2)\Big|_{E^2=E_m^2} = 0. \tag{4.45}$$

Differentiating $V_{1,2}(E^2)$ with respect to E^2 once and using eqs.(4.43), we note also that,

$$\left.\frac{dV_1}{dE^2}\right|_{E^2=0} = 0,$$

$$\left.\frac{dV_2}{dE^2}\right|_{E^2=0} \neq 0,$$

$$\left.\frac{dV_{1,2}}{dE^2}\right|_{E^2=E_m^2} \neq 0, \tag{4.46}$$

Further from eq.(4.43) and (4.44), it follows that $d^2V_1/d(E^2)^2$ is finite at $E^2 = 0$. This, together with the first of eqs.(4.46), guarantees that the point (0,0) in the $E^2 - V_E$ plane is mapped to the points $(\pm\infty, 0)$ in the $\xi - E^2$ plane. On the other hand, since $dV_2/d(E^2) \neq 0$ and $d^2V_2/d(E^2)^2$ is finite at $E^2 = E_0^2$ as well as at $E^2 = E_m^2$, the points $(E_0^2, 0)$ and $(E_m^2, 0)$ in the $E^2 - V_E$ plane are mapped into finite ξ points in the $\xi - E^2$ plane. Hence, the potentials $V_1(E^2)$ and $V_2(E^2)$ behave appropriately for the existence of soliton solutions for $E^2(\xi)$.

By detailed analysis of $V_{1,2}$, we find that when M takes values in the range $0 < M < M_{crit}$, only $V_1(E^2)$ is negative and, therefore, the solution for $E^2(\xi)$ in this range of Mach number values has only a single-hump: when M lies in the range $M_{crit} < M < M_{cut}$, V_2 is also negative, giving rise to double-hump soliton solutions for $E^2(\xi)$ which have non-zero E^2 at $\xi = 0$. The double-hump soliton solutions having $E_0 = 0$ (Nishikawa et al, 1974) correspond, in the present analysis, to the special case when both $V_1(E^2)$ and $V_2(E^2)$ are negative and the second equation in eqs.(4.45) is satisfied. Typical plots of $V_{1,2}(E^2)$ showing the above features of the solution for $E^2(\xi)$ are shown in Figure (8). The usual interpretation of 'particle motion in a potential well' can now be applied to the Sagdeev potentials $V_\Phi(\Phi)$ and $V_{1,2}(E^2)$ (ref. § I.6.3). Thus, these analyses confirm the smooth transition from single-hump Langmuir soliton solutions of Rudakov (1973) and Karpman (1975) to the double-hump Langmuir soliton solutions of Nishikawa et al (1974) as one increases the Mach number in the range $0 < M < 1$.

IV.5.3 Free Parameters: M and E_0

In the above discussion, the mathematical connection between different types of Langmuir soliton solutions obtained for different ranges of values of the Mach number has been brought out by taking the free parameters M and Δ which occurred in the equations of evolution,

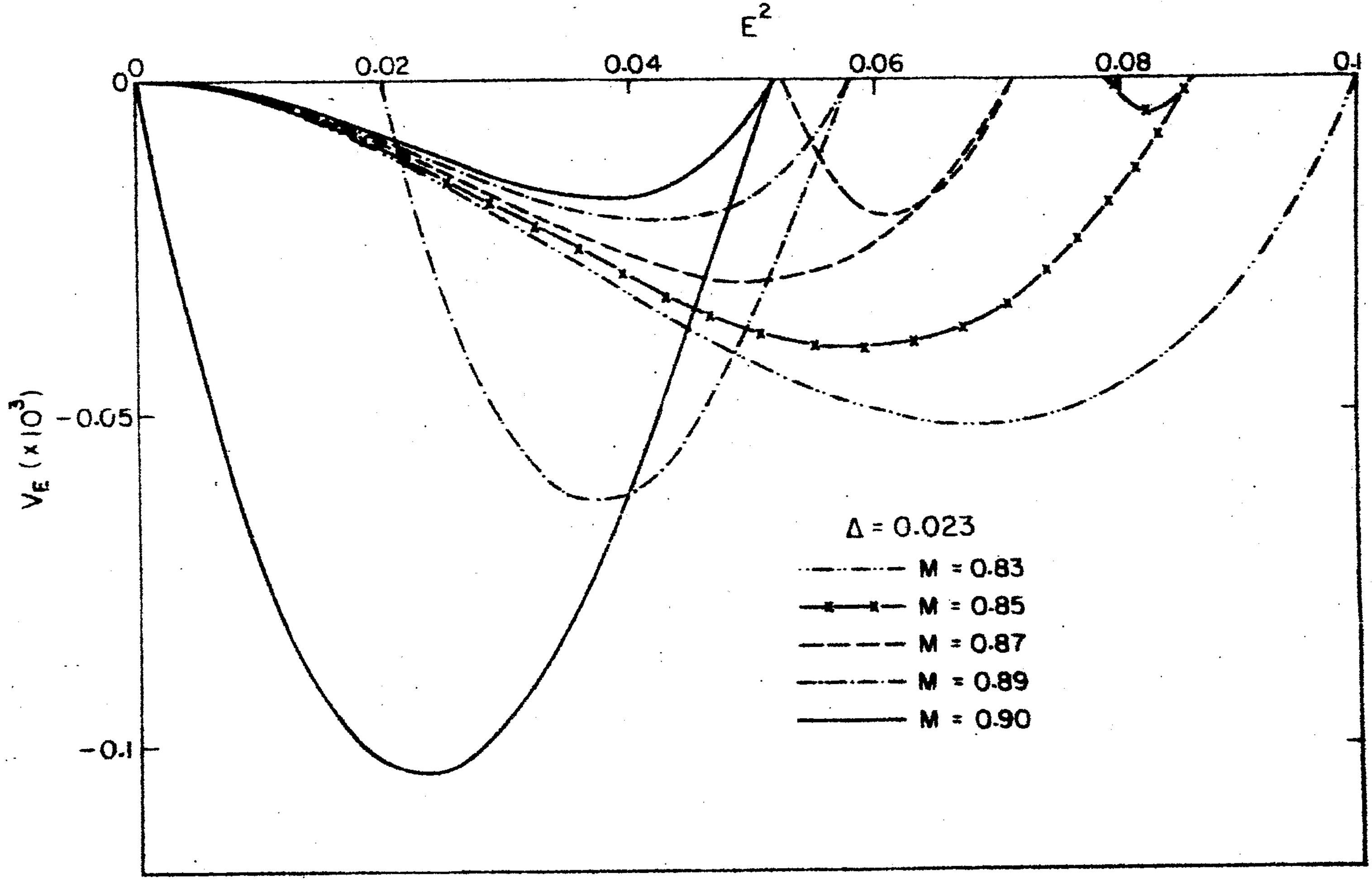

FIGURE (8). Sagdeev potential $V_E(E^2)$ as a function of E^2 (eq.(4.42)) for values of the parameters M & Δ as in Figure (1). Note the existence of two branches of $V_E(E^2)$ for double-hump solutions for $E^2(\xi)$.

namely, eqs. (4.20) and (4.21). In order to obtain a better physical understanding of these solutions, we now take M and E_0 as the free parameters, in which case Δ is no longer a free parameter and is determined by eq. (4.35) at $\xi = 0$, that is, by

$$E_0^2 = 4\left(b_1\theta_0 + b_2\theta_0^2\right), \tag{4.47}$$

where, from eq. (4.36), $\theta_0 = \beta_2(M, \Delta)$. Equation (4.47) then determines Δ for a given set of values of M and E_0.

Figures (9) and (10) show typical plots of finite amplitude Langmuir solitons given by eqs. (4.35) and (4.36) for $E_0^2 = 0.2$ and for different values of the Mach number. For this value of E_0^2, the solutions for $E^2(\xi)$ have single-hump structures when $M < 0.76$. $M = 0.76$ corresponds to the critical Mach number (M_{crit}) for $E_0^2 = 0.2$. For values of the Mach number $M > 0.76$, the solution for $E^2(\xi)$ develops a double-hump structure indicating increased concentration of Langmuir field intensity around $\xi = 0$. The solutions obtained by Nishikawa et al (1974) correspond, in our case, to taking $E_0 = 0$. Solutions of this type are found to exist only in the near-sonic values of the Mach number. Figure (11) shows the plot of these solutions obtained from the present analysis. For a given $E_0 \neq 0$, the corresponding critical Mach number can be calculated from the eqs. (4.37) and (4.47). The values of M_{crit} thus obtained are plotted as a function of E_0^2 in Figure (12).

