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SOLITARY WAVES IN PLASMAS

BY

MANNIL MOHAN

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To my uncle

M. ARAVINDAKSHA MENON

CONTENTS

CERTIFICATE

I hereby declare that the work presented in this thesis is original and has not formed the basis for the award of any degree or diploma by any University or institution.

M. MOHAN
(AUTHOR)

Certified by:

B. BUTI

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ABSTRACT

In this thesis we investigate the formation and evolution of solitary waves in laboratory and space plasmas under various situations. The electrostatic drift waves in a highly dispersive magnetized plasma can undergo non-linear self modulation and give rise to envelope solitons or envelope holes. The dependence of the modulational stability of these waves on the wave number, the angle of propagation, the magnetic field and density gradient is studied. In an inhomogeneous plasma an ion acoustic envelope soliton with width of the order of the density gradient scale length is found to split into two envelope solitons; one of which damps afterwards. But if the width is very small,

after a very long interval of time it attains a large amplitude and splits into many envelope solitons. An envelope hole, on the other hand develops two soliton-like humps whereas a periodic modulation develops a spectrum of envelope waves. A magnetized plasma with hot ions and cold electrons can sustain electron acoustic waves. These waves in the limit of weak nonlinearity and dispersion can give rise to solitary waves which undergo ion Landau damping or growth depending on whether they are moving faster or slower than the electron drift in the medium. Along with this damping or growth they develop tails. The rate of these processes with respect to changes in the angle of propagation and electron to ion temperature ratio are also studied. In the absence of ion Landau damping the electron acoustic waves can form finite amplitude solitary waves with an upper limit on the Mach number. The coupled ion acoustic-Langmuir solitons in a partially ionized plasma are found to be affected by collisional and viscous dissipations. The result of electron-ion and electron-neutral collisions is to make the Langmuir field to damp and let the ion density perturbations to radiate away. Whereas the ion-neutral collisions damp the ion density perturbations and let the Langmuir field to flow out. In the case of ion viscosity the ion density perturbations start radiating away followed by the Langmuir field.

Among these various processes, the effect of electron-ion and electron-neutral collisions is found to be stronger than the other two.

CHAPTER I

INTRODUCTION

Wave motion is so widespread in nature that the importance of its study can hardly be overemphasised. Waves in liquids, elastic waves, electromagnetic waves, waves in plasmas etc. are only a few of the representatives of an extensive family of waves in continuous media. The linear properties of most of these waves have been studied in great detail. But if the amplitude of a wave becomes very large then one has to deal with its nonlinear properties. Many interesting physical phenomena have been discovered in nonlinear media in recent years. This includes the effect of harmonic generation, self focussing and self contraction of wave packets, stimulated scattering, turbulence in plasmas, formation of solitary waves, anomalous heating, anomalous transport etc.

Here we present an investigation of some nonlinear wave phenomena in plasmas under different configurations. Plasmas can sustain a rich variety of waves many of which are unstable and grow to become nonlinear. One can explain the nonlinear development of a wave as follows:

Let us consider wave motion governed by the nonlinear equation,

$$\frac{\partial \phi}{\partial t} + C(\phi)\frac{\partial \phi}{\partial x} = 0, \tag{1.1}$$

where ϕ represents one of the fluctuating variables (density, electric field etc.) due to the wave. If the wave amplitude is not very large then this equation can be linearised as,

$$\frac{\partial \phi}{\partial t} + C(\phi_0)\frac{\partial \phi}{\partial x} = 0, \tag{1.2}$$

where ϕ_0 is the average equilibrium value of ϕ. Eq. (1.2) can have periodic wave solutions of the form,

$$\phi = a \sin(kx - \omega t) \tag{1.3}$$

with $a\ (\ll 1)$ as its amplitude and ω and k as its frequency and wave number. Substituting (1.3) into

(1.2) we get the linear dispersion relation of the wave,

$$\omega = C(\phi_0)k \tag{1.4}$$

The quantity $(\omega/k) = C(\phi_0)$ is called the phase velocity and the wave as a whole moves with this velocity without any change in its shape. But if the wave amplitude is large then the linear dispersion relation is not valid. Then to solve Eq. (1.1), we define (Witham, 1974)

$$C(\phi) = \frac{dx}{dt}, \tag{1.5}$$

and rewrite it as,

$$\frac{\partial \phi}{\partial t} + \frac{dx}{dt}\frac{\partial \phi}{\partial x} = \frac{d\phi}{dt} = 0, \tag{1.6}$$

which means that if we take (1.3) as the form of the wave at t = 0 (with the amplitude 'a' no longer very small), different portions of the wave will move with different constant velocities. For example, if we take $C(\phi) = \phi$, then the crest of the wave moves faster than its trough. This deforms the shape and leads to the steepening and the ultimate breaking of the wave. This is the nonlinear development of a wave described by Eq. (1.1). But if the system is dispersive then this equation will have higher order space derivative terms due to which the nonlinear steepening of the wave can be arrested. The presence of such terms will alter the linear dispersion relation (1.4); this in turn may lead to the relation,

$$\frac{\partial^2 \omega}{\partial k^2} \neq 0 \qquad (1.7)$$

Waves with this property are called dispersive (Witham, 1974) waves. In such cases (ω / k) will no longer be a constant and for an arbitrary initial perturbation, its different Fourier components will move with different phase velocities. Consequently the wave will spread out in space. This effect is opposite to that of nonlinear steepening. So if a wave motion is characterised by both nonlinearity and dispersion, these two effects can be balanced with each other.

To illustrate this, let us consider the equation,

$$\frac{\partial \phi}{\partial t} + \alpha \phi \frac{\partial \phi}{\partial x} + \beta \frac{\partial^3 \phi}{\partial x^3} = 0, \qquad (1.8)$$

which is the well known Korteweg-de Vries equation. This equation was originally derived for shallow water waves by Korteweg and deVries (1895). It has stationary solutions of the form,

$$\phi = \left(\frac{3u}{\alpha}\right) \mathrm{sech}^2 \left[\frac{1}{2}\left(\frac{u}{\beta}\right)^{1/2}(x - ut)\right], \qquad (1.9)$$

This represents a wave with a localized solitary hump moving with a constant velocity u without any change in the shape. The amplitude of this solitary wave is directly proportional to its velocity and its width is inversely proportional to the square root of its amplitude. Eq. (1.8) has been found to be applicable in many other physical situations also. Among the waves in plasmas, hydromagnetic waves, ion acoustic waves, drift waves etc. (Gardner and Morikawa, 1960; Washimi and Taniuti, 1966; Kever and Morikawa, 1969; Jeffrey and Kakutani, 1972; Jeffrey, 1973) are found to be governed by the K-dV equation or modified K-dV equations. Most of these modern applications began with the study of the problem of a nonlinear vibrating system by Fermi, Pasta and Ulam (1955) where a lack of thermalization of the energy in the system was observed. Later on it was found that such a nonlinear system can be modeled by the K-dV equation and there existed solitary waves in the system. These solitary waves were found to be stable under perturbations and collisions among themselves (Zabusky and Kruskal, 1965) which explained the lack of the energy thermalization. Because of their stability these solitary waves were named 'solitons'.

When a medium is strongly dispersive, then it is difficult to study the behaviour of an arbitrary wave form. This is because the difference in the phase velocities of its different Fourier components will be very large and the wave will quickly broaden. In such cases it is more appropriate to study the behaviour of the envelope of a single monochromatic wave like,

$$\phi = a \exp\left[i\left(kx-\omega t\right)\right] + c.c, \quad (1.9)$$

where 'a' represents the slowly varying amplitude of the wave envelope which is usually governed by an equation (Karpman, 1967, Karpman and Kruskal, 1969; Hasegawa, 1975)

$$i\left(\frac{\partial a}{\partial t} + V_g \frac{\partial a}{\partial x}\right)$$

$$+ P(k,\omega)\frac{\partial^2 a}{\partial x^2} + Q(k,\omega)|a|^2 a = 0, \quad (1.10)$$

where $V_g = (\partial\omega/\partial k)$ is the group velocity of the wave and $P = \frac{1}{2}\frac{\partial^2\omega}{\partial k^2}$. This is called the nonlinear Schrödinger equation in which the third and the fourth terms represent dispersion and nonlinearity. Comparing with the ordinary Schrödinger equation the nonlinear term represents a self generated potential seen by the quasi-particles whose wave function is 'a'. This potential is attractive if $PQ > 0$; in which case quasiparticles are trapped in it. This results in the increase of the quasi-particle density $|a|^2$ thereby increasing the strength of potential. Consequently more quasi particles will be trapped leading to an instability of the wave. This instability is called the modulational instability or self trapping instability (Hasegawa, 1970; 1971; 1975). However, after a certain stage the dispersive term of the equation will take over and balance off the further growth of the instability. Stationary solutions of this equation where this balance has been maintained are of the form,

$$|a| = a_0 \operatorname{sech}\left[\left(\frac{Q a_0}{2P}\right)^{1/2} (x - V_g t)\right], \qquad (1.11)$$

which, like Eq. (1.9) represents a solitary hump in the wave envelope or a wave packet moving with the group velocity V_g. It is called an envelope soliton (Karpman, 1967; Hasegawa, 1975; Mio et al., 1976). The width of this soliton is inversely proportional to the square root of its amplitude but the velocity is independent of its amplitude.

On the other hand when $PQ < 0$, the waves are modulationally stable. The corresponding stationary solution is given by

$$|a| = a_1 \left\{ 1 - \tilde{a}^2 \operatorname{sech}^2\left[\left(\left|\frac{Q}{2P}\right| a_1\right)^{1/2} \tilde{a}(x - V_g t)\right]\right\}^{1/2} \qquad (1.12)$$

where a_1 is the asymptotic value of $|a|$ and $\tilde{a}$ is the depth of the modulation (Hasegawa, 1975). This solution represents a localized depletion propagating with velocity v_g and it is called an envelope hole. When $\tilde{a}$ tends to unity, (1.12) becomes,

$$|a| = a_1 \, \mathrm{Tanh} \left[\left(\left| \frac{Q}{2P} \right| a_1 \right)^{1/2} (x - V_g t) \right], \quad (1.13)$$

which is called an envelope shock.

The Boussinesq equation (Boussinesq, 1872), the non-linear Klein-Gordon equation (Schiff, 1951), the nonlinear sine-Gordon equation (Barone et al., 1971), the Hirota equation (Hirota, 1973) etc. are a few more examples of nonlinear dispersive wave equations. In most of these cases, by the method of inverse scattering, one can show that any arbitrary initial wave will be decomposed into a series of solitons (Gardner et al., 1967; Lax, 1968; Zakharov and Shabat, 1972; Ablowitz et al., 1973; Scott et al., 1973; Miura, 1976). Apart from plasma physics, the nonlindar dispersive wave equations and soliton solutions have found applications in many other fields of Physics (Bishop and Schneider, 1978; Lonngren and Scott, 1978; Physica Scripta Vol.20, 1979).

The present thesis constitutes the study of the formation and time evolution of solitary waves in plasmas under different situations.

In chapter II we have investigated the envelope properties of drift waves (Kadomtsev, 1965; Krall, 1968) in a weakly inhomogeneous, collisionless, low β plasma. The nonlinear Schrödinger equation governing these waves has been derived by means of the Krylov-Bogoliubov-Mitropolsky perturbation method (Bogoliubov and Mitropolsky, 1961). This method can be briefly outlined as follows:

Usually any wave motion is governed by a system of nonlinear partial differential equations of the form (Taniuti and Wei, 1968)

$$\frac{\partial \Phi}{\partial t} + A\frac{\partial \Phi}{\partial x} + \left[\sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}\left(H_{\alpha}^{\beta}\frac{\partial}{\partial t} + K_{\alpha}^{\beta}\frac{\partial}{\partial x}\right)\right]\Phi = 0, \tag{1.14}$$

where Φ is a column vector with components ϕ_i (i = 1 to n) and A, H_{α}^{β} and K_{α}^{β} are n x n matrices the elements of which are functions of ϕ_i's , x and t in general. But in the following discussions we will assume them to be functions of ϕ_i's alone. These ϕ_i's represent fluctuating variables (like electric field, density, fluid velocity etc.) associated with the wave. In the limit of weak nonlinearity, each of the ϕ_i's can be expanded as,

$$\phi_i = \phi_i^{(0)} + \epsilon\,\phi_i^{(1)} + \epsilon^2\,\phi_i^{(2)} + \cdots, \tag{1.15}$$

where ϵ is a small parameter appropriate for the system. Then for one of the $\phi_i^{(1)}$'s say for $\phi_1^{(1)}$, we take a plane wave solution,

$$\phi_1^{(1)} = a \exp i\,(kx-\omega t)+c.c, \tag{1.16}$$

where the amplitude a depends on time and space through,

$$\frac{\partial a}{\partial t} = \epsilon A_1(a,\bar{a}) + \epsilon^2 A_2(a,\bar{a}) + \cdots \tag{1.17}$$

and

$$\frac{\partial a}{\partial x} = \epsilon B_1(a,\bar{a}) + \epsilon^2 B_2(a,\bar{a}) + \cdots \tag{1.18}$$

Then we substitute the expansions (1.15), (1.17) and (1.18) into the equation (1.14) and impose the condition that the equations determining the higher order solutions ($\phi_i^{(2)}$, $\phi_i^{(3)}$ etc.) are free of resonant and nonresonant secularities. From these conditions, one can determine the quantities A's and B's of Eqs. (1.17) and (1.18). The secularity removal condition to order ϵ^3 gives the relation,

$$\frac{\partial a}{\partial t_2} + V_g \frac{\partial a}{\partial x_2} + P(k,\omega)\frac{\partial^2 a}{\partial x_1^2} + Q(k,\omega)\,|a|^2 a + R(k,\omega)a = 0, \tag{1.19}$$

where $x_1 = \epsilon x$, $x_2 = \epsilon^2 x$ and $t_2 = \epsilon^2 t$.

This is a nonlinear Schrödinger equation.

In the case considered in chapter II, the wave is assumed to propagate at an arbitrary angle say α, with respect to the magnetic field. By computing the product PQ we have determined the values of k for which the waves are modulationally unstable. When $\alpha \simeq 0°$, the waves are ordinary ion acoustic waves and the modulational instability sets in only when $k\lambda_D > 1.47$ (Kakutani and Sugimoto, 1974), where λ_D is the electron Debye length. But when $\alpha \simeq 90°$, the waves are drift waves and the region of k-values for the case of modulational instability is found to be some-what complex (Mohan et al., 1978). However, one can study the variations in these regions with respect to changes in the magnetic field and the density gradient. This has been carried out for a typical Q-machine plasma and the magnetospheric plasma. It is seen that an increase in the density gradient increases the growth rate but decreases the region of instability. On the other hand, an increase in the magnetic field decreases the growth rate but increases the region of instability.

Chapter III deals with the modulational stability of ion acoustic waves in collisionless plasmas with density and electron temperature inhomogeneities. The electron temperature inhomogeneity is taken to be much smaller than the density inhomogeneity since the heat conductivity ($\sim T_e^{5/2}$) is very large at high temperatures. The ion-acoustic waves are found to be governed by the modified nonlinear Schrödinger equation which has some additional nonlinear nonlocal terms and damping terms introduced by the inhomogeneities (Mohan and Buti, 1979). Since it is difficult to analytically solve this modified nonlinear Schrödinger equation, we numerically compute the time evolution of various initial wave forms e.g., envelope solitons, envelope holes and periodic modulations. For the parameters appropriate for Q-machine plasmas, an envelope soliton with width comparable to the scale size of the density inhomogeneity is found to split into two envelope solitons. But afterwards the one in the front damps while the other one grows. In the case of an envelope hole two soliton-like humps start developing on either side of the central depression. And a periodic modulation excites other wave numbers and develops a spectrum. It has been found that a decrease in the inhomogeneity slows down these processes. For solar wind and solar corona the density gradients being much smaller, the envelope solitons,with widths ofthe order of a few Debye lengths,are found to grow into large amplitudes and attain saturation. Afterwards they split into many solitons.

A current carrying magnetized plasma with hot ions and cold electrons has been found to be unstable with respect to electron acoustic waves (Arefev, 1970; Lashmore-Davies and Martin, 1973). One encounters such plasmas in magnetron type devices, plasma accelerators, the ring current system in the magnetosphere etc. In chapter IV, we have studied the nonlinear properties of electrostatic electron acoustic waves propagating at an angle θ to the magnetic field (with Cos $\theta < (m_e/m_i)^{1/2}$). By making use of the reductive perturbation technique (Taniuti, 1974), we derive a modified KdV-equation for these waves (Mohan and Buti, 1980). Apart from the non-linearity and dispersion, this equation takes into account the ion Landau damping of the waves. Here we give a simplified description of the reductive perturbation scheme which is essentially a method of co-ordinate stretching. We assume that the wave is governed by a set of nonlinear partial differential equations like (1.14). This method is more suitable in the limit of small wave numbers and weak dispersion. That is for waves whose linear dispersion relation can be expanded like,

$$\omega = Vk + \alpha k^{p} + \cdots, \tag{1.20}$$

where $k \ll 1$ and p is a positive integer greater than unity. V is approximately the phase velocity of the wave. Carrying out a Galelian transformation,

$$X = x - Vt \tag{1.21}$$

and

$$T = t, \tag{1.22}$$

(1.20) becomes

$$\omega' = \alpha k^{p} + \cdots \tag{1.23}$$

If we denote the smallness of k by $\epsilon^{1/2}$ (ie; $k \sim \epsilon^{1/2}$) which is essentially a measure of the operator $\frac{\partial}{\partial x}\left(=\frac{\partial}{\partial X}\right)$ then the relation (1.23) tells us that in the moving frame the wave frequency or the measure of the operator $\frac{\partial}{\partial T}$ is of the order, $\epsilon^{p/2}$. Accordingly one can define two new space and time variables as,

$$\xi = \epsilon^{1/2} X = \epsilon^{1/2}(x - Vt) \tag{1.24}$$

and

$$\tau = \epsilon^{p/2} T = \epsilon^{p/2} t \tag{1.25}$$

In this new stretched co-ordinate system the wave numbers and frequencies are of the order of unity. For weak nonlinearities, we can expand the variables ϕ_i's as,

$$\phi_i = \phi_i^{(0)} + \epsilon^{\alpha_i} \phi_i^{(1)} + \epsilon^{\alpha_i + 1} \phi_i^{(2)} + \cdots, \tag{1.26}$$

where α_i's can be different for different variables. The values of α_i's are chosen so that when we substitute (1.26) and make use of (1.24) and (1.25) in the basic equations, and consider equations to different orders in the powers of ϵ, they are mathematically consistent with each other.

Then from the equations corresponding to the lowest order in the power of ϵ, we get a set of relationships among the $\phi_i^{(1)}$'s. Going over into next higher order equations and eliminating the second order quantities $\phi_i^{(2)}$'s, one usually gets an equation of the form,

$$\frac{\partial \phi_i^{(1)}}{\partial \tau} + b_i \phi_i^{(1)} \frac{\partial \phi_i^{(1)}}{\partial \xi} + c_i \frac{\partial^p \phi_i^{(1)}}{\partial \xi^p} = 0, \qquad (1.27)$$

where b_i's and c_i's are constants. For $p = 3$, Eq. (1.27) represents the well known K-dV equation.

Following the above procedure, we find that the electron acoustic waves are characterised by the modified K-dV equation. Treating the ion Landau damping as a small perturbation we then study the time evolution of solitons using the method of inverse scattering (Karpman and Maslov, 1977; Karpman, 1978; 1979). We observe that the electron acoustic solitons undergo damping or growth according as they are faster or slower than the electron drift in the medium. Along with this they develop tails also. The rates of damping, growth and the tail formation are found to increase with a decrease in the angle θ. But changes in the electron to ion temperature ratio is found to affect only the velocity of the soliton.

In all the problems discussed so far, the nonlinearity considered was rather weak. That is, the wave amplitudes were considered to be small and the basic equations were treated perturbatively. But it will be an interesting case if one can study large amplitude waves accounting for the complete nonlinearity in the system. This can be done only by sacrificing certain other features of the medium like Landau damping, collisional and viscous dissipations etc. However, one can afford to neglect these processes if their effects are very small. This is what we have done in chapter V. Accounting for the full electron and ion non-linearities, we reduce the original set of equations governing the electron-acoustic waves to a single equation (Buti et al., 1980). This equation is similar to the energy integral of a classical particle of unit mass. The potential energy of this particle is called the Sagdeev potential (Sagdeev, 1966). By analysing the Sagdeev potential, we show that supersonic finite amplitude electron acoustic solitary waves with density humps can exist. The upper limit on the Mach number for these waves is also determined.

Plasma is a medium which is prone to the excitation of turbulence very easily and it can sustain a multitude of waves simultaneously. The investigation of turbulence in plasma is of very great significance from the point of view of plasma heating, production of high electric and magnetic fields, shock waves etc. in laboratory as well as astrophysical situations. In an unmagnetized plasma with hot electrons and cold ions the most important modes to be considered are the high frequency Langmuir mode and the low frequency ion acoustic mode (Zakharov, 1972; Thornhill and ter Haar, 1978) which interact with each other. In chapter VI we have considered the effects of collisional and viscous dissipations on Langmuir-ion-acoustic interactions. We derive a pair of modified Zakharov equations which couples the two modes through the ponderomotive force of the Langmuir field. These equations have solitary wave solutions which represent localized ion density depressions with Langmuir field trapped in it moving with subsonic velocities. They are called coupled ion acoustic-Langmuir solitons. In order to find the effects of dissipations, we follow the time evolution of solitary waves according to the modified Zakharov equations. In the case of electron-neutral and electron-ion collisions, the Langmuir field damps at first. Then the ion density perturbations radiate away. In the case of ion neutral collisions, the ion density perturbation damps at first and the Langmuir field starts flowing out. Whereas the result of ion viscosity is to make the ion density perturbation to radiate away which will be followed by the Langmuir field. Of all the three dissipative mechanisms, electron-ion and electron-neutral collisions seem to be stronger than the other two.

CHAPTER II

STABILITY OF ELECTROSTATIC SOLITARY DRIFT WAVES

II.1 Introduction

Drift waves in a magnetized plasma arise whenever there is a density or temperature inhomogeneity across the magnetic field. These plasma oscillations move perpendicular to the magnetic field and the inhomogeneity. Inhomogeneous plasmas are quite common in all systems that use magnetic confinement and therefore the instability associated with these waves is called universal instability. (Galeev et al., 1963). In the linear limit, the drift waves are stable in the absence of dissipations and their phase velocity for long wave lengths is same as the velocity of the diamagnetic drift in the medium (Kadomtsev, 1965; Krall and Trivelpiece, 1973). A general study of these waves propagating obliquely to the magnetic field and the effects of finite ion Larmour radius, density variation, magnetic field variation, the presence of a neutral background in the plasma etc. on them has been carried out by various authors (Rudakov and Sagdeev, 1961; Rosenbluth et al., 1962; Krall, 1968; Timofeev and Shvilkin, 1976). In the presence of collisions, the drift waves can give rise to drift dissipative instability which has been considered to be a possible mechanism for the Bohm diffusion (Bohm et al., 1949) of the plasma particles. As far as their nonlinear properties are concerned, in the regime of weak nonlinearity and dispersion, they have been found to propagate as solitary waves (Orefice and Pozzoli, 1970; Nozaki and Taniuti, 1974; Todoroki and Sanuki, 1974).

In this chapter, we investigate the nonlinear self modulation of finite amplitude, monochromatic drift waves in a strongly dispersive medium. Following Krylov-Bogoliubov-Mitropolsky perturbation technique as described in chapter I, we derive a nonlinear Schrödinger equation for the amplitude of a wave propagating at an arbitrary direction with respect to the magnetic field. This equation determines the modulational stability and the envelope properties of the wave for long periods of time over large intervals in space. When the propagation direction is parallel to the magnetic field, the wave is ion acoustic and as it becomes perpendicular, the wave becomes a pure drift wave.

We also discuss the applications of our results in the context of Q-machine plasmas and magnetospheric plasmas. The values of the critical wave numbers which separate the regions of modulational stability and instability are computed numerically for various directions of propagation. The effects of the changes in the magnetic field and density gradient are also studied.

II.2 Nonlinear Schrödinger Equation

We consider a collisionless, low β plasma with cold ions and isothermal electrons. It has an external magnetic field in the z-direction and a weak density gradient in the x-direction. We make use of the following M.H.D. equations to illustrate the propagation of an electrostatic wave in this medium:

$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\vec{v}) = 0, \qquad (2.1)$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{e}{m_i}\vec{\nabla}\phi + \vec{v}\times\vec{\omega}_{ci} \tag{2.2}$$

and

$$\nabla^2\phi = -4\pi e\left[n - n_o \exp(e\phi/T_e)\right] \tag{2.3}$$

In these equations, the inertia of the electrons has been neglected and they have been described by a Boltzmann distribution. The quantities n, $n_o(x)$, $\vec{v}$, ϕ and ω_{ci} are the ion density, equilibrium plasma density, ion fluid velocity, the electric field potential and the ion gyrofrequency respectively.

We assume that the wave is planar and it propagates in the y-z plane making an angle α with the magnetic field. On defining a new space variable along the propagation direction, namely, $\xi = y \sin\alpha + z \cos\alpha$, Eqs. (2.1) - (2.3) become,

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \xi}(n v_\xi) + v_x V_o \omega_{ci} = 0, \tag{2.4}$$

$$\left(\frac{\partial}{\partial t} + v_\xi \frac{\partial}{\partial \xi}\right) v_x - \omega_{ci} v_y = 0, \tag{2.5}$$

$$\left(\frac{\partial}{\partial t} + v_\xi \frac{\partial}{\partial \xi}\right) v_y + \omega_{ci} v_x + \sin\alpha \frac{\partial \phi}{\partial \xi} = 0, \tag{2.6}$$

$$\left(\frac{\partial}{\partial t} + v_\xi \frac{\partial}{\partial \xi}\right) v_z + \cos\alpha \frac{\partial \phi}{\partial \xi} = 0 \tag{2.7}$$

and

$$\frac{\partial^2 \phi}{\partial \xi^2} + \left[n - \exp(\phi) \right] = 0, \tag{2.8}$$

where all the variations in the x-direction except that of the unperturbed equilibrium density have been neglected. In the above equations, n, v, ϕ, ξ and t are normalized with respect to the local equilibrium density n_o $(x = 0)$, ion acoustic velocity $(T_e/m_i)^{1/2}$, the characteristic potential (T_e/e), the local Debye length $\left\{T_e/4\pi n_o\ (x = 0)e^2\right]^{1/2}$ and the local ion plasma period $\left[m_i/4\pi n_o\ (x = 0)e^2\right]^{1/2}$. Further, $V_o = (c\ T_e/e\ B)\ n_o^{-1}\ (d\ n_o/dx)\ .\ (m_i/T_e)^{1/2}$ is the normalized diamagnetic drift velocity of the medium in the y-direction and $v_\xi = v_y \sin\alpha + v_z \cos\alpha$. Hereafter, we will be dealing with only the normalized quantities.

