APPLICATION OF DISTRIBUTION METHODS TO SYMMETRIES IN NUCLEI

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to my parents

CERTIFICATE

I hereby declare that the work presented in this thesis is original and has not formed the basis for the award of any degree or diploma by any University or Institution.

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STATEMENT

In the immensely complex system such as a nucleus, a large amount of information regarding general properties, can be extracted by an understanding of the symmetries manifested within such a system. The consequences of having a symmetry are far reaching. Besides giving rise to conservation laws, selection rules, and simple relationships between various observables, they provide an approximation scheme for the detailed study of the system. The study of group symmetry in nuclear structure is therefore of great importance.

Historically it was in nuclear physics that dynamical symmetries --i.e. those not associated with space-time invariance were first considered. In 1932 Heisenberg suggested that the neutron and proton may be regarded as the two charge states of a nucleon. Later it was found that nuclear forces are charge independent and this led to the very important notion of isospin. Wigner in 1937 extended the idea and introduced the group SU(4) in connection with the spin-isospin independence of the nuclear interaction. Later on various other groups, symplectic symmetry group Sp(N) for the study of pairing effects and SU(3) which describes rotational behaviour in nuclei were suggested. However, for a long time the real status of group symmetries was not clear in the sense that although one knew that some of these symmetries were not exact, one did not know to what extent they were "good" or "broken". Microscopic matrix methods, for studying this question though applicable in principle, become very difficult to handle in large vector spaces formed by the distribution of m nucleons in N single particle orbits. The advent of Spectral Distribution Methods,

introduced by French, paved the way for a systematic study of group symmetries in nuclei, and one now knows a good deal more about them.

The thesis describes an application of distribution methods to the study of symmetries in nuclei. We have considered symmetries associated with finite dimensional vector spaces, these being related to the unitary group in N dimensions U(N) and its subgroups.

Spectral averages of the nuclear interaction over states defined by the irreducible representation of various relevant groups have been evaluated. These have been used to determine ground state energies, low energy spectra and fractional occupancy of single particle orbits for nuclei in the (sd) and (fp)-shells. Results are compared with microscopic calculations and experiment and the agreement is good. It is found that the distribution method is able to predict binding energies and spectra in fairly large vector spaces (fp-shell) with remarkable accuracy.

The "goodness" of space symmetry for light (sd-shell) and heavier (fp-shell) nuclei has been investigated. When Wigner proposed the group SU(4), it was generally expected that the supermultiplet scheme would be confined to light nuclei. It was however, shown later by Franzini and Radicati that a mass formula, as derived from this symmetry model, seems to improve in validity with increasing mass number. We however find, that SU(4) symmetry in nuclei, is strongly mixed. It appears then, that this particular consequence arising out of space symmetry is not affected by strong mixing in the wave functions. Of course by the same argument the mass

relationship does not say anything significant about the goodness of symmetry either. The usefulness of SU(4) partial level densities in alpha transfer reactions has been briefly indicated.

The distribution methods have been used to investigate the average deformation (quadrupole moment) as a function of the excitation energy in a nucleus. We describe, how by defining a suitable configuration space and using these methods, one can obtain the energy variation of the average deformation. The average moment is of interest in heavy ion reaction processes where compound nuclear formation takes place. The configuration space used in the study of average deformation also enables one to calculate fixed angular momentum J averages. These have been used to determine binding energies and low lying spectra of nuclei.

Finally, some of the formal aspects regarding group averaging have been discussed. The spectral distribution method requires evaluation of the low moments of the Hamiltonian operator H over states belonging to an irreducible representation of a group. From the definition of these moments, it follows that for each moment only the group scalar part of the appropriate power of H would contribute. For evaluating these moments it is sometimes possible to construct new operators, one for each moment, which are group scalars and have the correct particle rank. Moreover these new operators reproduce the moments correctly and are much simpler to work with. It is therefore

very essential to extend this procedure for evaluating moments to as many groups as possible. We have described a way of evaluating moments over states defined by SU(4) symmetry and isospin, and also for the Elliott SU(3) group. Further, expressions for the calculation of spectral averages over states defined by the canonical chain of Unitary groups have been derived.

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CONTENTS

CHAPTER-I

INTRODUCTION

1.1 Symmetries and Statistics

A large amount of information regarding the behaviour of a many particle system can be obtained by a consideration of the underlying symmetries and an application of statistical ideas. The detailed study of a complex system such as a nucleus, being often too difficult, the relevance of these two general approaches, which provide an understanding of some broad features can be immediately appreciated. The characterization of a nuclear system in terms of its symmetry properties and statistical behaviour is important in the analysis of nuclear phenomena.

The existence of a symmetry has profound consequences. Occurrence of symmetries implies that the eigenstates of a system can be characterized by additional quantum numbers corresponding to the symmetry labels. Again, symmetry considerations are useful in finding conserved quantities which lead to selection rules, telling us that certain processes would not occur while others would be strongly favoured. Degeneracies in mass or energy and relationship between energies and transition probabilities are some of the obvious implications. Further, a symmetry may supply a starting point for constructing an approximation

scheme for the detailed study of a system.

Transformations which leave the Hamiltonian governing a quantum mechanical system invariant, lead to symmetry groups. Besides, the exact space-time symmetries of angular momentum and parity, arising out of the invariance under rotations and reflections in a three dimensional space, there are other symmetries which depend upon the specific nature of the Hamiltonian and act in abstract spaces. These are called 'dynamical' symmetries and were first applied in nuclear physics.

Heisenberg (1932) suggested that the neutron and proton may be regarded as the two charge states of a nucleon. He introduced a new variable later named by Wigner (1937) as isotopic spin. The usefulness of isotopic spin for describing charge independence of nuclear forces, first suggested by Guggenheim (1934) and Young (1935), was demonstrated by Cassen and Condon (1936). Feenberg and Wigner (1937) gave strong evidence of this charge independence which gives rise to isospin symmetry. This marked the beginning of the application of dynamical symmetries to nuclear physics. Wigner (1937) motivated by the idea of spin-isospin independence of the nuclear interaction, extended the isospin symmetry and introduced the group SU(4). The symplectic symmetry Sp(N), introduced

by Racah (1949) in the theory of complex spectra is related to the pairing effects in nuclei; and Elliott (1958) showed that the rotational behaviour in light nuclei is related to the group SU(3).

However, although these groups have been with us for só many years and have played an important role in the development of nuclear physics, the real status of group symmetries has not been clear. Except for the fact that some of these were not exact, one did not know to what extent they were 'good' or 'broken'. Of course, in principle, it is always possible to carry out matrix calculations by setting up the Hamiltonian matrix and diagonalizing it to obtain the eigenvalues and eigenfunctions. The eigenfunctions then provide information about the symmetry breaking. However, the vector spaces encountered in nuclear spectroscopy are often very large, and hence it is difficult to carry out such a programme. For these reasons symmetries in nuclei have by and large been studied in the past by truncating the many particle vector spaces according to some symmetry principle. Although this has been of considerable help in simplifying the problem, the validity of truncation has been examined only for systems involving few nucleons. In any case, it seems necessary to develop simple tests for the goodness of nuclear symmetries

which can give a measure of symmetry breaking to be expected before a detailed decomposition of a complicated many particle wave function into different symmetry types is carried out. It is equally necessary to know a priori the extent of breaking that can be tolerated before the symmetry itself ceases to be of any importance.

Historically, statistical considerations have also been employed in nuclear physics, particularly in the random matrix theory developed by Wigner (1957), Dyson (1963), Mehta (1967) and others. Random matrices, first encountered in mathematical statistics by Wishart (1928), Hsu (1939) and others found their application in nuclear physics when Wigner proposed that the local statistical behaviour of energy levels can be simulated by the eigenvalues of a random matrix.

In this very elegant formulation, the statistical element is introduced by considering an ensemble of Hamiltonians. All members of the ensemble are such that apart from being hermitian and time reversal invariant, the matrix elements are randomly distributed. Some general properties of nuclear spectra like distribution of eigenvalues and their spacings are then obtained by averaging over this ensemble. It is well known that at high excitation energies, because of the extremely high density of nuclear states, it would be impossible

to determine and explain the characteristics of every individual state. In such a situation one can talk about the average properties, these being much simpler to determine using the random matrix theory. This average behaviour is important in the study of nuclear reactions and this theory has been extensively applied in the analysis of neutron resonance reactions data.

Inspite of the success of this theory in describing the statistical properties of nuclear states like level spacings etc., it has one feature which is not in agreement with what one has learnt from spectroscopy in the ground state region of nuclei. The low lying levels in a nucleus can be understood by means of interactions which are only two-body and may be three-body in character. One does not need many-body interactions. On the other hand, the random matrix theory as usually applied to a complex nucleus involves predominantly many-body interactions.

It seems therefore that the application of group theory and statistics to nuclear physics in the past has not been completely satisfactory. The introduction of Spectral Distribution methods by French (1967) has provided a way of overcoming these difficulties. In this approach, one decomposes

the many particle spectroscopic space which can be arbitrarily large but finite, into subspaces defined by some symmetry(SU(4) for example) and one simultaneously considers statistical properties of states in the subspaces. The remarkable feature is that the operation of a Central Limit Theorem leads to a great statistical simplicity in the behaviour of the system. The density of states in a subspace defined by some symmetry is nearly normal (Gaussian). Thus the two broad underlying principles, that of symmetry and of statistics get combined and the close relationship between the two is demonstrated.

The merit of this method lies in the fact that one can now consider large spectroscopic spaces without making simple assumptions about the nature of the interaction. Simplicity in description arises from statistics; and further, this method makes explicit use of the fact that one has a many particle system with interactions of low particle rank. The spectral distribution method besides linking statistics and symmetry is free from the objections earlier encountered while applying symmetry and statistical ideas to nuclear systems.

In this thesis, the spectral distribution methods have been applied to the study of symmetries. Although statistical elements are indispensable for the success of the method and full use is made of them, the focus will be mainly on symmetries.

The Spectral distribution methods are reviewed in Sec. (1.2). Evaluation of low-order moments of a dynamical operator is described. Normality of distribution is discussed briefly. Application of these methods to the determination of binding energy, low-energy spectra of a nucleus, symmetry mixing etc is described in Sec.(1.2.4).

In the study of nuclear symmetries using distribution methods, we would be mainly dealing with symmetries associated with finite dimensional vector spaces. These are related to the unitary group in N dimensions U(N) and its subgroups. In Sec. (1.3), we discuss the direct-sum and direct-product subgroups of U(N).

Finally, in Sec.(1.4) the scope of this thesis is outlined.

1.2 Spectral Distribution Method

The main problem in quantum mechanics is the determination of the eigenvalues and eigenfunctions of a Hamiltonian operator which describes the system under consideration. In nuclear physics, often the matrix is set up in a finite vector space and is then diagonalized to obtain the energies and wave functions. Unfortunately, except for the simplest systems, an exact solution of such a model many body problem is out of question because of the computational limitation involved in

construction and diaganalization of very large matricies. For example, in the ds-shell which consists of 24 single particle states, the size of the largest matrix (JT) for $(ds)^4$ is 69, for $(ds)^8$ is 2268 and for $(ds)^{12}$ is 6706. The dimensionality of the Hamiltonian matrix increases very rapidly as the vector space is enlarged by increasing the number of single particle states or particles. In the fp-shell consisting of 40 single particle states, the largest (JT) matrix for $(fp)^4$ is 300, for $(fp)^8$ is about 10^5 and for $(fp)^{12}$ is of the order of 4×10^6.

One way of overcoming this difficulty is to choose an appropriate basis space such that the dimensionality of the Hamiltonian matrix is appreciably reduced. The vector space has usually been truncated by invoking symmetry arguments. However, this procedure is not entirely satisfactory because it assumes the goodness of symmetry. Secondly, it may so happen that severe truncations might lead to erroneous conclusions.

The spectral distribution methods introduced by French (1967) have proved to ba a powerful alternative to this conventional approach. These methods are well suited for studying general aspects of nuclear structure like distribution of levels and goodness of certain group symmetries. They also enable us to study details about the low lying levels in a

nucleus. Further, a knowledge obtained from such a study is helpful in suggesting good approximation methods for detailed problems.

In this method, instead of considering the detailed spectrum of various quantities, one deals with moments of their distribution in energy, configuration, isospin, SU(4) symmetry etc. The usefulness of the distribution method stems from the fact that due to a statistical simplicity one needs only the low order moments to describe systems with reasonable accuracy. This statistical simplicity in the behaviour of many particle systems where only low-order moments are most significant is attributed to the Central Limit Theorem. It is observed that the distribution of levels and various other quantities tends to a normal form as particle number increases. The reasons for normality of distributions will be discussed later.

In the next section, we discuss the distribution method formally. The moments of a distribution are defined and some comments regarding the spaces over which spectral averages can be easily evaluated are made.

1.2.1 Moments and Distributions

In spectroscopy we are interested in the solution of the eigenvalue problem

$$H\psi = E\psi \tag{1.2.1}$$

where H is the Hamiltonian operator and ψ_i and E_i denote the eigenfunctions and eigenvalues respectively.

The functions ψ are expanded in terms of a set of basis states ϕ_α.

$$\psi_i = \sum_\alpha B_{\alpha i} \phi_\alpha \tag{1.2.2}$$

The eigenvalue problem given in eq.(1.2.1) then reduces to the usual matrix problem. Let us consider the inverse problem where the ϕ's are expanded in terms of the eigenfunctions ψ's.

$$\phi_\alpha = \sum_i C_{i\alpha} \psi_i \tag{1.2.3}$$

A plot of $|C_{i\alpha}|^2$ vs E_i defines a distribution which can be studied via its energy moments, the p^{th} moment being defined as

$$M_\alpha^p = \sum_i |C_{i\alpha}|^2 E_i^p \tag{1.2.4}$$

$$= \langle \phi_\alpha | H^p | \phi_\alpha \rangle$$

Instead of considering the distribution of a single state α, let us choose a set of states $\underset{\sim}{\alpha}$ and define the average moments for the set $\underset{\sim}{\alpha}$ as:

$$M^{p}_{\underset{\sim}{\alpha}} = \frac{1}{d(\underset{\sim}{\alpha})} \sum_{\alpha \in \underset{\sim}{\alpha}} \langle \Phi_\alpha | H^p | \Phi_\alpha \rangle \qquad (1.2.5)$$

where $d(\underset{\sim}{\alpha})$ denotes the number of states (dimensionality) in the set $\underset{\sim}{\alpha}$. The first moment p = 1 defines the centroid energy $E_c(\underset{\sim}{\alpha}) \equiv M^{1}_{\underset{\sim}{\alpha}}$. In terms of the centroid energy the central moments μ_p are defined as:

$$\mu_p = \frac{1}{d(\underset{\sim}{\alpha})} \sum_{\alpha \in \underset{\sim}{\alpha}} \langle \Phi_\alpha | (H - M^{1}_{\underset{\sim}{\alpha}})^p | \Phi_\alpha \rangle \qquad (1.2.6)$$

The second central moment $\mu_2 \equiv \sigma^2(\underset{\sim}{\alpha})$ given by

$$\sigma^2(\underset{\sim}{\alpha}) = M^{2}_{\underset{\sim}{\alpha}} - (M^{1}_{\underset{\sim}{\alpha}})^2 \qquad (1.2.7)$$

is called the energy variance and describes the spreading of states about the centroid energy. In terms of the centroid energy and variance, a normal distribution is defined by the frequency function f(E) as:

$$f(E) = \frac{d(\underset{\sim}{\alpha})}{\sqrt{2\pi}\,\sigma(\underset{\sim}{\alpha})} \exp\left[- \frac{(E - E_c(\underset{\sim}{\alpha}))^2}{2\sigma^2(\underset{\sim}{\alpha})} \right] \qquad (1.2.8)$$

For a gaussian distribution the half width at half maximum is $1.18\,\sigma$ and one needs only two moments, the shape being unique.