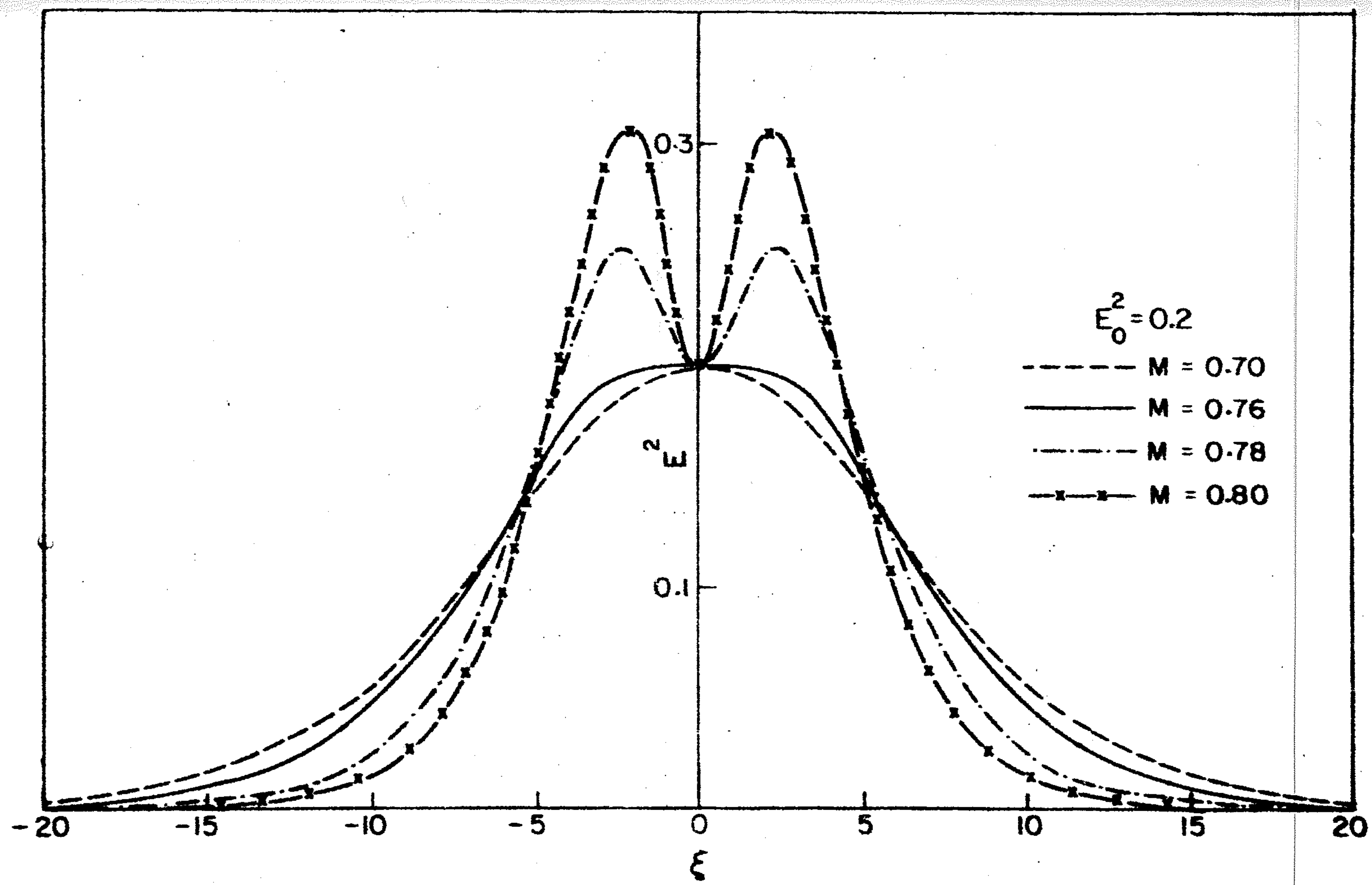

FIGURE (9). Plot of the Langmuir field intensity $E^2(\xi)$ (eq.(4.35)) for a given value of E_0^2 and for different values of the soliton Mach number M. For $E_0^2=0.2$, the transition from single-hump Langmuir solitons to the double-hump Langmuir solitons occurs at M=0.76, the corresponding Critical Mach number M_{crit}.

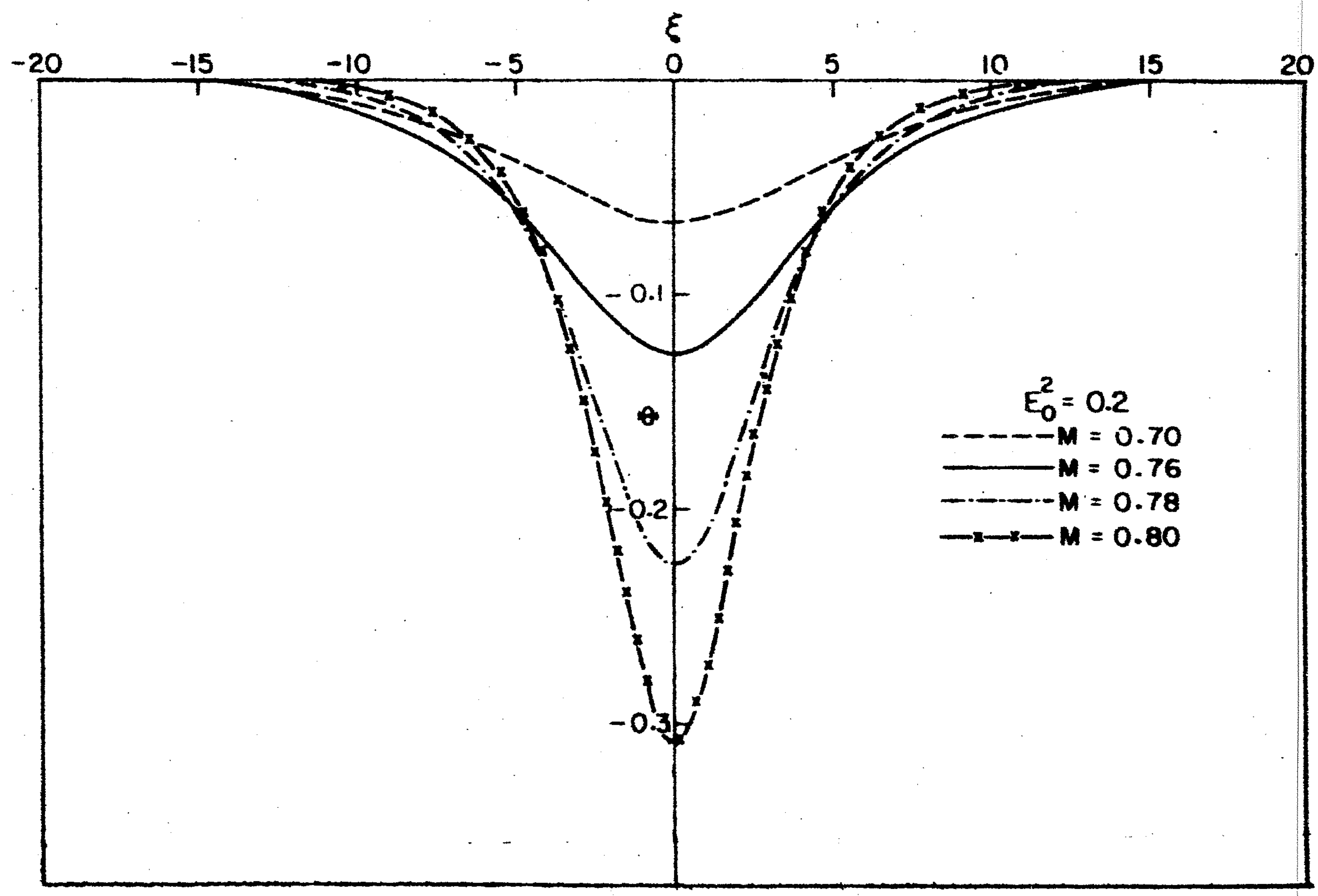

FIGURE (10). Low frequency ion wave potential $\Phi(\xi)$ (eq.(4.36)) for values of the parameters E_0^2 and M corresponding to the solutions plotted in Figure (9).