For a weakly nonlinear system, we expand all the quantities about the unperturbed uniform state as,

$$\begin{bmatrix} \phi \\ n \\ v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \epsilon \begin{bmatrix} \phi^{(1)} \\ n^{(1)} \\ v_x^{(1)} \\ v_y^{(1)} \\ v_z^{(1)} \end{bmatrix} + \epsilon^2 \begin{bmatrix} \phi^{(2)} \\ n^{(2)} \\ v_x^{(2)} \\ v_y^{(2)} \\ v_z^{(2)} \end{bmatrix} + \ldots.. \tag{2.9}$$

We now seek a monochromatic plane wave solution for $\phi^{(1)}$ of the form,

$$\phi^{(1)} = a e^{i\psi} + \bar{a} e^{-i\psi} , \qquad (2.10)$$

where $\psi = k\xi - \omega t$ is the phase factor and a is the complex amplitude which is slowly varying function of ξ and t through the relations:

$$\frac{\partial a}{\partial t} = \epsilon A_1(a,\bar{a}) + \epsilon^2 A_2(a,\bar{a}) + \cdots \qquad (2.11)$$

and

$$\frac{\partial a}{\partial \xi} = \epsilon B_1(a,\bar{a}) + \epsilon^2 B_2(a,\bar{a}) + \cdots \qquad (2.12)$$

Furthermore all the quantities in the expansion (2.9) are assumed to depend on ξ and t through a, $\bar{a}$ and ψ. The operators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \xi}$ are transformed as,

$$\frac{\partial}{\partial t} = \frac{\partial a}{\partial t}\frac{\partial}{\partial a} + \frac{\partial \bar{a}}{\partial t}\frac{\partial}{\partial \bar{a}} - \omega \frac{\partial}{\partial \psi} \qquad (2.13)$$

and

$$\frac{\partial}{\partial \xi} = \frac{\partial a}{\partial \xi}\frac{\partial}{\partial a} + \frac{\partial \bar{a}}{\partial \xi}\frac{\partial}{\partial \bar{a}} + k \frac{\partial}{\partial \psi} \qquad (2.14)$$

Putting (2.9) in Eqs. (2.4) - (2.8) and using (2.11) - (2.14) from the set of equations to order ϵ we get the following dispersion relation:

$$D(k,\omega) \equiv \frac{i}{\omega}\left\{(1+k^2)\omega^4 - \left[(1+k^2)\omega_{ci}^2 + k^2\right]\omega^2\right.$$

$$\left. + k V_0 \omega_{ci}^2 \sin\alpha\,\omega + k^2\omega_{ci}^2 \cos^2\alpha\right\} = 0 \qquad (2.15)$$

For an arbitrary angle α, this equation gives four roots for ω in terms of k. These correspond to an accelerated ion acoustic wave, a decelerated ion acoustic wave (Kadomtsev, 1965; Krall and Trivelpiece, 1973) and two ion cyclotron waves. For $\alpha \simeq 0°$, the accelerated ion acoustic wave becomes pure ion acoustic wave with the dispersion relation,

$$\omega = k\,(1+k^2)^{-1/2} \qquad (2.16)$$

But in the limit $\alpha \simeq 90°$, it becomes the drift wave which for low frequencies ($\omega \ll \omega_{ci}$) has the dispersion relation,

$$\omega = k\left\{V_0 + \left[\left(\frac{\pi}{2}-\alpha\right)^2/V_0\right]\left[(1+k^2)+(k^2/\omega_{ci}^2)\right]\right\}$$

$$\left[(1+k^2)+(k^2/\omega_{ci}^2)\right]^{-1} \qquad (2.17)$$

Then, solving for $n^{(1)}$, $v_x^{(1)}$, $v_y^{(1)}$ and $v_z^{(1)}$ in terms of $\phi^{(1)}$, we have,

$$n^{(1)} = C_n\,\phi^{(1)}, \qquad (2.18)$$

$$v_x^{(1)} = C_x\,\phi^{(1)}, \qquad (2.19)$$

$$v_y^{(1)} = c_y \, \phi^{(1)} \tag{2.20}$$

and

$$v_z^{(1)} = c_z \, \phi^{(1)} \tag{2.21}$$

with

$$c_n = \left[\frac{k^2 \cos^2\alpha}{\omega^2} + \frac{k^2 \sin^2\alpha}{(\omega^2 - \omega_{ci}^2)} - \frac{k \omega_{ci}^2 \sin\alpha V_0}{\omega(\omega^2 - \omega_{ci}^2)} \right]$$

$$c_x = \frac{i k \omega_{ci} \sin\alpha}{(\omega^2 - \omega_{ci}^2)}$$

$$c_y = \frac{\omega k \sin\alpha}{(\omega^2 - \omega_{ci}^2)}$$

$$c_z = \frac{k \cos\alpha}{\omega}$$

Now, going over to order ϵ^2, Eq. (2.4) - (2.8) yield the following equation for $\phi^{(2)}$:

$$\mathcal{L}\left\{ \phi^{(2)} \right\} + i \left(\frac{\partial D}{\partial \omega} A_1 - \frac{\partial D}{\partial k} B_1 \right) e^{i\psi}$$

$$+ a^2 f(k, \omega) e^{2i\psi} + c.\ c = 0, \tag{2.22}$$

where the operator $\mathcal{L}$ is defined as,

$$\mathcal{L} \equiv \omega^3 k^2 \frac{\partial^5}{\partial\psi^5} + \omega\left(\omega_{ci}^2 k^2 - \omega^2 + k^2\right)\frac{\partial^3}{\partial\psi^3}$$

$$+ \omega_{ci}^2\left(\frac{k^2\cos^2\alpha}{\omega} - \omega + kV_0\sin\alpha\right)\frac{\partial}{\partial\psi} \qquad (2.23)$$

and

$$\frac{\partial D}{\partial\omega} = \frac{i}{\omega^2}\Big[(1+k^2)(3\omega^2 - \omega_{ci}^2)\omega^2$$

$$- k^2(\omega^2 + \omega_{ci}^2\cos^2\alpha)\Big], \qquad (2.24)$$

$$\frac{\partial D}{\partial k} = \frac{i}{\omega k}\Big[(k^2-1)(\omega^2 - \omega_{ci}^2)\omega^2$$

$$- k^2(\omega^2 - \omega_{ci}^2\cos^2\alpha)\Big] \qquad (2.25)$$

and

$$f(k,\omega) = (\omega^2 - \omega_{ci}^2\cos^2\alpha)\Bigg\{\frac{(\omega_{ci}^2 - 4\omega^2)k^2}{\omega^3(\omega^2 - \omega_{ci}^2)}$$

$$\left[2(1+k^2)\omega^2 + k^2\cos^2\alpha\right] + \frac{k^3\sin\alpha}{\omega(\omega^2 - \omega_{ci}^2)^2}$$

$$\left[3V_0\omega\omega_{ci}^2 + 2k(2\omega^2 - \omega_{ci}^2)\sin\alpha\right]$$

$$- 2\omega(\omega_{ci}^2 - 4\omega^2)\Bigg\} \qquad (2.26)$$

In Eq. (2.22), the terms proportional to $e^{i\psi}$ would give rise to resonant secularity in the solution for $\phi^{(2)}$

(Kakutani and Sugimoto, 1974; Buti, 1976, 1977; Sharma and Buti, 1976, 1977). This can be removed by putting

$$A_1 + V_g B_1 = 0 , \qquad (2.27)$$

where $V_g = -(\partial D/\partial k)/(\partial D/\partial \omega)$ is the group velocity of the wave. Under the condition (2.27), the second order solutions are,

$$\phi^{(2)} = \frac{a^2 \chi}{2(16\Gamma_5 - 4\Gamma_3 + \Gamma_1)} e^{2i\psi} + B e^{i\psi} + c.c + \delta_\phi , \qquad (2.28)$$

$$n^{(2)} = a^2\left[\frac{1}{2} + \frac{(1+k^2)\chi}{6(5\Gamma_5 - \Gamma_3)}\right] e^{2i\psi} + \left[(1+k^2) B - 2ik B_1\right] e^{i\psi} + c.c + |a|^2 + \delta_\phi , \qquad (2.29)$$

$$v_x^{(2)} = \frac{-i a^2 k}{(\omega_{ei}^2 - 4\omega^2)} \Big[(\omega_{ei} C_y + 2\omega C_x) C_\xi + \frac{X \sin\alpha\, \omega_{ei}}{3(5\Gamma_5 - \Gamma_3)} \Big] e^{2i\psi} - \frac{1}{(\omega_{ei}^2 - \omega^2)} \Big\{ \Big[\sin\alpha\, \omega_{ei} - V_g (\omega_{ei} C_y + \omega C_x) \Big] B_1 + \omega_{ei} k \sin\alpha\, B \Big\} e^{i\psi} + c.c \qquad (2.30)$$

$$v_y^{(2)} = \frac{-a^2 k}{(\omega_{ei}^2 - 4\omega^2)} \Big[(\omega_{ei} C_x + 2\omega C_y) C_\xi + \frac{2}{3} \frac{\sin\alpha\, \omega X}{(5\Gamma_5 - \Gamma_3)} \Big] e^{2i\psi} + \frac{1}{(\omega_{ei}^2 - \omega^2)} \Big\{ i \Big[\omega \sin\alpha - V_g (\omega C_y + \omega_{ei} C_x) \Big] B_1 - \sin\alpha\, k \omega B \Big\} e^{i\psi} + c.c - \frac{2 k C_\xi C_x |a|^2}{\omega_{ei}} \qquad (2.31)$$

$$v_z^{(2)} = \frac{a^2 k}{\omega} \Big[\frac{C_\xi C_z}{2} + \frac{X \cos\alpha}{6(5\Gamma_5 - \Gamma_3)} \Big] e^{2i\psi} + \frac{1}{\omega} \Big\{ -i \Big[\cos\alpha - V_g C_z \Big] B_1 + k \cos\alpha\, B \Big\} e^{i\psi} + c.c + \delta_{z_1} \qquad (2.32)$$

where

$$C_\xi = C_y \sin\alpha + C_z \cos\alpha$$

$$X = -\Big\{ k C_\xi \Big[C_n (2\omega_{ei}^2 - 11\omega^2)$$

$$\frac{-k^2}{(\omega^2-\omega_{\ell i}^2)\omega^2}(\omega^4-2\omega^2\omega_{\ell i}^2\cos 2\alpha$$

$$+\omega_{\ell i}^4\cos^2\alpha)\Big]-\omega(\omega_{\ell i}^2-4\omega^2)\Big\},$$

$$\Gamma_1=\frac{k^2\cos^2\alpha\,\omega_{\ell i}^2}{\omega}-\omega\omega_{\ell i}^2+kV_0\sin\alpha\,\omega_{\ell i}^2,$$

$$\Gamma_3=\omega\omega_{\ell i}^2k^2-\omega^3+k^2\omega$$

$$\Gamma_5=\omega^3k^2.$$

B is an arbitrary complex quantity and δ_ϕ and δ_Z are real quantities all of which being independent of ψ. In order to find out δ_ϕ and δ_Z, we have to go to next higher order, namely to order, ϵ^3. The self consistency condition which ensures third order solutions free of nonresonant secularities (Kakutani and Sugimoto, 1974), yields,

$$\delta_\phi = g_\phi(k,\omega)\,|a|^2+\mu \tag{2.33}$$

and

$$\delta_Z = g_Z(k,\omega)\,|a|^2+\nu, \tag{2.34}$$

where

$$g_{\phi}(k,\omega) = (1-V_g^2)^{-1}\Big[V_g^2 - 2k(1+k^2)\frac{(\omega^2-\omega_{ei}^2\cos^2\alpha)}{\omega(\omega^2-\omega_{ei}^2)}V_g$$

$$-\frac{k^2(\omega^2-\omega_{ei}^2\cos^2\alpha)^2}{\omega^2(\omega^2-\omega_{ei}^2)^2}\Big] \tag{2.35}$$

$$g_z(k,\omega) = \cos\alpha\Big[\Big\{V_g\Big(\omega_{ei}+\frac{4k^3V_0\sin\alpha\cos^2\alpha}{\omega\,\omega_{ei}}\Big)$$

$$-\frac{k\cos^2\alpha}{\omega\,\omega_{ei}}\Big[2(1+k^2)\omega_{ei}^2+k^2(1+\sin^2\alpha)\Big]\Big\}$$

$$-\frac{k^2\omega_{ei}\cos\alpha}{\omega}(V_g-V_0\sin\alpha)\Big(\frac{\cos^2\alpha}{\omega}+\frac{2kV_g\sin^2\alpha}{\omega_{ei}^2}\Big)\Big]$$

$$\Big\{\omega_{ei}\Big[\cos^2\alpha-V_g(V_g-V_0\sin\alpha)\Big]\Big\}^{-1} \tag{2.36}$$

with μ and ν as absolute constants independent of a, $\bar{a}$ and ψ. Once again, from the set of equations to order ϵ^3, we get the following equation for $\phi^{(3)}$:

$$\mathcal{L}\{\phi^{(3)}\} - \left(\omega^2\frac{\partial^2}{\partial\psi^2} + \omega_{ci}^2\right)F$$

$$+(\omega + k\,\omega_{ci}\sin\alpha)\frac{\partial G}{\partial\psi} - \left(\omega_{ci}^2 V_0\right.$$

$$\left.+ k\sin\alpha\,\omega\frac{\partial^2}{\partial\psi^2}\right)H + k\frac{\cos\alpha}{\omega}\left(\omega^2\frac{\partial^2}{\partial\psi^2}\right.$$

$$\left.+\omega_{ci}^2\right)J - \omega\left(\omega^2\frac{\partial^3}{\partial\psi^3} + \omega_{ci}^2\frac{\partial}{\partial\psi}\right)L = 0, \quad (2.37)$$

where,

$$F = -\left[A_1\frac{\partial^2 n^{(2)}}{\partial a} + A_2\frac{\partial n^{(1)}}{\partial a} + B_1\frac{\partial}{\partial a}\left(n^{(1)}v_\xi^{(1)}\right.\right.$$

$$\left.+ v_\xi^{(2)}\right) + B_2\frac{\partial v_\xi^{(1)}}{\partial a} + c.c + k\frac{\partial}{\partial\psi}\left(n^{(1)}v_\xi^{(2)}\right.$$

$$\left.\left.+ n^{(2)}v_\xi^{(1)}\right)\right], \quad (2.38)$$

$$G = \left[A_1\frac{\partial v_x^{(2)}}{\partial a} + A_2\frac{\partial v_x^{(1)}}{\partial a} + v_\xi^{(1)}B_1\frac{\partial v_x^{(1)}}{\partial a}\right.$$

$$\left.+ c.c + k\,v_\xi^{(2)}\frac{\partial v_x^{(1)}}{\partial\psi} + k\,v_\xi^{(1)}\frac{\partial v_x^{(2)}}{\partial\psi}\right],$$

$$H = -\left[A_1\frac{\partial v_y^{(2)}}{\partial a} + A_2\frac{\partial v_y^{(1)}}{\partial a}\right. \quad (2.39)$$

$$+ v_\xi^{(1)}B_1\frac{\partial v_y^{(1)}}{\partial a} + \sin\alpha\left(B_1\frac{\partial\phi^{(2)}}{\partial a}\right.$$

$$\left.+ B_2\frac{\partial\phi^{(1)}}{\partial a}\right) + c.c + k\left(v_\xi^{(1)}\frac{\partial v_y^{(2)}}{\partial\psi}\right.$$

$$\left.\left.+ v_\xi^{(2)}\frac{\partial v_y^{(1)}}{\partial\psi}\right)\right], \quad (2.40)$$

$$J = \Big[A_1 \frac{\partial v_z^{(2)}}{\partial a} + A_2 \frac{\partial v_z^{(1)}}{\partial a} + v_\xi^{(1)} B_1 \frac{\partial v_z^{(1)}}{\partial a}$$

$$+ \cos\alpha \left(B_1 \frac{\partial \phi^{(2)}}{\partial a} + B_2 \frac{\partial \phi^{(1)}}{\partial a} \right) + c.c$$

$$+ k \left(v_\xi^{(1)} \frac{\partial v_z^{(2)}}{\partial \psi} + v_\xi^{(2)} \frac{\partial v_z^{(1)}}{\partial \psi} \right) \Big] \qquad (2.41)$$

and

$$L = - \Big[\left(B_1 \frac{\partial}{\partial a} + \bar{B}_1 \frac{\partial}{\partial \bar{a}} \right) \left(B_1 \frac{\partial}{\partial a} + \bar{B}_1 \frac{\partial}{\partial \bar{a}} \right) \phi^{(1)}$$

$$+ 2k \left(B_2 \frac{\partial}{\partial a} + \bar{B}_2 \frac{\partial}{\partial \bar{a}} \right) \frac{\partial \phi^{(1)}}{\partial \psi} + 2k \Big(B_1 \frac{\partial}{\partial a}$$

$$+ \bar{B}_1 \frac{\partial}{\partial \bar{a}} \Big) \frac{\partial \phi^{(2)}}{\partial \psi} - \left(\phi^{(1)} \phi^{(2)} + \frac{1}{6} \phi^{(1)3} \right) \Big] \qquad (2.42)$$

As was done in the case of Eq. (2.22), we remove the resonant secularity in the solution for $\phi^{(3)}$ by putting the coefficient of $e^{i\psi}$ in (2.37) equal to zero. Then we get,

$$i \left(A_2 + V_g B_2 \right) + P \left(B_1 \frac{\partial B_1}{\partial a} + \bar{B}_1 \frac{\partial B_1}{\partial \bar{a}} \right)$$

$$+ Q |a|^2 a + R a = 0, \qquad (2.43)$$

where

$$P = \frac{1}{2}\frac{dV_g}{dk} = \Big\{ V_g^2\big[k^2 - (1+k^2)(6\omega^2 - \omega_{ei}^2)\big] - V_g\big[8k\omega^3 - 4(1+\omega_{ei}^2)k\omega + V_0\omega_{ei}\sin\alpha\big] - \big[(\omega^2-\omega_{ei}^2)\omega^2 + (\omega_{ei}^2\cos^2\alpha - \omega^2)\big]\Big\}\Big\{4(1+k^2)\omega^3 - 2\omega\big[\omega_{ei}^2(1+k^2)+k^2\big] + kV_0\omega_{ei}^2\sin\alpha\Big\}^{-1}, \quad (2.44)$$

$$Q = \frac{ik}{(\partial D/\partial\omega)}\Big\{\Big[(\omega_{ei}^2 - 3\omega^2)(1+k^2) + \frac{k^2}{\omega^2}(\omega^2+\omega_{ei}^2\cos^2\alpha)\Big](h_1\sin\alpha + h_2\cos\alpha) + \frac{1}{\omega k}\big[\omega^2(\omega^2-\omega_{ei}^2) - k^2(\omega^2-\omega_{ei}^2\cos^2\alpha)\big] j(k,\omega) - \Big[(\omega^2-\omega_{ei}^2)(1+k^2) - \frac{k^2}{\omega^2}(\omega^2-\omega_{ei}^2\cos^2\alpha)\Big](u_1\sin\alpha + u_2\cos\alpha) + \frac{1}{k\omega}\big[\omega^2(\omega^2 - \omega_{ei}^2) - k^2(1+4k^2)(\omega^2-\omega_{ei}^2\cos^2\alpha)\big]\gamma_\phi - \frac{2k(\omega^2-\omega_{ei}^2\cos^2\alpha)}{\omega(\omega^2-\omega_{ei}^2)}\Big[\omega_{ei}(k\sin\alpha - V_0\omega)u_3 + (k\sin\alpha\,\omega - V_0\omega_{ei}^2)u_1 + \frac{k}{\omega}\cos\alpha(\omega^2-\omega_{ei}^2)u_2\Big] + \frac{1}{2k\omega}$$

$$\left[(\omega^2-\omega_{\ell i}^2)\omega^2-3k^2(\omega^2-\omega_{\ell i}^2\cos^2\alpha)\right]\Big\} \tag{2.45}$$

and

$$R=-k(\lambda_1\sin\alpha+\lambda_2\cos\alpha)+\left[\frac{k^2}{\omega}(\omega^2-\omega_{\ell i}^2\cos^2\alpha)\lambda_3-\omega(\omega^2-\omega_{\ell i}^2)\mu\right]\left[(\omega_{\ell i}^2-3\omega^2)(1+k^2)+\frac{k^2}{\omega^2}(\omega^2+\omega_{\ell i}^2\cos^2\alpha)\right]^{-1}. \tag{2.46}$$

The h's,u's and j appearing in (2.45) are defined as,

$$h_1=\left[S\cos\alpha(V_g^2-1)-k^2\sin\alpha(\omega^2-\omega_{\ell i}^2\cos^2\alpha)(\omega^2-\omega_{\ell i}^2)^{-2}(V_g^2+\cos^2\alpha)+NV_g\sin\alpha\right]\left[V_g(1-V_g^2)\right]^{-1},$$

$$h_2=\left[S\sin\alpha(1-V_g^2)-k^2\cos\alpha(\omega^2-\omega_{\ell i}^2\cos^2\alpha)\omega^{-2}(\omega^2-\omega_{\ell i}^2)^{-1}(V_g^2-\sin^2\alpha)+NV_g\cos\alpha\right]\left[V_g(1-V_g^2)\right]^{-1},$$

$$u_1=\frac{-k}{(\omega_{\ell i}^2-4\omega^2)}\left[\frac{(\omega_{\ell i}^2+2\omega^2)(\omega^2-\omega_{\ell i}^2\cos^2\alpha)}{\omega(\omega^2-\omega_{\ell i}^2)^2}k^2\sin^2\alpha+4\omega\sin\alpha\,\gamma_\phi\right],$$

$$u_2=\frac{k}{\omega}\left[\frac{(\omega^2-\omega_{\ell i}^2\cos^2\alpha)}{\omega^2(\omega^2-\omega_{\ell i}^2)}k^2\cos\alpha+\cos\alpha\,\gamma_\phi\right]$$

and

$$u_3 = \frac{-k}{(\omega_{ei}^2 - 4\omega^2)} \Big[3\omega_{ei} (\omega^2 - \omega_{ei}^2 \cos^2\alpha) k^2 \sin\alpha + 2\omega_{ei} \sin\alpha \, \gamma_\phi \Big],$$

$$j(k,\omega) = (1 - V_g^2)^{-1} \Big[V_g^2 - \frac{2k(1+k^2)}{\omega(\omega^2 - \omega_{ei}^2)} (\omega^2 - \omega_{ei}^2 \cos^2\alpha) V_g - \frac{k^2 (\omega^2 - \omega_{ei}^2 \cos\alpha)^2}{\omega^2 (\omega^2 - \omega_{ei}^2)^2} \Big], \quad (2.47)$$

with

$$S = \frac{k^3 \omega_{ei}^2 \sin 2\alpha \, V_g}{\omega (\omega^2 - \omega_{ei}^2)^2}$$

and

$$N = V_g - \frac{2k(1+k^2)(\omega^2 - \omega_{ei}^2 \cos^2\alpha)}{\omega(\omega^2 - \omega_{ei}^2)} \quad (2.48)$$

Further, the λ's appearing in (2.46) are absolute constants independent of a, $\bar{a}$ and ϕ.

In Eq. (2.43), A_2, B_2 and $\left(B_1 \frac{\partial B_1}{\partial a} + \bar{B}_1 \frac{\partial B_1}{\partial \bar{a}}\right)$ can be interpreted respectively as $\left(\frac{\partial a}{\partial t_2} - \frac{A_1}{\epsilon}\right)$, $\left(\frac{\partial a}{\partial \xi_2} - \frac{B_1}{\epsilon}\right)$ and $\frac{\partial^2 a}{\partial \xi_1^2}$ where the new time and space variables are defined as, $t_2 = \epsilon^2 t$, $\xi_2 = \epsilon^2 \xi$ and $\xi_1 = \epsilon \xi$. Further, on introducing the co-ordinate transformation $\eta = \epsilon(\xi - V_g t)$ and $\tau = \epsilon^2 t$ and by making use of relation (2.27), Eq. (2.43) simplifies to:

$$i\frac{\partial a}{\partial \tau} + P\frac{\partial^2 a}{\partial \eta^2} + Q|a|^2 a + Ra = 0 \qquad (2.49)$$

This is the required nonlinear Schrödinger equation for the complex amplitude a of the drift waves.

II.3 Modulational Stability of the Wave Envelopes

Eq. (2.49) describes the behaviour of the envelope of a monochromatic wave in the y-z plane. In order to study the stability of the envelope, let us express the complex amplitude a in terms of two real functions ρ and σ (Hasegawa, 1975) as,

$$a = \rho^{1/2}(\eta, \tau) \exp\{i\sigma(\eta, \tau)\} \qquad (2.50)$$

We eliminate the quantity R from Eq. (2.49) through a transformation, $a \rightarrow a \exp(iR\tau)$. Then substituting (2.50) for a, the real and imaginary parts of the resulting equation give,

$$\frac{\partial \rho}{\partial \tau} + 2P\frac{\partial}{\partial \eta}\left(\rho\frac{\partial \sigma}{\partial \eta}\right) = 0 \qquad (2.51)$$

and

$$\frac{\partial \sigma}{\partial \tau} + P\left(\frac{\partial \sigma}{\partial \eta}\right)^2 + \frac{P}{4\rho^2}\left(\frac{\partial \rho}{\partial \eta}\right)^2 - \frac{P}{2\rho}\frac{\partial^2 \rho}{\partial \eta^2} - QP = 0. \qquad (2.52)$$

For the linear stability of the envelope, we can take ρ and σ as,

$$\begin{bmatrix} \rho \\ \sigma \end{bmatrix} = \begin{bmatrix} \rho_0 \\ \sigma_0 \end{bmatrix} + \begin{bmatrix} \rho_1 \\ \sigma_1 \end{bmatrix} \exp i(K\eta - \Omega\tau) \tag{2.53}$$

Further, on linearising (2.51) and (2.52), we obtain the following dispersion relation:

$$\Omega^2 = (Q\rho_0 - PK^2)^2 - (Q\rho_0)^2 \tag{2.54}$$

This shows that the wave envelope is stable or unstable according as $PQ < 0$ or $PQ > 0$.

Looking for a stationary solution for $|a|$; we put $\frac{\partial |a|}{\partial \tau} = \frac{\partial \rho}{\partial \tau} = 0$ in Eq. (2.51) and integrate once over space to obtain,

$$\rho \frac{\partial \sigma}{\partial \eta} = C, \tag{2.55}$$

where C is a function of τ alone. Using Eqs. (2.52) and (2.55), we get,

$$\left(\frac{\frac{d^2 C}{d\tau^2}}{\frac{d(C^2)}{d\tau}} \right) = \frac{2P}{\rho^2} \frac{d\rho}{d\eta} = \text{Const.} \tag{2.56}$$

But the solution corresponding to the equation $\left(\frac{2P}{\rho^2}\frac{d\rho}{d\eta} = \text{const}\right)$, is physically inadmissible. So we have to choose,

$$C(\tau) = \text{constant} = C_1. \tag{2.57}$$

Eq. (2.55) can now be integrated to give,

$$\sigma = \int \frac{C_1}{\rho} d\eta + A(\tau). \tag{2.58}$$

Since by (2.55) $\frac{\partial \sigma}{\partial \eta}$ is a function of η alone, it follows from (2.52) that $\frac{\partial \sigma}{\partial \tau}$ must also be a function of η alone. So we take,

$$\frac{\partial \sigma}{\partial \tau} = \text{Const} = \Lambda \tag{2.59}$$

The solution (2.58) can therefore be written as,

$$\sigma = \int \frac{C_1}{\rho} d\eta + \Lambda \tau. \tag{2.60}$$

Substituting for σ in (2.52) and integrating after multiplying by $\frac{d\rho}{d\eta}$,

$$\left(\frac{d\rho}{d\eta}\right)^2 = -\frac{2Q}{P}\rho^3 + \frac{4\Lambda}{P}\rho^2 + \frac{C_2}{P}\rho - 4C_1^2, \tag{2.61}$$

where C_2 is a constant.