Departures from normality (defined by the first and second moment only) can be studied by evaluating higher moments ($p \geq 2$) and defining shape parameters, the skewness γ_1 and excess γ_2.

$$\gamma_1 = \mu_3/\mu_2^{3/2} \; ; \; \gamma_2 = \mu_4/\mu_2^2 - 3 \qquad (1.2.9)$$

In terms of these four moments, a frequency function for a distribution which is close to a gaussian is:

$$f(x) = \frac{1}{\sqrt{2\pi}}\left[1 + \frac{1}{6}\gamma_1 (x^3 - 3x) + \frac{1}{24}\gamma_2 (x^4 - 6x^2 + 3)\right] \exp\left(-x^2/2\right) \qquad (1.2.10)$$

where $x = (E - E_c)/\sigma$

It should be mentioned here that far away from the centroid this form is not positive definite and thus fails in one of the essential requirements for a frequency function. Therefore in using eq.(1.2.10) one must be certain that one is not in the negative domain.

In fig.(1.1) taken from Ratcliff (1971) four frequency functions which illustrate the meaning of γ_1 and γ_2 are displayed. For a gaussian distribution $\gamma_1 = \gamma_2 = 0$. γ_1 measures

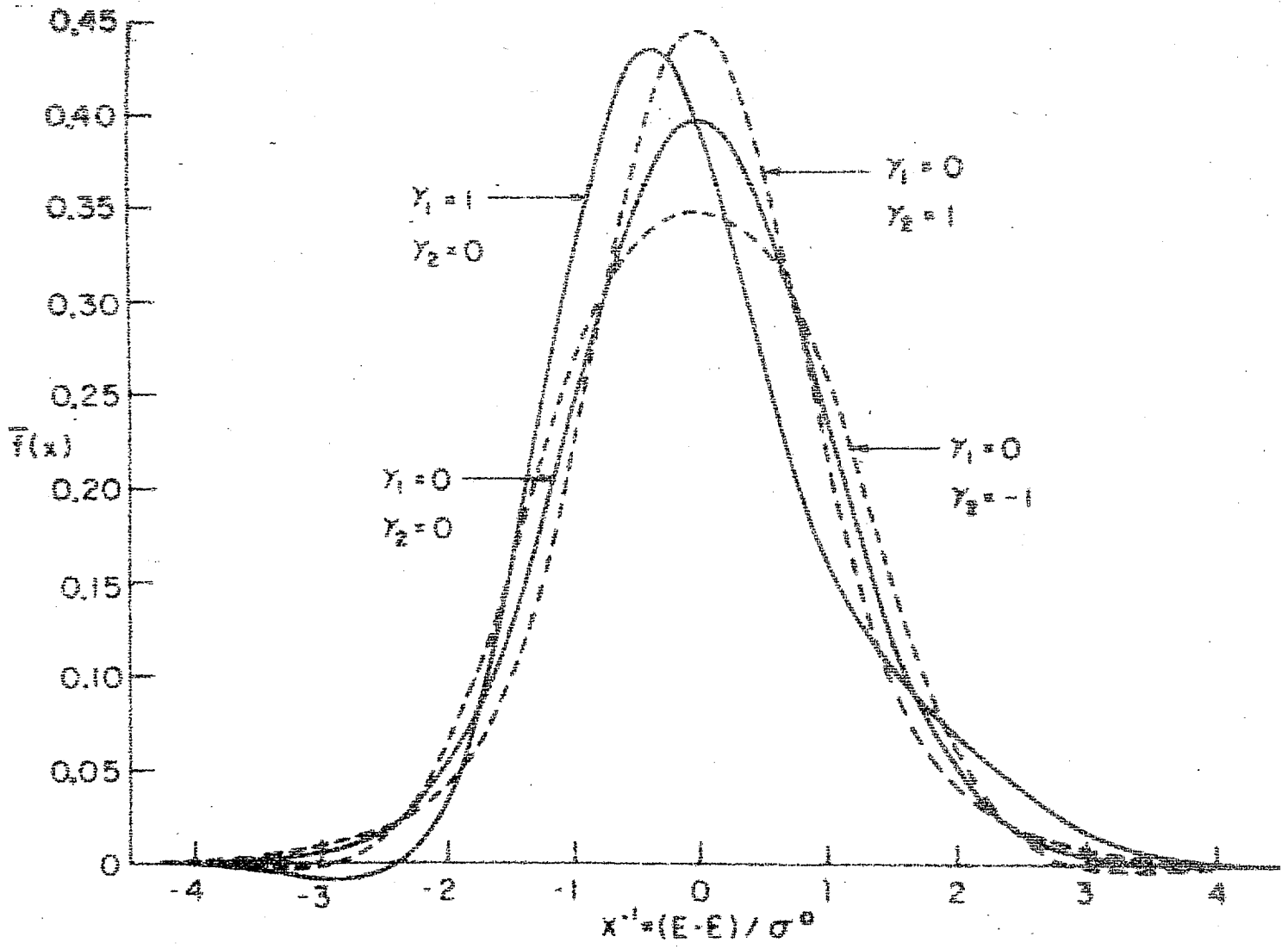

Fig. I.1 Four-moment frequency functions (taken from Ratcliff, 1971) for various large values of the Skewness (γ_1) and excess (γ_2) parameters.

the asymmetry of the distribution around the centroid and vanishes for a symmetrical distribution. Negative values of γ_2 indicate a flattened distribution. Positive values of γ_2 indicate a sharper central peak.

We can rewrite the variance $\sigma^2(\underset{\sim}{\alpha})$ given in eq.(1.2.7) by making an intermediate state expansion, as:

$$\sigma^2(\underset{\sim}{\alpha}) = \frac{1}{d(\underset{\sim}{\alpha})} \sum_{\substack{\alpha \in \underset{\sim}{\alpha} \\ \beta}} \langle \alpha | H | \beta \rangle \langle \beta | H | \alpha \rangle - E_c^2(\underset{\sim}{\alpha}) \qquad (1.2.11)$$

By grouping the intermediate state β into sets $\underset{\sim}{\beta}$ we get,

$$\sigma^2(\underset{\sim}{\alpha}) = \sigma^2(\underset{\sim}{\alpha}, \underset{\sim}{\alpha}) + \sum_{\underset{\sim}{\beta} \neq \underset{\sim}{\alpha}} \sigma^2(\underset{\sim}{\alpha}, \underset{\sim}{\beta}) \qquad (1.2.12)$$

where

$$\sigma^2(\underset{\sim}{\alpha}, \underset{\sim}{\alpha}) = \frac{1}{d(\underset{\sim}{\alpha})} \sum_{\substack{\alpha, \alpha' \\ \in \underset{\sim}{\alpha}}} \langle \alpha | H | \alpha' \rangle \langle \alpha' | H | \alpha \rangle - E_c^2(\underset{\sim}{\alpha})$$

and

$$\sigma^2(\underset{\sim}{\alpha}, \underset{\sim}{\beta}) = \frac{1}{d(\underset{\sim}{\alpha})} \sum_{\substack{\alpha \in \underset{\sim}{\alpha} \\ \beta \in \underset{\sim}{\beta} \\ \alpha \neq \beta}} \langle \alpha | H | \beta \rangle \langle \beta | H | \alpha \rangle \qquad (1.2.13)$$

The quantity $\sigma^2(\underset{\sim}{\alpha}, \underset{\sim}{\alpha})$ called the internal variance arises because states in the representation $\underset{\sim}{\alpha}$ are not all degenerate. The quantity $\sigma(\underset{\sim}{\alpha}, \underset{\sim}{\beta})$ called the external width describes the average r.m.s. matrix elements connecting representation $\underset{\sim}{\alpha}$ and the representation $\underset{\sim}{\beta}$.

From eq.(1.2.12) we observe that the width of one subset depends on interactions with the others. Hence to determine the width of a subspace, one needs to consider states outside this subspace also. We shall see later that these kinds of distributions in which the space is divided into subspaces are of major importance.

We next discuss questions regarding the spaces over which moments should be evaluated. The evaluation of widths for any arbitrary space $\underset{\sim}{\alpha}$ would involve a large number of matrix elements of complicated operators and is therefore extremely difficult. If however the space $\underset{\sim}{\alpha}$ is selected with symmetry consideration in mind, $\underset{\sim}{\alpha}$ defining irreducible representation (irrep) of some group, we shall find that these methods which rely upon the invariance properties of traces would become available for the evaluation of moments. In such a case the moments can be evaluated without having recourse to matrix methods. Obviously then,no restrictions regarding the size N of single particle states and the number of active nucleons m need be imposed. Also for such invariant subspaces the distributions are close to normal and therefore one needs a small number of moments to characterize them.

It is important that distributions be describable via their low moments and it is for subspaces defined by group symmetries that nearly gaussian distributions are expected.

It should be mentioned here that in choosing $\underset{\sim}{\alpha}$ which defines a certain symmetry, no assumptions regarding the goodness of this symmetry are made. Using symmetry subspaces to define distributions does not imply ignoring interactions which connect different subspaces. This is in contrast to earlier work where the spectroscopic space was truncated on the basis of symmetry being good. The choice of $\underset{\sim}{\alpha}$ helps us in learning about the goodness of group symmetries, broad distributions immediately telling us that such a symmetry is of no interest. Thus the distribution method combined with group theory seems a good way of investigating the goodness of group symmetries.

In the next section we describe how spectral moments can be evaluated.

1.2.2 Evaluation of Spectral Moments

The spectral distribution method seeks a direct way of calculating moments without evaluating the many-body matrix elements. In this section we derive an expression for the moment of a k-body operator in m-particle states belonging to the symmetry $\underset{\sim}{\alpha}$.

Following French (1967) we introduce the creation operators A_i and annihilation operators B_i for each single particle state ($i = 1, \ldots . N$). These satisfy the anticommutation relations

$$\left[A_i , B_j \right]_+ = \delta_{ij}$$

$$\left[A_i , A_j \right]_+ = \left[B_i , B_j \right]_+ = 0 \qquad (1.2.14)$$

The creation and destruction operators for m-particle states in state α are represented as $Z_\alpha(m)$ and $Z^\dagger_\alpha(m)$. We shall take α to be a determinantal state so that $Z_\alpha(m)$ and $Z^\dagger_\alpha(m)$ are product states of m A-operators and m B-operators respectively. An arbitrary m-particle state is then a linear combination of such determinantal states. The operator $Z_\alpha(k)\, Z^\dagger_{\alpha'}(k)$ is a k-body operator whose matrix element between the k-particle states β and β' is given by

$$\langle k\beta | Z_\alpha(k) Z^\dagger_{\alpha'}(k) | k\beta' \rangle = \delta_{\alpha\beta}\delta_{\alpha'\beta'} \qquad (1.2.15)$$

In terms of the state operators Z_α a k-body operator $O(k)$ can be written as:

$$O(k) = \sum_{\beta\beta'} \langle k\beta | O(k) | k\beta' \rangle Z_\beta(k) Z^\dagger_{\beta'}(k) \qquad (1.2.16)$$

The average of O(k) in m-particle states belonging to $\underset{\sim}{\alpha}$ is defined as:

$$\langle O(k) \rangle^{m\underset{\sim}{\alpha}} = \frac{1}{d(m\underset{\sim}{\alpha})^{\alpha \in \underset{\sim}{\alpha}}} \sum \langle m\alpha | O(k) | m\alpha \rangle \qquad (1.2.17)$$

where $d(m_{\underset{\sim}{\alpha}})$ is the total number of states over which the sum is taken. Instead of dealing with the average, let us consider the trace of $O(k)$ defined as:

$$\langle\langle O(k)\rangle\rangle^{m_{\underset{\sim}{\alpha}}} = d(m_{\underset{\sim}{\alpha}})\langle O(k)\rangle^{m_{\underset{\sim}{\alpha}}}$$

$$= \sum_{\alpha \in \underset{\sim}{\alpha}} \langle m\alpha | O(k) | m\alpha \rangle \qquad (1.2.18)$$

Using eq.(1.2.16) we write the trace of $O(k)$ as:

$$\langle\langle O(k)\rangle\rangle^{m_{\underset{\sim}{\alpha}}} = \sum_{\substack{\alpha \in \underset{\sim}{\alpha} \\ \beta\beta'}} \langle k\beta | O(k) | k\beta' \rangle \langle m\alpha | Z_{\beta}(k) Z^{\dagger}_{\beta'}(k) | m\alpha \rangle \qquad (1.2.19)$$

Now under a particle $\rightleftharpoons$ hole ($p \rightleftharpoons h$) transformation; $A_i \rightleftharpoons B_i$ and for the state operators we have the following correspondence:

$$Z_{\alpha}(m) \xrightarrow{p \rightleftharpoons h} \tilde{Z}_{\alpha}(m) = (-1)^{m(m-1)/2} Z^{\dagger}_{\alpha}(m)$$

$$Z^{\dagger}_{\alpha}(m) \xrightarrow{p \rightleftharpoons h} \tilde{Z}^{\dagger}_{\alpha}(m) = (-1)^{m(m-1)/2} Z_{\alpha}(m) \qquad (1.2.20)$$

Further, the $p \rightleftharpoons h$ transformation establishes a relationship between the state $(m\alpha)$ and the complementary state $(N\text{-}m\alpha_c)$ such that for the matrix element of O one has the equality:

$$\langle m\alpha | O(k) | m\alpha' \rangle = \langle N-m\alpha_c | \tilde{O}(k) | N-m\alpha_c' \rangle$$