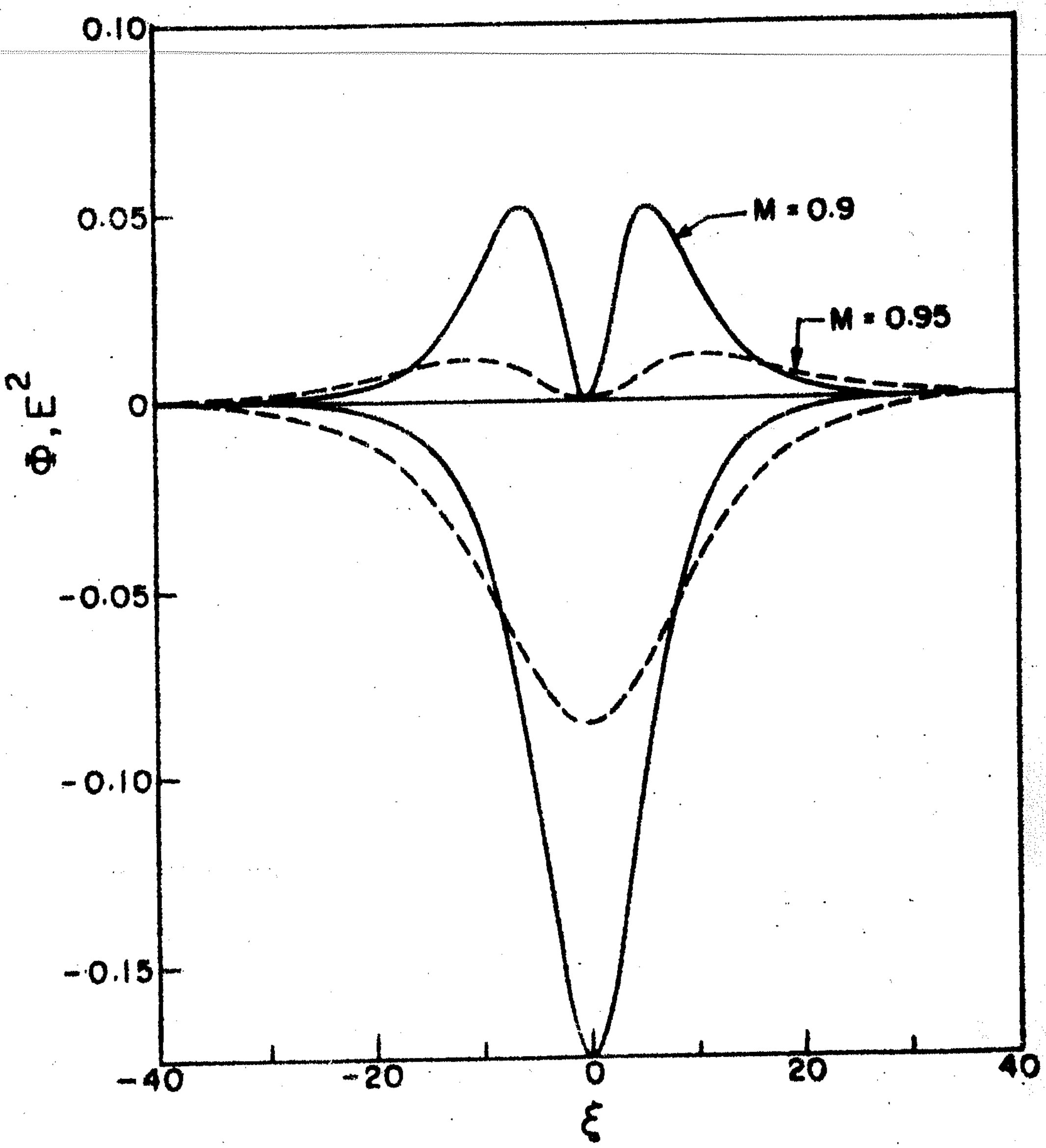

FIGURE (11). The double-hump Langmuir soliton solutions with $E_0^2 = 0$ (Nishikawa et al, 1974) as obtained from the present theory. These solutions exist for values of the parameters M & Δ corresponding to the curve $M=M_{cut}$ in the (M, Δ) parameter space shown in Figure (3).

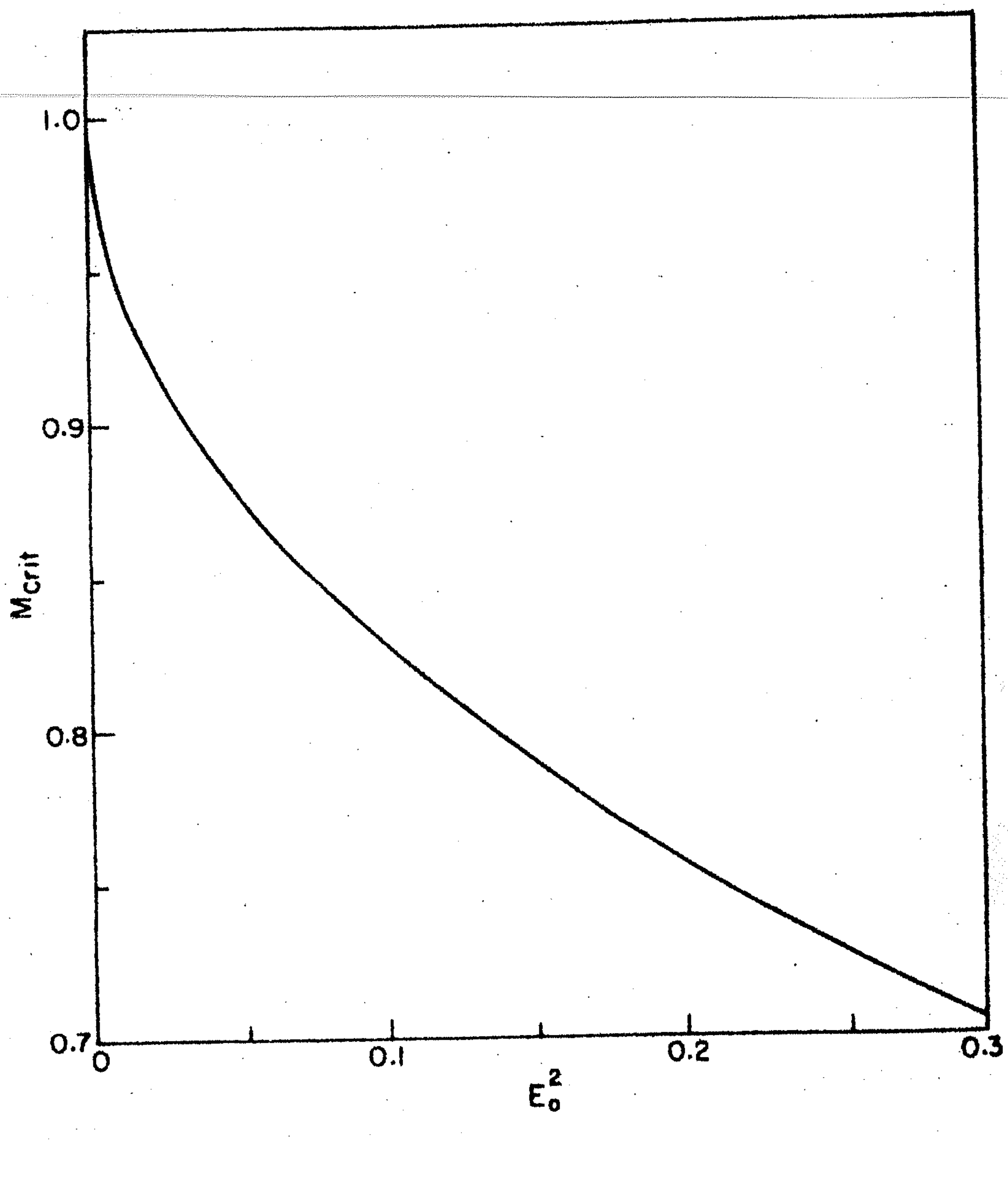

FIGURE (12). Plot of the Critical Mach number M_{crit} as a function of the Langmuir field intensity at the centre, E_0^2 (eqs.(4.37) & (4.47)).