For $PQ > 0$, Eq. (2.61) can be integrated with the choice of the constants $C_1 = C_2 = 0$ to give the following localized solution:

$$\rho = \rho_s \,\text{sech}^2\left[\left(\frac{Q\rho_s}{2P}\right)^{1/2} \eta\right], \quad \sigma = \Lambda \tau, \tag{2.62}$$

where, $\rho_s = (2\Lambda / Q)$.

This localized solution represents a soliton with amplitude $\rho_s^{1/2}$ and width $(2P/Q\,\rho_s)^{1/2}$.

When $PQ < 0$, the waves are modulationally stable and (2.61) can be integrated with the choice of the constants

$$C_1 = \rho_1\left(\frac{Q\rho_1 - \Lambda}{P}\right)^{1/2} \quad \text{and} \quad C_2 = 2\rho_1(3Q\rho_1 - 4\Lambda)$$

to give a solution:

$$\rho = \rho_1\left\{1 - \tilde{a}^2 \operatorname{sech}^2\left[\left(\left|\frac{Q}{P}\right|\frac{\rho_1}{2}\right)^{1/2}\tilde{a}\eta\right]\right\},$$

$$\sigma = \sin^{-1}\left\{\frac{\tilde{a}\operatorname{Tanh}\left[\left(\left|\frac{Q}{P}\right|\frac{\rho_1}{2}\right)^{1/2}\tilde{a}\eta\right]}{\left(1-\tilde{a}^2\operatorname{sech}^2\left[\left(\left|\frac{Q}{P}\right|\frac{\rho_1}{2}\right)^{1/2}\tilde{a}\eta\right]\right)^{1/2}}\right\} + \Lambda\tau + \left(\frac{Q\rho_1 - \Lambda}{P}\right)^{1/2}\eta. \qquad (2.63)$$

This solution represents a localized depletion in the wave amplitude from $\rho_1^{1/2}$ to $\left[\rho_1(1 - \tilde{a}^2)\right]^{1/2}$. This is called an envelope hole with $\left(\frac{2}{\rho_1}\left|\frac{P}{Q}\right|\right)^{1/2}\frac{1}{\tilde{a}}$ as its width.

II.4 Discussions

When the waves are modulationally unstable, for small values of K (Taniuti and Yajima, 1969) the growth rate is given by

$$\gamma = a_0 (2|PQ|)^{1/2} K, \qquad (2.64)$$

where a_0 is the initial amplitude of the wave.

With P and Q given by (2.44) and (2.45) and ω defined by (2.15), it is difficult to determine the values of k corresponding to stability and instability analytically. However, we have computed the critical values of k separating the stable and unstable regions for the Q-machine and the magnetospheric plasmas (Mohan et al.,1978).

1) Q-machine Plasma

Physical conditions for the occurance of drift waves are usually present in Q-machines (Motley, 1975). The magnetic field, density, density gradient scale lengths and electron temperature are typically of the order of 10^3 Gauss, 10^{10} cm^{-3}, 1 cm and 10^{-1}eV. In Figs. 1-4, the critical wave numbers k_c which separates the regions of modulational stability and instability are plotted against the angle of propagation. The hatched regions correspond to stability. When α is zero, (Figs. 1 and 2), the wave is ion acoustic and $k_{c\ min}$ ($k > k_{c\ min}$, unstable) becomes 1.47 which is in agreement with the result of Kakutani and Sugimoto (1974).

As the angle of propagation increases, a small region of instability developes near $k = 0$ and spreads to all values of k. This is due to the presence of the magnetic field since it is seen from Figs. 1 and 2 that this region becomes larger with an increase in the magnetic field and remains almost unaffected due to changes in the density gradient.

When the angle α approaches 90°, the wave becomes drift wave. Figs. 3 and 4 show the wave number ranges for which this wave is stable. In Fig. 3, the effect of the increase in the magnetic field is shown. We observe that as the magnetic field is increased keeping the density gradient fixed, the area of the region of stability decreases. In this case, the value of $k_{C\ max}$ ($k < k_{C\ max}$, unstable) at $\alpha = 90°$ increases with the magnetic field.

Fig.4 shows that for a fixed magnetic field, the area of the region of stability decreases with an increase in the density gradient scale length. However, the value of $k_{C\ max} = 0.57$ at $\alpha = 90°$ remains fixed.

The maximum value of $|PQ|$ in the region of instability for different angles of propagation are given in Tables 1 and 2 for changes in the magnetic field and density gradient respectively. Here we see that $|PQ|_{max}$ and hence the maximum growth rate γ_{max} decreases with an increase either in the magnetic field or in the inhomogeneity scale length. In the latter case, the growth rate is further reduced due to the following reason: For a drift wave, the density perturbation is caused by a transverse displacement in the plasma (Kadomtsev, 1965). Now, as the scale length of the inhomogeneity increases, for the same amount of displacement, the amplitude a_o of the density perturbation ($n^{(1)} \simeq (1 + k^2)\, \phi^{(1)}$ for $\omega \ll \omega_{ci}$) which appears in the growth rate (2.64) also decreases.

From Table 1, we observe that at $\alpha \sim 90^\circ$, the growth rate varies as B^{-p} $(1 \leq p \leq 2)$, however its variation with L is some what more complicated. For example at $\alpha = 87°$, $\gamma_{max} \propto L^{-1}$ but at $\alpha = \pi/2$, $\gamma_{max} \propto L^{-p}$ with p sometimes exceeding 3. From these two tables, it is quite clear that the variations of the growth rate of the instability on the density gradients and on magnetic field are very complex. For this reason it is difficult to give a simple physical interpretation to the results obtained. Neverthless, we may remark that since the cause of the instability under consideration is the inhomogeneity, the growth rate should increase with an increase in the inhomogeneity; this is exactly what we find. Regarding the variations with the magnetic field, if we consider the envelope soliton as a bunch of quasi particles moving with the group velocity V_g, whose motions get restricted with an increase in B since $V_g \propto B^{-1}$, the decrease in γ_{max} with an increase in B is expected. V_g also being proportional to L^{-1}, a similar interpretation for the increase of γ_{max} with a decrease in L would be justified.

ii) Magnetospheric Plasma

Recently electrostatic turbulence at frequencies between 1.7 and 56.2 Hz has been detected by the satellite Hawkeye 1 (Kintner and Gurnett, 1978). This turbulence was observed when the space craft crossed the plasma pause at altitudes higher than 3 R_E; R_E being the earth radius. There was no corresponding magnetic field disturbance indicating that the waves were electrostatic. They propagated across the density gradient and were interpreted as drift waves. Based on the values of the magnetospheric plasma parameters (T_e = 100 eV, B = 6 x 10^{-3} Gauss, n_o = 500 cm^{-3}, $L \sim 0.1\ R_E$), we have determined the values of k for the modulational stability and instability. These are shown in Figs. 5 and 6. The shapes of these regions are somewhat different from those in the case of Q-machine plasma. However, the variations in the areas of these regions with respect to changes in the magnetic field and density gradient remain the same. Again, from Tables 3 and 4 it is seen that at α = 90°, the behaviour of $|PQ|_{max}$ is same as in the case of Q-machine. But as α decreases,

$|PQ|$ becomes large very rapidly and the linear result (2.64) for γ becomes invalid. This rapid change is because of the factor $(\omega^2 - \omega_{ci}^2 \cos^2\alpha)$, which occurs very often in P and Q, changes sign for some value of α between 89° and 90° whereas in the case of Q-machine this happens only for α around 85°.

II.5 Summary and Conclusions

The modulational stability of electrostatic drift waves, in a weakly inhomogeneous, collisionless, low β plasma is investigated by deriving the nonlinear Schrödinger equation characterising them. The values of the critical wave number k_c, separating the regions of stability and instability are computed for different directions of propagation α, of the waves. When $\alpha = 0°$, the waves are ion acoustic and are unstable only for $k\lambda_D > 1.47$. But for $\alpha \simeq 90°$ the waves are drift waves and their behaviour is studied for Q-machine and magnetospheric plasmas. It is seen that an increase in the density gradient increases the growth rate but decreases the region of instability. On the other hand an increase in the magnetic field decreases the growth rate but increases the region of instability.

TABLE 1

Ω-machine plasma. Values of $|PQ|_{max}$ for increasing magnetic field but with constant inhomogeneity scale length (L = 40)

ω_{ci}	$\lvert PQ \rvert_{max}$			
	87°	88°	89°	90°
0.25	1.0×10^{-2}	8.0×10^{-3}	6.7×10^{-3}	7.7×10^{-3}
0.35	6.8×10^{-3}	4.9×10^{-3}	3.5×10^{-3}	3.7×10^{-3}
0.45	5.3×10^{-3}	3.7×10^{-3}	2.4×10^{-3}	2.1×10^{-3}
0.55	4.5×10^{-3}	2.9×10^{-3}	1.8×10^{-3}	1.3×10^{-3}

TABLE 2

Q-machine plasma. Values of $|PQ|_{max}$ for increasing inhomogeneity scale length but with constant magnetic field ($\omega_{ci} = 0.35$)

L	$\|PQ\|_{max}$			
	87°	88°	89°	90°
30	9.4×10^{-3}	7.2×10^{-3}	5.6×10^{-3}	6.5×10^{-3}
40	6.8×10^{-3}	4.9×10^{-3}	3.5×10^{-3}	3.7×10^{-3}
50	5.5×10^{-3}	3.9×10^{-3}	2.6×10^{-3}	1.6×10^{-3}
60	4.7×10^{-3}	3.3×10^{-3}	2.2×10^{-3}	1.1×10^{-3}

TABLE 3

Magnetospheric plasma. Values of $|PQ|_{max}$ for increasing magnetic field but constant inhomogeneity scale length ($L = 0.1\ R_E$)

ω_{ci}	$\|PQ\|_{max}$	
	89°	90°
2×10^{-2}	5.264×10^{-4}	6.020×10^{-6}
2.25×10^{-2}	5.548×10^{-4}	4.648×10^{-6}
2.50×10^{-2}	5.741×10^{-4}	3.842×10^{-6}
2.75×10^{-2}	5.930×10^{-4}	3.182×10^{-6}

TABLE 4

Magnetospheric plasma. Values of $|PQ|_{max}$ for increasing inhomogeneity scale length but constant magnetic field ($\omega_{ci} = 2 \times 10^{-2}$)

L	$\|PQ\|_{max}$	
	89°	90°
0.1 R_E	5.264×10^{-4}	6.020×10^{-6}
0.125 R_E	5.712×10^{-4}	3.851×10^{-6}
0.150 R_E	6.084×10^{-4}	2.675×10^{-6}
0.175 R_E	6.348×10^{-4}	1.964×10^{-6}

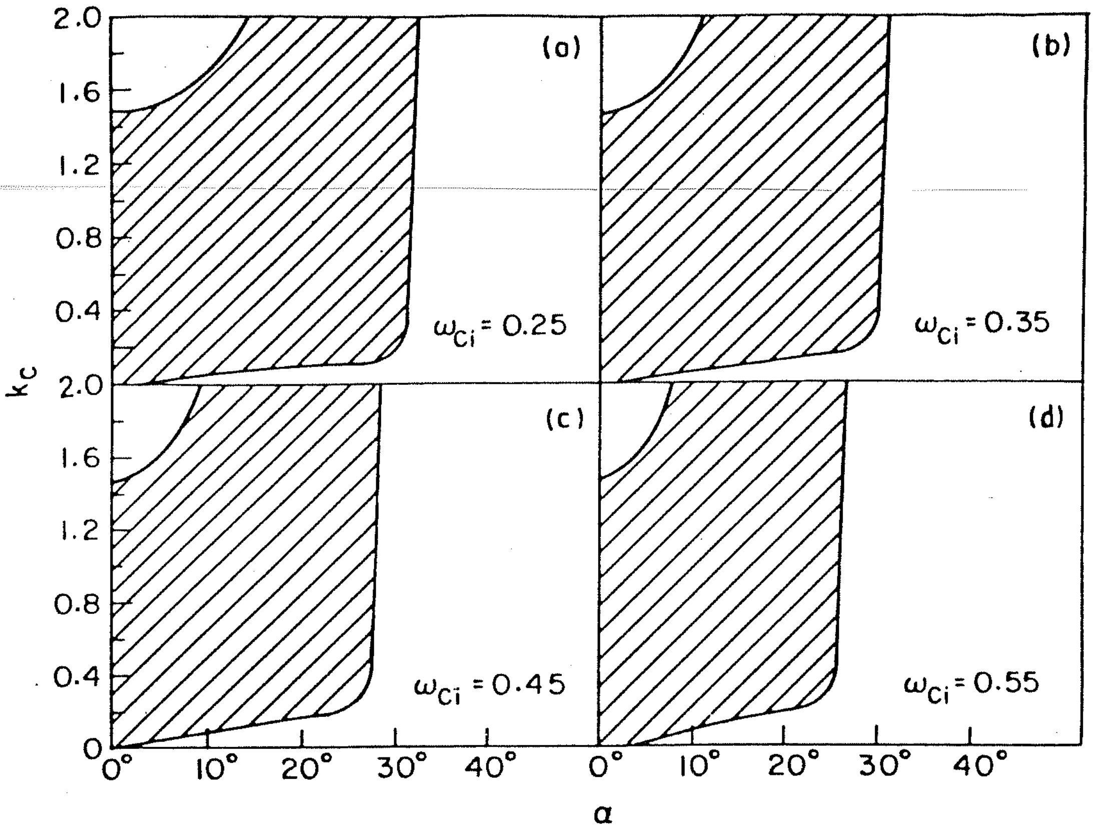

Fig.1: (Q-machine). Regions of stability and instability in the ion acoustic regime for a constant inhomogeneity scale length (L = 40) and for ω_{ci} = 0.25, 0.35, 0.45 and 0.55.

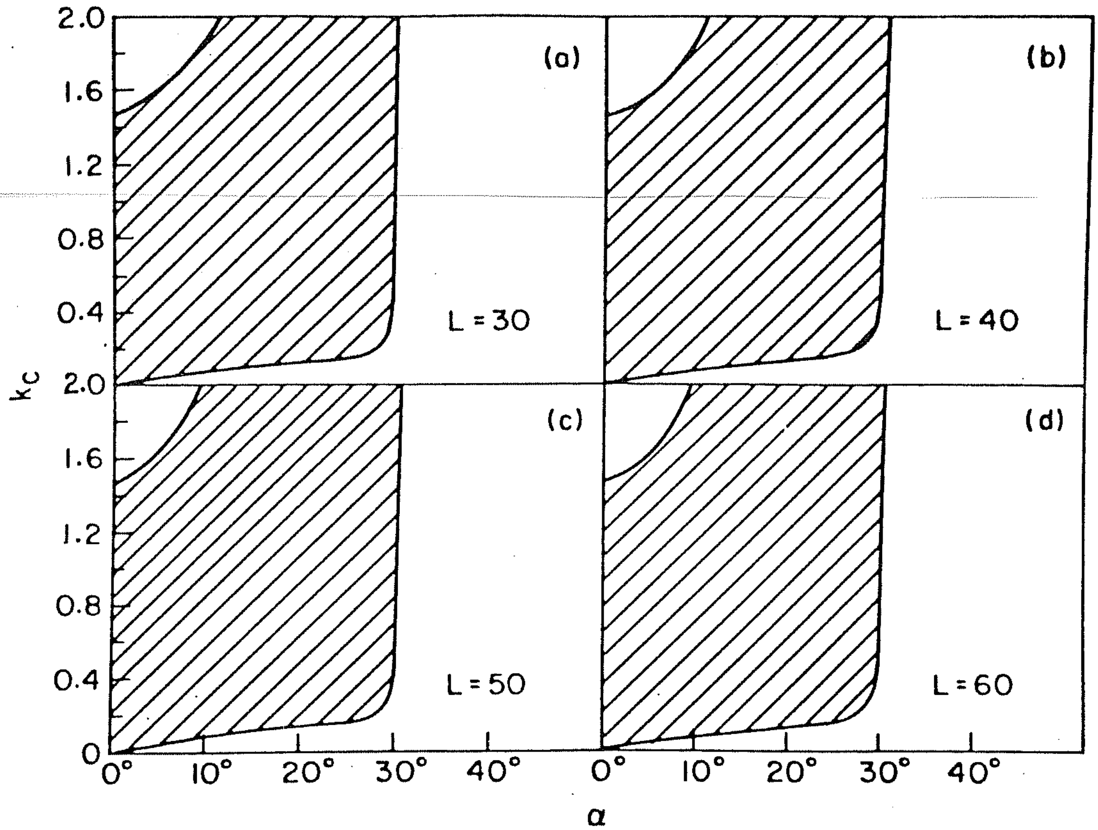

Fig.2: (Q-machine). Regions of stability and instability in the ion acoustic regime for a constant magnetic field (ω_{ci} = 0.35) and for L = 30, 40, 50 and 60.

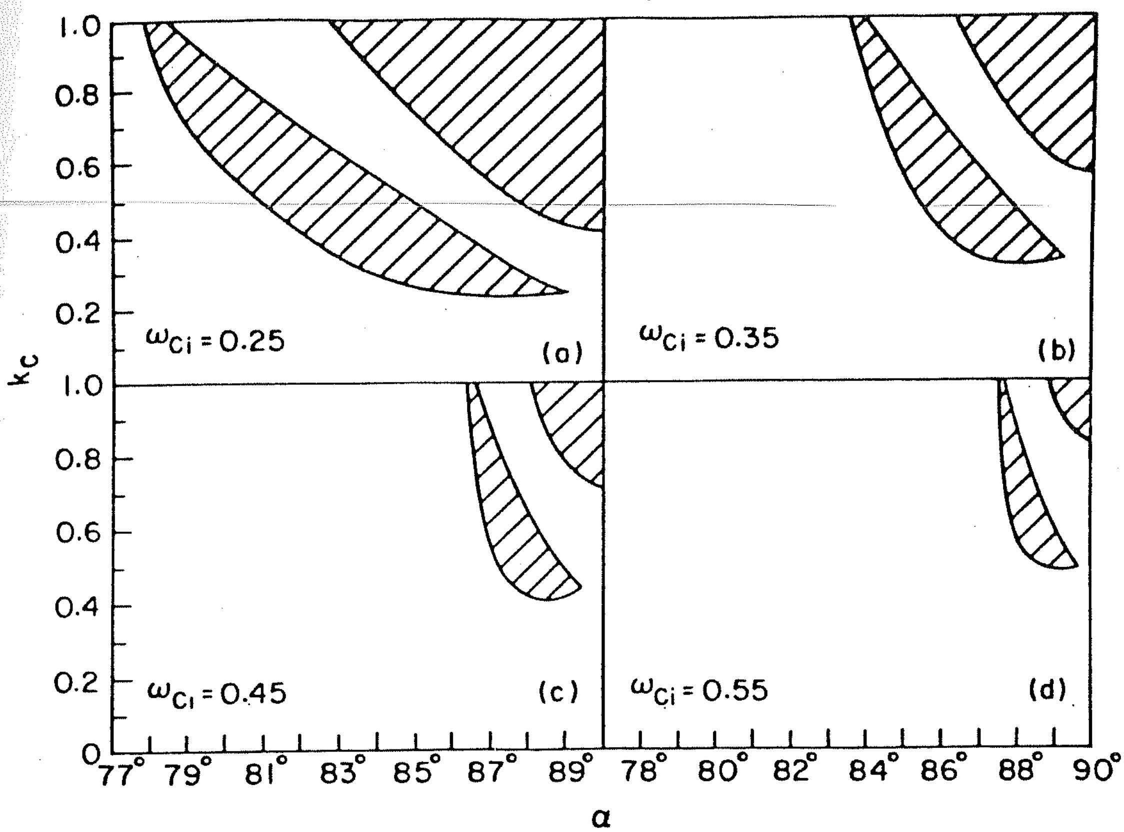

Fig.3; (Q-machine). Regions of stability and instability in the drift wave regime for a constant inhomogeneity scale length (L = 40) and for ω_{ci} = 0.25, 0.35, 0.45 and 0.55.

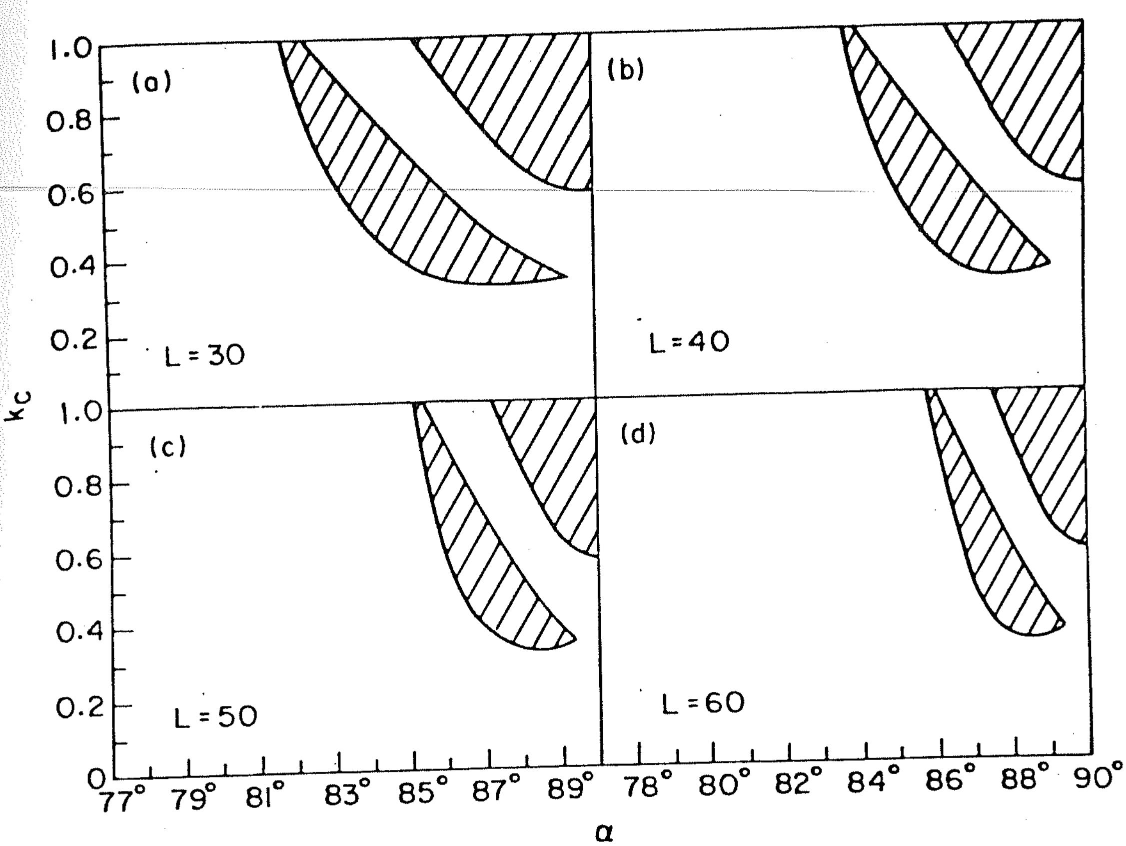

Fig.4: (Q-machine). Regions of stability and instability in the drift wave regime for a constant magnetic field (ω_{ci} = 0.35) and for L = 30, 40, 50 and 60.

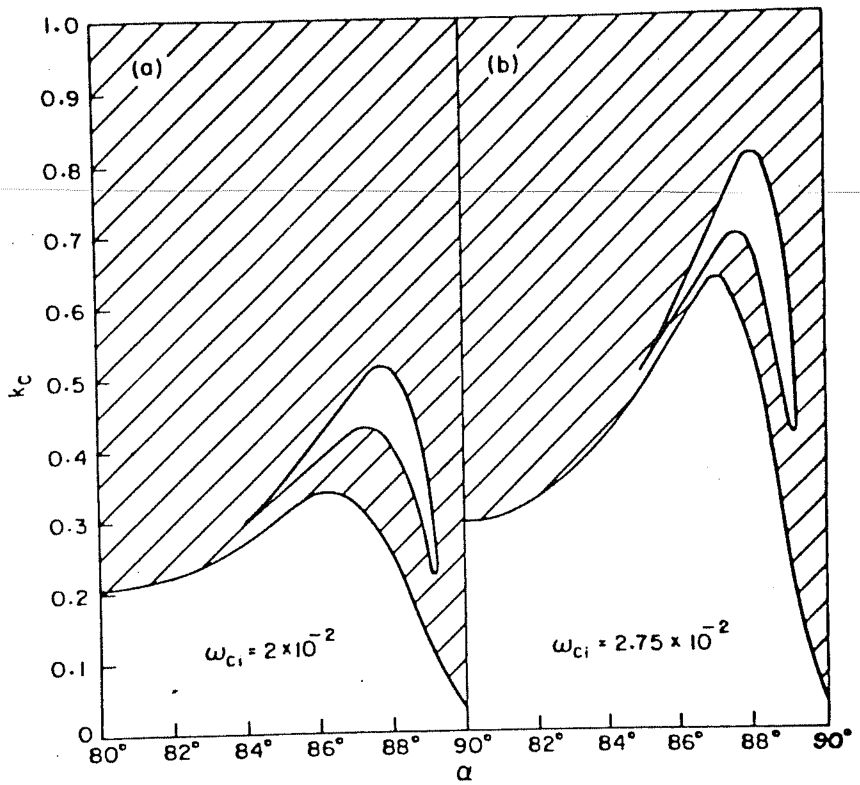

Fig. 5: (Magnetosphere). Regions of stability and instability of drift waves for a constant inhomogeneity scale length ($L = 0.1\ R_E$) and for $\omega_{ci} = 2 \times 10^{-2}$ and 2.75×10^{-2}.