(1.2.21)

where

$$O \xrightarrow{p \rightleftharpoons h} \tilde{O}$$

Eq.(1.2.19) can be rewritten as:

$$\langle\langle O(k) \rangle\rangle^{m\alpha}_{\tilde{\alpha}} = \sum_{\substack{\alpha \in \alpha \\ \beta\beta'}} \langle k\beta | O(k) | k\beta' \rangle \langle 0 | Z^{\dagger}_{\alpha}(m) Z_{\beta}(k) Z^{\dagger}_{\beta'}(k) Z_{\alpha}(m) | 0 \rangle$$

$$= \sum_{\substack{\alpha \in \alpha \\ \beta\beta'}} \langle k\beta | O(k) | k\beta' \rangle \langle Z^{\dagger}_{\alpha}(m) Z_{\beta}(k) Z^{\dagger}_{\beta'}(k) Z_{\alpha}(m) \rangle_0$$

(1.2.22)

where $|0\rangle$ is the particle vacuum and $\langle\ \rangle_0$ denotes the vacuum expectation value (vev). Using eq.(1.2.21) we get,

$$\langle\langle O(k) \rangle\rangle^{m\alpha}_{\tilde{\alpha}} = \sum_{\substack{\alpha \in \alpha \\ \beta\beta'}} \langle k\beta | O(k) | k\beta' \rangle \langle Z^{\dagger}_{\alpha_c}(N-m) Z^{\dagger}_{\beta}(k) Z_{\beta'}(k) Z_{\alpha_c}(N-m) \rangle_0$$

Making use of the result that

$$\langle Z^{\dagger}_{\alpha}(m) Z^{\dagger}_{\beta}(k) Z_{\beta'}(k) Z_{\alpha'}(m) \rangle_0 = \langle Z^{\dagger}_{\beta}(k) Z^{\dagger}_{\alpha}(m) Z^{(m)}_{\alpha'} Z^{(k)}_{\beta'} \rangle_0$$

the trace of $O(k)$ becomes

$$\langle\langle O(k) \rangle\rangle^{m\underset{\sim}{\alpha}} = \sum_{\beta\beta'} \langle k\beta | O(k) | k\beta' \rangle \langle Z^{\dagger}_{\beta}(k) \Big[\sum_{\alpha_c \in \underset{\sim}{\alpha}} Z^{\dagger}_{\alpha_c}(N-m) Z_{\alpha_c}(N-m) \Big] Z_{\beta'}(k) \rangle_0$$

(1.2.23)

We now define a density operator $\rho(m\underset{\sim}{\alpha})$ given by

$$\rho(m\underset{\sim}{\alpha}) = \sum_{\alpha \in \underset{\sim}{\alpha}} Z_{\alpha}(m) Z^{\dagger}_{\alpha}(m) \qquad (1.2.24)$$

$\rho(m\underset{\sim}{\alpha})$ is an m-particle operator and therefore gives zero on all k-particle states with $k<m$. Further, $\rho(m\underset{\sim}{\alpha})$ gives unity when acting on m-particle states belonging to $\underset{\sim}{\alpha}$, and zero for all other m-particle states. It therefore acts as a projection operator. Assuming that the matrix elements of $O(k)$ are real or considering the hermitian nature of ρ and using eq.(1.2.24) we obtain for the trace of $O(k)$:

$$\langle\langle O(k) \rangle\rangle^{m\underset{\sim}{\alpha}} = \sum_{\beta} \langle\langle \tilde{\rho}(N-m\underset{\sim}{\alpha}) O(k) \rangle\rangle^{k\beta}$$

$$= \sum_{\underset{\sim}{\beta}} \langle\langle \tilde{\rho}(N-m\underset{\sim}{\alpha}) O(k) \rangle\rangle^{k\underset{\sim}{\beta}} \qquad (1.2.25)$$

where the k-particle states have been divided into sets $\underset{\sim}{\beta}$ and summed over all $\underset{\sim}{\beta}$.

If we consider some subspace decomposition of the many nucleon states where the density operator ρ for each subspace is diagonal and its matrix element a multiple of unity, for such cases eq.(1.2.25) can be written as:

$$\langle\langle O(k)\rangle\rangle^{m\underset{\sim}{\alpha}} = \sum_{\underset{\sim}{\beta}} \langle \tilde{\rho}(N-m\underset{\sim}{\alpha})\rangle^{k\underset{\sim}{\beta}} \langle\langle O(k)\rangle\rangle^{k\underset{\sim}{\beta}} \qquad (1.2.26)$$

Eq.(1.2.26) relates the trace of a k-body operator in various k-particle spaces ($k\underset{\sim}{\beta}$) to the trace of O(k) in the space ($m\underset{\sim}{\alpha}$). Since a k-body operator is completely specified by its matrix elements in k-particle states (defining space), we see that the trace in the defining space 'propagates' to other spaces by means of the density operator ρ.

Using the identity:

$$d(k\underset{\sim}{\beta})\langle \tilde{\rho}(N-m\underset{\sim}{\alpha})\rangle^{k\underset{\sim}{\beta}} = d(m\underset{\sim}{\alpha})\langle \rho(k\underset{\sim}{\beta})\rangle^{m\underset{\sim}{\alpha}}$$

and eq.(1.2.26) the average of O(k) is given by

$$\langle O(k)\rangle^{m\underset{\sim}{\alpha}} = \sum_{\underset{\sim}{\beta}} \langle m\underset{\sim}{\alpha}|\rho(k\underset{\sim}{\beta})|m\underset{\sim}{\alpha}\rangle \langle O(k)\rangle^{k\underset{\sim}{\beta}} \qquad (1.2.27)$$

For the purpose of evaluating averages, the operator O(k) is

then equivalent to an operator $\hat{O}(k)$ defined as:

$$\hat{O}(k) \equiv \sum_{\underset{\sim}{\beta}} \rho(k\underset{\sim}{\beta}) \langle O(k) \rangle^{k\underset{\sim}{\beta}} \qquad (1.2.28)$$

$\hat{O}(k)$ is called the linear trace equivalent of $O(k)$. Whenever $\rho(k\underset{\sim}{\beta})$ are diagonal with respect to subspaces $\underset{\sim}{\beta}$ and behave like a multiple of unity in each subspace; then these projection operators $\rho(k\underset{\sim}{\beta})$ are scalars under the transformations of the group over which the average is carried out. Thus $\hat{O}(k)$ is a group scalar and only the group scalar part of any operator $O(k)$ contributes to the average.

If the operator whose average we wish to evaluate does not have a definite particle rank, one could decompose it into operators of definite rank, each of which would then propagate according to eq.(1.2.27). Instead of proceeding this way, French and Ratcliff (1971) have given an alternate method of evaluating averages of such operators.

Having derived the formal result for evaluating averages, we illustrate the averaging method by an example. Let us consider a vector space $S(N,m)$ obtained by distributing m fermions over N single particle states. The dimensionality of this space is $\binom{N}{m}$. Let us consider the case of averaging for the k-body operator $O(k)$ over all m-particle states. This

average over all m-particle states called 'Scalar' averaging can be considered to be an average over the irreducible representation $[1^m]$ of U(N). In this case French (1967) has shown that the density operator $\rho(k) = \binom{n}{k}$ where n is the number operator and for the average we get

$$\langle O(k) \rangle^m = \binom{m}{k} \langle O(k) \rangle^k \qquad (1.2.29)$$

The binomial coefficient $\binom{m}{k}$ propagates the average of O(k) in the defining space to the rest of the space. This result can be generalized to operators of mixed particle rank. Consider the operator O with mixed particle rank going from 0 to a maximum ν . The average is given by,

$$\langle O \rangle^m = \sum_{t=0}^{\nu} \binom{\nu - m}{\nu - t} \binom{m}{t} \langle O \rangle^t \qquad (1.2.30)$$

Eq.(1.2.30) can be rewritten as:

$$\langle O \rangle^m = \sum_{t=0}^{\nu} \rho_t^{\nu}(m) \langle O \rangle^t \qquad (1.2.31)$$

$\rho_t^{\nu}(m)$ are the density operators and possess the following properties:

1. $\rho_t^{\nu}(m)$ is a v degree polynomial in m

2. $\rho_t^{\nu}(m) = \delta_{mt}, \quad 0 \leq m \leq \nu$

which means that $\rho_t^{\nu}(m)$ act as projection operators for various states in the defining representation of the operators.

The propagation formula for scalar averaging requires input information $(\langle o \rangle^t)$ for $t = 0 \rightarrow \nu$ particles. The input space can be regarded as a net where information at some points on the net is fed in. The propagation net discussed above is called an 'elementary net' and is not the most convenient net to use. There are two reasons for this. Firstly, one needs ν-body averages and these might be difficult to evaluate and secondly, the averages in the defining space need to be calculated very accurately in order that the averages near the closed shell are produced correctly. It is, therefore, better to replace some of the points in the elementary net by hole representations. This is called the 'optimum net'.

Averaging over all m-particle states is easy, the density operators being simple to construct. While dealing with finer averages (i.e. over states defined by subgroups of U(N)) the density operators might be harder to construct but the principle essentially remains the same as in scalar averaging. The advantage of averaging over states defined by the irreps of a group arises from the fact that the density operators in such

cases are group scalars and hence can often be expressed as polynomials in the various Casimir invariants. The explicit construction of these polynomials involves inversion of a set of linear equations and this being simple, one can in these cases entirely get away from the tedious matrix method.

Essentially then, given the Hamiltonian H which describes interactions of particles, the distribution method aims at calculating moments of H which define the distribution. The moments of operators in m-particle space S(N,m) are evaluated by expressing them as linear combinations of averages over simpler subspaces involving only few particles. Let H possess no 3-body or more complicated parts. Then the matrix elements of H in the (0+1+2) particle space called the 'input' space completely specify the operator. Knowledge about H in the input space is expressed in terms of traces over basis subsets of this space and propagation laws for the traces in terms of density operators are derived giving moments for many particle states.

From the definition of these moments eq.(1.2.27) it follows that for each moment only the group scalar part of the appropriate power of H would contribute. For evaluating these averages, it is sometimes possible to construct new operators, one for each moment, which are group scalars and have the correct

particle rank. Moreover, these new operators reproduce the moments correctly and are much simpler to work with.

It is essential that one should be able to describe distributions in terms of low-moments. Higher moments of operators might be difficult to evaluate. In the next section we present evidence to show that distributions are normal, only two moments being then sufficient to define them.

1.2.3 Normality of Distributions

Analysis of shell model calculations has shown that the shell model spectrum is essentially gaussian. The accuracy with which a two-moment and a four-moment distribution reproduces the eigen-energies in many particle spectra has been studied by Chang and Zuker (1972). Exact shell model calculations in the (ds)-shell of dimensionalities ranging upto 1200 using a realistic interaction have been compared with moment method results and excellent agreement is obtained.

A detailed numerical study regarding the origin of normality of distribution has been carried out by French and Wong (1970) and by Bohigas and Flores (1971) suggesting that the gaussian nature of the spectrum is connected with the two body nature of the effective interaction and the direct product nature of the m-particle states. The direct product nature means that if the single particle state is represented by ψ,

the two-body states are quadratic in ν and the k-particle states are of the k^{th} order. French and Wong choose a finite spectroscopic space defined by distributing a given number of particles in a finite set of single particle states. The Hamiltonian matrix is set up in this space by making random selections of the two body matrix elements. Many such matricies are constructed and the ensemble average is taken to determine the distribution of eigenvalues. The distribution for this two body random ensemble (TBRE) is a gaussian. Further, when three-body, four-body and many-body parts are added to TBRE one after another, the distribution gradually departs from a gaussian and eventually changes into a semi-circular type as predicted by the random matrix theory. In their study, French and Wong have also shown that just by setting to zero the matrix elements between many particle states which cannot be connected by a two-body interaction and making random selections according to some probability law for the rest, does not lead to a gaussian distribution of the level density. The normality is a consequence of the two-body character of the interaction giving rise to strong correlations between matrix elements, and the nature of the spectroscopic space.

The normality of distribution is now quite rigorously proved (for states belonging to a fixed exact symmetry) using two body Gaussian orthogonal ensembles by Mon (1973). It is

expected that extensions of the central limit theorem give rise to the normality of distributions over states of fixed unitary symmetry which is not exact.

1.2.4 Application of the Spectral Distribution Method to Nuclear Spectroscopy

The spectral distribution method which deals with the averages of H and powers of H over various subspaces of the complete space can be used for the study of quantities of physical interest like binding energies, low lying spectra and questions related to group symmetries. Having pointed out that distributions are described via their low-moments, and having indicated how one may in some cases evaluate these moments; we show in this section how a knowledge of centroid energies, variances and the dimensionalities of irreps of some group can provide detailed information about nuclear spectroscopy.

The method is based on making a continuous approximation to the discrete eigenvalue spectrum of a given Hamiltonian by defining a frequency function f(mE) given by,

$$f(mE) = \sum_{\underset{\sim}{\alpha}}{}' f(mE\underset{\sim}{\alpha}) = \sum_{\underset{\sim}{\alpha}} \frac{d(m\underset{\sim}{\alpha})}{\sqrt{2\pi}\sigma(\underset{\sim}{\alpha})} \exp\left[- \frac{(E - E_c(m\underset{\sim}{\alpha}))^2}{2\sigma^2(m\underset{\sim}{\alpha})} \right] \tag{1.2.32}$$

The centroid energy $E_c(m\underset{\sim}{\alpha})$ and variance $\sigma^2(m\underset{\sim}{\alpha})$ for m-particle states belonging to the symmetry $\underset{\sim}{\alpha}$ have been defined through eqs. (1.2.5 to 1.2.7) and the frequency function for a gaussian distribution was given in eq.(1.2.8). $f(mE\underset{\sim}{\alpha})$ is a normal distribution with area under the curve equal to the dimensionality $d(m\underset{\sim}{\alpha})$. f(mE) which is obtained by summing over all subspaces for a given m describes the number of states per unit energy interval.

In terms of scalar moments $E_c(m)$ and $\sigma^2(m)$, we can define a gaussian frequency function $f_o(mE)$ as:

$$f_o(mE) = \frac{d(m)}{\sqrt{2\pi}\,\sigma(m)} \exp\left[-\frac{(E - E_c(m))^2}{2\sigma^2(m)}\right] \qquad (1.2.33)$$

It is observed that f(mE) which is a sum of gaussians defined by $f(mE\underset{\sim}{\alpha})$ corresponding to the symmetry subspace $\underset{\sim}{\alpha}$ is itself very close to being a gaussian $f_o(mE)$ given in eq.(1.2.33). This stability under addition of gaussians is not yet understood.