IV.6 Some Limiting Cases

In the previous section, we have considered general, two-term solutions for the equations of evolution (eqs.(4.20) and (4.21)) and have obtained the parameter regions in the (M, Δ) parameter space for the existence of different types of Langmuir soliton solutions. We consider, in this section, some limiting cases and recover explicitly the solutions obtained earlier by other authors (Rudakov, 1973; Karpman, 1975; Nishikawa et al, 1974; Schamel et al, 1977). In all these analyses (barring Schamel et al, 1977), the solution for $\Phi(\xi)$ (or $\Theta(\xi)$) is obtained by taking only the first two terms on the right hand side of eq.(4.30). Taking, therefore, the limit $\alpha_3 \longrightarrow 0$, we obtain from eqs.(4.30), (4.32) and (4.33) the following solution for $\Theta(\xi)$:

$$\Theta(\xi) = -\beta_2' \cdot \mathrm{sech}^2(k\xi), \tag{4.48}$$

where,

$$\beta_2' = \alpha_1'/\alpha_2 \, , \quad \alpha_1' = 3\alpha_1/2 \, . \tag{4.49}$$

IV.6.1 The Linear Case - Zakharov's Equations

The Zakharov equations correspond to the case when only the linear response of the ions in the low frequency waves is considered. These equations which have been used to obtain the Rudakov soliton and the Karpman soliton solutions can be obtained from the present theory by retaining only the first term in the expansion (4.34), so that,

$$E^2 = 4 b_1 \theta . \qquad (4.50)$$

Substituting the expression for b_1 given by eq. (4.29) in eq. (4.50) and using the resulting expression for E^2 in eq. (4.10), we obtain, in the linear approximation,

$$\delta n_e = - \frac{E^2}{4(1-M^2)} . \qquad (4.51)$$

Equations (4.12) and (4.51), then, constitute the set of equations first obtained by Zakharov (1972) for a stationary solution moving with a Mach number M. Since δn_e is negative and finite for a Langmuir soliton, it follows from eq. (4.51) that $M^2 < 1$. Equations (4.12) and (4.51) give, on integration, the near-static Rudakov soliton for $M << 1$ and the near-sonic Karpman soliton for $M \lesssim 1$. Both these solutions have single-hump structures for the Langmuir field intensity $E^2(\xi)$ with a corresponding depression in the density given by eq. (4.51). As discussed in § IV.5.1, such single-hump solutions for $E^2(\xi)$ exist in our analysis for all values of the Mach number in the range $0 < M \lesssim M_{crit}$ (ref. Figure (3)).

IV.6.2 Weakly Nonlinear Case

Zakharov's analysis assumes charge-neutrality for the low frequency motion where the ion motion is governed by the linear wave equation. The importance of ion nonlinearity together with dispersion of the low frequency ion waves (through Poisson's equation) in the dynamics of Langmuir solitons for near-sonic velocities was first shown frequency Langmuir waves is governed, as usual, by the Schrodinger-like equation (4.12), the low frequency ion motion is described by a weakly nonlinear equation of the K-dV type which is driven by the ponderomotive force due to Langmuir waves. Very much unlike the near-sonic solutions of Karpman (1975), solutions of these equations have a node at the centre for Langmuir field amplitude and, therefore, a double-hump structure (with $E_0 = 0$) for the plasmon number density. The low frequency ion wave potential, however, has only one bottom-well structure as in the linear case. As discussed earlier, these solutions are obtained in the present theory when the parameters M and Δ take values corresponding to the curve $M = M_{cut}$ in the (M, Δ) parameter space shown in Figure (3).

To recover explicitly the above solutions from our analyses, we consider the solutions (4.35) and (4.48), namely,

$$E^2(\xi) = 4\left(b_1\theta + b_2\theta^2\right), \tag{4.35'}$$

$$\theta(\xi) = -\beta_2'\cdot \operatorname{sech}^2(k\xi), \tag{4.48'}$$

and, take the free parameter E_0 to be equatl to zero. Equation (4.35)' then yields,

$$b_1 + b_2\theta_0 = 0, \tag{4.52}$$

where θ_0 is now given by eq. (4.48)' at $\xi = 0$:

$$\theta_0 = -\beta_2'(M, \Delta). \tag{4.53}$$

Using eq. (4.48)' in eq. (4.35)' along with eqs. (4.49), (4.52) and (4.53), we obtain the following solutions for $E(\xi)$ and $\Phi(\xi)$:

$$E(\xi) = \pm(-6\alpha_1 b_1/\alpha_2)^{1/2} \cdot \operatorname{sech}(k\xi) \cdot \tanh(k\xi), \quad (4.54)$$

$$\Phi(\xi) = -(3M^2\alpha_1/2\alpha_2) \cdot \operatorname{sech}^2(k\xi), \quad (4.55)$$

where k is given by the second of eqs.(4.33). Solutions (4.54) and (4.55) have the same functional form as the solutions obtained by Nishikawa et al (1974). It may be noted here that the solutions (4.54) and (4.55) have only one free parameter which can be either the Mach number M or the nonlinear frequency shift Δ. The other parameter can, then, be determined through eqs.(4.52) and (4.53).

IV.6.3 Finite Amplitude, Quasi-Neutral Case

Schamel et al (1977) have considered finite amplitude Langmuir waves together with the full ion nonlinearity in the low frequency ion waves. However, these authors assume charge neutrality for the low frequency response to close their system of equations (that is, $\delta n_i' = \delta n_e'$, in the notation of eqs.(4.4)). Even though they recover the single-hump Langmuir solitons of Rudakov (1973) and Karpman (1975) in the small amplitude limit, they have not discussed the form of their solutions for finite amplitude waves but have, instead, obtained a set of 'existence relations' through Sagdeev potential analysis. While the assumption of charge neutrality is not quite consistent with taking full ion nonlinearity into account, the 'existence relations' obtained by these authors are not valid, as we shall show below, for intermediate as well as for near-sonic values of the Mach number in the range, $0 \leq M < 1$.

Since the evolution equations (4.20) and (4.21) derived in the present theory are fully nonlinear in the ion motion and, further, take into account completely the space charge effects in the low frequency response, we can obtain the 'existence relations' of Schamel et al (1977) by forcing the assumption of charge neutrality for the low frequency ion waves. This assumption implies, through the eq. (4.20), the following relation between E^2 and Φ :

$$M\left(M^2-2\Phi\right)^{-1/2}=\exp\left(\Phi-E^2/4\right) \tag{4.56}$$

If, because of charge neutrality, $n_i(0) = n_e(0) = N$, be the common ion and electron number density at $\xi = 0$, then, from eq. (4.56), we obtain

$$M\left(M^2-2\Phi_0\right)^{-1/2}=\exp\left(\Phi_0-E_0^2/4\right)=N. \tag{4.57}$$

Also, from the 'energy integral' given by eq. (4.24), we get

$$1+M^2=M\left(M^2-2\Phi_0\right)^{1/2}-\frac{1}{4}(\lambda-1)E_0^2 + \exp\left(\Phi_0-E_0^2/4\right), \tag{4.58}$$

since, $dE/d\xi = d\Phi/d\xi = 0$ at $\xi = 0$. From eqs. (4.57) and (4.58), we easily obtain the following equations for λ and M^2 in terms of E_0^2 and N :

$$M^2=2\left(\ln N+E_0^2/4\right)\left(1-N^{-2}\right)^{-1}, \tag{4.59}$$

$$\lambda = 1+(N-1)\left(1-M^2/N\right)/\left(E_0^2/4\right). \tag{4.60}$$

Equations (4.59) and (4.60) are the 'existence relations' derived by Schamel et al (1977).

We now examine the validity of the existence relations (4.59) and (4.60) with respect to the Mach number. Equation (4.56) yields, on expansion, the following expression for E^2 in terms of Φ :

$$\frac{E^2}{4} = (M^2-1)\left(\frac{\Phi}{M^2}\right) + \frac{1}{2}(M^4-3)\left(\frac{\Phi}{M^2}\right)^2 + \cdots, \qquad (4.61)$$

This solution can, then, be compared with our solution for E^2 given by eq. (4.28), namely,

$$\frac{E^2}{4} = \left[(M^2-1) + \left\{-\frac{4}{3}M^2\left(2\Delta + \epsilon^2 M^2/3\right)\right\}\right]\left(\frac{\Phi}{M^2}\right) + b_2\left(\frac{\Phi}{M^2}\right)^2 + \cdots, \qquad (4.62)$$

where, in obtaining eq. (4.62), we have made use of the expression for b_0 and b_1 given by eqs. (4.29). Obviously, the solution (4.61) departs from the general solution (4.62) already in the linear term; only in the limit $M \ll 1$ does eq. (4.62) reduce to eq. (4.61) while for larger values of M, the additional term being proportional to $(8\Delta/3)M^2$ can be quite large. Thus, the existence relations (4.59) and (4.60) obtained by Schamel et al (1977) are not valid for intermediate as well as near-sonic values of the Mach number in the range $0 < M < 1$. On the other hand, the explicit solution for $E^2(\xi)$ and $\Phi(\xi)$ given respectively by eqs. (4.35) and (4.36) are valid, as discussed in § IV.5, in the range of the Mach number $0 < M < M_{..}$

IV.7 Langmuir Wave and Ion Wave Energy

In this section, we evaluate the total energy associated with the high frequency Langmuir waves and the low frequency ion waves which are defined, respectively, by

$$N_P = \int_{-\infty}^{+\infty} E^2(\xi)\, d\xi, \tag{4.63}$$

$$N_\Phi = \int_{-\infty}^{+\infty} \Phi(\xi)\, d\xi, \tag{4.64}$$

where $E^2(\xi)$ and $\Phi(\xi)$ are given by eqs.(4.35) and (4.36). The quantity N_p is sometimes called the total plasmon number, being the total number of quasi-particles associated with the Langmuir field oscillations.