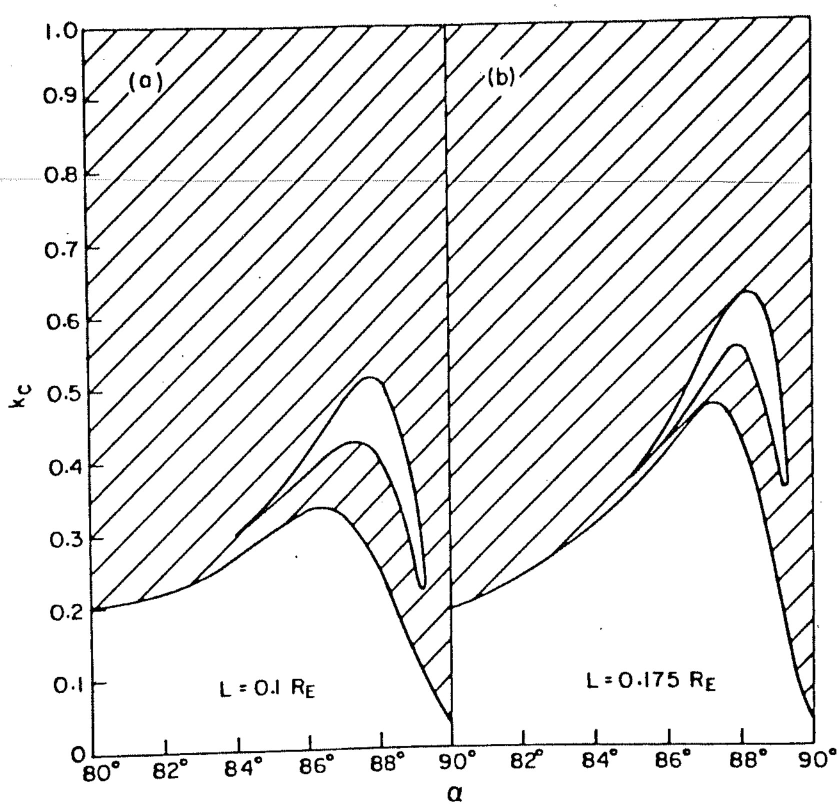

Fig.6: (Magnetosphere). Regions of stability and instability of drift waves for a constant magnetic field (ω_{ci} = 2 x 10^{-2}) and for L = 0.1 R_E and 0.175 R_E.

CHAPTER III

MODULATED ION ACOUSTIC WAVES IN INHOMOGENEOUS PLASMAS

III.1 Introduction

Ion acoustic waves are one of the most common modes in plasmas. They are electrostatic oscillations sustained by a balance between the pressure of the hot electrons and the inertia of the relatively cold ions. The existence and behaviour of ion acoustic waves in the linear as well as in the weakly nonlinear regimes have been verified in many laboratory experiments (Wong and D'Angelo, 1964; Ikezi et al., 1970; Ikezi and Kiwamoto, 1971). They have important practical applications such as a diagnostic tool to measure the electron temperature in a plasma, heating of the ions by Landau damping etc.

In a weakly dispersive and homogeneous plasma, nonlinear ion acoustic waves are governed by a K-dV equation (Washimi and Taniuti, 1966; Davidson, 1972) which have stationary solitary wave solutions travelling with velocities near the ion sound speed, $C_s = (T_e/m_i)^{1/2}$. The presence of density or temperature inhomogeneity however modifies the K-dV equation (Nishikawa and Kaw, 1975; Goswami and Sinha, 1976). This changes the propagation characteristics of the ion acoustic solitons. The amplitude of a soliton decreases as it moves towards increasing density.

In a highly dispersive but homogeneous plasma, the self modulation of a monochromatic ion acoustic wave is governed by a nonlinear schrödinger equation (Shimizu and Ichikawa, 1972; Kako, 1974; Kakutani and Sugimoto, 1974). In this chapter we study how this equation will get modified if the plasma has inhomogeneities in density and electron temperature. We consider the gradient in electron temperature to be much smaller than that in the density. This is because a high temperature plasma cannot sustain large gradients in temperature since the conductivity goes as $T^{5/2}$. As in chapter II, the Krylov-Bogoliubov-Mitropolsky perturbation scheme (Bogoliubov and Mitropolsky, 1961; Kakutani and Sugimoto, 1974) is employed to derive the modified nonlinear schrödinger equation. Assuming a plane wave solution, the growth rate for the wave is calculated. Then we study the time evolution of different envelope waves for long intervals of time. It is seen that for conditions suitable for Q-machine plasmas, an envelope soliton climbing up a density gradient slows down and splits into two envelope solitons. Then the one in the front damps while the other grows. In the case of an envelope hole two asymmetric soliton-like humps develop on either side of the central depression with the larger hump in the front. For a periodic modulation, other wave numbers are excited nonlinearly giving rise to a spectrum.

Apart from laboratory plasmas, ion acoustic waves are observed in space plasmas too. Plasma wave frequency measurements by the spacecrafts Helios 1 and 2 and wave length measurements by the spacecraft Imp 6 now provide strong evidence of the presence of short wavelength ion acoustic waves in the solar wind plasma near the earth (Gurnett and Frank, 1978). But the density gradient in this plasma is extremely small. However, an envelope soliton with a very small width after very long intervals of time is found to grow into a large amplitude and attain saturation. Then it starts splitting into many soliton-like humps. The same phenomenon has been found to take place in the case of solar corona plasma also.

III.2 The Modified Nonlinear Schrödinger Equation

We consider a collisionless plasma with cold ions and hot electrons. It has weak gradients in density and electron temperature. These gradients are maintained by a zero order electric field E_o and an external force F say, due to gravitational field. In the analysis that follows, we neglect the inertia of the electrons. The relevant one dimensional fluid equations considered for the present situations are,

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n v) = 0, \tag{3.1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{e}{m_i} E + \frac{F}{m_i}, \tag{3.2}$$

$$\frac{\partial E}{\partial x} = 4\pi e(n - n_e) \tag{3.3}$$

and

$$0 = -e n_e E - \frac{\partial}{\partial x}(n_e T_e) \tag{3.4}$$

where n and n_e are the ion and electron densities, v is the ion fluid velocity, E is the electric field and F is the external force. We normalize these quantities to the local equilibrium values of the plasma density n_o, ion acoustic velocity $(T_e/m_i)^{1/2}$ and the characteristic electric field $(T_e/e\,\lambda_D)$ and the space and time variables with respect to the local electron Debye length and ion plasma period. Then after eliminating the electron density from Eqs. (3.3) and (3.4) we get,

$$\frac{\partial n}{\partial t} + (\alpha + \beta/2)\, n v + \frac{\partial}{\partial x}(n v) = 0, \tag{3.5}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{\beta}{2} v^2 - E - F = 0 \tag{3.6}$$

and

$$nE - E\frac{\partial E}{\partial x} - \frac{1}{2}(\alpha+\beta)E^2 + (\alpha+\beta)\, n$$

$$+ \frac{\partial n}{\partial x} - \frac{3}{2}(\alpha+\beta)\frac{\partial E}{\partial x} - \frac{\partial^2 E}{\partial x^2} = 0, \tag{3.7}$$

where $\alpha = n^{-1}\,(dn/dx)$ and $\beta = T_e^{-1}(dT_e/dx)$ represent the density and electron temperature gradients. In the above equations, the terms containing α and β are due to the normalization with respect to the local equilibrium parameters which are themselves functions of space.

We expand E, n and v about the unperturbed state as,

$$\begin{bmatrix} E \\ n \\ v \end{bmatrix} = \begin{bmatrix} E_o \\ 1 \\ 0 \end{bmatrix} + \epsilon \begin{bmatrix} E^{(1)} \\ n^{(1)} \\ v^{(1)} \end{bmatrix} + \epsilon^2 \begin{bmatrix} E^{(2)} \\ n^{(2)} \\ v^{(2)} \end{bmatrix} + \ldots\ldots \qquad (3.8)$$

As in chapter II, we choose a plane wave solution for $E^{(1)}$ namely

$$E^{(1)} = a e^{i\psi} + \bar{a} e^{-i\psi}, \qquad (3.9)$$

where $\psi = kx - \omega t$, a is the complex amplitude and ω and k are related by the dispersion relation:

$$D(k,\omega) \equiv (\omega^2 - k^2 + \omega^2 k^2) = 0. \qquad (3.10)$$

With the choice $\alpha = \epsilon \rho$ and $\beta = \epsilon^2 \nu$ (ρ, ν being of order unity), on following the Krylov-Bogoliubov-Mitropolsky perturbation scheme, from Eqs. (3.5) - (3.7) to order ϵ, we can show that

$$n^{(1)} = \frac{ik}{\omega^2}\left(a e^{i\psi} - \bar{a} e^{-i\psi}\right) \qquad (3.11)$$

and

$$v^{(1)} = \frac{i}{\omega}\left(a e^{i\psi} - \bar{a} e^{-i\psi}\right) \qquad (3.12)$$

For $E^{(2)}$ Eqs. (3.5)-(3.7) to order ϵ^2 yields the equation,

$$(\omega^2-k^2)\,E^{(2)}-\omega^2k^2\frac{\partial^2E^{(2)}}{\partial\psi^2}+i\Big[\Big(\frac{\partial D}{\partial\omega}\Big)A_1$$

$$-\Big(\frac{\partial D}{\partial k}\Big)B_1+a\rho\frac{\omega^2}{k}(4+k^2)\Big]e^{i\psi}$$

$$+\frac{i a^2}{\omega^2 k}(\omega^4-3k^4)\,e^{2i\psi}+c.c=0 \qquad (3.13)$$

The removal of resonant secularity in $E^{(2)}$ in Eq.(3.13) demands that

$$A_1+V_g\,B_1+h\,a=0, \qquad (3.14)$$

where, $V_g = -(\partial D/\partial k)/(\partial D/\partial\omega)=(\omega^3/k^3)$ is the ion acoustic group velocity and $h = \rho\omega^3\,(4+k^2)/2k^3$. On replacing A_1 and B_1 by $\partial a/\partial t_1$, and $\partial a/\partial x_1$, where $t_1 = \epsilon t$ and $x_1 = \epsilon x$ and on using the substitution,

$$a = A\,e^{-h\,t_1} \qquad (3.15)$$

(3.14) can be rewritten as,

$$\frac{\partial A}{\partial t_1}+V_g\frac{\partial A}{\partial x_1}=0 \qquad (3.16)$$

This means that A depends on t and x only through $\xi = (x_1 - V_g t_1) = \epsilon\,(x - V_g t)$. So the wave amplitude 'a', as a whole damps with a rate h in a frame moving with velocity V_g.

Under the condition (3.14), we have the following secular free second order solutions:

$$E^{(2)} = i\frac{(3k^4-\omega^4)}{3k^3\omega^4}a^2e^{2i\psi} + b\,e^{i\psi} + c.c \tag{3.17}$$

$$n^{(2)} = -\frac{\left[3(\beta\omega^2+1)k^4-\omega^4\right]}{6\omega^6k^2}a^2e^{2i\psi}$$

$$+\left[\frac{(k^2-1)}{k^2}B_1 + \frac{ik}{\omega^2}b + a\rho\frac{(k^2-2)}{k^2}\right]e^{i\psi}$$

$$+c.c + \delta_n \tag{3.18}$$

and

$$v^{(2)} = -\frac{\left[3(\omega^2+1)k^4-\omega^4\right]}{6\omega^5k^3}a^2e^{2i\psi}$$

$$-\left[\frac{\omega}{k^3}B_1 - \frac{i}{\omega}b + a\frac{\rho\omega(4+k^2)}{2k^3}\right]e^{i\psi}$$

$$+c.c + \delta_v \tag{3.19}$$

where b is an arbitrary complex quantity independent of ψ. δ_n and δ_v are real constants which are determined as in chapter II using the condition that the third order solutions are free from secularities arising due to constant terms. They are found to be,

$$\delta_n = -(k^2+2)\,(1+k^2)^{3/2}\left[2k^4(k^4+3k^2+3)\right]^{-1}$$

$$\Big\{\left[2\alpha_+ + (6+k^2)\rho\right] e^{\alpha_+ \xi_1} \int_{-\infty}^{\xi_1} |a|^2 e^{-\alpha_+ \xi_1'} d\xi_1'$$

$$-\left[2\alpha_- + (6+k^2)\rho\right] e^{\alpha_- \xi_1} \int_{-\infty}^{\xi_1} |a|^2 e^{-\alpha_- \xi_1'} d\xi_1'\Big\}$$

$$+\left(\frac{2k}{\omega^3}\,V_g + 1\right)(V_g{}^2-1)^{-1}|a|^2 \tag{3.20}$$

and

$$\delta_v = -(k^2+2)\left[2\,k^4(k^4+3k^2+3)\right]^{-1}$$

$$\Big\{\left[\left(\frac{k^6}{\omega^6}+1\right)\alpha_+ + (2k^6+6k^4+7k^2+6)\rho\right]$$

$$e^{\alpha_+ \xi_1} \int_{-\infty}^{\xi_1} |a|^2 e^{-\alpha_+ \xi_1'} d\xi_1' - \left[\left(\frac{k^6}{\omega^6}+1\right)\alpha_- + (2k^6$$

$$+6k^4+7k^2+6)\rho\right] e^{\alpha_- \xi_1} \int_{-\infty}^{\xi_1} |a|^2 e^{-\alpha_- \xi_1'} d\xi_1'\Big\}$$

$$+\left(\frac{2k}{\omega^3} + V_g\right)(V_g{}^2-1)^{-1}|a|^2 \tag{3.21}$$

with

$$\alpha_\pm = \rho\left[k^2(k^4+3k^2+3)\right]^{-1}\Big[-(2k^6+6k^4$$

$$+5k^2-2) \pm (k^2+2)\,(k^2+1)^{3/2}\Big] \tag{3.22}$$

In determining the above relations it has been assumed that $|a|^2$ is bounded and all its derivatives vanish as $|\xi_1| \to \infty$.

Removing the resonant secularity from the third order solutions, we obtain the relation,

$$i\frac{\partial a}{\partial \tau} + i\left[\frac{1}{\epsilon}\frac{\omega^3}{2k^3}(4+k^2)\rho + \frac{3}{2}\frac{\omega^3}{k^3}\nu\right]a$$

$$-\rho\frac{\omega^5}{2k^4}(k^2+10)\frac{\partial a}{\partial \xi_1} + P\frac{\partial^2 a}{\partial \xi_1^2} + a\Big\{Q|a|^2$$

$$+\Big[X_+ e^{\alpha_+\xi_1}\int_{-\infty}^{\xi_1}|a|^2 e^{-\alpha_+\xi_1'}d\xi_1'$$

$$-X_- e^{\alpha_-\xi_1}\int_{-\infty}^{\xi_1}|a|^2 e^{-\alpha_-\xi_1'}d\xi_1'\Big]\Big\}$$

$$+Ra = 0, \tag{3.23}$$

where

$$P = \frac{1}{2}\frac{dV_g}{dk} = -\frac{3}{2}\frac{\omega^5}{k^4}, \tag{3.24}$$

$$Q = -(\omega^3/12k^6)(k^4+3k^2+3)^{-1}$$

$$(3k^{10}+6k^8-6k^6-29k^4-30k^2-12), \tag{3.25}$$

$$R = \frac{3}{8}\frac{\omega^5}{k^4}(k^2-8)\rho^2 a \tag{3.26}$$

and

$$X_{\pm} = (\rho/4k^5)(k^2+2)(k^4+3k^2+3)^{-2}$$

$$\Big[(k^{10}+9k^8+29k^6+44k^4+32k^2+8)$$

$$\pm 2(k^2+2)(k^2+1)^{5/2}(k^4+2k^2+2)\Big] \qquad (3.27)$$

The time variable in Eq.(3.23) is defined as $\tau = \epsilon^2 t$. Eq. (3.23) is the required modified nonlinear schrödinger equation. Apart from the usual nonlinearity and dispersion the additional features of this equation are the damping terms and the nonlinear nonlocal terms introduced by the inhomogeneities. So if we put $\rho = \nu = 0$, it reduces to the ordinary nonlinear schrödinger equation considered earlier (Kakutani and Sugimoto, 1974).

III.3 Discussions

The modified nonlinear schrödinger equation changes the characteristics of the time evolution of ion acoustic wave envelopes. In the following part of the discussion, first we give the linear and then the nonlinear analysis of different kinds of wave envelopes and their applications to laboratory plasmas, solar wind and solar corona.

i) <u>Linear Stability</u>

For a small amplitude plane wave solution for (3.23) namely,

$$a = a_0 \exp\left[i(K\xi - \Omega\tau)\right] \tag{3.28}$$

we get the dispersion relation,

$$\Omega = PK^2 + \left\{Q - (X_+/\alpha_+) + (X_-/\alpha_-)\right\} |a_0|^2 - i\left\{\frac{1}{\epsilon}(\omega^3/2k^3)(4+k^2)\rho + (\omega^5/2k^4) + (k^2+10)K\rho + (3\omega^3/2k^3)\nu\right\}, \tag{3.29}$$

where a_0 is a complex constant and K and Ω are wave number and frequency of the wave envelope. According to Eq. (3.29), the damping rate of the wave is given by

$$\gamma = -\left[(\omega^3/2k^3)(4+k^2)\alpha + (\omega^5/2k^4) + (k^2+10)K'\alpha + (3\omega^3/2k^3)\beta\right] \tag{3.30}$$

where $K' = \epsilon K$ ($\ll k$) is the envelope wave number as seen in the (x, t) co-ordinate system. The last two terms in (3.30) are very small compared to the first term. Consequently we observe that the effect of density gradient dominates over that of the temperature gradient for small values of K'. This makes γ almost same as the damping rate described by (3.15). If the wave is moving towards decreasing density, γ is + ve and the wave grows. Then it becomes modulationally unstable. This happens even for $k < k_c = 1.47$ where k_c

is the critical wave number above which the modulational instability sets in if the plasma is homogeneous. On the other hand when the wave is moving towards increasing density, it damps.

ii) Long Time Behaviour of the Wave Envelopes

The linear theory helps us to study only the initial stages of the development of the waves. But numerically solving Eq. (3.23), the long time behaviour of different finite amplitude wave envelopes can be studied (Mohan and Buti, 1979). For this we adopt the DuFort-Frankel scheme (Richtmyer and Morton, 1967; Smith, 1969) with asymptotic and periodic boundary conditions for the solutions. The accuracy of the procedure is checked by varying the time and space step sizes. Computational results are shown in Figs. 1-9. Figs. 1 and 2 show the evolution of an envelope soliton as it travels towards increasing density and temperature. The various parameters are same as those given in chapter II for Q-machine plasmas. For these figures we have taken the initial wave form which corresponds to the stationary solution of the nonlinear Schrödinger equation when $PQ > 0$. This is given by,

$$a = a_0 \operatorname{sech}\left[\left|\frac{Q}{2P}\right|^{1/2} a_0 \xi\right] \tag{3.31}$$

and is shown in Fig. 3 by the curve A. Curve B corresponds to the potential in an ordinary nonlinear schrödinger equation and curve C corresponds to the modified potential in the initial stages of the evolution when the nonlinear-nonlocal terms are included. In plotting C, the function $|a|^2$ in the nonlocal term is replaced by an equal-area rectangle with height equal to $(a_0^2/2)$. This causes the total effective attractive potential to shift to the left as shown by the curve D. As a result, some of the plasmons contained in the initial wave packet moves to the left, giving rise to a wave deformation which in turn leads to a further change in the potential and hence an additional shift. As more and more plasmons accumulate over the shifting potential trough, a new soliton is formed while the initial one damps. Further, as described by Eq. (3.30), the wave undergoes an initial damping as seen in Figs. 1 and 2. This is because the plasmons in the envelope encounter a force due to the density gradient namely, $-(\partial\omega/\partial x) \simeq -\frac{1}{2}\,\omega^3\alpha$ opposite to the direction of propagation. The wave loses part of its energy in overcoming this force which results in its damping. The energy thus spent is used in nonlinearly exciting more plasmons represented by the nonlinear nonlocal term in

(3.23). this causes the wave amplitude to grow afterwards. It is also seen from Fig.2 that an increase in the inhomogeneity scale lengths slows down the above mentioned processes.

An analytical treatment of envelope Langmuir solitons in plasmas with very weak inhomogenities has been done by Chen and Liu (1976,1978). They showed that the solitons undergo nonuniform acceleration without any change in their amplitudes and widths. Following the method of inverse scattering it was shown by Karpman (1979) that an envelope soliton evolving according to a pertburbed non linear Schrodinger equation does not develop tails or undergo splitting. This is true for a broad class of perturbations. But it should be noted that in the Eq.(3.23) the additional terms introduced by the inhomogeneities are not very small to be considered as perturbations and a direct comparison of the results cannot be made. A numerical study of a modified equation with a non-local term introduced by the Landau damping has also been carried out by Yajima et al.(1978). They find an asymmetric broadening of an envelope soliton due to the resonant particle interation. But again it does not split into two as in our case.

Figures 4and 5 show the time evolution of an envelope hole. This is a stationary wave envelope in a homogeneous plasma when $PQ < 0$. The initial wave form in this case is taken to be

$$a = a_0 \left\{ 1 - \tilde{a}^2 \operatorname{sech}^2\left[\left|\frac{Q}{2P}\right|^{1/2} \tilde{a}\, a_0 \xi\right]\right\}^{1/2}$$

$$\exp\left\{ i \sin^{-1}\left[\frac{\tilde{a}\, \operatorname{Tanh}\left[\left|\frac{Q}{2P}\right|^{1/2} \tilde{a}\, a_0 \xi\right]}{\left\{1 - \tilde{a}^2 \operatorname{sech}^2\left[\left|\frac{Q}{2P}\right|^{1/2} \tilde{a}\, a_0 \xi\right]\right\}^{1/2}}\right]\right.$$

$$\left. + i a_0 \left[\frac{Q}{P}\left(1 - \tilde{a}^2\right)\right]^{1/2} \xi \right\}, \qquad (3.32)$$

where $\tilde{a} < 1$ is a real quantity. This wave, after an initial damping is found to grow into two soliton-like humps. The contribution from the nonlocal term of (3.23) to the total potential in this case is shown by curve C in Fig.6. Once again, this is calculated by approximating $\tilde{a}^2 \operatorname{Sech}^2\left[\left|\frac{Q}{2P}\right|^{1/2} \tilde{a}\, a_0 \xi\right]$ appearing in $|a|^2$ of the nonlocal term by an equal area rectangle of hight, $(\tilde{a}^2/2)$. The effect of this is to lower and form an attractive region in the total potential as shown by the curve D. This causes the wave to have a larger amplitude on the right side than on the left side. However for large values of $|\xi|$, the amplitude of the wave decreases. This is because in this limit a becomes constant in space and according to (3.23), $\partial |a|^2/\partial \tau$ becomes negative. In this case also, we find that with a decrease in the inhomogeneity, these processes slow down.

In Figures 7 and 8 is shown the behaviour of an initial periodic disturbance given by,

$$a = a_0 + \tilde{a} \cos(l \xi) \tag{3.33}$$

This wave after an initial damping, nonlinearly excites other wave numbers and develops a spectrum. As a result, the initial perturbation grows into shapes with the same periodicity as the original wave. Within each period, there are two pulses which have the appearance of solitons. As in the case of envelope soliton, the time evolution of an envelope hole as well as a periodic disturbance in the presence of inhomogeneities are also different from those described by Yajima et al. (1978).

The time scales involved for these processes in the Q-machine are of the order of 10^{-6} Secs $\left[\sim 10^{2} (\omega_{pi})^{-1}\right]$ ($\tau = 1.2$ in Fig. 1 corresponds to $t \sim 10^{-6}$ Sec.).

In the cases we have discussed so far, we had taken $\lambda \sim L$, where λ is the wave length of the envelope and L is density gradient scale length. But if $\lambda \sim \epsilon L$, the initial part of the evolution of envelope solitons over very short intervals of time (in terms of τ) is same as the ones shown in Figs. 1 and 2. However, afterwards the amplitudes grow into large values and saturates. Then they are found to split into many soliton-like humps. This is shown in Figs. (9a) and (9b) for the cases of solar wind and solar corona with the following parameters:

i) <u>Solar Wind</u>

$T_e = 2 \times 10^5\ °K$, $n = 8\ cm^{-3}$, $\lambda_{De} = 6 \times 10^2$ cms,
$L = 10^{10}\ \lambda_{De}$, $\omega_{pi} = 3.7 \times 10^3\ Sec^{-1}$.

ii) <u>Solar Corona</u>

$T_e = 10^6\ °K$, $n = 10^6\ cm^{-3}$, $\lambda_{De} = 7$ cms,
$L = 10^5\ \lambda_{De}$, $\omega_{pi} = 1.3 \times 10^6\ Sec^{-1}$

Since the density gradient scale lengths in these plasmas are very large, the values of ϵ are extremely small. Because of this, although the time intervals involved in these processes are very small in terms of τ, in real time ($t = \epsilon^{-2}\tau$) they are quite large. For example, for the solar corona the time taken for the initial soliton to split into many solitons is of the order of 0.1 Sec $\left[\sim 10^5 (\omega_{pi})^{-1}\right]$. Whereas in the case of solar wind it turns out to be the order of 10^6 Secs $\left[\sim 10^9 (\omega_{pi})^{-1}\right]$ which is of the order of a month. Assuming that the ion acoustic instability in the solar wind arises around 0.9 AU it will take about 10^4 Secs ($\tau \sim 10^{-12}$ See - Fig. 9a) to reach the earth and by that time the initial soliton would not have started splitting. So it is rather difficult to observe the splitting of solitons in the solar wind near the earth. It should be further noted that even though the wave amplitudes attain large values, in the real space and time variables the wave amplitudes ($\sim \epsilon a$) still remain less than unity (cf. Eqs. (3.8) and (3.9)).

III.4 Summary

A modified nonlinear schrödinger equation, governing the envelope properties of ion acoustic waves in inhomogeneous plasmas is derived. For small amplitudes the wave gets damped while propagating towards increasing density. A finite amplitude wave, say an envelope soliton in a typical Q-machine plasma with width of the order of the density inhomogeneity scale length splits into two envelope solitons, one of which damps afterwards. In the case of an envelope hole, two soliton-like humps develop on either side of the central depression whereas a periodic modulation excites other wave numbers and develops a spectrum. An increase in the inhomogeneity scale lengths slows down these processes. The plasmas of solar wind and solar corona have extremely weak density gradients; however, envelope solitons with very small widths after long intervals of time are found to attain large amplitudes and then split into many solitons.

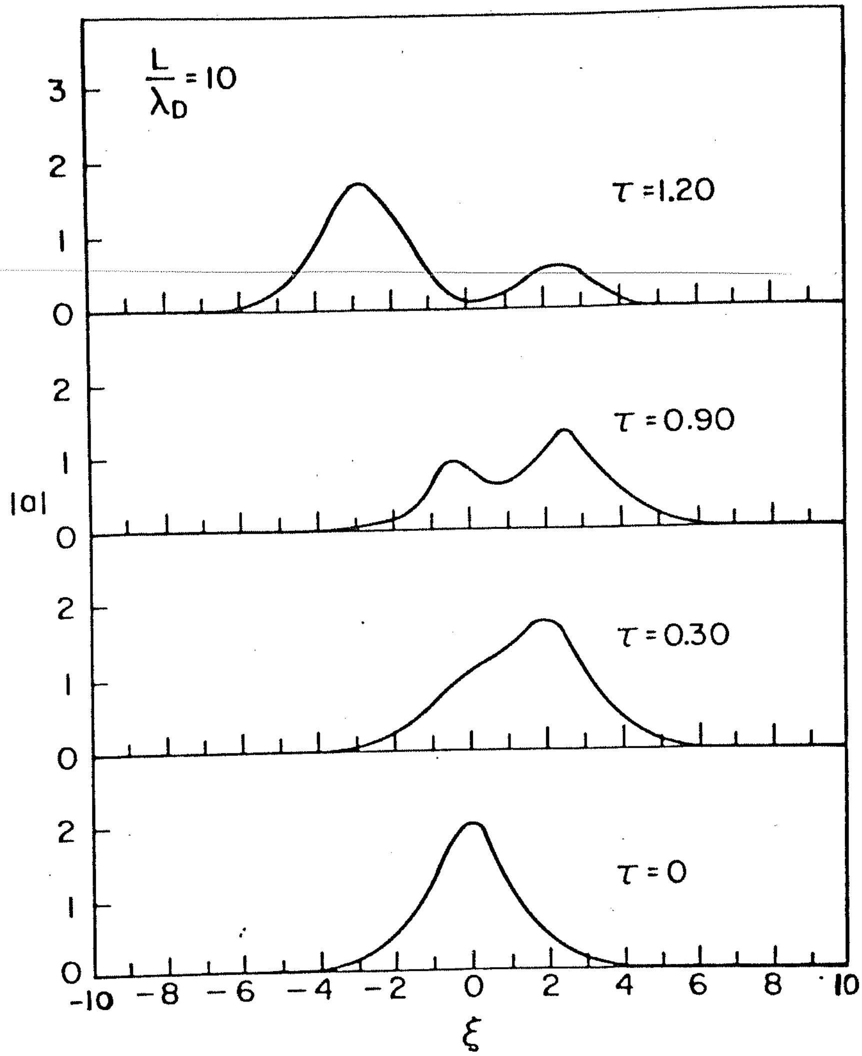

Fig. 1: (Q-machine). Evolution of an envelope soliton (PQ>0) for $\rho = \alpha/\epsilon = 1$, $\nu = \beta/\epsilon^2 = 1$, $k = 2$ and $a_o = 2$ with the ratio of the density gradient scale length to the Debye length, $L/\lambda_D = \alpha^{-1} = 10$.