In terms of the frequency function f(mE), we define a distribution function F(mE) as:

$$F(mE) = \int_{-\infty}^{E} f(mE')\,dE'$$

$$= \sum_{\underset{\sim}{\alpha}} \frac{d(m\underset{\sim}{\alpha})}{\sqrt{2\pi}\,\sigma(m\underset{\sim}{\alpha})} \int_{-\infty}^{E} \exp\left[-\frac{(E - E_c(m\underset{\sim}{\alpha}))^2}{2\sigma^2(m\underset{\sim}{\alpha})}\right] dE' \qquad (1.2.34)$$

The distribution function F(mE) gives the area under the curve upto energy E. With these definitionsof the frequency function and the distribution function we now show how one can determine various quantities of interest in nuclear structure.

a. Binding Energies and Low Energy Spectra

To determine the ground state energy of a nucleus we follow the procedure suggested by Ratcliff (1971). Since F(mE) essentially gives the area and therefore the number of states upto energy E, the ground state (non-degenerate) will be determined at an energy E_g where F(mE) jumps discontinuously from zero to one. Now the degeneracy g_o of a ground state with angular momentum J_o and isospin T_o is given by

$$g_o = (2J_o + 1)\quad(2T_o + 1) \qquad (1.2.35)$$

The position of the ground state is then estimated at an energy E_g such that

$$F(mE_g) = g_o/2 \qquad (1.2.36)$$

this being a natural way to define the value at the discontinuity. In a similar way knowing the (JT) sequence of levels the i^{th} excited state can be located, this being given by,

$$F(mE_{ex}) = \sum_{n=1}^{i-1} (2J_n+1)(2T_n+1) + \frac{1}{2}(2J_i+1)(2T_i+1) \qquad (1.2.37)$$

Often it is advisable, first to locate an excited state which lies higher up in the spectrum. This state is used as a reference state and the binding energy of a nucleus is then obtained by subtracting from the energy of the reference state the observed experimental energy difference between this and the ground state. For the low lying levels, comparison with states below the reference state is left aside.

This way of doing things has two advantages. Firstly, one escapes from the ground state region where uncertainties might exist. Whereas a gaussian has tails going to infinity, the exact distribution has no tail as it is cut off at the lowest and highest eigenstate of the system. Secondly, by moving closer to the centroid one makes less stringent demands on the distribution.

b. Symmetry Mixing

In locating the ground state using the distribution function F(mE), one simultaneously gets information about the relative contribution of the various symmetries $\underset{\sim}{\alpha}$ to the total area under the curve. Accordingly we define intensities $I(m\,\underset{\sim}{\alpha}\,E)$ given by

$$I(m\,\underset{\sim}{\alpha}\,E) = \frac{1}{F(mE)} \int_{-\infty}^{E} f(m\,\underset{\sim}{\alpha}\,E')\,dE' \qquad (1.2.38)$$

with the normalization condition

$$\sum_{\underset{\sim}{\alpha}} I(m\,\underset{\sim}{\alpha}\,E) = 1$$

The intensities $I(m\,\underset{\sim}{\alpha}\,E_g)$ may be interpreted as relative mixing intensities of the different symmetries in the ground state region, ignoring the possibility of interweaving of pure symmetries, an assumption which may often be reasonable on physical grounds.

Besides providing information about mixing of symmetries, the relative intensities can also be used to study the energy dependence of the expectation value of some operator O. Assuming that O does not connect two representations and the expectation value of O does not vary much over the width of

the representation, the energy variation can be written as:

$$\langle O \rangle_E^m = \sum_{\underset{\sim}{\alpha}} I(m \underset{\sim}{\alpha} E) \langle O \rangle^{m \underset{\sim}{\alpha}} \qquad (1.2.39)$$

These considerations have been used by French and Chang (1971) to determine the spin cut-off factor in the level density theory, and also for calculating occupancies of spherical orbits for various nuclei as a function of energy.

We intend using the distribution method to study symmetries in nuclei. Since our interest here is in unitary symmetries, in the next section we discuss briefly the unitary group and its subgroups.

1.3. The Unitary Group and its Subgroups

The starting point in nuclear spectroscopy is the consideration of a finite number N of single particle states. The group U(N), which describes unitary transformations amongst these N single particle states, underlies the whole discussion. Averaging over all m-particle states is essentially an average over the irrep $[1^m]$ of U(N). This is because the Pauli Principle allows only the completely antisymmetric representation $[1^m]$ for m particles. The nuclear states described in terms of a restricted set of N single particle states, then have a definite U(N) symmetry. The goodness

of this symmetry implies that the model space so constructed is indeed reasonable. A large amount of discussion of group symmetries in nuclear physics deals with the group U(N) and its subgroups which correspond to partitions of N into subsets.

Instead of considering the N single particle states as a single set, one can divide this space into ℓ subspaces, the i^{th} subspace having a dimensionality N_i. The partition is given by,

$$N = \sum_{i=1}^{\ell} N_i \qquad (1.3.1)$$

The set of all unitary transformations which operate in each subspace independently also form a group This direct-sum group is a subgroup of U(N) and each subset N_i may be thought of as defining an orbit. This subgroup decomposition is given by

$$U(N) = \sum_{i=1}^{\ell} U(N_i) \qquad (1.3.2)$$

Having partitioned N into orbits N_i, one can assign m_i particles to each of these. The configuration $(m_1\ m_2 \cdots m_i\ m_\ell)$ with the constraint

$$m_1 + m_2 + \cdots\cdots + m_\ell = m \qquad (1.3.3)$$

defines an irrep for the subgroup $\sum_{i=1}^{\ell} U(N_i)$.

Besides this direct-sum group, one can also deal with direct-product subgroups of U(N). A decomposition of the N basis states distinguishing between the spin-orbital subspace and isospin subspace corresponds to the subgroup U(N/2) x U(2) of U(N). One can also consider the direct-product subgroup U(N/4) x U(4) where the decomposition is according to the N/4 spatial orbits and the four spin-isospin states of a single nucleon.

The various subgroups of U(N) for which the distribution method has been applied to study general features of nuclei are listed below. Spectral averages have also in some cases been used to study the goodness of symmetries defined by these decompositions.

a) Configuration;

$$U(N) \supset U(N_1) + U(N_2) + \ldots + U(N_\ell) = \sum_{i=1}^{\ell} U(N_i)$$

b) Isospin;

$$U(N) \supset U(N/2) \times U(2)$$

c) Configuration-Iospin;

$$U(N) \supset \sum_{i=1}^{\ell} U(N_i/2) \times U(2)$$

d) Wigner Supermultiplet.

$$U(N) \supset U(N/4) \times U(4)$$

$$U(N) \supset U(N/4) \times \left[SU_S(2) \times SU_T(2) \right]$$

The configuration averages have been obtained by French and Ratcliff (1971) and by Chang, French and Thio (1971). These are very important in shell model calculations where the effects of high lying shell model orbits on effective interactions need to be investigated. Also in the formation of a compound nucleus one needs to know the importance of various configurations at different excitation energies. Isospin distributions in nuclei have been studied by French (1969) and configuration-isospin distributions by Chang, French and Thio (1971).

1.4 Scope of the Thesis

The elegence of the distribution method arising out of a combination of group theory and statistics will become apparent when we deal with specific applications in the subsequent chapters.

In the next two chapters the Wigner SU(4) symmetry will be considered. Spectral averages of the nuclear interaction over states defined by the irrep of SU(4) (Parikh, 1973) and SU(4)-Isospin (Haq and Parikh, 1974) are used to determine ground state energies and low-lying spectra of nuclei. The goodness of space symmetry is investigated and the usefulness of SU(4) partial level densities in alpha transfer reactions is briefly indicated.

In Chapter IV, spectral distribution methods are used to study the average deformation (quadrupole moment) of a nucleus as a function of excitation energy. This average deformation may be of interest in heavy ion reaction processes where compound nuclear formation takes place. The configuration space chosen for the above study enables one to calculate fixed angular momentum J-averages (Jacquemin, 1973) which are used to determine the binding energies and spectra of nuclei.

In Chapter V, some of the formal aspects regarding group averaging are discussed. A way of evaluating moments over states defined by SU(4) symmetry and isospin and for the Elliott SU(3) group is described. Further, expressions for the calculation of spectral averages over states defined by the canonical chain of unitary groups are derived. Concluding remarks and scope for future work is given in Chatper VI.

CHAPTER II

SPACE SYMMETRY IN LIGHT NUCLEI

2.1 Introduction

Spectral distribution methods which deal with low order moments of dynamical operators have been discussed in Chapter I. These methods are well suited for studying symmetries in nuclei and in this chapter we use them to study SU(4) symmetry. The Supermultiplet theory is introduced in section (2.2) and evidences of space symmetry in nuclei are discussed. Earlier work regarding the study of space symmetry using the distribution method is briefly reviewed.

In section (2.3) we discuss formally the group U(N) and its direct product subgroup U(N/4) X U(4). A brief discussion about the representations of U(N/4) and U(4) is given. The supermultiplet group SU(4) is introduced in section (2.4) and we discuss in detail its infinitesimal generators, representations and Casimir invariants. Some remarks regarding the charge-spin (TS)-structure of the supermultiplet are made.

Section (2.5) deals with SU(4) isospin-spin averages. Evaluation of spectral moments is discussed and expressions for centroid energy and variance are derived. These are used to calculate (SU(4)TS) moments in the 2s-1d shell for some interactions. Having obtained these moments, we use them to determine ground state energies and low energy spectra of nuclei

in this shell. Goodness of SU(4) symmetry is investigated and a preliminary application of SU(4) partial level densities to alpha transfer reactions is made. Concluding remarks are presented in section (2.6).

2.2 The Supermultiplet Theory

It was suggested by Wigner (1937) that in addition to charge independence of nuclear forces, it may be a useful approximation to neglect the forces involving ordinary spin. The Hamiltonian which describes the interaction between nucleons, is then a function of only the space coordinates. Under this assumption, nuclear states can be classified according to irreducible representations (irreps) of the group SU(4) which describes unitary unimodular transformations among the four charge-spin states of a single nucleon. In this case the four single nucleon states are completely equivalent and irreps of SU(4) which contain several spin-isospin multiplets (a supermultiplet) are eigenstates of the system.

The supermultiplet theory is based on the assumption that spin-isospin independent interactions between nucleons can provide a reasonable wave function of the nucleus. Both the total spin S and isotopic spin T are good quantum numbers. The validity of S and T implies that the $(2S+1)\,(2T+1)$ states of a set of isobars are degenerate in energy; S_Z and T_Z values of these states range independently from $-S$ to S and from $-T$ to T

respectively.

2.2.1 Evidences of space symmetry in nuclei

SU(4) symmetry has many interesting manifestations in nuclei. For the low-lying states of Li^6 (Lauritsen and Ajzenberg-Selove, 1966) which are assumed to arise due to the two active nucleons in the p orbit $((1s)^4(1p)^2)$, all space symmetric (S,D) states are known and all the space antisymmetric (P) states lie much higher in energy. Such a situation indicates that the effective interaction has a sizeable Majorana part, an interaction which is attractive in space symmetric states and repulsive in space antisymmetric states.

In a many particle system the assumption that the two body interaction favours space symmetric states over space antisymmetric states leads to pronounced energy effects. Consider a set of levels such that each level can accomodate only two protons and two neutrons (Blatt and Weisskopf; 1952, Chapter VI). The wave function for a four particle system in this case is symmetric in space part and antisymmetric in spin-isospin. Now for an m-particle system, the binding energy would be maximized if one has a maximum number of symmetric pairs consistent with the Pauli Exclusion Principle. Therefore, nucleons would tend to occupy the single particle levels in such a way that one has a maximum number of symmetric pairs. This has the consequences that - a) for m particles there would be a four structure with k fully

saturated levels and $k<4$ nucleons in an unsaturated level. The space symmetry can be written as $[4^k, r]$ and b) nuclei with fully saturated levels would be tightly bound. An examination of binding energies of various nuclei confirms this four periodicity which can be easily interpreted in terms of space symmetry considerations.

It is also observed that the difference $\Delta E(\Delta T = 1)$ in energy between the ground state of a nucleus assuming its isospin $T_g = (N-Z)/2$ and the lowest state with isospin $T_g + 1$ is large for nuclei with $A=4m$, $4m+1$, $4m+3$ particles but small for $A=4m+2$ nuclei. The supermultiplet scheme provides an easy explanation for this "$4m+2$" effect. For $A=4m$, $4m+1$ and $4m+3$ nuclei, the most space symmetric representation allows only one value of isospin and therefore ΔE corresponds to the difference in energy between two different space symmetries. On the other hand for $4m+2$ nuclei, the space symmetric representation allows two values of isospin namely $T=0$ and $T=1$ and hence ΔE in this case is comparatively smaller.

When Wigner proposed the group SU(4), it was generally expected that the supermultiplet scheme would be confined to light nuclei. This was based on the argument that for heavy nuclei, the two body interaction cannot be assumed to be independent of the spin and isospin of two nucleons. However, Franzini and Radicati (1963) later derived from this symmetry model, a relation-

ship between binding energies of isobars which seems to be valid throughout the periodic table and appears to improve in validity with increasing mass number.

2.2.2 Study of SU(4) symmetry using distribution method

Although the supermultiplet scheme has very striking consequences, the SU(4) symmetry model cannot be exact because nuclear forces do depend on spin and isospin (Elliott; 1966). It is therefore necessary to understand how this symmetry manifests itself in nuclei. Parikh (1973) has studied space symmetry in light nuclei using spectral distribution methods. Averages of the Hamiltonian H and H^2 over states belonging to a definite SU(4) symmetry are evaluated. These spectral averages are used to determine ground state energies and low-lying spectra of nuclei in the 2s-1d shell using Ratcliff's procedure (c.f. sec.1.2). Goodness of SU(4) symmetry was investigated by determining the mixing intensities of various SU(4) representations in the ground state region.

Application of SU(4) distributions to the determination of ground state energies and spectra has been very successful. Although discrepancies with experimental results as well as exact calculations do exist, it is expected that a finer averaging over definite SU(4) symmetry, spin and isospin would improve the results. As far as goodness of space symmetry is concerned, Parikh has observed that there is a substantial admixture of SU(4) represen-

tations in the ground state. Results of calculations for nuclei with $T = T_z = (N-Z)/2$ have shown that with realistic interactions, there is an almost equal mixture of the lowest two symmetries. However, it is surprising that the strong manifestation and usefulness of the symmetry arising as a consequence of the relationship predicted by the Wigner model hold good to a much greater extent than would be expected on the basis of symmetry breaking.

In view of this and the fact that SU(4) moments enable one to study only states of lowest isospin for a given mass number A, in this and the next chapter, we study space symmetry in nuclei with neutron excess i.e. $T > (N-Z)/2$. This is done by first decomposing the group SU(4) into a direct-product subgroup $SU_T(2) \times SU_S(2)$; where $SU_T(2)$ and $SU_S(2)$ describe special unitary transformations in isospin and spin spaces respectively. Spectral averages are then evaluated for states having a definite SU(4) symmetry $\underset{\sim}{f}$, isospin T and spin S. This way one hopes to study goodness of space symmetry in the wave function for neutron excess nuclei and also the validity of the Franzini-Radicati mass formula.