Substituting for $E^2(\xi)$ and $\Phi(\xi)$ from eqs.(4.35) and (4.36) in eqs.(4.63) and (4.64), the integrals in the latter equations can be reduced to the form (Rao & Varma, 1981)

$$N_P = -2k^{-1}\beta_1|\beta_2|^{1/2}\cdot\left[2b_1 + (\beta_1 - |\beta_2|)b_2\right] J^{(1)} + 4b_2\beta_1|\beta_2|/k, \tag{4.65}$$

$$N_\Phi = -M^2\beta_1|\beta_2|^{1/2} J^{(1)}/k, \tag{4.66}$$

where $J^{(1)}$ is the integral defined by,

$$J^{(1)} = \int_{\beta_1}^{\beta_1+|\beta_2|} \left[\eta^2(\eta-\beta_1)\right]^{-1/2} \cdot d\eta , \tag{4.67}$$

and can be evaluated explicitly as,

$$J^{(1)} = \begin{cases} 4\beta_1^{-1/2}\left[\tan^{-1}\left\{\exp\left(\cosh^{-1}\left(1+|\beta_2| \cdot \beta_1^{-1}\right)^{1/2}\right)\right\} - \frac{\pi}{4}\right]; \ \beta_1 > 0, \\ -2|\beta_1|^{-1/2} \cdot \ln\left[\dfrac{|\beta_1|^{1/2}+|\beta_2|^{1/2}}{|\beta_1|^{1/2}-|\beta_2|^{1/2}}\right]; \ \beta_1 < 0. \end{cases} \tag{4.68}$$

Figures (13) and (14) show the plot of N_p and N_Φ (obtained from eqs. (4.65) - (4.68)) as functions of Mach number (M) for different values of the nonlinear frequency shift (Δ). These figures should be compared with Figures (4) and (5) where the corresponding energy densities are plotted. It follows from these figures that the Langmuir solitons with larger Langmuir (ion) wave amplitudes have larger total Langmuir (ion) field energy for the same values of M and Δ. In Figure (15), we plot N_p as a function of M for different values of N_Φ. Clearly, for a given N_Φ, the Langmuir field energy decreases with the increase in the Mach number. Thus, if one were to consider the Langmuir soliton as a "particle", then, the total energy associated with the Langmuir

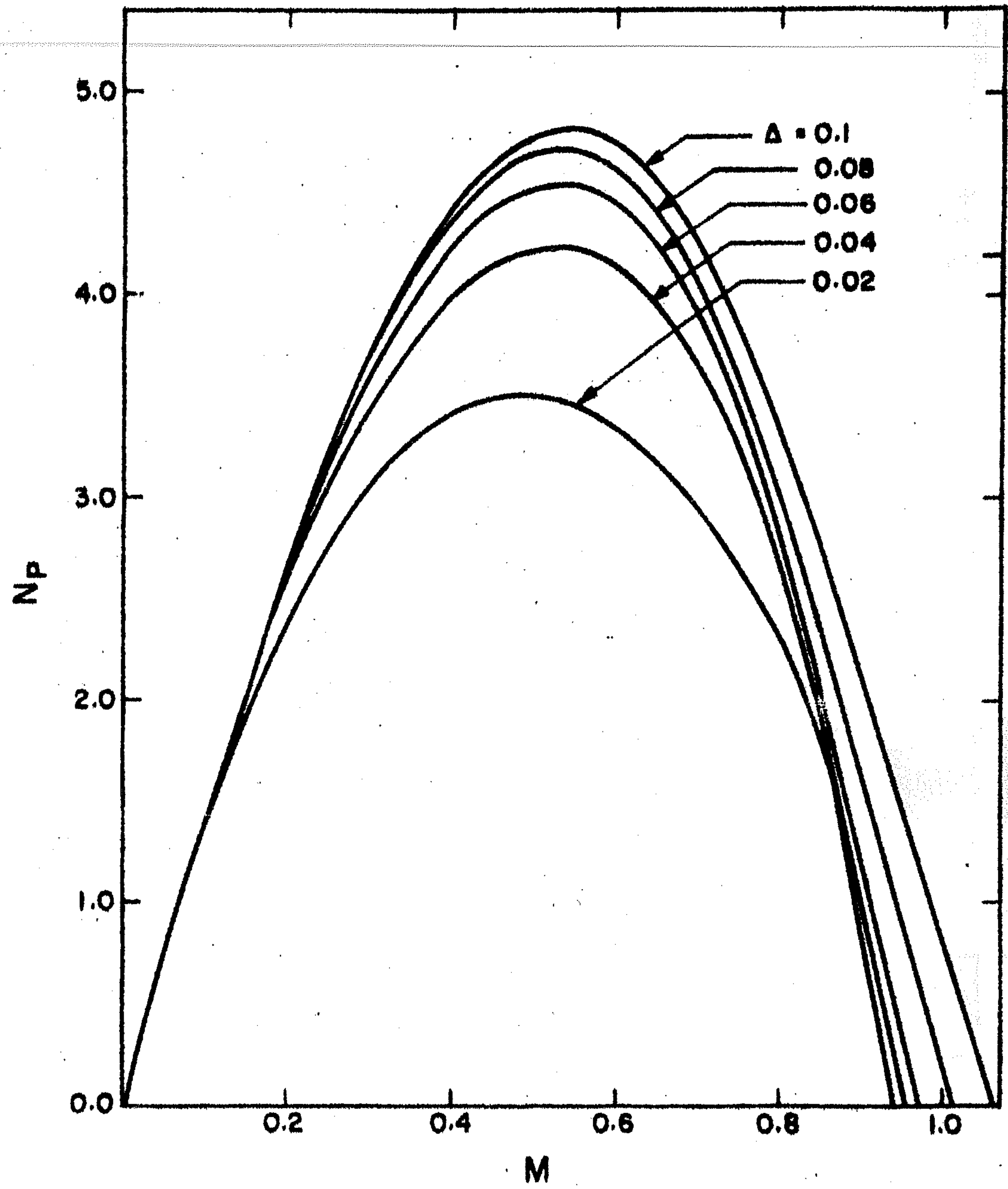

FIGURE (13). Total plasmon number N_p (eq.(4.65)) as a function of the soliton Mach number M for different values of the non-linear frequency shift Δ.