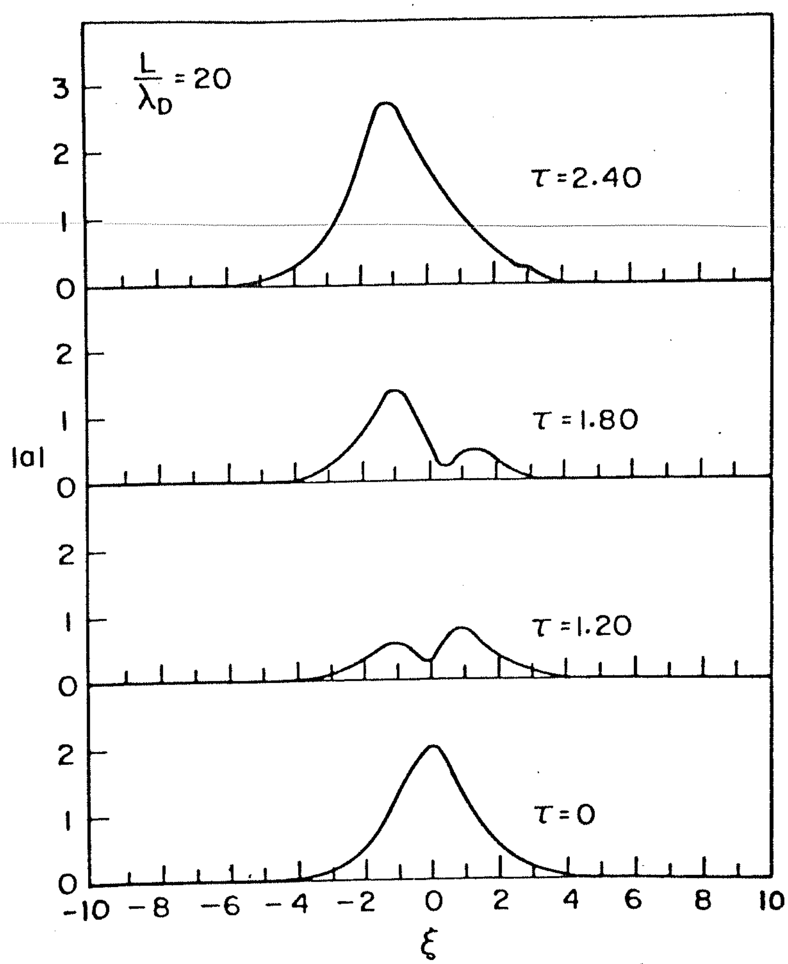

Fig.2: (Q-machine). Same as Fig.1 but for $L/\lambda_D = 20$.

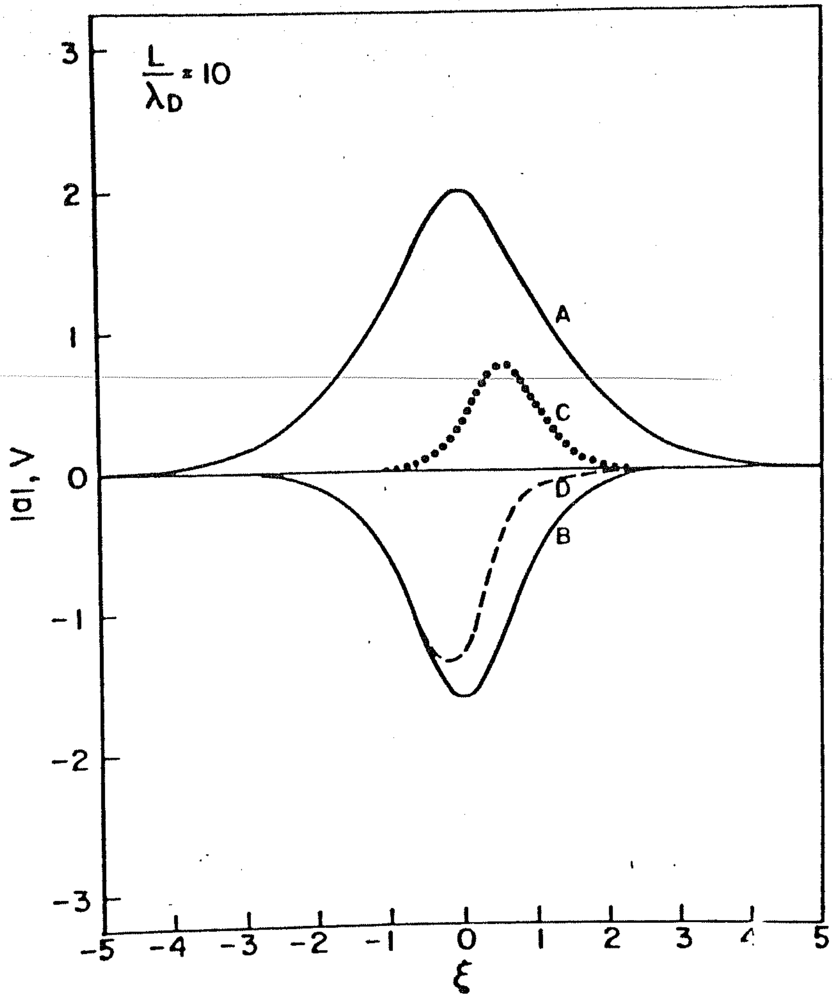

Fig.3; (Q-machine). The initial envelope soliton and the total potential V given by the nonlinear terms of (3.23) for $\rho = \alpha/\epsilon = 1$, $\nu = \beta/\epsilon^2 = 1$, $k = 2$, $a_0 = 2$ and $L/\lambda_D = 10$. The curves A and B correspond to the initial wave packet and the unmodified potential respectively. The curve C gives the contribution from the nonlocal term whereas the curve D represents the total effective potential.

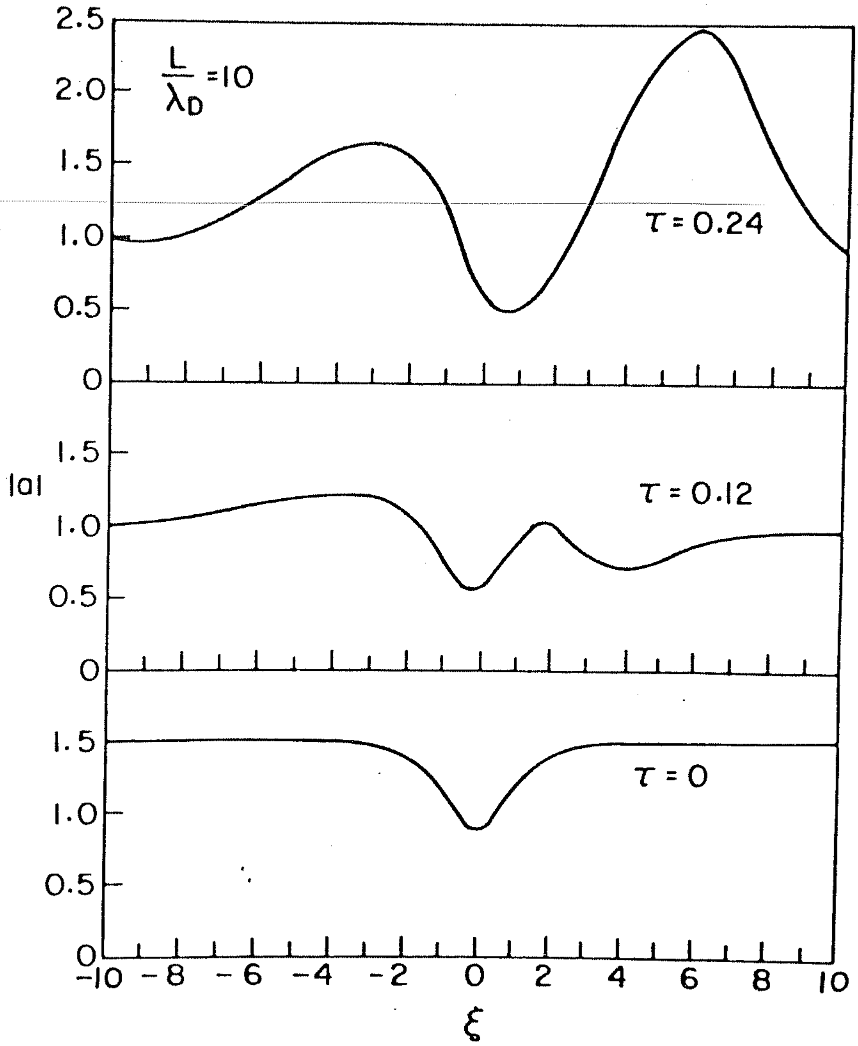

Fig.4: (Q-machine). Evolution of an envelope hole (PQ < 0) for $\rho = \alpha/\epsilon = 1$, $\nu = \beta/\epsilon^2 = 1$, $k = 1$, $a_o = 1.5$, $\tilde{\alpha} = 0.4$ and $L/\lambda_D = 10$.

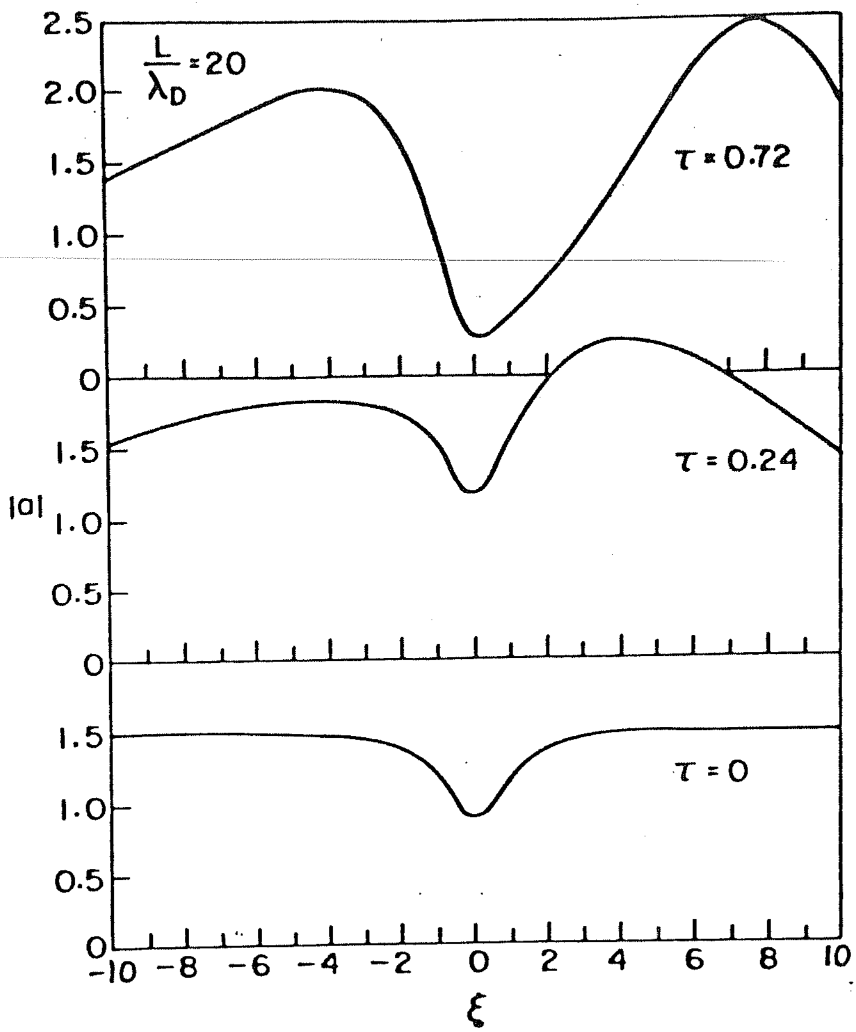

Fig.5: (Q-machine). Same as Fig.4 but for $L/\lambda_D = 20$.

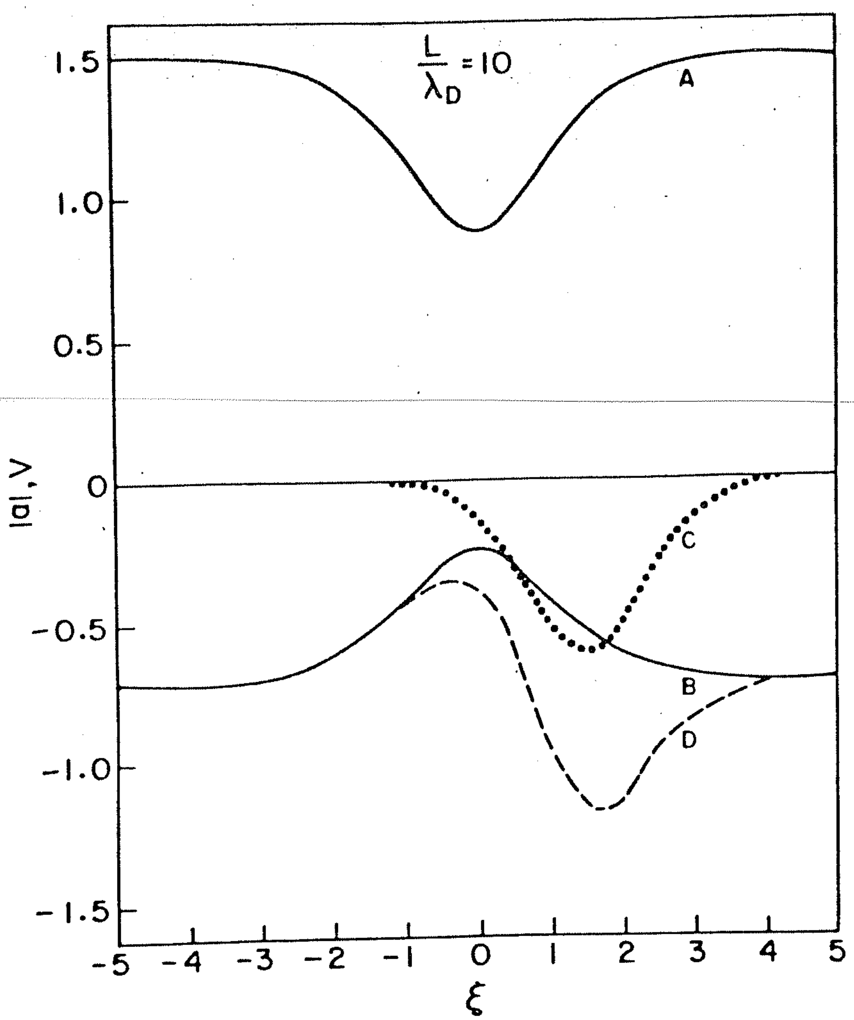

Fig. 6: (Q-machine). The initial envelope hole ($PQ < 0$) and the total potential V, given by the nonlinear terms of (3.23) for $\rho = \alpha/\epsilon = 1$, $\nu = \beta/\epsilon^2 = 1$, $k = 1$, $a_0 = 1{\cdot}5$, $\tilde{a} = 0{\cdot}4$ and $L/\lambda_D = 10$. The curve A represents the initial wave and the curve B represents the unmodified potential. The curves C and D give the contribution from the nonlocal term and the total effective potential respectively.

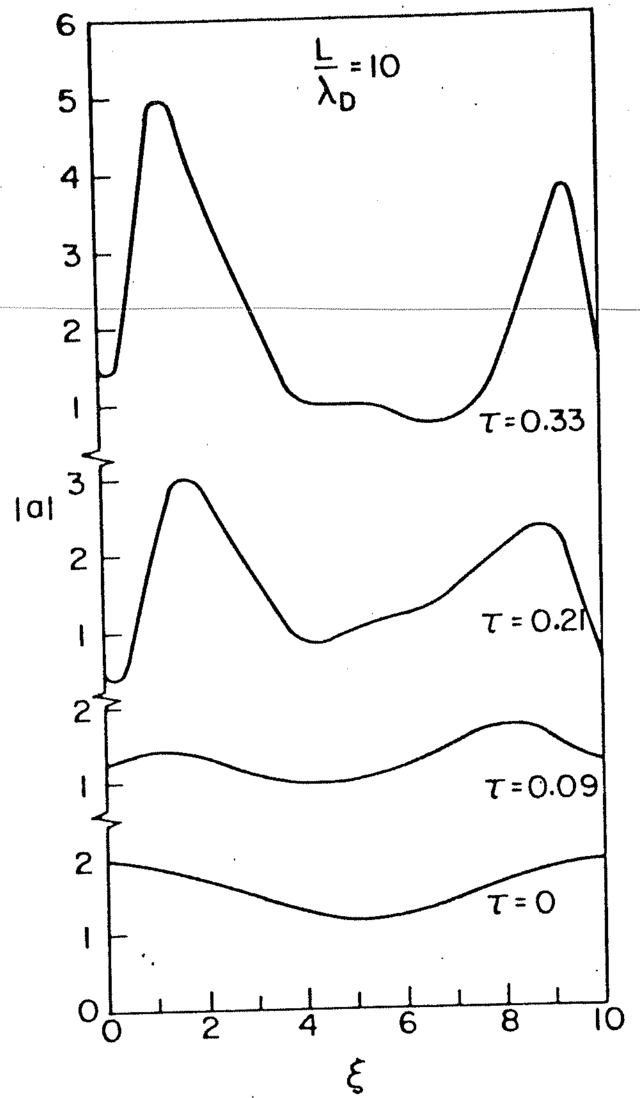

Fig.7: (Q-machine). Evolution of a periodic modulation (PQ <0) for $\rho=\alpha/\epsilon=1$, $\nu=\beta/\epsilon^2=1$, $k=1$, $a_0=1.6$, $\tilde{a}=0.4$, $\ell=\pi/5$ and $L/\lambda_D = 10$.

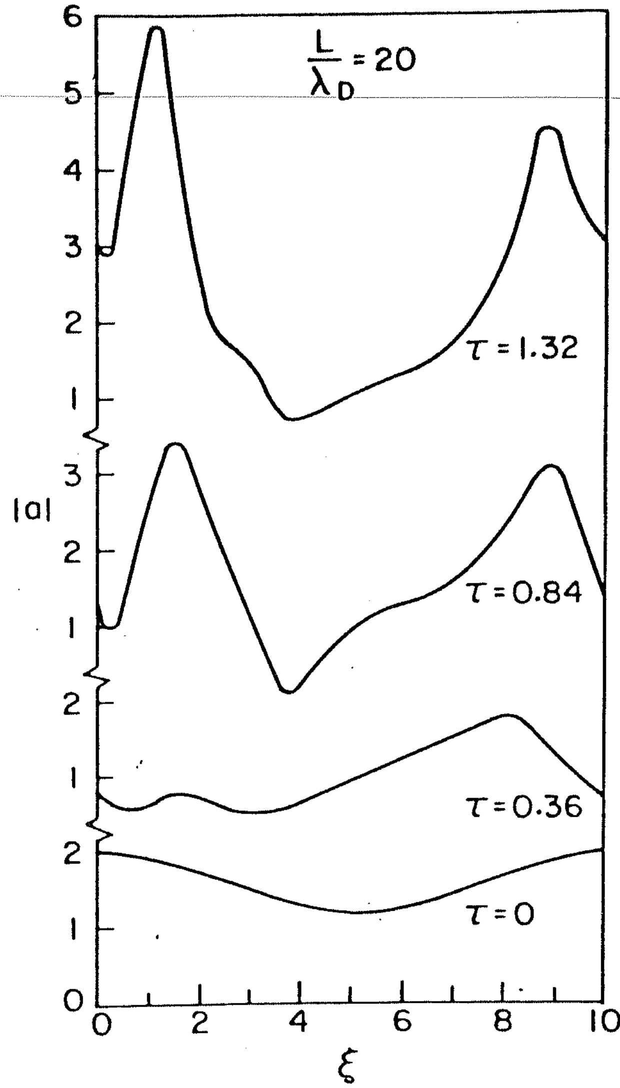

Fig.8: (Q-machine). Same as Fig.7 but for $L/\lambda_D = 20$.

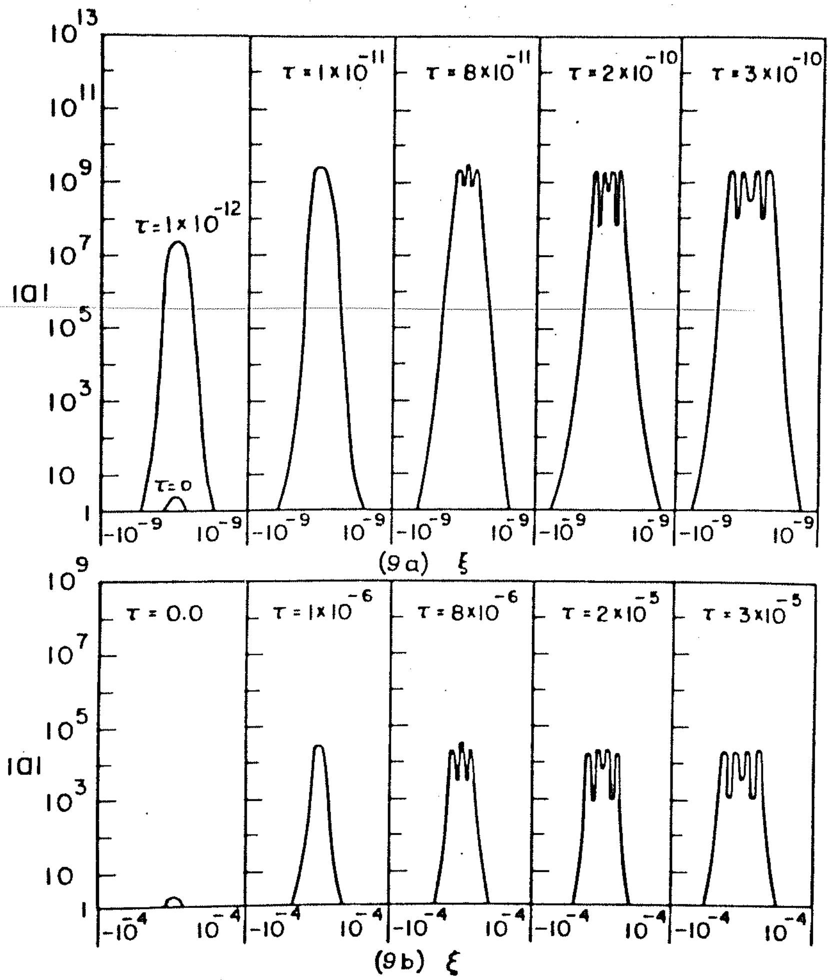

Fig.(9a): (Solar Wind). Evolution of an envelope soliton ($PQ > 0$) for $\rho = \alpha/\epsilon = 1$, $\nu = 0$, $k = 2$, $a_o = 2$ and $L/\lambda_D = 10^{10}$.

Fig.(9b): (Solar Corona). Same as Fig.(9a) but for $L/\lambda_D = 10^5$.

CHAPTER IV

ELECTRON ACOUSTIC SOLITONS IN CURRENT-CARRYING MAGNETIZED PLASMAS

IV.1 Introduction

In a low β magnetized plasma, if the ions are much hotter than the electrons, then the ions can be considered to be nonmagnetized and the electrons to be magnetized. That is, for oscillations across the magnetic field with frequencies lying between the electron and ion gyrofrequencies, the electrons are rather tightly bound to the magnetic field whereas the ions move quite freely. In other words, the effective mass of an electron increases, the latter being $m_e^* = (m_e/\text{Cos}^2\,\theta)$, where m_e is the electron mass and θ is the angle of propagation of the oscillations with the magnetic field. For $\text{Cos}\,\theta < (m_e/m_i)^{1/2}$, the effective mass of the electron becomes larger than that of an ion. In analogy with the ion acoustic waves, one finds that these electron acoustic waves propagate with a phase velocity $v_p = (T_i\,\text{Cos}^2\theta\,/m_e)^{1/2}$; T_i being the ion temperature. Their wave lengths lie between the electron and ion Larmour radii. In most of the cases the thermal velocity of the ions is not very large compared to the electron acoustic phase velocity which makes these waves to undergo ion Landau damping. But if there are external electric fields or inhomogeneities in the plasma, then the electrons will acquire an averave drift across the magnetic field. The velocity of this drift adds to the phase velocity of the waves vectorially. Then there will be two kinds of electron acoustic waves: those which are faster than the electron drift and those which are slower than the electron drift. The faster waves undergo Landau damping whereas the slower ones are negative energy waves and they undergo Landau growth (Lashmore-Davies and Martin, 1973).

Magnetized plasmas with hot ions and cold electrons are expected to occur in magnetron type devices, plasma accelerators, collisionless shocks, in the terrestrial magnetosphere etc. Especially in the case of magnetospheric plasmas, there are some observations on the ring current around 7 R_E. Russel and Thorne (1970) and Frank (1971) have shown that the plasmas in this region have protons and electrons with energies of the order of 40 keV and 1-4 keV respectively. In the magnetotail also some times the ions are found to be hotter than the electrons (Frank et al., 1976)

The linear properties of electrostatic and electromagnetic electron acoustic waves have been studied by many authors (Arefev, 1970; Lashmore-Davies and Martin, 1973 ; Sizonenko and Stepanov, 1969) to explain certain phenomena such as the rapid turbulent heating, anomalous conductivity, anomalous diffusion, radiation in theta pinches etc. Goedbloed and coworkers (1973) have extended their theory to the nonlinear regime in the electromagnetic case. By the method of averaging over random phases; they have estimated the level of turbulence attained by the field fluctuations.

In In this chapter, we consider the propagation of electrostatic electron acoustic waves in a weakly nonlinear plasma. The nonlinearity, dispersion and ion Landau damping are systematically taken into account. By means of reductive perturbation technique (Washimi and Taniuti, 1966), we show that these nonlinear waves are governed by a modified K-dV equation. An analysis of this equation by the method of inverse scattering (Karpman and Maslov, 1977; Karpman,1978; 1979) gives the time evolution of electron acoustic solitons.