Besides the interest arising out of such physical considerations, there is also a technical reason for carrying out averages over states belonging to irreducible representations in the following chain decomposition:

$$U(N) \supset U(N/4)XU(4) \supset U(N/4)X(SU_T(2)XSU_S(2))$$

This is because it would be one of the first examples of the group averaging procedure where the subgroup is one step further removed from U(N) than in earlier cases. One hopes to learn more about the averaging method from this example.

In the study by Parikh, the SU(4) distribution is assumed to be a Gaussian. Further, there is a stability under addition in the sense that the sum of all SU(4) frequency functions for a given particle number is itself nearly a Gaussian and quite similar to the 'normal' frequency function obtained in scalar averaging. By carrying out this finer averaging one will also be able to investigate the Gaussian nature of SU(4) distribution.

2.3 U(N) and the direct product subgroup U(N/4) X U(4)

As pointed out before (sec.1.3) the group U(N) enters quite naturally in the discussion of a system with m particles distributed over N single particle states. The creation A_i and annihilation operator $B_i (i = 1,.....N)$ for these single particle states are fermion operators and obey the following anticommutation relations:

$$\left[A_i, A_j\right]_+ = 0; \quad \left[B_i, B_j\right]_+ = 0$$

$$\left[A_i, B_j\right]_+ = \delta_{ij} \qquad (2.3.1)$$

Out of these operators A_i and B_i one can construct 1-body operators U_{ij} where

$$U_{ij} = A_i B_j \qquad i,j = 1,\ldots\ldots, N \tag{2.3.2}$$

The set of N^2 such operators U_{ij} is closed under commutation;

$$[U_{ij}, U_{kl}] = \delta_{jk} U_{il} - \delta_{il} U_{kj} \tag{2.3.3}$$

These are the infinitesmial generators of U(N). One can consider a subset of unitary transformations of U(N). For the study of space symmetry (SU(4) symmetry) one is interested in the direct-product subgroup given by,

$$U(N) \supset U(N/4) \times U(4) \tag{2.3.4}$$

The subgroup is defined by the requirement that the infinitesimal generators of U(N/4) commute with those of U(4).

For the study of this direct-product subgroup, a convenient notation is to introduce creation $A_{\alpha i}$ and destruction operators $B_{\alpha i}$ where α refers to space quantum numbers (nlm) of the single particle states and i stands for the four charge-spin states of a single nucleon. These operators being fermion operators obey the anticommutation rules, i.e.

$$\left[A_{\alpha i}, A_{\beta j}\right]_+ = 0; \left[B_{\alpha i}, B_{\beta j}\right]_+ = 0$$
$$\left[A_{\alpha i}, B_{\beta j}\right]_+ = \delta_{\alpha\beta}\delta_{ij} \qquad (2.3.5)$$

In terms of these operators, one can as before construct 1-body operators $U_{\alpha\beta}$ and U_{ij}; specifically,

$$U_{\alpha\beta} = \sum_{i=1}^{4} A_{\alpha i} B_{\beta i} \qquad \alpha, \beta = 1, \ldots N/4$$
$$U_{ij} = \sum_{\alpha=1}^{N/4} A_{\alpha i} B_{\alpha j} \qquad i, j = 1, \ldots 4 \qquad (2.3.6)$$

The $(N/4)^2$ operators in the set $\{U_{\alpha\beta}\}$ and the sixteen operators in the set $\{U_{ij}\}$ are the infinitesimal generators of U(N/4) and U(4) respectively. They satisfy the following relations,

$$\left[U_{\alpha\beta}, U_{\gamma\delta}\right] = \delta_{\beta\gamma} U_{\alpha\delta} - \delta_{\alpha\delta} U_{\gamma\beta}$$
$$\left[U_{ij}, U_{kl}\right] = \delta_{jk} U_{il} - \delta_{il} U_{kj}$$

and

$$\left[U_{\alpha\beta}, U_{ij}\right] = 0 \qquad (2.3.7)$$

2.3.1 Representations of U(N/4) and U(4)

The irreducible representations of U(N) are labelled by Young diagrams $\underset{\sim}{f} = [f_1 f_2 \ldots f_N]$ with utmost N rows such that

$$f_1 \geq f_2 \geq f_3 \cdots\cdots \geq f_N \geq 0$$

and

$$f_1+f_2+f_3+ \cdots\cdots + f_N = m \qquad (2.3.8)$$

The dimensionality of the irrep $\underset{\sim}{f}$ of U(N) is given by,

$$\left[\dim \underset{\sim}{f}\right]_{U(N)} = \prod_{i<j} \frac{(l_i - l_j)}{(N-1)!\,(N-2)!\cdots 1!}$$

where

$$l_i = f_i + N - i \qquad (2.3.9)$$

According to Pauli Exclusion Principle an m-particle state has to be completely antisymmetric. It can therefore belong only to the irrep $[1^m] \equiv [11 \cdots\cdots 1]$ of U(N). This condition places certain restrictions on irreps of U(N/4) and U(4) that can be allowed. Denoting the irrep of U(4) by $\underset{\sim}{f} = [f_1 f_2 f_3 f_4]$, the only allowed irrep of U(N/4) is that which is obtained by changing rows into colums in $\underset{\sim}{f}$. This is called a conjugate representation denoted by $\underset{\sim}{\tilde{f}}$. The consequence of this is that an irrep of U(4) cannot have more than N/4 columns and the irrep of U(N/4) not more than four columns. For such a direct-product subgroup then, one needs to specify the irrep of either U(N/4) or U(4).

In terms of the representations of U(N/4) one can understand the four periodicity in the binding energy of nuclei

mentioned in section (2.1). For an antisymmetric spin-isospin wave function, because of Pauli Principle, the space part of the wave function has to be completely symmetric. However, as mentioned before there are certain restrictions on the irreps of U(N/4) and U(4). Therefore, for $m = 4k+r$ $(r < 4)$ particles the most space symmetric representation is $[4^k, r]$.

2.4 The Supermultiplet group SU(4)

Following the notation of Hecht and Pang (1969), the four spin-isospin $|m_s m_t\rangle$ states of a single nucleon can be written as:

$$|1\rangle = |1/2\ 1/2\rangle\ ; \quad |2\rangle = |1/2\ -1/2\rangle$$
$$|3\rangle = |-1/2\ 1/2\rangle\ ; \quad |4\rangle = |-1/2\ -1/2\rangle \qquad (2.4.1)$$

The infinitesimal operators which generate unitary transformations in this four dimensional space can be built from the creation $A_{\alpha i}$ and annihilation operators $B_{\alpha i}$ defined as

$$|\alpha i\rangle = A_{\alpha i}|0\rangle\ ; \quad |0\rangle = B_{\alpha i}|\alpha i\rangle \qquad (2.4.2)$$

Instead of the sixteen one-body operators U_{ij} defined as

$$U_{ij} = \sum_{\alpha} A_{\alpha i} B_{\alpha j} \qquad (2.4.3)$$

it is convenient to take linear combinations and use these more physical operators as generators of the group U(4). These new operators are defined as:

$$\mathcal{N} = U_{11} + U_{22} + U_{33} + U_{44}$$

$$S_z = (U_{11} + U_{22} - U_{33} - U_{44})/2$$

$$T_z = (U_{11} - U_{22} + U_{33} - U_{44})/2$$

$$E_{oo} = (U_{11} - U_{22} - U_{33} + U_{44})/2$$

$$S_+ = (U_{13} + U_{24})/\sqrt{2}$$

$$S_- = (U_{31} + U_{42})/\sqrt{2}$$

$$T_+ = (U_{12} + U_{34})/\sqrt{2}$$

$$T_- = (U_{21} + U_{43})/\sqrt{2}$$

$$E_{10} = (U_{13} - U_{24})/\sqrt{2}$$

$$E_{-10} = (U_{31} - U_{42})/\sqrt{2}$$

$$E_{01} = (U_{12} - U_{34})/\sqrt{2}$$

$$E_{0-1} = (U_{21} - U_{43})/\sqrt{2}$$

$$E_{11} = U_{14}$$

$$E_{-1-1} = U_{41}$$

$$E_{1-1} = U_{23}$$
$$E_{-11} = U_{32} \qquad (2.4.4)$$

Here $\mathcal{N}$ is the number operator, $S_{\pm}$, S_Z are the three spin operators and $T_{\pm}$, T_Z are the three isospin operators. The nine components of the operator E are given by,

$$\underset{\sim}{E} = \sum_{\alpha i j} \langle i | \sigma \tau | j \rangle A_{\alpha i} B_{\alpha j} \qquad (2.4.5)$$

where σ and τ are the usual Pauli operators for spin and isospin. These are the sixteen generators of the group U(4) which is a Lie group of rank four. The operators $\mathcal{N}$, S_Z, T_Z and E_{oo} commute with each other. The fifteen operators S, T and E are generators of the group SU(4) which has a rank three. In this case the three operators S_Z, T_Z and E_{oo} commute with each other and the single nucleon states $|i\rangle$ are eigenstates of these operators. The remaining twelve operators act as step-up or step-down operators and transform one single particle state into another.

The eigenvalues of the three simultaneously commuting operators S_Z, T_Z and E_{oo} are used to label the states within a representation. Let the eigenvalues be m_1, m_2 and m_3 respectively. Then (m_1, m_2, m_3) is called the weight of a state. A weight

(m_1, m_2, m_3) is said to be higher than the weight (m_1', m_2', m_3') if the first non zero number in the sequence $(m_1 - m_1', m_2 - m_2', m_3 - m_3')$ is positive. The state of highest weight labels the irrep.

2.4.1 Representation of SU(4)

The irreps of U(4) are labelled by the Young diagram $\underset{\sim}{f} = [f_1 f_2 f_3 f_4]$ with utmost four rows. The irreps of SU(4) are the same as those of U(4) but representations differing only in completed columns are identical with respect to SU(4). Thus irreps of SU(4) are characterized by the set of three numbers $[f_1 - f_4, f_2 - f_4, f_3 - f_4]$. With respect to U(4), representations differing in completed columns are distinguished by the number operator $\mathcal{N}$. In the state of the highest weight for the symmetry $[f_1 f_2 f_3 f_4]$ there are f_1 particles in the spin-isospin state $|1\rangle$, f_2 particles in $|2\rangle$, f_3 particles in $|3\rangle$ and f_4 particles in $|4\rangle$. Wigner has used the three numbers (PP'P'') for the highest weights for SU(4) and these are related to $(f_1 f_2 f_3 f_4)$ in the following manner:

$$\begin{aligned} P &= (f_1 + f_2 - f_3 - f_4)/2 \\ P' &= (f_1 - f_2 + f_3 - f_4)/2 \\ P'' &= (f_1 - f_2 - f_3 + f_4)/2 \end{aligned} \qquad (2.4.6)$$

where P gives the maximum value of S_z and therefore S contained

in the irrep, P' is the maximum value of T_Z and therefore T in the irrep consistent with $S_Z = S$ and P'' is the maximum value of E_{oo} for a state with $S_Z = S$ and $T_Z = T$. Wigner labels the irrep of SU(4) by the three quantum numbers (PP'P'').

2.4.2 Casimir Invariants of SU(4)

The group SU(4) is a semi-simple Lie group of rank three and has therefore three Casimir invariants. A Casimir invariant is defined to be a polynomial function of generators of the group such that it commutes with all the generators. The three Casimir operators for SU(4) are,

$$G_2 = \sum_{ij} U_{ij} U_{ji}$$

$$G_3 = \sum_{ijk} U_{ij} U_{jk} U_{ki} \qquad (2.4.7)$$

$$G_4 = \sum_{ijkl} U_{ij} U_{jk} U_{kl} U_{li}$$

which are quadratic, cubic and quartic in the generators. These operators commute with the generators, i.e.,

$$\left[G_\alpha , U_{ij} \right] = 0 \qquad \alpha = 2,3 \text{ and } 4 \qquad (2.4.8)$$

$$i,j = 1,2,3 \text{ and } 4$$

The eigenvalues of these operators completely specify an irrep and all states in a given representation are eigenstates of these operators. The eigenvalue of a Casimir operator is determined by letting it act on the highest weight state. All step-up operators when acting on such a state give zero and this fact is exploited by using the commutation relation given in eq.(2.3.7) to bring all step up operators to the right. Finally, only the diagonal operators survive. Since in the state of highest weight for the irrep $\underset{\sim}{f}$ there are f_i particles in the state $|i\rangle$ the eigenvalue of the Casimir operator can be immediately obtained. Explicitly,

$$\langle G_2 \rangle_{\underset{\sim}{f}} = \sum_{i=1}^{4} f_i^2 + 3f_1 + f_2 - f_3 - 3f_4$$

$$\langle G_3 \rangle_{\underset{\sim}{f}} = \sum_{i=1}^{4} f_i^3 + 6f_1^2 + 3f_2^2 - 3f_4 - \sum_{k>i} f_i f_k + 9f_1 - f_2 - 5f_3 - 3f_4$$

$$\langle G_4 \rangle_{\underset{\sim}{f}} = \sum_{i=1}^{4} f_i^4 + 9f_1^3 + 5f_2^2 + f_3^3 - 3f_4^3 - \sum_{i<j} f_i^2 f_j - \sum_{i<j} f_i f_j^2$$

$$+ 27f_1^2 + 5f_2^2 - 5f_3^2 - 3f_4^2 - 8f_1f_2 - 6f_1f_3 - 4f_1f_4$$

$$- 4f_2f_3 - 2f_2f_4 + 27f_1 - 11f_2 - 13f_3 - 3f_4$$

(2.4.9)

2.4.3 The charge-spin structure of the supermultiplet

The spin-isospin group U(4) can be decomposed into a direct-product subgroup $SU_T(2) \times SU_S(2)$. The generators of $SU_T(2)$ and $SU_S(2)$ are the three isospin and the three spin operators respectively.

A representation $\underset{\sim}{f}$ of U(4) will give rise to many irreps of the direct-product subgroup. The irreps of this subgroup are labelled by the total isospin T and spin S of the states. These (TS) multiplets occuring for a given symmetry $\underset{\sim}{f}$ are determined using a chain process. Starting from a known structure and using Littlewood's rules and adding the isospin and spin vectorially, one can obtain the multiplets for any other symmetry $\underset{\sim}{f}'$. Details of this method with tables are given in the appendix.

2.5 SU(4) isospin-spin averages

A straight forward way of studying space symmetry in nuclei would be to diagonalize the Hamiltonian operator H in a space symmetry representation. However, this can only be done for a few nucleons in a reasonably small vector space since the matrix dimensionalities increase very rapidly. In order to study SU(4) symmetry without putting any restrictions on the number of active nucleons m or the size of the vector space formed by distributing these into N single particle states, we use the spectral distribution methods.