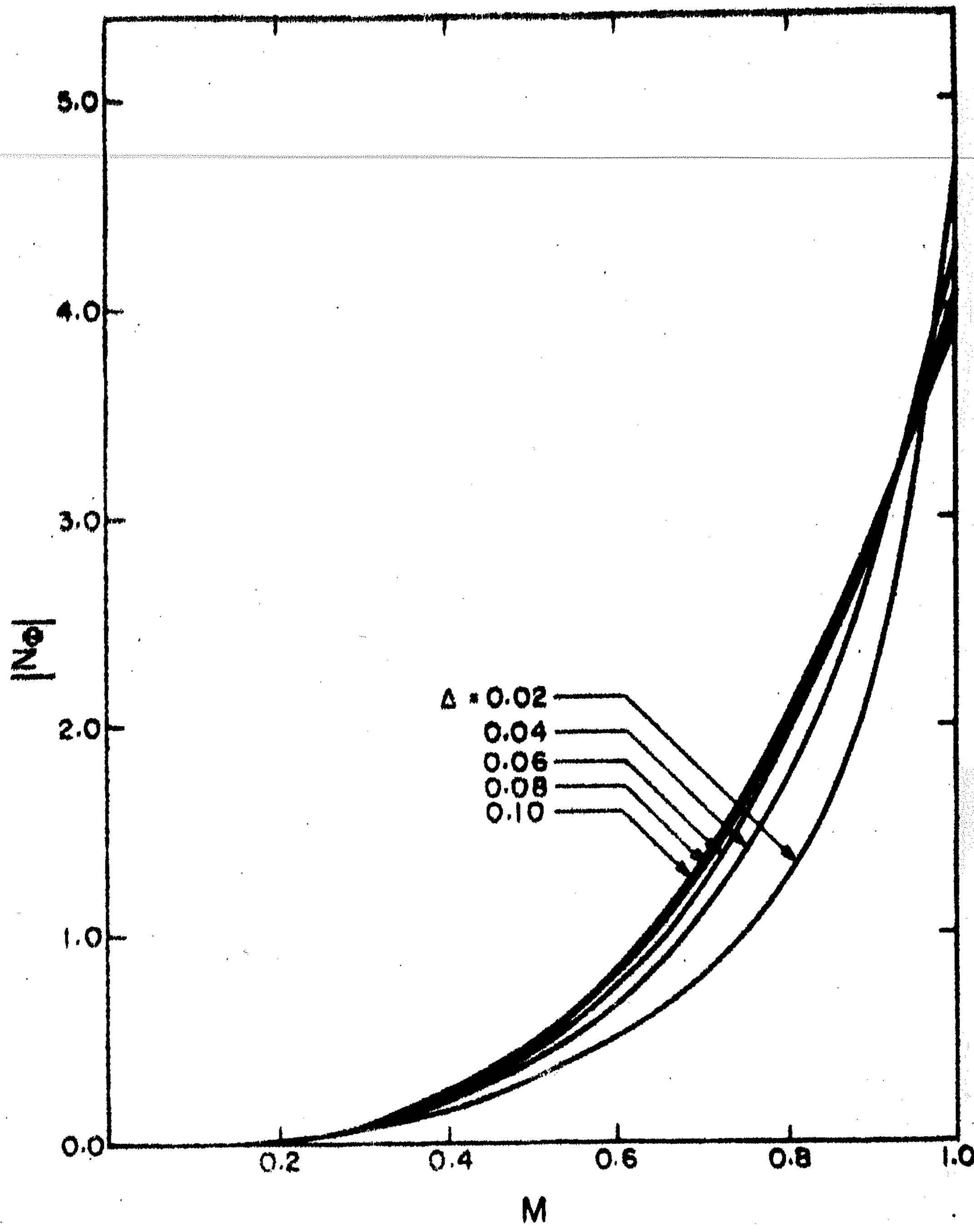

FIGURE (14). Total ion field energy N_Φ (eq.(4.66)) for values of the parameters M and Δ as in Figure (13).

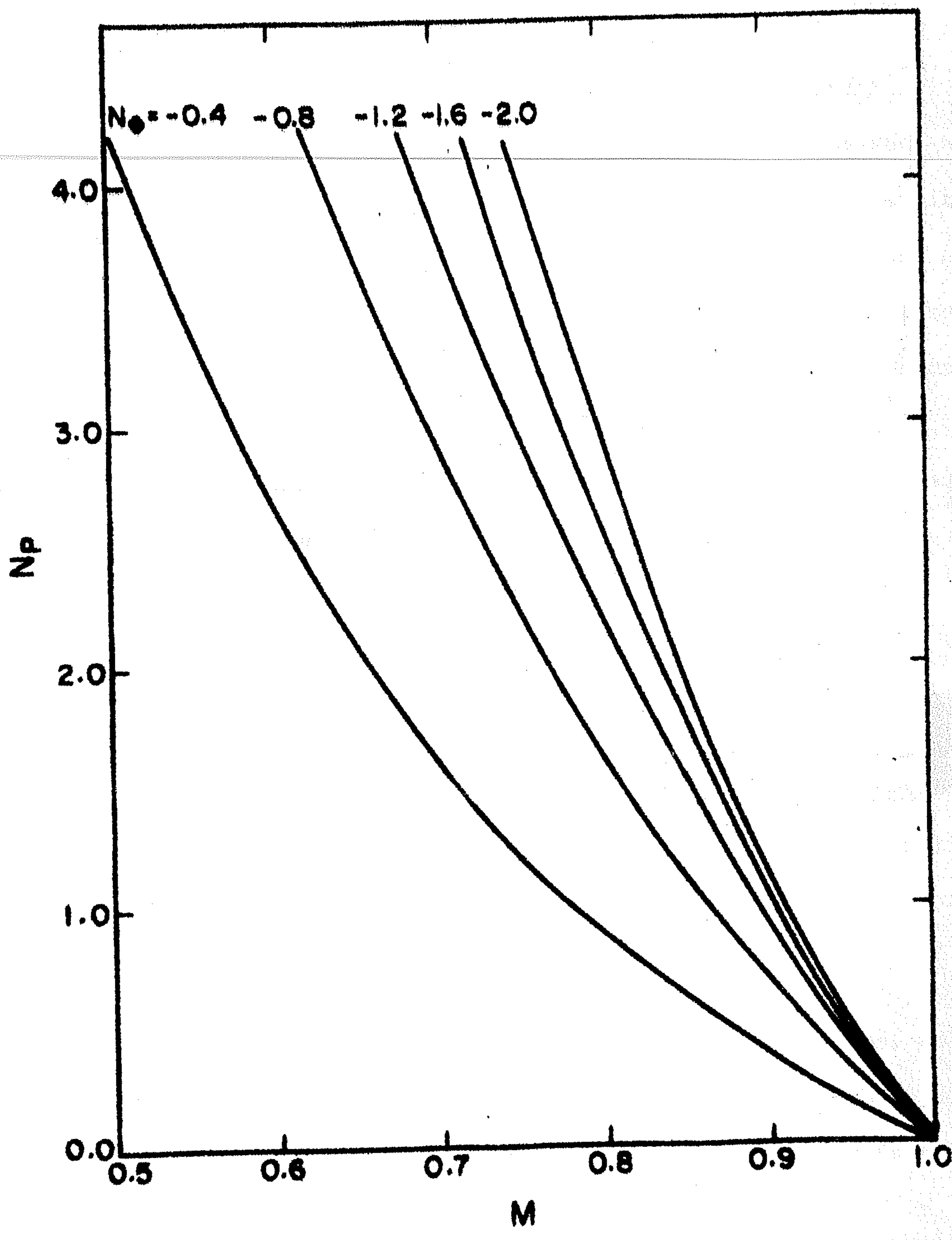

FIGURE (15). Plot of the total plasmon number N_p as a function of the soliton Mach number M for different, given values of the total ion field energy N_Φ.

field oscillations can be identified as the "mass" of the "particle". Figure (16) shows the variation of $|N_\Phi|$ with respect to the Mach number when N_P is kept constant. It follows from this figure that the solitons with larger Mach number have greater total ion wave energy for same amount of Langmuir field energy. Similar results are obtained from the Zakharov equations also (see, for instance, Gibbons et al, 1977). We conclude this section with Figures (17) and (18) wherein lines of constant N_P and N_Φ are respectively plotted in the (M, Δ) parameter space. The corresponding critical and cut-off Mach numbers are also plotted in these figures.

IV.8 Summary and Conclusions

We have developed, in this Chapter, a theory for the nonlinear, amplitude modulated Langmuir waves and the associated ion acoustic waves. The significant features and the main results of our investigation of the problem can be summarized as follows:

(a) A set of governing equations for the Langmuir solitons, valid in the entire range of the Mach number, namely, $0 \leq M < 1$ has been derived without using the charge neutrality condition for the low frequency ion waves and any a priori ordering schemes. An analytic method is, then, developed for solving these coupled set of nonlinear equations. The method is capable of taking into account any arbitrary degree of ion nonlinearity consistent with the nonlinearity retained in the Langmuir field amplitude.