We observe the damping or growth of these solitons along with the appearance of the tails. The rates of these processes increase with a decrease in the angle of propagation.

IV.2 Linear Dispersion Relation

We consider a low β, collisionless plasma. The magnetic field is in the z-direction and the electrons have an average drift V_o in the y-direction. The ion temperature (T_i) is taken to be much larger than the electron temperature (T_e). Let the propagation direction lie in the y-z plane making an angle θ with the magnetic field. The angle θ is such that the phase velocity in the z-direction is greater than the electron thermal velocity, $V_{Te} = (T_e/m_e)^{1/2}$. Consequently for the electron dynamics, electron inertia is not negligible and they are governed by fluid-equations. But for the ions, we use Vlasov equation so that the ion-Landau damping can be taken into account. In this equation the magnetic field is ignored since the ions are considered to be nonmagnetized. Thus the basic equations for the system are,

$$\frac{\partial n_e}{\partial t} + \vec{\nabla} \cdot (n_e \vec{v}_e) = 0, \tag{4.1}$$

$$\frac{\partial \vec{v}_e}{\partial t} + (\vec{v}_e \cdot \vec{\nabla})\,\vec{v}_e = \frac{e}{m_e}\vec{\nabla}\phi - \frac{e}{m_e c}\,\vec{v}_e \times \vec{B} - \frac{T_e}{n_e m_e}\,\vec{\nabla} n_e, \qquad (4.2)$$

$$\nabla^2 \phi = 4\pi e\left[n_e - \int_{-\infty}^{\infty} f\,dv\right] \qquad (4.3)$$

and

$$\frac{\partial f}{\partial t} + (\vec{v} \cdot \vec{\nabla}) f - \frac{e}{m_i}\vec{\nabla}\phi \cdot \frac{\partial f}{\partial \vec{v}} = 0 \qquad (4.4)$$

In these equations, n_e is the electron density, $\vec{v}_e$ is the electron fluid velocity, ϕ is the electric field potential, $\vec{B}$ is the magnetic field and f is the ion distribution function.

Defining a new space variable in the direction of propagation namely $\xi = v_y \sin\theta + v_z \cos\theta$ and using the following normalizations:

$$\xi \rightarrow \xi / \left[(T_i/T_e)^{1/2}\rho_e\right],\ t \rightarrow t\,\omega_{ce}\cos\theta_0,$$

$$v_{ej} \rightarrow v_j / \left[(T_i/m_e)^{1/2}\cos\theta_0\right];\ (j = x, y, z),$$

$$n_e \rightarrow n_e/n_0,\ f \rightarrow f/n_0,\ v \rightarrow \left[v/(T_i/m_i)^{1/2}\right],$$

and $$\phi \rightarrow -e\phi/T_i, \qquad (4.5)$$

where ρ_e is the electron-Larmour radius, n_o the equilibrium electron and ion density and θ_0 is specified by a fixed direction, Eqs.(4.1) to (4.4) can be rewritten as

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial \xi}(n_e v_\xi) = 0, \tag{4.6}$$

$$\frac{\partial v_x}{\partial t} + v_\xi \frac{\partial v_x}{\partial \xi} + (v_y - V_0)\frac{1}{\cos\theta_0} = 0, \tag{4.7}$$

$$\frac{\partial v_y}{\partial t} + v_\xi \frac{\partial v_y}{\partial \xi} + \frac{\sin\theta}{\cos^2\theta_0}\frac{\partial \phi}{\partial \xi} - \frac{v_x}{\cos\theta_0} + \frac{T_e}{T_i}\frac{\sin\theta}{\cos^2\theta_0}\frac{1}{n_e}\frac{\partial n_e}{\partial \xi} = 0, \tag{4.8}$$

$$\frac{\partial v_z}{\partial t} + v_\xi \frac{\partial v_z}{\partial \xi} + \frac{\cos\theta}{\cos^2\theta_0}\frac{\partial \phi}{\partial \xi} + \frac{T_e}{T_i}\frac{\cos\theta}{\cos^2\theta_0}\frac{1}{n_e}\frac{\partial n_e}{\partial \xi} = 0, \tag{4.9}$$

$$\left(\frac{\omega_{ce}}{\omega_{pe}}\right)^2 \frac{\partial^2 \phi}{\partial \xi^2} = \int_{-\infty}^{\infty} f\, dv - n_e \tag{4.10}$$

and

$$\left(\frac{C_s}{V_{Ti}}\right)\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial \xi} + \frac{\partial \phi}{\partial \xi}\frac{\partial f}{\partial v} = 0. \tag{4.11}$$

Here $v_\xi = v_y \sin\theta + v_z \cos\theta$ is the component of the electron fluid velocity in the ξ - direction.

Linearising Eqs. (4.6) - (4.11), keeping in mind that $0 < (\cos\theta_0\ ,\ \cos\theta\ ) \ll (m_e/m_i)^{1/2}$ and $c_s < v_{Ti}$, we get the dispersion relation,

$$(\omega - k V_0 \sin\theta)^2 \Big[1 + i\,(1 + k^2 \sin^2\theta)(\pi/2)^{1/2} \left(\frac{\omega}{k}\right) \frac{C_s}{V_{Ti}} \Big] = k^2 (1 + k^2 \sin^2\theta)\, \frac{\cos^2\theta}{\cos^2\theta_0} \qquad (4.12)$$

This gives the frequency and growth rate in the small wave number limit as,

$$\omega_r = k V_0 \sin\theta \pm k \frac{\cos\theta}{\cos\theta_0} \pm \frac{1}{2} k^3 (\sin^2\theta \cos\theta / \cos\theta_0) \qquad (4.13)$$

and

$$\gamma = \mp (\pi/8)^{1/2} \frac{\cos\theta}{\cos\theta_0}\, k \left(\frac{C_s}{V_{Ti}}\right) \left(V_0 \sin\theta \mp \frac{\cos\theta}{\cos\theta_0} \right) \qquad (4.14)$$

The - ve and + ve signs correspond to the phase velocity being greater or less than $V_c \sin\theta$. Since $\sin\theta \simeq 1$, this means that the waves will damp or grow according as they are faster or slower than the electron drift. Moreover from (4.14) we see that $|\gamma|$ increases as θ decreases.

IV.3 Modified K-dV Equation

Next, we consider the propagation of small but finite amplitude electron acoustic waves. Following the method of reductive perturbation (discussed in chapter I), we introduce the stretched variables,

$$\eta = \epsilon^{1/2}(\xi - u t) \quad \text{and} \quad \tau = \epsilon^{3/2} t \tag{4.15}$$

Here, the smallness parameter ϵ is taken to be the same as $\cos^2\theta_0$ and $u = V_0 \sin\theta \pm (\cos\theta / \cos\theta_0)$, is the wave phase velocity. Further, we define,

$$\cos\theta = \alpha_1 \epsilon^{1/2}, \; (C_s / V_{Ti}) = \alpha_2 \Delta,$$

$$(T_e/T_i) = \alpha_3 \epsilon \quad \text{and} \quad (\omega_{ce}/\omega_{pe})^2 = \alpha_4 \tag{4.16}$$

where α_1, α_2, α_3 and α_4 are of order unity and Δ is such that $\epsilon < \Delta < 1$. For a weakly nonlinear system, we use the following expansions:

$$n_e = 1 + \epsilon n^{(1)} + \epsilon^2 n^{(2)} + \cdots, \tag{4.17}$$

$$v_x = \epsilon v_x^{(1)} + \epsilon^2 v_x^{(2)} + \cdots, \tag{4.18}$$

$$\phi = \epsilon\phi^{(1)} + \epsilon^2 \phi^{(2)} + \cdots, \tag{4.19}$$

and

$$f = f_0 + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots, \tag{4.20}$$

with $f_0 = (2\pi)^{-1/2} \exp(-v^2/2)$. As far as the v_y and v_z

expansions are concerned, we must satisfy the inequality, $v_y - V_o < v_x < v_z$. This restriction can be easily understood by going into the reference frame of the drifting electrons: The fluctuating electrostatic field of the wave is almost in the y-direction. So in the linear limit, it cannot produce any fluid motion in the y-direction because of the external magnetic field. Any fluid motion in the y-direction can only be due to the nonlinearity. The second part of the inequality follows from the requirement that the wave frequency is much less than the electron gyrofrequency. Further to make the expansions (4.17) - (4.20) to be consistent with (4.15) and (4.16), we choose the following expansions for v_y and v_z:

$$v_y = V_0 + \epsilon^2 v_y^{(1)} + \epsilon^3 v_y^{(2)} + \cdots \qquad (4.21)$$

and

$$v_z = \epsilon^{1/2} v_z^{(1)} + \epsilon^{3/2} v_z^{(2)} + \cdots \qquad (4.22)$$

Making use of (4.17) - (4.22), Eqs. (4.6) - (4.11) to the lowest order in ϵ yield,

$$n^{(1)} = \phi^{(1)}, \qquad (4.23)$$

$$v_x^{(1)} = \frac{\partial \phi^{(1)}}{\partial \eta} \qquad (4.24)$$

$$v_y^{(1)} = (u - V_0) \frac{\partial^2 \phi^{(1)}}{\partial \eta^2}, \qquad (4.25)$$

$$v_z^{(1)} = \frac{\alpha_1}{(u - V_0)} \phi^{(1)} \tag{4.26}$$

and

$$f^{(1)} = f_0 \phi^{(1)} \tag{4.27}$$

To the next order in ϵ on using the results (4.23) - (4.27), we obtain,

$$2 \frac{\partial \phi^{(1)}}{\partial \tau} + (a\alpha_3 - \alpha_1^2 V_0) \frac{\partial \phi^{(1)}}{\partial \eta} + 3a \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \eta} + a \frac{\partial^3 \phi^{(1)}}{\partial \eta^3} + a \frac{\partial}{\partial \eta} (\phi^{(2)} - n^{(2)}) = 0, \tag{4.28}$$

$$\alpha_4 \frac{\partial^2 \phi^{(1)}}{\partial \eta^2} = \int_{-\infty}^{\infty} f^{(2)} dv - n^{(2)} \tag{4.29}$$

and

$$\frac{\partial}{\partial \eta} \left(\phi^{(2)} - \int_{-\infty}^{\infty} f^{(2)} dv\right) = -\phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \eta} + \Delta \alpha_2 u (2\pi)^{-1/2} P \int_{-\infty}^{\infty} \frac{\partial \phi^{(1)}}{\partial \eta} \frac{d\eta'}{(\eta - \eta')} \tag{4.30}$$

where $a = u - V_0$ and P denotes the principal part. (4.30) is similar to the corresponding relationship derived earlier by Ott and Sudan (1969) in the case of ion acoustic waves. Eliminating $\phi^{(2)}$, $f^{(2)}$ and $n^{(2)}$ from these equations, we get the following modified K-dV equation:

$$\frac{\partial \phi}{\partial \tau} + A\frac{\partial \phi}{\partial \eta} + a\phi\frac{\partial \phi}{\partial \eta} + B\frac{\partial^3 \phi}{\partial \eta^3} + \Delta D P\int_{-\infty}^{\infty}\frac{\partial \phi}{\partial \eta'}\frac{d\eta'}{(\eta-\eta')} = 0, \tag{4.31}$$

where,

$$A = \frac{a}{2}\left(\alpha_3 - a V_0\right), \tag{4.32}$$

$$B = \frac{a}{2}\left(1+\alpha_4\right) \tag{4.33}$$

and

$$D = a\alpha_2 u (8\pi)^{-1/2} \tag{4.34}$$

In Eq.(4.31) $\phi^{(1)}$ has been replaced by ϕ.

IV.4 Discussions

The last term in Eq.(4.31) represents the ion-Landau damping. To check this, we linearise this equation and then Fourier analyse it (Davidson, 1972). Let us define,

$$\phi(\eta,\tau) = \int_{-\infty}^{\infty}\phi_{k'}(\tau)\, e^{ik'\eta}\, dk' \tag{4.35}$$

By making use of the inverse transformation (Lighthill, 1964; Ott and Sudan, 1969) namely,

$$i\int_{-\infty}^{\infty}(\mathrm{sign}\, k')\, e^{ik'\eta}\, dk' = -2P\frac{1}{\eta}, \tag{4.36}$$

and solving for $\phi_{k'}(\tau)$, we obtain,

$$\phi_{k'}(\tau) = \phi_{k'}(\tau=0)\exp\left[i k'^3 \tau/2 + \hat{\gamma}_{k'}\tau\right] \tag{4.37}$$

Here $\hat{\gamma}_{k'}$ is the growth rate of a wave with wave number k'. In (ξ, t) frame of reference, this becomes,

$$\gamma_k = \mp\left(\frac{\pi}{8}\right)^{1/2} \frac{\cos\theta}{\cos\theta_0}\left(\frac{C_s}{V_{T_i}}\right)\left(V_0 \sin\theta \mp \frac{\cos\theta}{\cos\theta_0}\right)k, \tag{4.38}$$

which is same as (4.14).

However, if the wave has finite amplitude even if it is small, the nonlinearity is not negligible. Eq. (4.31) without the last term has a stationary solution of the form,

$$\phi = \phi_0 \operatorname{sech}^2\left[\frac{1}{L}(\eta - V\tau)\right], \tag{4.39}$$

where ϕ_0 is the amplitude, $L = (\phi_0 a/12B)^{-1/2}$ is the width and $V = (\phi_0 a/3 + A)$ is the velocity.

Now, applying the inverse scattering analysis (Karpman and Maslov, 1977; Karpman, 1978, 1979) on Eq. (4.31) we obtain the time evolution of waves which are initially of the form (4.39) (Mohan and Buti, 1980). We rewrite Eq. (4.31) as,

$$\frac{\partial \Phi}{\partial T} - 6\Phi\frac{\partial \Phi}{\partial X} + \frac{\partial^3 \Phi}{\partial X^3} = \Delta R[\Phi], \tag{4.40}$$

where we have used the transformations,

$$\Phi = -\frac{a}{6} B^{-1/3}\phi \tag{4.41}$$

$$X = B^{-1/3}(\eta - A\tau), \tag{4.42}$$

$$T = \tau \tag{4.43}$$

and

$$R[\Phi] = -D B^{1/3} P \int_{-\infty}^{\infty} \frac{\partial \Phi}{\partial X'} \frac{dX'}{(X-X')} \tag{4.44}$$

Eq.(4.40) can further be rewritten in an operator form (Lax, 1968; Scott et al., 1973) as,

$$i \frac{\partial \hat{L}}{\partial T} + [\hat{L}, \hat{A}] = i \Delta R[\Phi], \tag{4.45}$$

where

$$\hat{L} = -\frac{\partial^2}{\partial X^2} + \Phi \tag{4.46}$$

and

$$\hat{A} = -4i \frac{\partial^3}{\partial X^3} + 3i \frac{\partial \Phi}{\partial X}$$

$$+ 3i \Phi \frac{\partial}{\partial X} \tag{4.47}$$

Consider now the eigenvalue problem,

$$\hat{L}\{\Phi(X,T)\}\psi(X,T) = \lambda(T)\psi(X,T), \tag{4.48}$$

where $\Phi(X,t)$ is the solution of Eq.(4.40). After differentiating (4.48) with respect to T and using (4.45), we get,

$$(\hat{L} - \lambda)\left(\frac{\partial \psi}{\partial T} + i \hat{A} \psi\right) = -\Delta R[\Phi]\psi$$

$$+ \frac{\partial \lambda}{\partial T}\psi \tag{4.49}$$

For the continuous region of the eigenvalue spectrum, $\frac{\partial \lambda}{\partial T} = 0$. But to find the time variation in the discrete region, we multiply (4.49) with ψ^* and integrate over X; this leads to,

$$\frac{d\lambda}{dT} = \Delta \frac{\int_{-\infty}^{\infty} \psi^* R[\Phi] \psi \, dx}{\int_{-\infty}^{\infty} \psi^* \psi \, dx} \tag{4.50}$$

Now, since each of the discrete eigenvalues changes in time, the shapes of the corresponding solitons are no longer invariant. However, the rate of the time variation is very small, since $\Delta < 1$. So for time intervals, $T \ll (1/\Delta)$, it is justified to look for solutions of Eq. (4.40) of the form,

$$\Phi(X,T) = \Phi_s\left[(X - \bar{V}(T)T), T\right] + \delta\Phi\left[(X - \bar{V}(T)T), T\right], \tag{4.51}$$

where

$$\Phi_s = \Phi_0(T)\, \mathrm{sech}^2\left[\frac{1}{\bar{L}(T)}(X - \bar{V}(T)T)\right] \tag{4.52}$$

and

$$\delta\Phi = \Phi_0(T)\, W\left[(X - \bar{V}(T)T), T\right], \tag{4.53}$$

with

$$\bar{L}(T) = \left[-2/\Phi_0(T)\right]^{1/2}, \tag{4.54}$$

$$\bar{V}(T) = -2\Phi_0(T) \tag{4.55}$$

and

$$\delta \Phi << \Phi_s \tag{4.56}$$

At T = 0, W vanishes and Φ_s corresponds to the solution (4.39).

To solve the eigenvalue equation (4.48), we approximate Φ occuring in $\hat{L}$ by Φ_s ; this gives,

$$\psi(X,T) = \frac{1}{2}\, \mathrm{sech}\left[\frac{1}{L(T)}(X - \bar{V}(T)\,T)\right] \tag{4.57}$$

and

$$\lambda(T) = \left[\Phi_0(T)/\alpha\right]. \tag{4.58}$$

Substituting (4.57) and (4.58) in (4.50), to the first order in Δ , we get,

$$\frac{d\Phi_0}{dT} = \Delta \int_{-\infty}^{\infty} R\left[\Phi_s\right] \mathrm{sech}^2 Z\, dZ, \tag{4.59}$$

where $Z = (X - \bar{V}(T)\,T)/L(T)$. From this we immediately obtain,

$$\Phi_0(T) = \Phi_0(0) + \Delta\, G\, T, \tag{4.60}$$

where

$$G = - D\, B^{-1/3}\, \Phi_0\, P\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d}{dZ'}\left(\mathrm{sech}^2 Z'\right) \frac{\mathrm{sech}^2 Z}{(Z - Z')}\, dZ'\, dZ \tag{4.61}$$

Integration of (4.40) over X, gives

$$\frac{d}{dT} \int_{-\infty}^{\infty} \Phi \, dX = 0 \tag{4.62}$$

This shows that the total area under the curve Φ remains invariant. However, the area of the soliton part of the solution Φ as given by (4.51) changes with time. To compensate this the area corresponding to $\delta\Phi$ also has to change. Substituting (4.51) in (4.62) and using (4.52), (4.53) and (4.61), we get

$$\frac{d}{dT}\left[\int_{-\infty}^{\infty} \Phi_0 W \, dz\right] = -2\Delta G,$$

that is,

$$\int_{-\infty}^{\infty} \Phi_0 W \, dz = 2\Delta G T \tag{4.63}$$

Hence the area of the tail is proportional to time. But since $\delta\Phi \ll \Phi_s$, the nonlinearity in $\delta\Phi$ is very small and it will lag behind the soliton. This gives it the appearance of a tail. The formation of such tails has been observed numerically in an earlier study by Watanabe (1977) in the case of KdV equations modified to include the effects of collisions and viscosity.

By dividing (4.63) by the distance travelled by the soliton in time T, we get the approximate height of the tail as,

$$H \simeq -\frac{\Delta G}{\Phi_0} . \tag{4.64}$$

To compute the integral G, we rewrite it in the following form (Ott and Sudan, 1969):

$$G = -\frac{D B^{-1/3}}{2} \Phi_0 \int_{-\infty}^{\infty} |k| \, F(k) \, F(-k) \, dk , \tag{4.65}$$

where $F(k) = (\pi/2)^{1/2} k \text{ Cosech } (k\pi/2)$, is the fourier transform of $\text{Sech}^2 Z$. Since F(k) is symmetric in k, (4.65) can be evaluated to yield (Gradshteyn and Rizhik, 1965),

$$G = -D B^{-1/3} \Phi_0 \left(\frac{\pi}{2}\right) \int_0^{\infty} k^3 \, \text{Cosech}^2 \left(\frac{k\pi}{2}\right) dk , \tag{4.66}$$

$$= -D B^{-1/3} \Phi_0 \left(\frac{6}{\pi^3}\right) \zeta(3) , \tag{4.67}$$

where ζ (3) is a Reimann Zeta function.

Transforming back into the η, τ frame of reference, we find that for $\tau \ll (1/\Delta)$, the amplitude, width and velocity of the soliton as well as the length and height of its tail at any time are given by,

$$\phi_0(\tau) = \phi_0(0) \left[1 - 2 \Omega \tau \right] , \tag{4.68}$$

$$L(\tau) = L(0)\left[1 + \frac{\Omega\tau}{2}\right], \tag{4.69}$$

$$V(\tau) = \frac{a}{3}\phi_0(\tau) + A, \tag{4.70}$$

$$l = \frac{a}{3}\phi_0(\tau)\,\tau \tag{4.71}$$

and

$$h = \left[a(1+\alpha_4)/2\right]^{1/3} \frac{3\Omega L(0)}{16\,L(\tau)} \tag{4.72}$$

where,

$$\Omega = \left[\Delta\, 12 a \alpha_2 u\, \zeta(3)\right]\left[2^{1/2}\pi^{7/2}L(0)\right]^{-1} \tag{4.73}$$

The growth rate of the soliton amplitude given by (4.68) is,

$$\Gamma = -2\Omega \tag{4.74}$$

We have computed Γ for various angles of propagation for damping as well as growing solitons. These are given in Table 1. We find that, as in the linear case, here also Γ increases when the angle of propagation decreases. But if there are changes in the electron to ion temperature ratio, the pressure in the medium gets affected and it changes the soliton velocity according to (4.70) while its amplitude and width remain unchanged.

IV.5 Conclusions

Collisionless magnetized plasmas with low β with hot ions and cold electrons can sustain electron acoustic waves. In case of weak nonlinearity and dispersion they are described by a modified K-dV equation. Initial perturbations in the form of solitons undergo ion Landau damping or growth; simultaneously giving rise to tails. The rates of these processes increase as the angle of propagation of the wave with the magnetic field decreases.

Table 1

Damping and growth rate of the soliton for various angles of propagations with $\alpha_2 = 3$, $V_o = 4$ and $L(0) = 1$.

$\alpha_1 = \frac{\cos\theta}{\cos\theta_o}$	(Rate of Damping) $\times \Delta^{-1}$	(Rate of Growth) $\times \Delta^{-1}$
0.5	0.6266	0.4874
0.75	0.9922	0.6789
1.0	1.3926	0.8353
1.25	1.828	0.9572
1.5	2.298	1.462

CHAPTER V

EXACT ELECTRON ACOUSTIC SOLITARY WAVES

V.1 Introduction

In chapter IV, we discussed the formation and evolution of electron acoustic waves in a weakly nonlinear plasma. There the electrons were described by means of fluid equations and the ions by means of the Vlasov equation which helped us to consider the ion-Landau damping. As a consequence of the ion-Landau damping, the waves were seen to undergo deformations such as damping, growth and tail formations. This prevented us from finding stationary solutions of the KdV equation governing the electron acoustic waves.

On the other hand, if the thermal velocity of the ions is much larger than the electron acoustic phase velocity, then the ion-Landau damping is completely negligible. The ions can then very well be described by a Boltzmann distribution instead of the Vlasov equation. In this chapter, we consider this case. Then taking into account the complete electron and ion nonlinearities we reduce the original set of equations into a single equation. This equation is similar to the energy integral of a classical particle of unit mass. The potential energy of the particle is referred to as the Sagdeev potential. A careful analysis of the Sagdeev potental reveals some important characteristics of stationary solitary wave solutions to the basic equations. The main advantage in this scheme is that the wave amplitudes need no longer be small since the full nonlinearity in the system has been included. This kind of analysis was first carried out by Sagdeev (1966) for ion acoustic waves. He found that finite amplitude ion acoustic solitary waves can exist with an upper limit on their amplitudes. Recently Buti (1980) studied these nonlinear ion-acoustic waves in a two-electron-temperature plasma and showed the possible existence of solitons and holes (corresponding to density dips in the plasma).

In case of electron acoustic waves also, we show that there can be both solitons and holes. However, the hole solutions have widths much less than the characteristic length scale in the system which invalidates their description in terms of a fluid theory.

V.2 Basic Equations and the Sagdeev Potential

The plasma considered is same as the one described in chapter IV except that we assume that there are no zero order electron drifts. Further we replace the Poisson's equation with the quasi-neutrality condition. That is, the electron and ion density perturbations are taken to be the same. This is justified as long as the wavelengths are much larger than the characteristic length scales $((T_i/T_e)^{1/2}\rho_e$, where ρ_e is the electron-Larmour radius) in the medium. Besides, with this condition it becomes possible to integrate the basic equations governing the system exactly. The basic equations are,

$$\frac{\partial n_e}{\partial t} + \vec{\nabla}\cdot\left(n_e \vec{v}_e\right) = 0, \tag{5.1}$$

$$\frac{\partial \vec{v}_e}{\partial t} + \left(\vec{v}_e\cdot\vec{\nabla}\right)\vec{v}_e = \frac{e}{m_e}\vec{\nabla}\phi - \frac{e}{m_e c}\vec{v}_e\times\vec{B} - \frac{T_e}{n_e m_e}\vec{\nabla} n_e \tag{5.2}$$

and

$$n_e = n_i = n_0 \exp\left[-(e\phi/T_i)\right] \tag{5.3}$$

We assume that the wave is propagating in the y-z plane making an angle θ with the magnetic field and define a new space variable $\xi = y \sin\theta + z \cos\theta$. We normalize these equations in the following way:

$$\xi \to \xi / \left[(T_i/T_e)^{1/2} \rho_e\right], \; t \to t\,\omega_{ce} \cos\theta,$$

$$v_j \to v_j / \left[(T_i/m_e)^{1/2} \cos\theta\right]; (j = x, y, z),$$

$$n \to n_i/n_0 = n_e/n_0, \; \phi \to -e\phi/T_i.$$

Then we have,

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \xi}(n v_\xi) = 0, \tag{5.4}$$

$$\frac{\partial v_x}{\partial t} + v_\xi \frac{\partial v_x}{\partial \xi} + \frac{v_y}{\cos\theta} = 0, \tag{5.5}$$

$$\frac{\partial v_y}{\partial t} + v_\xi \frac{\partial v_y}{\partial \xi} + \frac{\sin\theta}{\cos^2\theta} \frac{\partial \phi}{\partial \xi} - \frac{v_x}{\cos\theta}$$

$$+ \alpha \frac{\sin\theta}{\cos^2\theta} \frac{1}{n} \frac{\partial n}{\partial \xi} = 0, \tag{5.6}$$

$$\frac{\partial v_z}{\partial t} + v_\xi \frac{\partial v_z}{\partial \xi} + \frac{1}{\cos\theta} \frac{\partial \phi}{\partial \xi} + \frac{\alpha}{\cos\theta} \frac{1}{n} \frac{\partial n}{\partial \xi} = 0 \tag{5.7}$$

and

$$n = n_i = \exp(\phi) \tag{5.8}$$

n and n_i are the normalized electron and ion densities, $\alpha = (T_e/T_i)$ and $V_\xi = V_y \sin\theta + V_z \cos\theta$.