2.5.1 Evaluation of spectral moments

a) Centroid energies

Averages of the Hamiltonian operator H and H^2 over states belonging to a definite SU(4) symmetry $\underset{\sim}{f}$, isospin T and spin S are evaluated. If we assume the normality of distribution of states belonging to ($\underset{\sim}{f}$,T,S) the following two moments (c.f. eq. (1.2.5)) characterize the spectral distribution,

$$\langle H \rangle^{m\underset{\sim}{f}TS} = \frac{1}{d(m\underset{\sim}{f}TS)} \sum_{i \in \underset{\sim}{f}TS} \langle i|H|i\rangle \qquad (2.5.1)$$

$$\langle H^2 \rangle^{m\underset{\sim}{f}TS} = \frac{1}{d(m\underset{\sim}{f}TS)} \sum_{i \in \underset{\sim}{f}TS} \langle i|H^2|i\rangle \qquad (2.5.2)$$

where $d(m\underset{\sim}{f}TS)$ is the total number of states over which the sum is taken.

$$d(m\underset{\sim}{f}TS) = (2T+1)(2S+1) \times d(m\tilde{\underset{\sim}{f}})_{U(N/4)} \qquad (2.5.3)$$

Here $\tilde{\underset{\sim}{f}}$ is the conjugate representation of $\underset{\sim}{f}$ and $d(m\tilde{\underset{\sim}{f}})$ is the dimensionality of U(N/4). The first moment is the centroid energy $E_c(m\underset{\sim}{f}TS)$ and the second moment determines the variance (c.f. eq.(1.2.7)) defined as:

$$\sigma^2(m\underset{\sim}{+}TS) = \langle H^2 \rangle^{m\underset{\sim}{+}TS} - E_c^2(m\underset{\sim}{+}TS) \qquad (2.5.4)$$

To determine these averages using the propagation technique mentioned in Chapter I (c.f. sec.(1.2.2); eq. (1.2.27 - 1.2.28)) one needs to construct density operators which propagate information from the input space to the rest of the space. The density operators are scalars under transformations of the group and they may be written as polynomial functions of the Casimir operators of the group, provided there are sufficient number of scalar operators. In otherwords one should be able to construct density operators in terms of the Casimir operators of the group for all irreps in the defining space. For SU(4) averaging one is able to construct density operators in terms of the scalar operators n, G_2, G_3 and G_4.

However, instead of evaluating the expectation values of the density operators, which are expressible in terms of eigenvalues of the Casimir operators of U(4) in irreps of U(4); it is often advantageous to follow a different procedure. In the averaging process it is clear that only the group scalar part of an operator survives which means that only the group scalar part of the operator is relevant for evaluating averages. Therefore, if one can construct a new scalar operator $\hat{H}$, the linear trace equivalent of H with the correct maximum particle rank and which

reproduces the averages of H in the defining space, it will reproduce them in all representations.

Now for averaging over states defined by ($\underset{\sim}{f}$,T,S), one can write down the average of H in terms of the number operator and various other operators which are scalar with respect to the group SU(4), isospin and spin. The operator $\hat{H}$ called the linear trace equivalent of H (valid for evaluating centroid energies) can be written as

$$\hat{H} = P_2(n) + P_o(n)\, G_2 + P_o(n)\, T^2 + P_o(n)\, S^2 \qquad (2.5.5)$$

where $P_k(n)$ is a polynomial of degree k in the number operator n and G_2 is the bilinear Casimir operator of SU(4). Further, for the equivalence to be faithful there should be as many ($\underset{\sim}{f}$,T,S) representations in the (0,1,2)-particle space (defining space) as there are unknown parameters in eq.(2.5.5). We see from table (II.1) that this is in fact the case. The unknown parameters in eq. (2.5.5) can be evaluated by using as inputs averages in the defining space which are listed in table (II.1).

The expression for centroid energy can be written as:

$$E_c(m\,\underset{\sim}{f}\,TS) = \frac{1}{2}(m^2-3m+2)\,E_c(0\underset{\sim}{0}00) - m(m-2)\,E_c(1\underset{\sim}{1}\tfrac{1}{2}\tfrac{1}{2})$$
$$+ \frac{1}{8}\left[-9m+2m^2+\langle G_2\rangle+2T(T+1)+2S(S+1)\right]E_c(2\underset{\sim}{2}11)$$
$$+ \frac{1}{8}\left[3m+m^2-\langle G_2\rangle-2T(T+1)+2S(S+1)\right]E_c(2\underset{\sim}{1}01)$$

$$+\frac{1}{8}\left[3m+m^2-\langle G_2\rangle+2T(T+1)-2S(S+1)\right]E_c(2\underset{\sim}{1}1+0)$$

$$+\frac{1}{8}\left[-m+\langle G_2\rangle-2T(T+1)-2S(S+1)\right]E_c(2\underset{\sim}{2}00)$$

(2.5.6)

This polynomial expression propagates the input information given by the average energy over 1, 2 particles with different spin and isospin to the rest of the space.

It turns out however that we cannot define an equivalent operator for H^2 in terms of n, G_2, G_3 and G_4, S^2 and T^2 (G_3 and G_4 are the cubic and quartic Casimir operators of SU(4)). This is because the most general polynomial operator with particle rank ≤ 4 constructed from the operators 1, n, G_2, G_3, G_4, S^2, T^2 has 23 unknown parameters whereas the number of input irreps in the defining space is 28. It is therefore not possible to construct an expression for variance similar to the one for the centroid. In principle, one can derive an expression for the variance by including so called 'mixed' operators but this will be discussed separately. Some comments, however, on ($\underset{\sim}{mf}$T) averaging are given at the end. For the present we have defined and evaluated what can be termed an 'equivalent' width.

Table (II.1)

Centroid energies and widths for the defining space using various interactions

m	$\tilde{f}$	T	S	E_c (m $\tilde{f}$ T S)			σ (m $\tilde{f}$ T S)		
				Rosenfeld	KB(12.5) for particles	KB(12.5) for holes	Rosenfeld	KB(12.5) for particles	KB(12.5) for holes
1	[1]	1/2	1/2	-2.300	-2.311	-265.478	2.312	2.313	3.880
2	[2]	0	0	-1.230	-3.109	-242.351	3.213	3.355	5.273
	[2]	1	1	-3.976	-4.152	-243.394	3.213	3.355	5.273
	[11]	0	0	-7.934	-7.727	-246.969	3.365	3.556	5.470
	[11]	1	0	-6.600	-6.563	-245.805	3.365	3.556	5.470

b) Equivalent width

Let us assume that for a given particle number m and $\underset{\sim}{f}$ all widths for various (TS) multiplets are equal. Such an approximation seems reasonable in view of the fact that we have observed a near constancy of widths (Parikh, 1973) belonging to different irreps $\underset{\sim}{f}$ for a given m. Further if the distribution is assumed to be a Gaussian, this constant width can be determined from the condition that the sum of K Gaussians, one for each $(T_i S_i)$ having its centre at $E_c(m\underset{\sim}{f}T_i S_i)$ and width σ should add upto a Gaussian having an SU(4) centroid and width. Explicitly, then we require that the equation

$$\frac{d(m\underset{\sim}{f})}{\sqrt{2\pi}\,\sigma(m\underset{\sim}{f})}\exp\left[-\frac{(E-E_c(m\underset{\sim}{f}))^2}{2\sigma^2(m\underset{\sim}{f})}\right]$$

$$=\sum_{i=1}^{k}\frac{d(m\underset{\sim}{f}(TS)_i)}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{(E-E_c(m\underset{\sim}{f}(TS)_i))^2}{2\sigma^2}\right] \qquad (2.5.7)$$

be satisfied at $E = E_c(m\underset{\sim}{f})$ where $E_c(m\underset{\sim}{f})$, $\sigma(m\underset{\sim}{f})$ and $d(m\underset{\sim}{f})$ are the SU(4) centroid, width and dimensionality respectively. The only unknown quantity σ in eq.(2.5.7) is then determined numerically by using an iterative procedure. Examples are shown in figures (II.1) and (II.2).

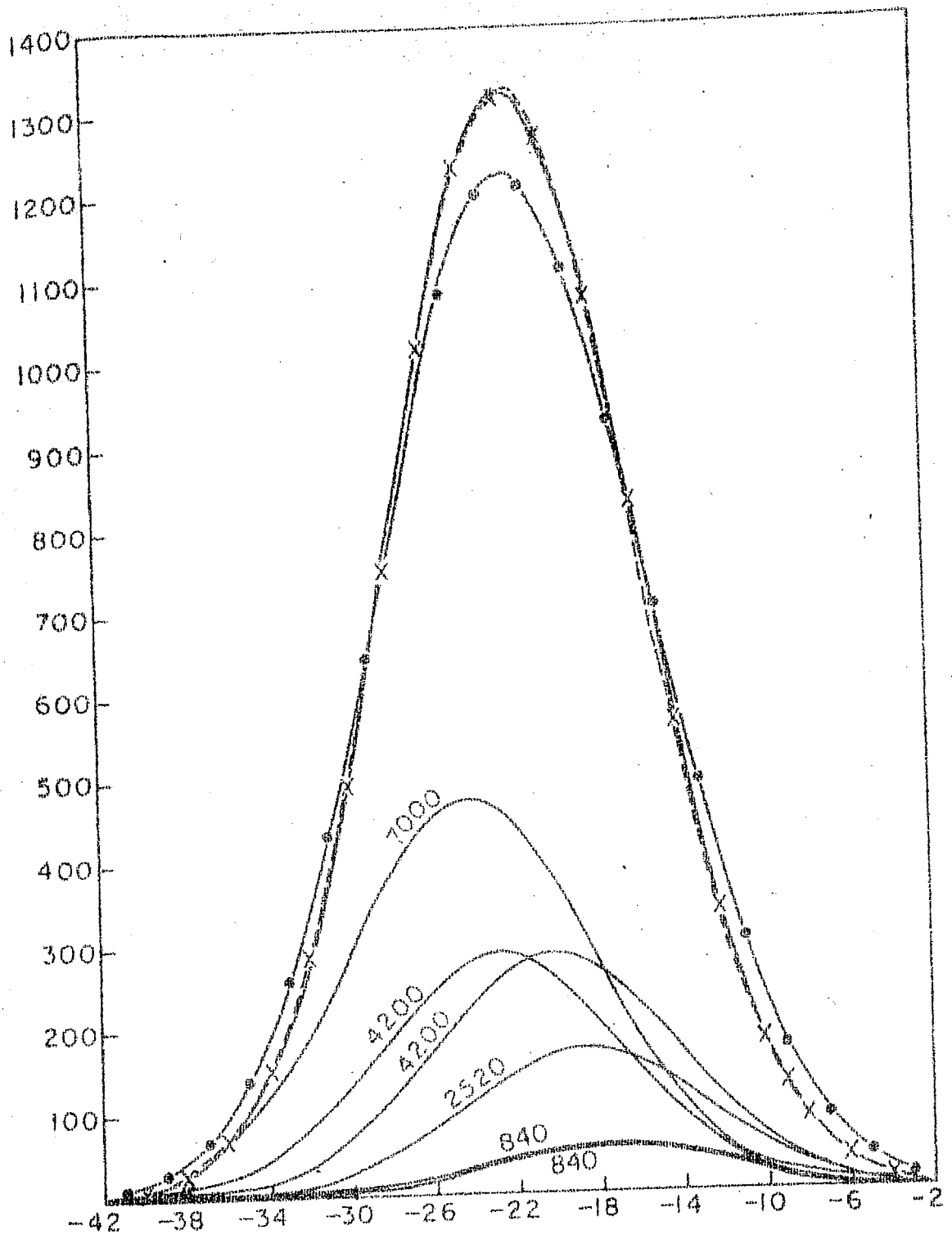

Fig. II. 1 The normal distribution derived from SU(4) moments (dashed curve) compared with the sum of normal distributions (marked by an x) derived from SU(4) isospin-spin moments using the equivalent width approximation. The solid curve is the sum of (SU(4) TS) curves assuming that each (SU(4) TS) distribution has a width equal to SU(4) width. Rosenfeld interaction was used and the SU(4) representation is [411] for 6 particles. Individual (SU(4) TS) curves are also given.

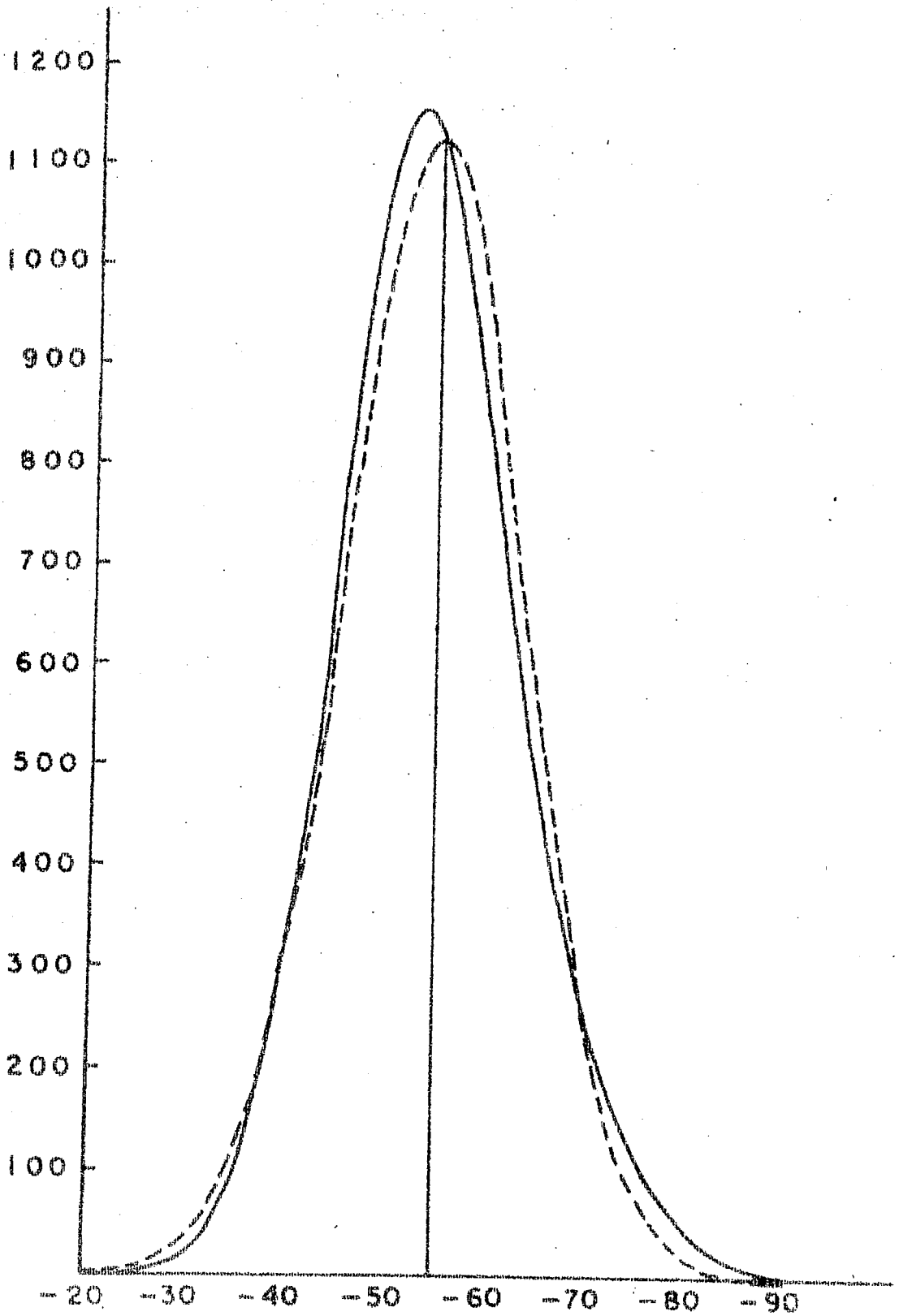

Fig. II.2 The normal distribution derived from SU(4) moments (dashed line) is compared with the sum of 35 normal distributions (solid line) derived from SU(4) isospin-spin moments. SU(4) representation is $[6510]$ for 12 particles.