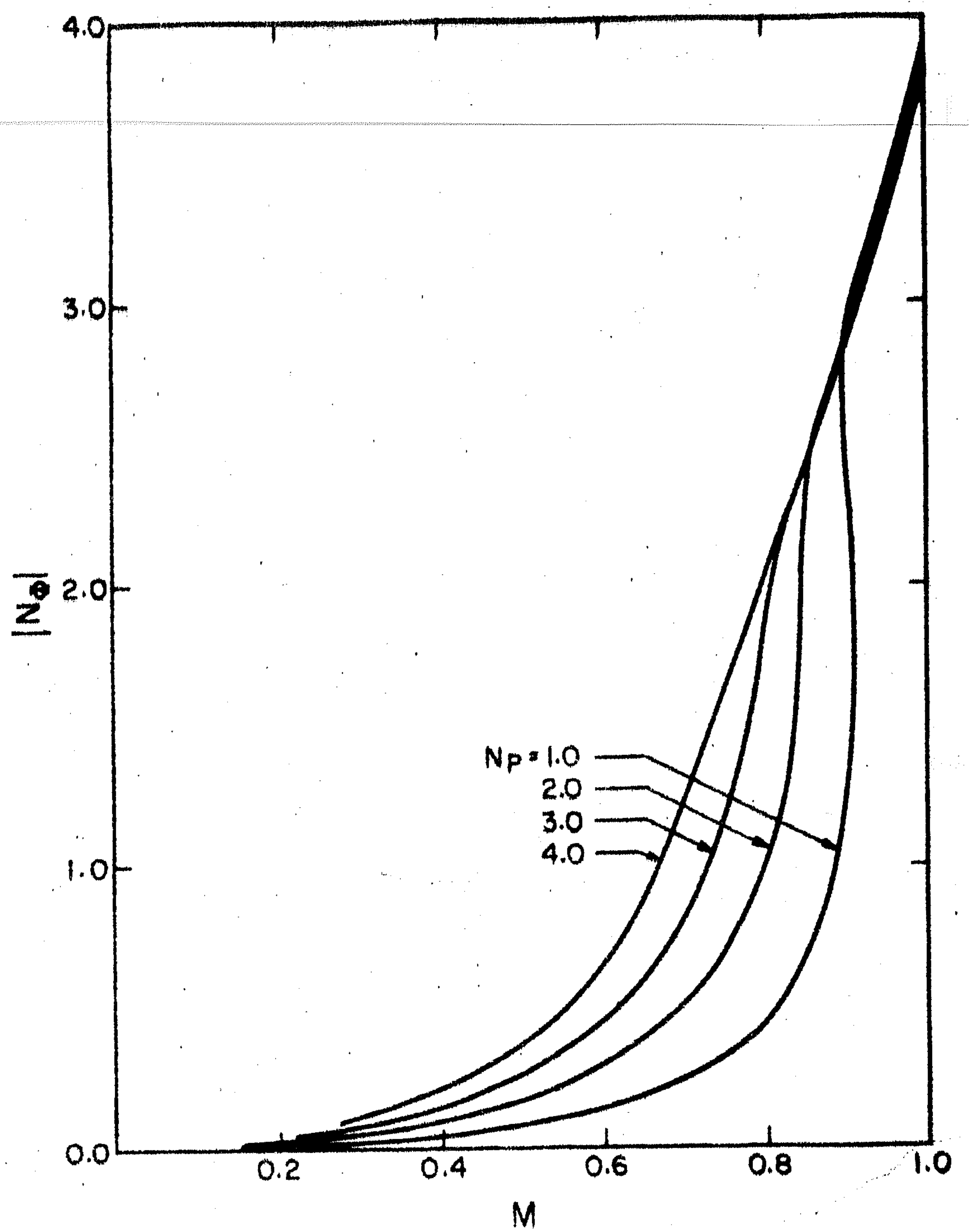

FIGURE (16). Plot of the ion field energy N_{Φ} as a function of the soliton Mach number M for different, given values of the total plasmon number N_p .

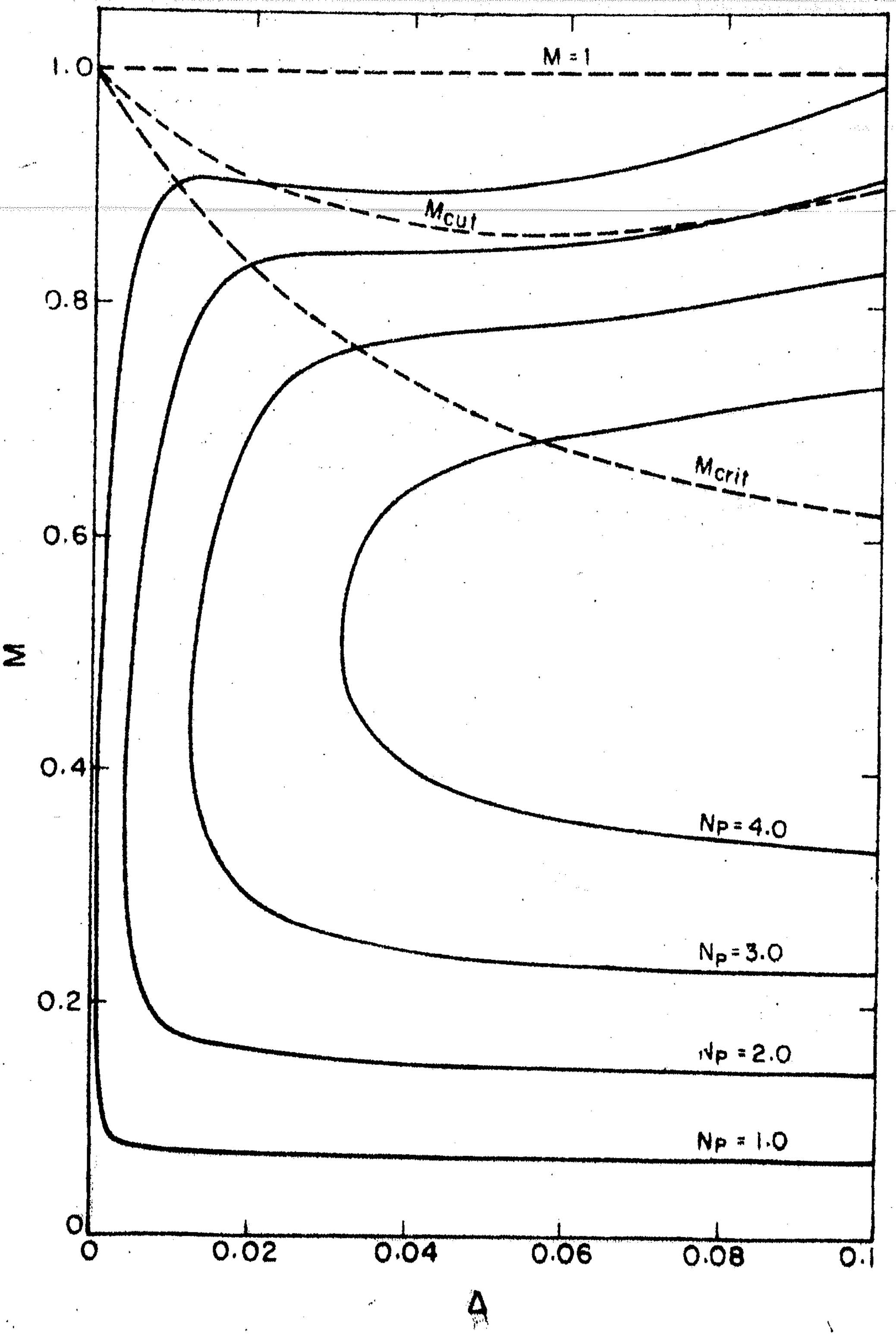

FIGURE (17). Constant plasmon number trajectories N_p in the (M, Δ) parameter space. Corresponding Critical and Cut-off Mach numbers (M_{crit} and M_{cut} respectively) are also plotted.