We now look for stationary solutions of Eqs.(5.4) - (5.8) which depend on ξ and t only through the variable, $\eta = K\xi - \Omega t$. These represent waves moving with constant velocity (Ω/K) without any deformation. The operators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \xi}$ can the be replaced by $-\Omega\frac{\partial}{\partial \eta}$ and $K\frac{\partial}{\partial \eta}$ respectively. Now Eqs. (5.4) and (5.7) can be integrated once which together with Eq. (5.8) yield,

$$K\left(v_y \sin\theta + v_z \cos\theta\right) = \left(1 - \frac{1}{n}\right)\Omega \tag{5.9}$$

and

$$v_z = \frac{K}{\cos\theta}\,\frac{(1+\alpha)}{\Omega}\,(n-1) \tag{5.10}$$

In deriving these and further relations, we impose asymptotic boundary conditions on the waves; namely

$$\left.\begin{array}{l} n \to 1 \\ v \to 0 \\ \phi \to 0 \\ (dn/d\eta) \to 0 \end{array}\right] \quad \text{as} \quad \eta \to \pm\infty$$

From Eqs. (5.5) and (5.6) we obtain,

$$-\frac{\Omega}{n}\frac{dv_x}{d\eta} + \frac{v_y}{\cos\theta} = 0 \tag{5.11}$$

and

$$\frac{\Omega}{n}\frac{dv_y}{d\eta}+\frac{K}{n}\frac{\sin\theta}{\cos^2\theta}(1+\alpha)\frac{dn}{d\eta}$$
$$=\frac{v_x}{\cos\theta} \qquad (5.12)$$

Eliminating v_x, v_y and v_z, Eqs. (5.9) - (5.12) can be reduced to,

$$\frac{1}{2}\left(\frac{dn}{d\eta}\right)^2+\psi(n)=0, \qquad (5.13)$$

where

$$\psi(n)=\frac{n^2}{K^2\cos^2\theta\, M^2}\left[\frac{(1+\alpha)}{\cos^2\theta}-\frac{M^2\sin\theta}{n^2}\right]^{-2}$$
$$\left\{\frac{(n-1)^2}{2n^2}\sin^2\theta\, M^4-\frac{\sin^2\theta}{\cos^2\theta}(1+\alpha)\left[(n-1)\right.\right.$$
$$\left.-\ln N+\frac{\cos^2\theta}{\sin^2\theta}\frac{(n-1)^2}{n}\right]M^2$$
$$\left.\frac{(1+\alpha)^2(n-1)^2}{2\cos^2\theta\sin^2\theta}\right\} \qquad (5.14)$$

The quantity M $(=\Omega/K\sin\theta)$, is the Mach number of the wave in the y-direction.

Eq. (5.13) is in the form of the energy equation of a classical particle of unit mass whose total energy is zero. First term on the left hand side corresponds to its kinetic energy and $\psi(n)$ corresponds to its potential energy which is called the Sagdeev potential.

V.3 Solitary Waves

Equation (5.13) will have real solutions (Buti et al., 1980) only if,

$$\psi(n) < 0 \tag{5.15}$$

To check this, first we analyse ψ (n) near n = 1. From (5.14), retaining terms up to order $(n - 1)^2$ we get,

$$\psi(n \simeq 1) = \frac{(1+X)^2 (n-1)^2}{2M^2K^2} \left[M^2 - (1+\alpha)(1+X)\right] \left[M^2 X - (1+\alpha)(1+X)^2\right]^{-1}, \tag{5.16}$$

where $X = (\cos^2\theta/\sin^2\theta), (<<1)$. To satisfy the condition (5.15) we must have,

$$(1+\alpha)(1+X) < M^2 < (1+\alpha)(1+X)^2/X \tag{5.17}$$

Moreover, from expression (5.16), it is apparent that n = 1 is a double root of the equation ψ (n) = 0. Consequently, $\left.\frac{d\psi}{dn}\right|_{n=1} = 0$ which means the oscillating particle in the potential trough slowly comes to rest at the point n = 1. This corresponds to the asymptotic boundary conditions of the wave. That is, the density in the wave becomes n = 1 as $\eta \to \pm\infty$.

The equation ψ (n) = 0 will have one more root say, at n = N corresponding to the maximum displacement of the oscillating particle so that the amplitude of the wave will be (N - 1). By putting ψ (N) = 0 we get,

$$M^4 - 2B_0 M^2 - C_0 = 0, \tag{5.18}$$

where

$$B_0 = \frac{N^2}{(N-1)^2} \frac{(1+\alpha)(1+X)}{X} \left[(N-1-\ln N) + \frac{X}{N}(N-1)^2 \right] \tag{5.19}$$

and

$$C_0 = \frac{N^2}{X}(1+\alpha)^2(1+X)^3 \tag{5.20}$$

Solving Eq. (5.18), we obtain,

$$M^2 = B_0 \pm (B_0^2 - C_0)^{1/2} \tag{5.21}$$

Let us examine the root with the positive sign. For N $\gtrsim$ 1, M^2 is very large since X $\ll$ 1 and this case is inadmissible because our basic equations are valid only for wave velocities smaller than the ion thermal velocity. This puts an upper limit on M namely, $M^2 < M_u^2 \equiv [(m_e/m_i)/\cos^2\theta]$. However, for N $\ll$ 1, one can achieve this condition and (5.21) can be rewritten as,

$$M^2 \simeq \frac{N^2}{X}(1+X)(1+\alpha)\left\{ -\ln N \pm \left[(\ln N^{-1})^2 - \frac{X}{N^2} \right]^{1/2} \right\}, \tag{5.22}$$

with the restrictions that, $\ln N^{-1} > (x^{1/2}/N)$

For $N < 1$, defining $N = 1 - \delta N$ and using the series expansions for $\ln N$, in the root with the negative sign we have,

$$M^2 < \frac{C_0}{2B_0} = \frac{(1+\alpha)(1+x)^2}{(1+2x+2F)}, \tag{5.23}$$

where

$$F = (\tfrac{1}{3}+x)\delta N + (\tfrac{1}{4}+x)\delta N^2 + \cdots \tag{5.24}$$

so that, $M^2 < (1+\alpha)(1+x)$. This violates the inequality (5.17) and hence this case is also ruled out. Henceforth we will consider only the root with the positive sign for $N < 1$ with $\ln N^{-1} > (x^{1/2}/N)$ and the root with the negative sign for $N > 1$.

Now we consider the nature of the potential $\psi(n)$ near $n = N$. Here the condition (5.15) demands,

$$\left(\frac{d\psi}{dn}\right)_{n=N} < 0 \qquad \text{for} \qquad N < 1 \tag{5.25}$$

and

$$\left(\frac{d\psi}{dn}\right)_{n=N} > 0 \qquad \text{for} \qquad N > 1 \tag{5.26}$$

Combining (5.17), (5.21), (5.25) and (5.26), we finally obtain the following inequalities:

$$N(1+\alpha)(1+X) > \left[M^2 = B_o - (B_o^2 - C_o)^{1/2}\right]$$

$$> (1+\alpha)(1+X) \qquad \text{for} \quad N > 1 \qquad (5.27)$$

and

$$\frac{(1+\alpha)(1+X)^2}{X} > \left[M^2 = B_o + (B_o^2 - C_o)^{1/2}\right] >$$

$$\frac{N^2(1+\alpha)(1+X)^2}{X} \text{ for } \quad N < 1;\ \ln N^{-1} > (X^{1/2}/N) \qquad (5.28)$$

Usually we have $X < (m_e/m_i)$. Hence the requirements $N \ll 1$ and $\ln N^{-1} > (X^{1/2}/N)$ can be simultaneously satisfied. Under these conditions, relation (5.28) with the help of Eq. (5.22) can be rewritten as,

$$(1+X)/N^2 > \ln N^{-1} > (1+X) \qquad (5.29)$$

which is indeed satisfied for $N \ll 1$. So we conclude that electron holes can exist for $1 \gg N \gtrsim X^{1/2}/\ln N^{-1}$. Our numerical computations confirm their existence. For $\alpha = 0.1$ and $X = 10^{-4}$, electron holes could occur for $3 \times 10^{-3} < N < 8 \times 10^{-3}$. The value of $|\psi(n)|$ at $n = n_1 = (1+N)/2$ (midpoint between $n = 1$ and $n = N$), is $\sim 10^8$ Using this in Eq. (5.13), we find that

$$\frac{n_1}{W} \sim \left[-2\,\psi(n_1)\right]^{1/2}, \qquad (5.30)$$

where W is the width of the hole. This gives the value of W:

$$W \sim n_1 \left[-2\,\psi(n_1) \right]^{-1/2}$$
$$\sim 10^{-3} \tag{5.31}$$

This means that the holes will have spatial extensions much smaller than $(T_i/T_e)^{1/2}\rho_e$ which we have used to normalize the lengths scale in the system. Because of this our fluid model is not quite adequate. Consequently the existence of electron holes may not be realizable. A satisfactory answer can be obtained only by including kinetic effects in our approach.

V.4 Small Amplitude Limit

In the small amplitude limit, we can take $n = 1 + \delta n$; $\delta n \ll 1$. On retaining terms up to δn^3, Eq. (5.13) reduces to,

$$\left(\frac{d\delta n}{d\eta}\right)^2 + \chi_1\,\delta n^2 + \chi_2\,\delta n^3 = 0, \tag{5.32}$$

where

$$\chi_1 = \frac{(1+x)^2}{M^2K^2}\left[M^2 - (1+\alpha)(1+x)\right]$$
$$\left[M^2 x - (1+\alpha)(1+x)^2\right] \tag{5.33}$$

and

$$\chi_2 = \frac{2}{M^2 K^2}(1+x)^2 \left[M^2 x - (1+\alpha)(1+x)^2\right]^{-2}$$

$$\left[2M^4 x - M^2(1+\alpha)(1+x)\left(\frac{2}{3}+3x\right) + (1+\alpha)^2(1+x)^3\right] \tag{5.34}$$

We can easily show that Eq. (5.32) has a square hyperbolic ~~secant type solution, namely,~~

$$\delta n = \delta n_0 \operatorname{sech}^2(\eta/L), \tag{5.35}$$

where

$$\delta n_0 = -\chi_1/\chi_2 \tag{5.36}$$

and

$$L^2 = (-\chi_1/4)^{-1} \tag{5.37}$$

In order that L is real, we must have $\chi_1 < 0$ which is ensured if (5.17) is satisfied, i.e., if $\psi < 0$. Moreover, with $\chi_1 < 0$, the amplitude δn_o will be positive if $\chi_2 > 0$ which demands that,

$$M^2 < \frac{3}{2}(1+\alpha) \tag{5.38}$$

In calculating (5.38), we have neglected terms of order X in (5.34). When $M^2 > \frac{3}{2}(1+\alpha)$, δn_o is negative. But in that case, the small amplitude limit calculation has to be abandoned because by putting $N = (1+\delta n_o)$ in (5.21) and expanding, we can show that $\delta n_o > \frac{3}{4}$ for $M^2 > \frac{3}{2}(1+\alpha)$ so that the wave amplitude is no longer very small.

We conclude this section with a remark that in the small amplitude limit, we can have only solitons and not the electron holes.

Next, we compare the small amplitude limit solution with the solution obtained in chapter IV in the absence of the ion-Landau damping. The solution (4.39) in terms of the density perturbation to the first order in ϵ can be written in the (ξ, t) frame of reference as,

$$\delta n = \epsilon n^{(1)} = \delta n_0 \operatorname{sech}^2\left[\left(\frac{\delta n_0}{6}\right)^{1/2}(\xi - Mt)\right], \quad (5.39)$$

where we have put $V_o = 0$, $\alpha_4 = 0$ and $\cos\theta = \cos\theta_o$.
Evaluating the coefficients χ_1, and χ_2 in Eq. (5.32) after neglecting χ and α, we have,

$$\chi_1 = -\frac{2}{3}\frac{\delta n_0}{K^2} \quad (5.40)$$

and

$$\chi_2 = \frac{2}{3}\frac{1}{K^2} \quad (5.41)$$

which give the solution (5.35) as,

$$\delta n = \delta n_0 \operatorname{sech}^2\left[\left(\frac{\delta n_0}{6}\right)^{1/2}(\xi - Mt)\right]. \quad (5.42)$$

This is same as (5.39)

V.5 Conclusions

For $X = 10^{-4}$ and $\alpha = 0.1$, the region of Mach numbers corresponding to solitary waves with density hump is determined numerically and it is shown in Fig.1. We observe that only supersonic electron acoustic solitons can occur. The upper limit of the amplitudes in this case turns out to be $M_u = 2.3$.

In summery, we find that in the absence of Landau damping, supersonic, finite amplitude electron acoustic solitary waves with density humps can exist in a magnetoplasma.

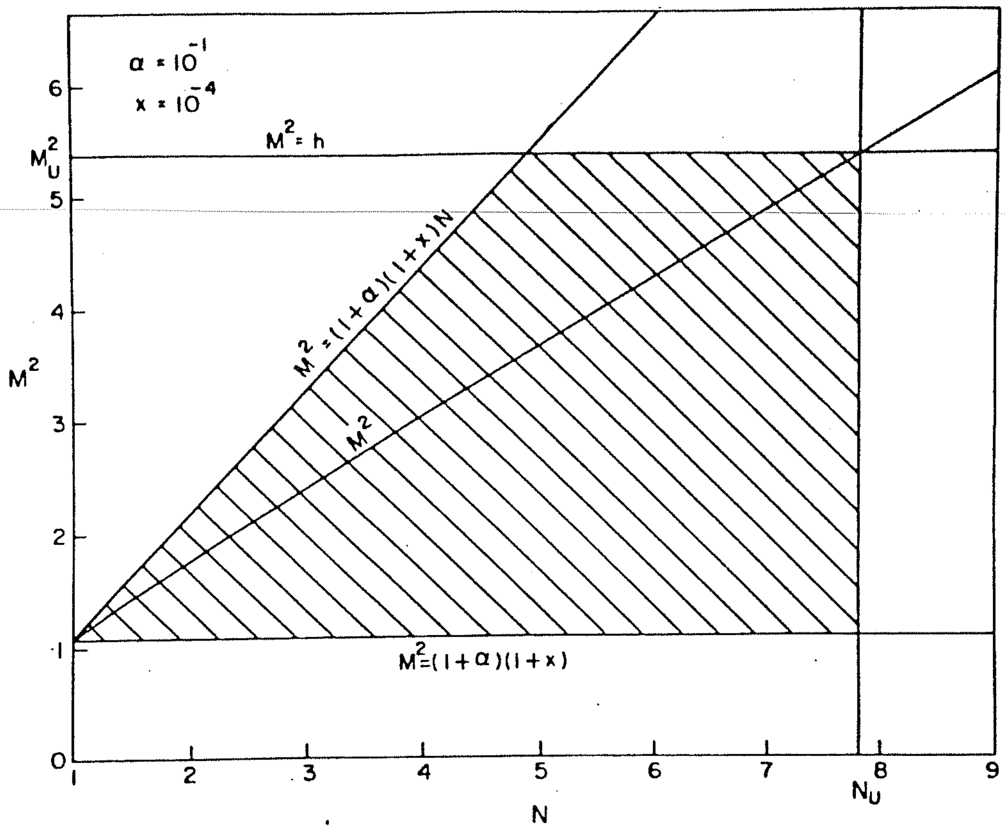

Fig.1: M^2 versus N. Solitary waves with density hump for $\alpha = 0.1$, $X = 10^{-4}$ can exist in the shaded region.

CHAPTER VI

EFFECTS OF COLLISIONAL AND VISCOUS DISSIPATIONS ON THE ION ACOUSTIC-LANGMUIR INTERACTIONS

VI.1 Introduction

Quite frequently the plasmas of laboratory as well as of astrophysical situations are in a state of turbulence. Any small noise or oscillations in a plasma may develop and set off turbulence in it. The main characteristic of a turbulent system is that its total energy will be exchanged and shared among the different modes that it can sustain simultaneously (Kadomtsev, 1965; Tsytovich, 1970, 1972;Davidson, 1972;

Ichimaru, 1973). Turbulence in plasmas has a very important role to play in various phenomena like heating, anomalous diffusion of particles, anomalous conductivity, transport of energy, nonlinear excitation of waves etc.

In an unmagnetized turbulent plasma with hot electrons and cold ions the most important modes to be considered are the Langmuir and the ion acoustic modes (Thornhill and ter Haar, 1977). These two modes are coupled to each other in the following way: The energy density of the high frequency Langmuir oscillations has a pressure associated with it (similar to the radiation pressure). Whenever this pressure develops a spatial gradient, it gives rise to a force called the Miller force (Gaponov and Miller, 1958) or the ponderomotive force. This force acts on the ions in the plasma and drives low frequency ion acoustic perturbations which in turn trap the Langmuir waves in them. The coupled oscillations thus produced move with velocities less than the ion acoustic velocity because of the resistence offered by the trapped Langmuir field. In the limit of small Mach numbers (near zero) the nonlinearity in the ion perturbations is very small and the waves are described by the Zakharov equations (Zakharov, 1972). These are a pair of coupled nonlinear Schrödinger equation for the electric field of the Langmuir field and a wave equation for the ion density perturbations of the ion acoustic mode. These equations have been found to have stationary solitary wave solutions (Gurovich and Karpman, 1970; Karpman, 1975; Gibbons et al., 1977). But as the Mach number increases (approaches unity), the ion density perturbations grow larger and steeper which makes the associated ion nonlinearity an important factor. In such cases, the wave equation for the ion is to be replaced by the K-dV equation or the Boussinesq equation (Ikezi et al., 1975; Kaw and Nishikawa, 1976; Makhankov, 1974). In this case also, there are stationary solitary solutions (Nishikawa et al., 1974) with the amplitude of the ion density perturbations larger than that in the previous case.

Such coupled oscillations have been observed in various laboratory experiments and computer simulations (Morales and Lee, 1974; Abdulloev et al., 1975; Degtyarev et al., 1975; Ikezi et al., 1974; Appert and Vaclavik,1977).

In this chapter, we discuss the effects of dissipations on the ion acoustic-Langmuir interactions. The types of dissipative processes considered are the inter-particle collisions and ion viscosity. These dissipative processes are important in plasmas, with neutral backgrounds which are quite common in laboratory experiments such as rotating plasma devices like the Homopolar, plasma guns, plasma-neutral gas-impact experiments etc. (Danielsson, 1970; 1973; Alfven, 1960; Bratenahl et al., 1960; Wilcox et al., 1964).

In certain fusion devices also, the regions of plasma near the boundaries are relatively cold and not fully ionized.

Taking into account these dissipative processes, we derive a pair of modified Zakharov equations. Since it is not possible to find an analytic solution to these equations, the time evolution of solitary waves according to these equations are studied numerically. We see that the effect of electron-ion and electron-neutral collisions is to damp the Langmuir oscillations. As the Langmuir field amplitude goes down, the ion density perturbation starts radiating away. On the other hand the effect of ion-neutral collisions is to damp the ion density perturbations. Consequently the trapped Langmuir field starts to flow out. In the case of ion viscosity the ion density perturbations radiate away at first which is followed by the Langmuir field. As far as the relative strengths of these processes are concerned, the effect of electron-ion and electron-neutral collisions dominate over the other two.

These results are of considerable significance from the point of view of modulational instability and the subsequent collapse of solitons in a three dimensional Langmuir turbulent plasma (Thornhill and ter Haar, 1978; Wong and Quon, 1975; Buti, 1977; Pereira et al., 1977). The main consequence of collapse is the electric fields and the ion density perturbations growing into narrow and very large amplitude waves called spikons and cavitons respectively. But if there are dissipative mechanisms such as the ones considered here present in the system, the process of collapse can be slowed down. A similar effect was actually observed in the case of Landau damping by Khakimov and Tsytovich (1976). This slowing down can affect the energy distribution among the different wave numbers in the system or in other words modify the spectrum of Langmuir turbulence.

VI.2 Modified Zakharov Equations of the Ion Acoustic-Langmuir Waves

Let us consider an unmagnetized, partially ionized plasma with hot electrons and cold ions and non zero ion viscosity. Two kinds of oscillations can occur in this plasma: One is the high frequency Langmuir oscillations with a characteristic time period $(\omega_{pe})^{-1}$ and the other is the low-frequency ion acoustic perturbations with a characteristic time period $(\omega_{pi})^{-1}$ where,

$$\omega_{pe,i} = (4\pi n_0 e^2/m_{e,i})^{1/2}$$

The governing equations for the electrons are:

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e v_e) = 0, \tag{6.1}$$

$$\frac{\partial v_e}{\partial t} + v_e \frac{\partial v_e}{\partial x} = \frac{e}{m_e}\frac{\partial \phi}{\partial x} - \frac{\gamma_e T_e}{n_e m_e}\frac{\partial n_e}{\partial x}$$

$$- \nu_{en} v_e - \nu_{ei}(v_e - v_i) \qquad (6.2)$$

and

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi e(n_e - n_i), \qquad (6.3)$$

where ν_{en} is the electron-neutral collision frequency, ν_{ei} is the electron-ion collision frequency and γ_e is the ratio of specific heats for the electrons. Since the electrons are light, they respond to both kinds of oscillations and so we assume that their motion has two components with time scales $(\omega_{pe})^{-1}$ and $(\omega_{pi})^{-1}$. Whereas the ions are heavy and they respond only to the low frequency oscillations. Accordingly we take,

$$n_e = n_o + \delta n_s + \delta n_f, \qquad (6.4)$$

$$n_i = n_o + \delta n_i, \qquad (6.5)$$

$$v_e = v_s + v_f \qquad (6.6)$$

and

$$\phi = \phi_s + \phi_f \qquad (6.7)$$

with

$$\frac{\partial}{\partial t}(\delta n_s, \delta n_i, v_s, v_i, \phi_s) \ll \frac{\partial}{\partial t}(\delta n_f, v_f, \phi_f) \tag{6.8}$$

and

$$\frac{\partial}{\partial x}(\delta n_s, \delta n_i, v_s, v_i, \phi_s) \ll \frac{\partial}{\partial x}(\delta n_f, v_f, \phi_f) \tag{6.9}$$

The subscripts s and f refer to the slow and fast oscillations. Further, we assume that $v_i \simeq v_s$ and $\delta n_s \simeq \delta n_i = \delta n$ as far as the fast oscillations are concerned. But the difference $(\delta n_s - \delta n_i)$ will be accounted for slow oscillations since it is responsible for the slow part ϕ_s of the potential ϕ (Thornhill and ter Haar, 1978).

We substitute Eqs. (6.4) - (6.7) in Eqs. (6.1) - (6.3) and neglect the nonlinear terms in the fast oscillations. Then by making use of the conditions (6.8) and (6.9) and after eliminating δn_f and v_f, we get,

$$\left(\nu_e + \frac{\partial}{\partial t}\right)\frac{\partial E_f}{\partial t} + \omega_{pe}^2\left(1 + \frac{\delta n}{n_0}\right)E_f - \frac{\gamma_e T_e}{m_e}\frac{\partial^2 E_f}{\partial x^2} = 0, \tag{6.10}$$

where $\nu_e = (\nu_{en} + \nu_{ei})$ and $E_f = -(\partial\phi/\partial x)$ is the electric field of the high frequency oscillations which are defined (Zakharov, 1972) as,

$$E_f = \frac{1}{2}\left[E(x,t)\, e^{-i\omega_{pe}t} + c.c\right], \tag{6.11}$$

where the amplitude E (x,t) is a slowly varying function of space and time. Substituting (6.11) in (6.10) and neglecting the second order time variations in E (for $\nu_e \ll \omega_{pe}$), we obtain

$$2i\frac{\partial E}{\partial t} + \frac{\gamma_e T_e}{m_e \omega_{pe}}\frac{\partial^2 E}{\partial x^2} + i\nu_e E = \omega_{pe}\frac{N}{n_0}E, \tag{6.12}$$

which is a modified nonlinear Schrödinger equation. In this equation, δn has been replaced by N which represents the potential trough created by the ponderomotive force of the Langmuir field which in turn is trapped in it. To calculate this ponderomotive force, we average equation (6.2) over the fast oscillations; this leads to:

$$\frac{\partial v_s}{\partial t} + v_s\frac{\partial v_s}{\partial x} + \nu_{en} v_s = \frac{e}{m_e}\frac{\partial \phi_s}{\partial x} - \frac{\partial}{\partial x}\left[\frac{e^2|E|^2}{4m_e^2\omega_{pe}^2}\right] - \frac{\gamma_e T_e}{m_e(n_0+\delta n_s)}\frac{\partial \delta n_s}{\partial x} \tag{6.13}$$

In deriving Eq. (6.13), we have used,

$$v_f \simeq \frac{e}{2m_e\omega_{pe}}\left[iEe^{-i\omega_{pe}t} + c.c\right] \tag{6.14}$$

and

$$\langle n_f, v_f, \phi_f \rangle = 0, \tag{6.15}$$

where $\langle\,\rangle$ denotes the time average over the fast time scale, $(\omega_{pe})^{-1}$. Eq. (6.13) tells us that on the slow time scale, apart from the force of the slowly varying electric field, the electrons feel an additional force $\left[\partial/\partial x(e^2|E|^2/4\,m_e\omega_{pe}^2)\right]$ This is the ponderomotive force. Further, since the electrons are thermalized, one can neglect the electron inertia and integrate (6.13) to obtain the electron density; this is given by

$$(n_0 + \delta n_s) = n_0 \exp\left[\left(e\phi_s - \frac{e^2|E|^2}{4m_e\omega_{pe}^2}\right)\frac{1}{T_e}\right] \tag{6.16}$$

Next, we consider the low frequency ion motion. The governing equations are,

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v_i) = 0, \tag{6.17}$$

$$\frac{\partial v_i}{\partial t} + v_i\frac{\partial v_i}{\partial x} = -\frac{e}{m_i}\frac{\partial \phi_s}{\partial x} - \nu_{in} v_i - \nu_{ie}(v_i - v_s) + \eta\frac{\partial^2 v_i}{\partial x^2} \tag{6.18}$$

and

$$\frac{\partial^2 \phi_s}{\partial x^2} = 4\pi e\,(n_e - n_i), \qquad (6.19)$$

where ν_{in} is the ion-neutral collision frequency, ν_{ie} is the ion-electron collision frequency and η is the coefficient of ion viscosity. We assume that $\nu_{ie} \lesssim \nu_{in}$ which is in fact the case for the plasmas we are going to consider subsequently. Since $v_i \simeq v_s$, we can neglect the term $\nu_{ie}(v_i - v_s)$ in comparison with $\nu_{in} v_i$. Hence Eq. (6.18) after replacing ν_{in} by ν_i becomes,

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} = \frac{-e}{m_i}\frac{\partial \phi_s}{\partial x} - \nu_i v_i + \eta \frac{\partial^2 v_i}{\partial x^2} \qquad (6.20)$$

Following the reductive perturbation method outlined in chapter I and IV, we introduce the following stretched variables:

$$\xi = \epsilon^{1/2}(x - C_s t), \quad \tau = \epsilon^{3/2} t, \qquad (6.21)$$

where $C_s = (T_e/m_i)^{1/2}$ is the ion acoustic phase velocity. We expand n_i, v_i and ϕ_s as,

$$n_i = n_0 + \epsilon n^{(1)} + \epsilon^2 n^{(2)} + \cdots, \qquad (6.22)$$

$$v_i = \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \cdots, \qquad (6.23)$$

and

$$\phi_s = \epsilon \phi^{(1)} + \epsilon^2 \phi^{(2)} + \cdots \tag{6.24}$$

Here the smallness parameter ϵ is chosen in such a way that the order of δn^2 $(\delta n \simeq \epsilon n^{(1)})$ which is a measure of ion nonlinearity, is same as that of $|E|^2$ occuring in the ponderomotive force (Nishikawa et al., 1974). Let us define,

$$|E| = \epsilon \rho |E'|, \tag{6.25}$$

$$\nu_i = \epsilon^p \beta \nu_i' \tag{6.26}$$

and

$$\eta = \epsilon^q \gamma \eta', \tag{6.27}$$

where ρ, β and γ are of the order unity and p and q are positive numbers.