Another way of evaluating an equivalent width is to assume that for a given m, all representations with the same isospin T have a constant width. This constant width $\sigma(m\underset{\sim}{f}TS)$ can be determined by requiring that

$$\frac{d(mT)}{\sqrt{2\pi}\,\sigma(mT)} \exp\left[-\frac{(E-E_c(mT))^2}{2\sigma^2(mT)}\right]$$

$$= \sum_{i=1}^{k} \frac{d(mT(\underset{\sim}{f}S)_i)}{\sqrt{2\pi}\,\sigma(m\underset{\sim}{f}TS)} \exp\left[-\frac{(E-E_c(mT(\underset{\sim}{f}S)_i))^2}{2\sigma^2(m\underset{\sim}{f}TS)}\right]$$

(2.5.8)

be satisfied at $E = E_c(mT)$ where $d(mT)$, $E_c(mT)$ and $\sigma(mT)$ are the dimensionality, centroid and width for isospin distribution (Chang, French and Thio, 1971). Once again an iterative procedure is adopted to determine $\sigma(m\underset{\sim}{f}TS)$.

In order to test the equivalent width approximation, the following method has been used. The SU(4) isospin-spin width $\sigma(m\underset{\sim}{f}TS)$ is first determined using eq. (2.5.8). Now by averaging over the isospin and spin quantum numbers, the SU(4) width $\sigma(m\underset{\sim}{f})$ is evaluated. $\sigma(m\underset{\sim}{f})$ is related to $\sigma(m\underset{\sim}{f}TS)$ by the following equations:

$$\langle H^2 \rangle^{m_f TS} = \sigma^2(m_f TS) + E_c^2(m_f TS)$$

$$\langle H^2 \rangle^{m_f} = \frac{\sum_{TS}(2T+1)(2S+1)\langle H^2 \rangle^{m_f TS}}{\sum_{TS}(2T+1)(2S+1)}$$

$$E_c(m_f) = \sum_{TS}(2T+1)(2S+1)E_c(m_f TS) \Big/ \sum_{TS}(2T+1)(2S+1)$$

$$\sigma^2(m_f) = \langle H^2 \rangle^{m_f} - E_c^2(m_f) \qquad (2.5.9)$$

The numbers so obtained are then compared in tables (II.2) and (II.3) with exact calculation of $\sigma(m_f)$ by Parikh (1973). The agreement is excellent.

In the next section, we use the expression for centroid energy and width obtained here to evaluate these moments in the 2s-1d shell.

2.5.2 SU(4) TS moments in 2s-1d shell

The expressions (2.5.6) - (2.5.8) are used to determine centroid energies and equivalent widths in 2s-1d shell. Input traces for propagation are evaluated by means of explicit shell model calculations upto two particles in this shell. The Oak Ridge Rochester Shell model program (J.B. French, E.C. Halbert,

Table (II.2)

Comparison of the exact and equivalent width for 4 and 12 particles in 2s-1d shell.

Particle No. m	SU(4) symmetry $\underset{\sim}{f}$	Rosenfeld Interaction $\sigma(\underset{\sim}{mf})$	Rosenfeld Interaction $\sigma_{exact}(\underset{\sim}{mf})$	KB(12,5) Interaction $\sigma(\underset{\sim}{mf})$	KB(12,5) Interaction $\sigma_{exact}(\underset{\sim}{mf})$
4	[4000]	5.06	5.06	4.97	4.98
	[3100]	5.00	4.88	5.27	5.20
	[2200]	5.09	4.90	5.45	5.35
	[2110]	4.96	4.88	5.43	5.49
	[1111]	5.30	5.32	5.74	6.12
12	[6600]	10.84	10.77	10.01	10.19
	[6510]	9.30	9.12	9.40	9.34
	[6420]	8.26	8.04	9.01	8.83
	[6411]	8.13	7.94	9.13	9.00
	[6330]	7.73	7.51	8.82	8.60
	[6321]	7.47	7.33	8.94	8.79
	[6222]	7.07	7.02	8.93	8.82
	[5520]	8.13	7.95	9.13	9.02
	[5511]	8.06	7.92	9.28	9.22
	[5430]	7.47	7.34	8.94	8.82
	[5421]	7.38	7.34	9.14	9.11
	[5331]	7.07	7.09	9.08	9.09
	[5322]	7.05	7.15	9.19	9.28
	[4440]	7.07	7.03	8.93	8.88
	[4431]	7.05	7.15	9.19	9.29
	[4422]	7.10	7.28	9.33	9.53
	[4332]	7.02	7.29	9.37	9.64
	[3333]	7.21	7.64	9.61	10.08

Table (II.3)

Exact and equivalent SU(4) widths for 4 holes using KB(12.5) interaction

Number of holes m	SU(4) symmetry $\underset{\sim}{f}$	KB(12.5) $\sigma(\underset{\sim}{mf})$	Interaction $\sigma_{exact}(\underset{\sim}{mf})$
4	[4000]	6.77	6.92
	[3100]	7.16	7.15
	[2200]	7.38	7.30
	[2110]	7.45	7.45
	[1111]	7.82	8.00

J.B. McGrory and S.S.M. Wong, 1969) calculates matrix elements in jj representation. In order to obtain wave functions in a space symmetry (SU(6)) representation one constructs a transformation operator O(T) (Parikh and Wong (1972)):

$$O(T) = -110P_x - 2G_2(SU(3)) + 35S^2 + 0.3L^2$$

where P_x is the space exchange operator, $G_2(SU(3))$ is the Casimir operator of the Elliott SU(3) group and S^2 and L^2 are the squares of the total spin and orbital angular momentum operators. Although for the present purpose, one is not interested in SU(3) and R(3) (L) symmetries, the Casimir operators for these groups are included in the transformation operator to eliminate some of the degeneracies in the eigenvalues. The parameters in O(T) are chosen such that after diagonalizing it in an (mJT) space, its eigenvalues are ordered according to space symmetry (SU(6)), with the states of highest symmetry being lowest in energy. The eigen functions of O(T) are then used as transformation matrices U to transform H into the space symmetry representation

$$H \longrightarrow \tilde{H} = UHU^{-1}$$

By taking suitable partial traces in H, one can obtain the necessary input information for the propagating equations.

The Hamiltonian $H = H^{(1)} + H^{(2)}$ where $H^{(1)}$ is the one body part and $H^{(2)}$ the two body part. $H^{(2)} = \sum_{(i)} V_{ij}$ and for V_{ij} three different choices have been made.

i) A central Rosenfeld exchange mixture - The Rosenfeld interaction has a strength of 40 MeV and a Yukawa radial shape with range a = 1.37 fm. The harmonic oscillator length parameter $\nu = (m\omega/\hbar)^{1/2}$ is taken to be 0.370 fm^{-2} so that the dimensionless parameter $\lambda = a(\nu/2)^{1/2} = 0.589$. The single particle energies used with this interactions are taken from O^{17}. These are $\epsilon_{d_{5/2}} = -4.15$ MeV; $\epsilon_{s_{1/2}} = -3.28$ MeV and $\epsilon_{d_{3/2}} = 0.93$ MeV.

ii) Effective interaction matrix elements of Kuo and Brown KB(12.5) (Kuo and Brown, 1966) with the following single particle energies;

$$\epsilon_{d_{5/2}} = -4.14928, \quad \epsilon_{d_{3/2}} = 0.93070, \quad \epsilon_{s_{1/2}} = -3.28063$$

iii) Modified surface delta interaction MSDI - The MSDI interaction (Wildenthal, McGrory, Halbert and Graber, 1971) is expressed as

$$V_T(ij) = -4\pi A_T \delta(r_i - r_j) f_{ij} + B_T$$

where A_T and B_T are strengths depending only on isospin T and f_{ij} is an operator such that the integral over radial wave functions is equal to $(-1)^{n_a+n_b+n_c+n_d}$. For the present calculations the following values of A_T, B_T and single particle energies were chosen:

$$A_{T=0} = 0.646; \qquad A_{T=1} = 0.906$$

$$B_{T=0} = -1.47; \qquad B_{T=1} = 0.77$$

$$\epsilon_{d_{5/2}} = -7.56; \quad \epsilon_{d_{3/2}} = -3.96; \quad \epsilon_{s_{1/2}} = -6.03$$

The input centroids for various interactions are shown in table (II.1). For the calculation of equivalent width, SU(4) and (mT) widths were taken from Parikh (1973) and Thio (1970) respectively. Averages for four particles and four holes using various interactions are shown in table (II.4 - II.5). Centroid energies and widths for twelve particles belonging to a particular symmetry $\underset{\sim}{f}$ are shown in table (II.6). In table (II.7) these averages for 12 particles and isospin 1 are listed.

For a fixed SU(4) symmetry $\underset{\sim}{f}$, we see from table (II.6) that the centroid span for various (TS) multiplets contained in $\underset{\sim}{f}$ is about 30 and 15 MeV for Rosenfeld and KB(12.5) interactions. The width for the latter case is about 2 MeV larger than the

Table (II.4)

SU(4) TS centroids and widths for 4 particles using Rosenfeld and KB(12.5) interactions. I and II denote that the equivalent width was evaluated using SU(4) and (mT) widths respectively

m	$\underset{\sim}{f}$	(2T+1,2S+1)	Rosenfeld Interaction $E_c(\underset{\sim}{mfTS})$	$\sigma_I(\underset{\sim}{mfTS})$	$\sigma_{II}(\underset{\sim}{mfTS})$	KB(12.5) Interaction $E_c(\underset{\sim}{mfTS})$	$\sigma_I(\underset{\sim}{mfTS})$	$\sigma_{II}(\underset{\sim}{mfTS})$
4	[4000]	1,1	2.872	4.267	5.295	-3.293	4.880	5.740
	[3100]	1,3	-8.585	4.457	5.295	-11.426	5.064	5.740
	[2200]	1,1	-11.209	4.578	5.295	-13.836	5.189	5.740
	[2200]	1,5	-17.329	4.578	5.295	-17.147	5.189	5.740
	[2110]	1,3	-17.913	4.812	5.295	-18.454	5.471	5.740
	[1111]	1,1	-25.201	5.324	5.295	-24.379	6.114	5.740
	[4000]	3,3	-0.036	4.267	4.791	-4.336	4.880	5.326
	[3100]	3,1	-7.252	4.457	4.791	-10.261	5.064	5.326
	[3100]	3,3	-9.292	4.457	4.791	-11.365	5.064	5.326
	[3100]	3,5	-13.371	4.457	4.791	-13.572	5.064	5.326
	[2200]	3,3	-13.956	4.578	4.791	-14.870	5.189	5.326
	[2110]	3,1	-16.580	4.812	4.791	-17.290	5.471	5.326
	[2110]	3,3	-18.619	4.812	4.791	-18.394	5.471	5.326
	[4000]	5,5	-5.455	4.267	4.084	-6.422	4.880	4.656
	[3100]	5,3	-10.704	4.457	4.084	-11.243	5.064	4.656
	[2200]	5,1	-13.328	4.578	4.084	-13.654	5.189	4.656

Table (II.5)

Centroids and widths for 4 holes using KB(12.5) and MSDI Interactions

m	$\underset{\sim}{f}$	(2T+1,2S+1)	KB(12.5) Interaction $E_c(\underset{\sim}{mfTS})$	$\sigma_I(\underset{\sim}{mfTS})$	$\sigma_{II}(\underset{\sim}{mfTS})$	MSDI Interaction $E_c(\underset{\sim}{mfTS})$	$\sigma_{II}(\underset{\sim}{mfTS})$
4	[4000]	1,1	-194.687	6.851	7.816	-210.761	3.811
	[3100]	1,3	-202.819	7.055		-213.200	
	[2200]	1,1	-205.230	7.180		-214.756	
	[2200]	1,5	-208.541	7.180		-214.084	
	[2110]	1,3	-209.848	7.433		-215.863	
	[1111]	1,1	-215.774	8.011		-218.751	
	[4000]	3,3	-195.730	6.851	7.341	-208.527	3.570
	[3100]	3,1	-201.655	7.055		-211.414	
	[3100]	3,3	-202.759	7.055		-211.191	
	[3100]	3,5	-204.966	7.055		-210.742	
	[2200]	3,3	-206.273	7.180		-212.522	
	[2110]	3,1	-208.684	7.433		-214.078	
	[2110]	3,3	-209.788	7.433		-213.854	
	[4000]	5,5	-197.816	6.851	6.393	-204.059	3.158
	[3100]	5,3	-202.637	7.055		-207.171	
	[2200]	5,1	-205.048	7.180		-208.727	

Table (II.6)

The centroids and widths for 12 particles belonging to the symmetry [6510] using the Rosenfeld and KB(12.5) interactions.