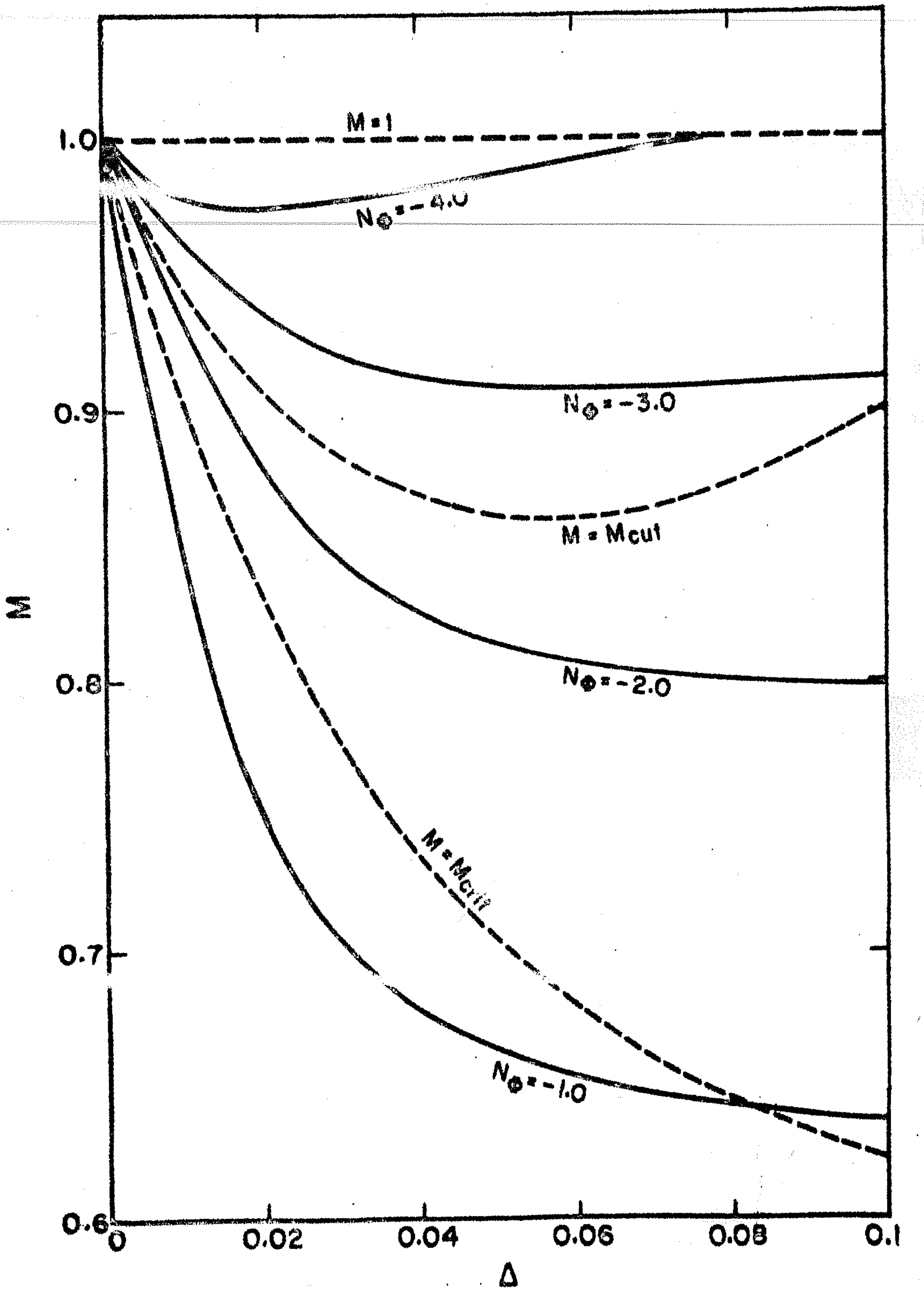

FIGURE (18). Constant ion field energy trajectories N_{Φ} in the (M, Δ) parameter space. Corresponding Critical and Cut-off Mach numbers (M_{crit} and M_{cut} , respectively) are

(b) A class of double-hump Langmuir soliton solutions having non-zero Langmuir field intensity at the centre of the solitons is found for a range of values of the Mach number M and the nonlinear frequency shift Δ. Using these solutions, a smooth transition from single-hump soliton solutions of Rudakov (1973) and Karpman (1975) to the double-hump soliton solutions of Nishikawa et al (1974) has been established with respect to the Mach number. The regions of parameter values for the existence of different types of soliton solutions are explicitly obtained in the (M, Δ) parameter space. The theory developed here yields, under appropriate limiting conditions, various soliton solutions obtained earlier by other authors.

(c) The existence of soliton solutions for the Langmuir field amplitude as well as for the ion wave has been discussed through Sagdeev potential analyses. These analyses, further, confirm the smooth transition from single-hump Langmuir solitons to the double-hump Langmuir solitons with respect to the Mach number.

(d) The discussion of the existence and structure of Langmuir soliton solutions presented above has been carried out by considering terms up to second order in the low frequency ion wave potential (ref. § IV.5). However, this can be extended so as to incorporate higher order nonlinearities in the ion wave potential. In this connection, a conjecture is made about the existence of many-hump Langmuir solitons corresponding to higher order nonlinearities in the low frequency ion dynamics.

APPENDIX

We give here the first four coefficients b_1, b_2, b_3 and b_4 in the expansion (4.28):

$$b_1 = \alpha + M^2, \tag{A.1}$$

where, as defined in the text,

$$\alpha = -1 - 4\lambda M^2/3\,, \quad \lambda = 2\Delta + \epsilon^2 M^2/3\,. \tag{A.2}$$

Also,

$$b_2 = \frac{F_2}{F_1}\,, \quad b_3 = \frac{F_4}{F_3}\,, \quad b_4 = \frac{F_6}{F_5}\,, \tag{A.3}$$

where

$$F_1 = 9b_1 - 36\lambda M^2,$$

$$F_2 = -\frac{9}{2} b_1 (3 - \alpha^2) + 3M^2(\alpha^2 - M^2) + 6M^2 \alpha b_1 + 6M^4(1+\alpha) + 4\lambda M^6,$$

$$F_3 = 9b_1^2 - 96\lambda M^2 b_1,$$

$$F_4 = -24M^2(b_2 B_2 - \lambda b_2^2) - 9\left[b_1^2\left(\frac{5}{2} - \alpha b_2 + \frac{1}{6}\alpha^3\right) + 6b_1 b_2\left(\frac{3}{2} - A_2\right) + 12 b_2^2 (1+\alpha)\right]$$

$$+6M^2\left[b_1\left(-\frac{M^2}{2}+\alpha b_2-\frac{\alpha^3}{6}\right)+4b_2B_2-4\lambda B_2^2\right]$$

$$-6M^2\left[b_1(b_1A_2+b_2A_1)+4b_2(b_1A_1+b_2\lambda)+4\lambda b_2^2\right]$$

$$+6M^4\left[b_1\left(\frac{3}{2}-A_2\right)+3b_2(1+\alpha)\right]$$

$$+4\left(b_1A_1+2\lambda b_2\right),$$

$$F_5=9b_1^3-180\lambda M^2b_1^2,$$

$$F_6=-24M^2\Big[3b_3(b_1B_2+b_2B_1)$$

$$+b_2(b_1B_3+b_2B_2+b_3B_1)\Big]$$

$$-9\Big[b_1^3\left\{\frac{35}{8}-\frac{1}{2}(b_2^2+2\alpha b_3)+\frac{\alpha^2}{2}b_2-\frac{\alpha^4}{24}\right\}$$

$$+6b_1^2b_2\left(\frac{5}{2}-A_3\right)+(12b_1b_2^2+9b_1^2b_3)$$

$$\cdot\left(\frac{3}{2}-A_2\right)+(1+\alpha)(36b_1b_2b_3+8b_2^3)\Big]$$

$$+6M^2\Big[b_1^2\Big\{-\frac{5}{8}M^2+\frac{1}{2}(b_2^2+2\alpha b_3)-\frac{\alpha^2}{2}b_2$$

$$+\frac{1}{24}\alpha^4\Big\}+4b_1b_2B_3+12B_1b_2b_3$$

$$+B_2(4b_2^2+6b_1b_3)\Big]$$

$$-6M^2\Big[b_1^2(b_1A_3+b_2A_2+b_3A_1)$$
$$+4b_1b_2(b_1A_2+b_2A_1+b_3\lambda)$$
$$+(4b_2^2+6b_1b_3)(b_1A_1+b_2\lambda)+12\lambda b_1b_2b_3\Big]$$
$$+6M^4\Big[b_1^2\left(\frac{5}{2}-A_3\right)+3b_1b_2\left(\frac{3}{2}-A_2\right)$$
$$+(1+\alpha)(2b_2^2+4b_1b_3)\Big]$$
$$+4\Big[b_1^2A_2+2b_1b_2A_1+\lambda(b_2^2+2b_1b_3)\Big], \qquad \text{(A.4)}$$

where the following notations are used:

$$A_1=-\alpha,$$
$$A_2=-b_2+\frac{1}{2}\alpha^2,$$
$$A_3=-b_3+\alpha b_2-\frac{1}{6}\alpha^3,$$
$$A_4=-b_4+\frac{1}{2}(b_2^2+2\alpha b_3)-\frac{1}{2}\alpha^2b_2+\frac{1}{24}\alpha^4,$$
$$B_1=-\lambda b_1,$$
$$B_2=-\frac{1}{2}M^2-\lambda b_2+\frac{1}{2}\alpha^2,$$

$$B_3 = -\frac{1}{2} M^2 - \lambda b_3 + \alpha b_2 - \frac{1}{6}\alpha^3,$$

$$B_4 = -\frac{5}{8} M^2 - \lambda b_4 + \frac{1}{2}(b_2^2 + 2\alpha b_3) - \frac{1}{2}\alpha^2 b_2 + \frac{1}{24}\alpha^4.$$

Similarly, higher order coefficients can be obtained. Then, for any given set of values of the parameters M and Δ, all the coefficients b_n (or a_n) can be evaluated explicitly.

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