Substituting (6.21) - (6.27) in (6.17), (6.19) and (6.20) from the lowest order in ϵ (cf. chapter III) we get the equations,

$$-c_s \frac{\partial n^{(1)}}{\partial \xi} + \frac{\partial}{\partial \xi}(n_o v^{(1)}) = 0, \tag{6.28}$$

$$-c_s \frac{\partial v^{(1)}}{\partial \xi} = \frac{-e}{m_i} \frac{\partial \phi^{(1)}}{\partial \xi} - \epsilon^{(p-\frac{1}{2})} \beta \nu_i' v^{(1)} + \epsilon^{(q+\frac{1}{2})} \gamma \eta' \frac{\partial^2 v^{(1)}}{\partial \xi^2} \tag{6.29}$$

and

$$0 = n_0 \frac{e\phi^{(1)}}{T_e} - n^{(1)} \tag{6.30}$$

Now we choose the numbers p and q in such a way that the lowest order equations are not affected by collisions and viscosity. Hence we take p = (3/2) and q = (1/2) so that the last two terms on the right hand side of Eq. (6.29) will go to the next higher order equation in ϵ . Solving (6.28)-(6.30), we get

$$n^{(1)} = \frac{n_0}{C_s} v^{(1)} = \frac{n_0 e\phi^{(1)}}{T_e} \tag{6.31}$$

From the equations corresponding to next higher order in ϵ, after eliminating $n^{(2)}$, $v^{(2)}$ and $\phi^{(2)}$ and using (6.31), we get

$$\frac{\partial n^{(1)}}{\partial \tau} + \frac{C_s}{n_0} n^{(1)} \frac{\partial n^{(1)}}{\partial \xi_1} + \frac{C_s T_e}{8\pi n_0 e^2} \frac{\partial^3 n^{(1)}}{\partial \xi_1^3}$$

$$+ \frac{\beta}{\alpha} \nu_i' n^{(1)} - \frac{\gamma}{2} \eta' \frac{\partial^2 n^{(1)}}{\partial \xi_1^2}$$

$$= - \frac{C_s n_0 e^2 \rho^2}{8 T_e m_e \omega_{pe}^2} \frac{\partial}{\partial \xi_1} |E'|^2 \tag{6.32}$$

Transforming back into the (x,t) frame of reference and using Eqs. (6.25)-(6.27), Eq. (6.32) becomes,

$$\frac{\partial N}{\partial t} + C_s \frac{\partial N}{\partial x} + \frac{C_s}{n_0} N \frac{\partial N}{\partial x}$$

$$+ \frac{C_s T_e}{8\pi n_0 e^2} \frac{\partial^3 N}{\partial x^3} + \frac{\nu_i}{2} N - \eta \frac{1}{2} \frac{\partial^2 N}{\partial x^2}$$

$$= -\frac{C_s n_0 e^2}{8 T_e m_e \omega_{pe}^2} \frac{\partial}{\partial x} |E|^2, \qquad (6.33)$$

where we have replaced $\epsilon n^{(1)}$ $(\simeq \delta n)$ by N. This equation is a modified K-dV equation which is coupled to Eq. (6.12) through the ponderomotive force term.

On using the following normalizations,

$$x \to x(\sqrt{3}\lambda_{De})^{-1}, \quad t \to t\,(C_s/\sqrt{3}\lambda_{De}),$$

$$N \to N(n_0)^{-1}, \quad \phi \to \phi(e/T_e),$$

$$v \to v(C_s)^{-1}, \quad \nu_e \to \nu_e(1/\omega_{pe}),$$

$$\nu_i \to \nu_i(\sqrt{3}/\omega_{pi}), \quad \eta \to \eta(\sqrt{3}\,C_s\lambda_{De})^{-1},$$

$$E \to E\,(16\pi n_0 T_e)^{-1/2}, \qquad (6.34)$$

Eqs. (6.12) and (6.33) reduce to,

$$2i\epsilon \frac{\partial E}{\partial t} + \frac{\partial^2 E}{\partial x^2} + i\nu_e E = NE \qquad (6.35)$$

and

$$\frac{\partial N}{\partial t} + \frac{\partial N}{\partial x} + N\frac{\partial N}{\partial x} + \frac{1}{6}\frac{\partial^3 N}{\partial x^3} + \frac{\nu_i}{2} N - \eta\frac{\partial^2 N}{\partial x^2} = -\frac{1}{2}\frac{\partial}{\partial x}|E|^2, \tag{6.36}$$

where $\varepsilon = (m_e/3m_i)^{1/2}$ and we have used $\gamma_e = 3$. These are the modified coupled Zakharov equations.

VI.3 Discussions

For the case $\nu_e = \nu_i = \eta = 0$, Eqs. (6.35) and (6.36) have the following stationary solitary wave solutions (Nishikawa et al., 1974):

$$E = a \operatorname{sech} bX \operatorname{Tanh} bX \exp\{i[\varepsilon M x + (\varepsilon M^2 + \Omega)t]\} \tag{6.37}$$

and

$$N = h \operatorname{sech}^2 bX, \tag{6.38}$$

where

$$X = x - Mt, \tag{6.39}$$

$$a = \frac{6\sqrt{3}}{5}(1-M), \tag{6.40}$$

$$b = \sqrt{\frac{3}{10}}(1-M)^{1/2}, \tag{6.41}$$

$$h = \frac{-9}{5}(1-M) \tag{6.42}$$

and

$$\Omega = \frac{1}{2\varepsilon}\left(b^2 - \varepsilon^2 M^2\right), \tag{6.43}$$

with M as the Mach number of the wave.

For nonzero values of ν_e, ν_i and η, the system is dissipative and hence it is not possible to find stationary solutions to (6.35) and (6.36). In order to find the effects of the various dissipative mechanisms, we study numerically the time evolution of an initial wave given by (6.37) and (6.38) according to the modified Zakharov equations. We have used the following three-time-level Zabusky-Kruskal method (Zabusky and Kruskal, 1965; Appert and Vaclavik, 1977) for the ion density equation (6.36):

$$\begin{aligned}
&\left(N_\ell^{j+1} - N_\ell^{j-1}\right)\frac{1}{T} + \left(N_{\ell+1}^{j} - N_{\ell-1}^{j}\right)\frac{1}{2H} \\
&+\left(N_{\ell+1}^{j} + N_\ell^{j} + N_{\ell-1}^{j}\right)\left(N_{\ell+1}^{j} - N_{\ell-1}^{j}\right)\frac{1}{3H} \\
&+\left(N_{\ell+2}^{j} - 2N_{\ell+1}^{j} + 2N_{\ell-1}^{j} - N_{\ell-2}^{j}\right)\frac{1}{6H^3} \\
&+\frac{\nu_i}{6}\left(N_{\ell+1}^{j} + N_\ell^{j} + N_{\ell-1}^{j}\right) - \eta\left(N_{\ell+1}^{j} - 2N_\ell^{j} + N_{\ell-1}^{j}\right)\frac{1}{H^2} + \left(|E|_{\ell+1}^{j} + |E|_\ell^{j}\right. \\
&\left.+|E|_{\ell-1}^{j}\right)\left(|E|_{\ell+1}^{j} - |E|_{\ell-1}^{j}\right)\frac{1}{3H} = 0,
\end{aligned} \tag{6.44}$$

where ℓ and j refer to the spatial and temporal points and H and T to the space and time step sizes.

The electric field equation (6.35) was discretized according to the Crank-Nicholson (1965) scheme as follows:

$$i\mathcal{E}\left(E_{\ell}^{j+1}-E_{\ell}^{j}\right)\frac{1}{T}+\left(E_{\ell+1}^{j+1}-2E_{\ell}^{j+1}\right.$$
$$\left.+E_{\ell-1}^{j+1}+E_{\ell+1}^{j}-2E_{\ell}^{j}+E_{\ell-1}^{j}\right)\frac{1}{2H^2}$$
$$+\frac{i\nu_e}{2}\left(E_{\ell}^{j}+E_{\ell}^{j+1}\right)-\frac{1}{2}\left(N_{\ell}^{j+1}E_{\ell}^{j+1}\right.$$
$$\left.+N_{\ell}^{j}E_{\ell}^{j}\right)=0. \tag{6.45}$$

The values of T and H are chosen so that the solutions(6.37) and (6.38) remain stationary for very long intervals of time when we put $\nu_e=\nu_i=\eta=0$.

The values of ν_e, ν_i and η appropriate for some rotating plasma devices (Danielsson, 1973) have been determined. Such plasmas have neutral density $\sim$ plasma density $(n_o)\sim 10^{11}\text{cms}^{-3}$, electron temperature$(T_e)\sim 10\times$ion temperature$(T_i)\sim 100$ eV. The calculated values of the electron-neutral and electron-ion collision cross sections are,

$$\sigma_{en}\sim\sigma_{ei}\sim 10^{-15}\,cm^2$$

and the ion-neutral collision cross section is,

$$\sigma_{in} \sim 10^{-14}\ cm^2$$

Since $\sigma_{en} \sim \sigma_{ei}$, the electron-neutral and the electron-ion collision frequencies are,

$$\nu_{en} \sim \nu_{ei} = n_0 \sigma_{en} (T_e/m_e)^{1/2} \sim 10^7\ sec^{-1}$$

and the ion neutral collision frequency is,

$$\nu_i = n_0 \sigma_{in} (T_i/m_i)^{1/2} \sim 10^6\ sec^{-1}$$

The electron and ion plasma frequencies are,

$$\omega_{pe} = (4\pi n_0 e^2/m_e)^{1/2} \sim 10^9\ sec^{-1}$$

and

$$\omega_{pi} = (4\pi n_0 e^2/m_i)^{1/2} \sim 10^7\ sec^{-1}$$

respectively. And the ion viscosity (Braginskii, 1965) is,

$$\eta = 1.28 (T_i/m_i \nu_i) \sim 10^6\ cm^2\ sec^{-1}$$

Figs. (1-5) show the effects of the various dissipative processes on an ion acoustic-Langmuir soliton in the reference frame moving with its initial velocity. Fig.1 corresponds to the case of electron-ion and electron-neutral collisions alone. As was observed in the case of electron-ion collisions in an earlier work (Gurovich and Karpman, 1970; Karpman, 1975) here also we see that at first the Langmuir oscillations damp. But as the Langmuir field amplitude goes down, the associated ponderomotive force decreases. Now the ion density depression is no more stable and it starts to emit ion acoustic radiation. The latter part of the evolution is similar to the one observed by Appert and Veclavik (1977) for an ion density depletion without any Langmuir field in it.

Fig.2 shows the soliton evolution in the presence of ion-neutral collisions alone. In this case the ion density amplitude damps at first. This means a decrease in the potential trough containing the Langmuir field. Hence the Langmuir field starts to flow out. Eventually all the ion density perturbations will damp out.

Fig.3 shows the effect of ion viscosity. In this case the wave undergoes slight deformation. Then the ion density perturbation starts radiating away. Since the Langmuir oscillations are not affected directly by the ion viscosity, the Langmuir field flows out rather slowly.

In Figs. 4 and 5 we show the time evolution of the initial solitary wave when all the three dissipative processes are operative. It is seen that the effect of electron-neutral and electron-ion collisions dominates over the other two in spite of ν_e (normalized) being much smaller than the other two.

VI.4 Summary

A pair of modified Zakharov equations is derived to study the effects of collisional and viscous dissipations on the ion acoustic-Langmuir interactions. Using these equations, the time evolution of coupled sonic-Langmuir solitons are investigated for some typical rotating device plasma parameters. The effect of electron-ion and electron-neutral collisions is to make the Langmuir field damp at first and then to let the ion density perturbations radiate away. In the case of ion neutral collisions, the ion density perturbation damps at first and the Langmuir field starts to flow out. Whereas in the case of ion viscosity, the ion density perturbations radiate away followed by the Langmuir field. Of the various processes studied, the effects of electron-ion and electron-neutral collisions are found to be stronger than the other two.

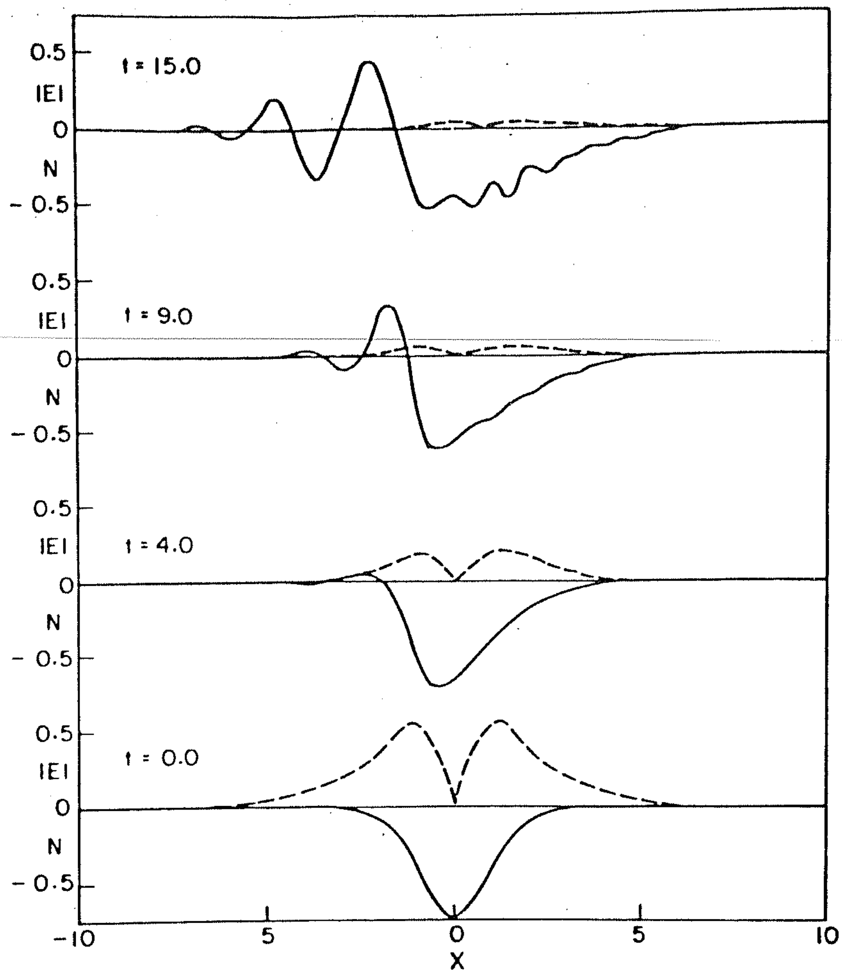

Fig. 1: Evolution of a coupled ion acoustic Langmuir Soliton. Full lines represent the ion density perturbation (N) and the dotted lines represent the envelope of the Langmuir field (|E|). The various quantities are, $\nu_e = 1.02 \times 10^{-2}$, $\nu_i = 0.0$, $\eta = 0.0$, $M = 0.5$, $a = 1.0$, $h = -0.75$.

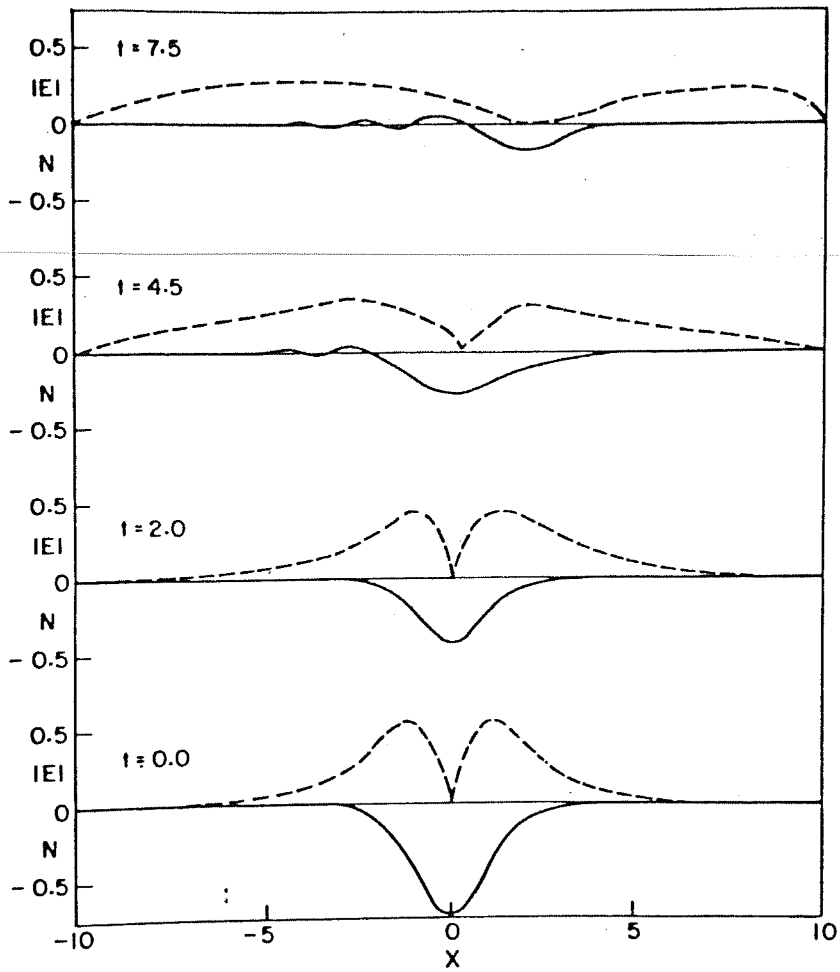

Fig.2; Same as Fig.1 with $\nu_e = 0.0$, $\nu_i = 1.93 \times 10^{-1}$, $\eta = 0.0$, M = 0.5, a = 1.0, h = -0.75.

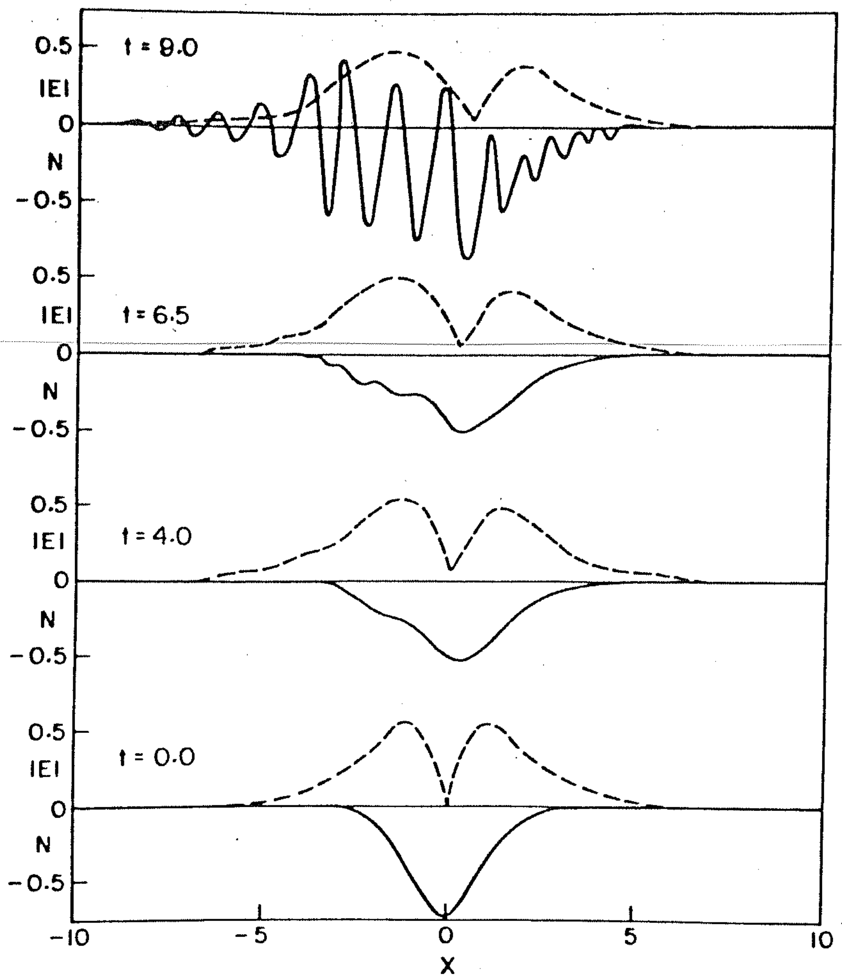

Fig.3; Same as Fig.1. with $\nu_e = 0.0$, $\nu_i = 0.0$, $\eta = 5.18 \times 10^{-1}$, M = 0.5, a = 1.0, h = -0.75.

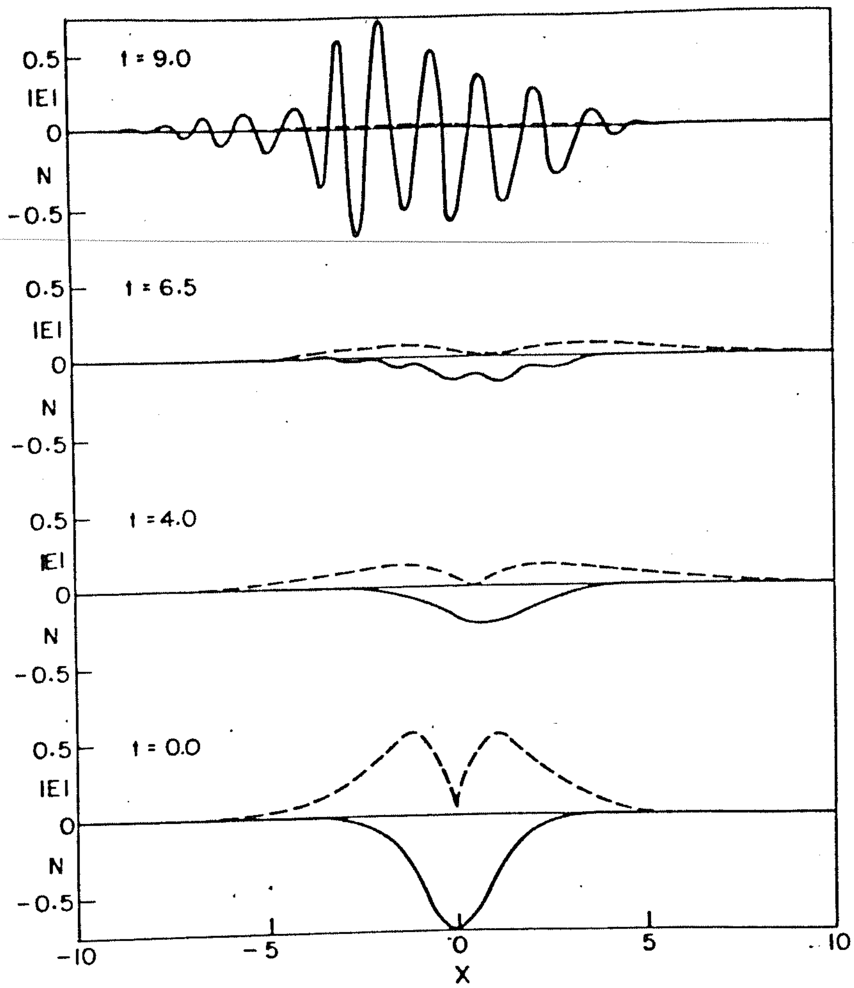

Fig.4: Evolution in the presence of all the dissipative effects; $\nu_e = 1.02 \times 10^{-2}$, $\nu_i = 1.93 \times 10^{-1}$, $\eta = 5.18 \times 10^{-1}$, M = 0.5, a = 1.0, h = -0.75.

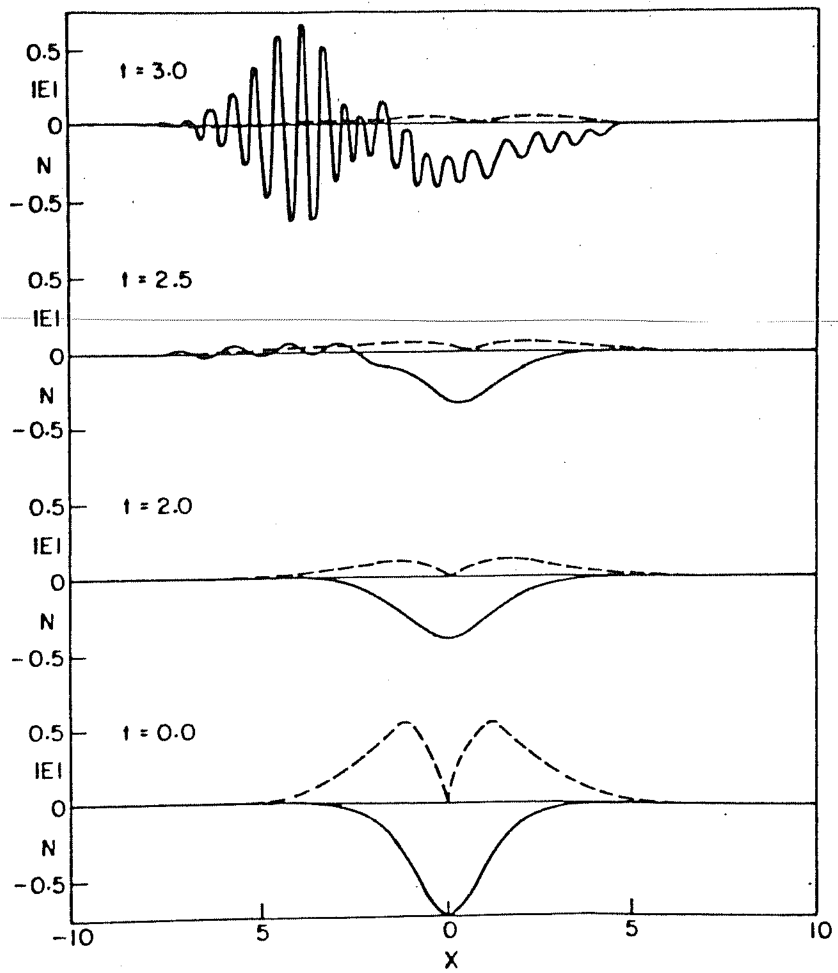

Fig.5: Same as Fig.4 with, $\nu_e = 2.74 \times 10^{-2}$, $\nu_i = 2.37 \times 10^{-1}$, $\eta = 6.91 \times 10^{-1}$, M = 0.5, a = 1.0, h = -0.75.

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