$\underset{\sim}{f}$	dim($\underset{\sim}{f}$)	(2T+1, 2S+1)	Rosenfeld Int. E_c(mfTS)	Rosenfeld Int. σ(mfTS)	KB(12.5) Int. E_c(mfTS)	KB(12.5) Int. σ(mfTS)
[6510]	105	1,3	-42.73	6.24	-60.23	8.16
	105	3,1	-41.39		-59.07	
	175	1,5	-46.80		-62.44	
	175	5,1	-42.80		-58.95	
	245	1,7	-52.92		-65.75	
	245	7,1	-44.92		-58.76	
	315	1,9	-61.08		-70.16	
	315	9,1	-47.74		-58.52	
	385	1,11	-71.28		-75.68	
	385	11,1	-51.28		-58.22	
	315	3,3	-43.43		-60.17	
	315	3,3	-43.43		-60.17	
	525	3,5	-47.51		-62.38	
	525	5,3	-44.84		-60.05	
	525	3,5	-47.51		-62.38	
	525	5,3	-44.84		-60.05	
	735	3,7	-53.63		-65.69	
	735	7,3	-46.96		-59.87	
	735	3,7	-53.63		-65.69	
	735	7,3	-46.96		-59.87	
	945	3,9	-61.79		-70.10	
	945	9,3	-49.78		-59.62	
	945	3,9	-61.79		-70.10	
	945	9,3	-49.78		-59.62	
	1155	3,11	-71.98		-75.62	
	1155	11,3	-53.32		-59.32	
	875	5,5	-48.92		-62.26	
	875	5,5	-48.92		-62.26	
	1225	5,7	-55.04		-65.57	
	1225	7,5	-51.04		-62.07	
	1225	5,7	-55.04		-65.57	
	1225	7,5	-51.04		-62.07	
	1575	5,9	-62.20		-69.98	
	1575	9,5	-53.86		-61.83	
	1715	7,7	-57.16		-65.39	

Table (II.7)

Centroids and widths for 12 particles and isospin 1 using the Rosenfeld, KB(12.5) and MSDI interactions

$\tilde{f}$	(2T+1, 2S+1)	dim($\tilde{f}$)	Rosenfeld interaction $E_c(\tilde{mf}TS)$	Rosenfeld interaction $\sigma(\tilde{mf}TS)$	KB(12.5) Interaction $E_c(\tilde{mf}TS)$	KB(12.5) Interaction $\sigma(\tilde{mf}TS)$	MSDI interaction $E_c(\tilde{mf}TS)$	MSDI interaction $\sigma(\tilde{mf}TS)$
[6600]	3,3	9	-29.44	6.92	-49.63	9.30	-95.48	5.67
	3,7	21	-39.64		-55.15		-94.36	
	3,11	33	-57.99		-65.08		-92.35	
[6510]	3,1	105	-41.39		-59.07		-99.70	
	3,3	315	-43.43		-60.17		-99.48	
	3,3	315	-43.43		-60.17		-99.48	
	3,5	525	-47.51		-62.38		-99.03	
	3,5	525	-47.51		-62.38		-99.03	
	3,7	735	-53.63		-65.69		-98.36	
	3,7	735	-53.63		-65.69		-98.36	
	3,9	945	-61.79		-70.10		-97.46	
	3,9	945	-61.79		-70.10		-97.46	
	3,11	1155	-71.98		-75.62		-96.34	
[6420]	3,3	1701	-52.76		-67.20		-102.14	
	3,3	1701	-52.76		-67.20		-102.14	
	3,3	1701	-52.76		-67.20		-102.14	
	3,5	2835	-56.84		-69.41		-101.69	
	3,5	2835	-56.84		-69.41		-101.69	
	3,5	2835	-56.84		-69.41		-101.69	
	3,7	3969	-62.96		-72.72		-101.02	
	3,7	3969	-62.96		-72.72		-101.02	
	3,7	3969	-62.96		-72.72		-101.02	
	3,9	5103	-71.11		-77.13		-100.13	
[6411]	3,1	840	-55.38		-69.61		-103.69	
	3,3	2520	-57.42		-70.71		-103.47	
	3,5	4200	-61.50		-72.92		-103.03	
	3,5	4200	-61.50		-72.92		-103.03	
	3,7	5880	-67.62		-76.23		-102.35	
	3,9	7560	-75.78		-80.65		-101.46	

Table (II.7)(contd)

[6330]	3,1	525	-55.38	-69.61	-103.70
	3,3	1575	-57.42	-70.71	-103.48
	3,5	2625	-61.50	-72.92	-103.03
	3,5	2625	-61.50	-72.92	-103.03
	3,7	3675	-67.62	-76.23	-102.35
[6321]	3,1	2688	-62.38	-74.88	-105.70
	3,3	8064	-64.42	-75.98	-105.47
	3,3	8064	-64.42	-75.98	-105.47
	3,5	13440	-68.49	-78.19	-105.02
	3,5	13440	-68.49	-78.19	-105.02
	3,7	18816	-74.61	-81.50	-104.35
[6222]	3,3	4410	-71.41	-81.26	-107.47
[5520]	3,3	2520	-57.42	-70.71	-103.47
	3,1	840	-55.38	-69.61	-103.69
	3,5	4200	-61.50	-72.92	-103.03
	3,5	4200	-61.50	-72.92	-103.03
	3,7	5880	-67.62	-76.23	-102.35
	3,9	7560	-75.78	-80.65	-101.46
[5511]	3,3	3645	-62.08	-74.23	-104.81
	3,7	8505	-72.28	-79.75	-103.69
[5430]	3,1	2688	-62.38	-74.88	-105.69
	3,3	8064	-64.42	-75.99	-105.47
	3,3	8064	-64.42	-75.99	-105.47
	3,5	13440	-68.49	-78.19	-105.02
	3,5	13440	-68.49	-78.19	-105.02
	3,7	18816	-74.61	-81.50	-104.35
[5421]	3,1	11025	-69.37	-80.15	-107.69
	3,3	33075	-71.41	-81.26	-107.47
	3,3	33075	-71.41	-81.26	-107.47
	3,5	55125	-75.49	-83.47	-107.02
	3,5	55125	-75.49	-83.47	-107.02
	3,7	77175	-81.61	-86.78	-106.35
[5331]	3,3	35721	-76.08	-84.77	-108.80
	3,3	35721	-76.08	-84.77	-108.80
	3,5	59535	-80.16	-86.98	-108.35

Table (II.7) (contd)

[5322]	3,1	17010	-78.70	-87.18	-110.36
	3,3	51030	-80.74	-88.29	-110.13
	3,5	85050	-84.82	-90.49	-109.68
[4440]	3,3	4410	-71.41	-81.26	-107.47
[4431]	3,1	17010	-78.70	-87.18	-110.36
	3,3	51030	-80.74	-88.29	-110.13
	3,5	85050	-84.82	-90.49	-109.68
[4422]	3,3	60480	-85.40	-91.80	-111.46
[4332]	3,1	34020	-88.03	-94.21	-113.02
	3,3	102060	-90.07	-95.32	-112.79

former. From our results in table (II.7) we observe that for 12 particles having isospin 1, the centroid span for the two inter actions is about 60 and 45 MeV respectively. The widths in the two cases being 6.92 and 9.30 MeV, one would, on this basis expect that the realistic interaction of Kuo and Brown would give rise to greater mixing as compared with the central Rosenfeld interaction.

2.5.3 Ground state energies and low energy spectra

As mentioned in Chapter I(sec 1.2.4) spectral moments can be used to determine ground state energies following the procedure suggested by Ratcliff. The ground state for the nucleus with isospin T is located at an energy E_g where the cumulative distribution F(E,T) (c.f. eq.(1.2.34)) written as:

$$F(ET)=\sum_i \frac{d(mT(\tilde{\pm}s)_i)}{\sqrt{2\pi}\sigma(mT(\tilde{\pm}s)_i)} \int_{-\infty}^{E} \exp\left[-\frac{(E'-E_c(mT(\tilde{\pm}s)_i))^2}{2\sigma^2(mT(\tilde{\pm}s)_i)}\right] dE' \qquad (2.5.10)$$

satisfies the relation

$$F(E_g,T_o) = (2J_o + 1)(2T_o + 1)/2 \qquad (2.5.11)$$

where J_o and T_o denote the angular momentum and isospin of the ground state.

We can also locate the excited states (c.f.eq.(1.2.37)), the i^{th} excited state being given by,

$$F(E;T) = (2T+1)\left[\sum_{n=0}^{i-1}(2J_n+1) + (2J_i+1)/2\right] \qquad (2.5.12)$$

In view of the discussion in Chapter I (sec.1.2) for locating the ground state, we shall as far as possible make use of a reference state. A comparison with empirical energies (Garvey, Gerace, Jaffe, Talmi and Kelson, 1969) (after making Coulomb correction) is given in table (II.8) and table (II.9). Except for a few cases, predictions of ground state energies differ from observed binding energies by $\sim$ 4-5 MeV. A part of the larger disagreement as we go higher in isospin can be attributed to the lack of observed excited reference states. Also for very high isospin, the interaction used may be inadequate. Small differences could also arise due to the approximation of an equivalent width and questions relating to the determination of an exact width have been investigated in Chapter V. On the whole, however, it appears that we are able to determine ground state energies for nuclei far away from the stability line nearly as well as in the stable region. In order to do better one needs distribution

Table (II.8)

Ground state energies (relative to O^{16}) for $4 \leq m \leq 12$ in the 2s-1d shell.

A	T	Rosenfeld I	Rosenfeld II	Kuo-Brown I	Kuo-Brown II	B.E.
20	0	-37.05	-37.39	-37.85	-37.86	-40.70
	1	-30.17	-30.14	-31.63	-31.38	-30.30
	2	-23.49	-22.79	-24.97	-24.19	-23.90
21	1/2	-43.78	-44.01	-46.01	-45.69	-47.56
	3/2	-38.22	-27.75	-41.19	-40.41	-38.40
	5/2	-27.94	-26.40	-30.83	-29.06	-24.94
22	0	-54.35	-54.77	-57.99	-57.70	-58.40
	1	-54.20	-53.89	-57.94	-57.34	-57.96
	2	-44.05	-43.11	-48.38	-47.09	-41.80
	3	-35.54	-32.78	-39.99	-36.78	-29.95
23	1/2	-65.32	-64.73	-70.34	-69.75	-70.82
	3/2	-60.15	-59.10	-66.05	-64.84	-63.22
	5/2	-49.71	-48.31	-55.65	-53.43	-48.70
	7/2	-38.44	-34.94	-43.74	-39.64	-31.96
24	0	-78.50	-77.69	-84.47	-83.89	-87.40
	1	-72.14	-71.28	-78.97	-78.16	-77.93
	2	-69.30	-67.37	-77.53	-75.42	-71.97
	3	-54.07	-52.32	-61.23	-58.25	-52.56
	4	-43.21	-38.20	-49.62	-43.75	-36.57
25	1/2	-86.19	-85.14	-94.33	-93.60	-94.74
	3/2	-81.71	-80.26	-91.04	-89.63	-86.91
	5/2	-73.55	-71.21	-83.30	-80.44	-76.06
	7/2	-58.45	-56.12	-66.89	-62.71	-57.41
	9/2	-44.61	-38.68	-51.41	-44.65	-37.68
26	0	-95.95	-92.94	-107.81	-105.07	-106.00
	1	-97.27	-95.93	-107.72	-106.67	-105.93
	2	-87.63	-85.75	-98.75	-95.80	-93.22
	3	-79.42	-76.72	-90.96	-86.88	-83.13
	4	-63.10	-59.76	-72.88	-67.09	-60.73
	5	-47.96	-40.73	-55.44	-47.04	-42.23
27	1/2	-107.97	-106.80	-119.87	-119.16	-119.10
	3/2	-102.32	-100.99	-114.46	-113.20	-112.28
	5/2	-93.32	-90.93	-106.05	-103.42	-100.32
	7/2	-81.05	-78.48	-93.31	-88.48	-86.48
	9/2	-64.29	-60.40	-74.38	-67.88	-65.33
	11/2	-43.20	-39.13	-49.43	-44.74	-
28	0	-122.52	-121.67	-135.27	-134.84	-136.20
	1	-115.43	-114.34	-129.53	-128.75	-126.89

Table (II.8) (contd)

A	T	Rosenfeld		Kuo-Brown		B.E.
		I	II	I	II	
28	2	-112.52	-110.18	-126.47	-124.49	-120.83
	3	-98.99	-96.43	-113.86	-109.86	-105.12
	4	-86.87	-83.34	-100.80	-93.99	-92.52
	5	-65.16	-60.48	-75.06	-68.27	-66.40
	6	-41.52	-41.52	-47.31	-47.31	-44.34

Table (II.9)

Ground state energies(relative to O^{16}) using KB(12.5) interaction for $12 \leq m \leq 20$ in the 2s-1d shell

A	T	KBI	KBII	B.E.
29	1/2	-145.27	-144.90	-144.79
	3/2	-141.43	-140.00	-136.23
	5/2	-132.11	-129.26	-126.42
	7/2	-118.42	-113.21	-111.96
	9/2	-100.28	-92.95	-94.39
	11/2	-71.10	-67.72	-69.59
30	0	-156.97	-156.82	-156.00
	1	-157.34	-157.49	-155.47
	2	-148.98	-146.96	-143.17
	3	-141.12	-136.81	-134.60
	4	-120.54	-114.47	-115.17
	5	-104.76	-95.14	-98.93
31	1/2	-169.76	-164.36	-168.31
	3/2	-167.14	-165.85	-161.96
	5/2	-156.89	-154.03	-150.91
	7/2	-142.42	-137.37	-137.37
	9/2	-123.36	-116.04	-129.26
32	0	-186.07	-185.84	-183.50
	1	-180.52	-179.95	-176.31
	2	-177.98	-175.75	-171.20
	3	-161.67	-157.80	-155.18
	4	-148.94	-141.41	-142.96
33	1/2	-195.78	-195.66	-192.26
	3/2	-193.13	-192.02	-186.31
	5/2	-180.99	-178.11	-171.20
	7/2	-165.77	-160.61	-161.54
34	0	-207.60	-207.81	-203.80
	1	-207.00	-206.63	-203.74
	2	-199.22	-197.48	-192.82
	3	-188.83	-184.28	-184.07
35	1/2	-219.37	-219.52	-216.44
	3/2	-215.19	-214.42	-210.65
	5/2	-203.15	-200.42	-201.07
36	0	-237.49	-237.61	-231.90
	1	-230.22	-230.11	-225.04
	2	-225.16	-222.96	-220.58

with higher moments.

Using Ratcliff's procedure one can compare the spectra (low lying levels) obtained by using distributions with either shell model results (where available) or with the experimentally observed level sequence. For reasons outlined earlier, the ground state which comes at a substantially lower energy should be ignored in the comparison. In figures (II.3), (II.4) and (II.5) we compare the low lying levels of some nuclei in the first half of the 2s-1d shell obtained using various interactions.

There is a remarkably good agreement, particularly since the two moment distribution would not be expected to predict close lying levels in the low energy region. For such cases one needs higher moments or fixed J distributions to make detailed comparisons.

For the second half of the shell we use the MSDI interaction to obtain spectra of nuclei. Comparison with shell model calculations of Wildenthal et al (1971) has been made in figures (II.6), (II.7), (II.8) and (II.9). Again, apart from the ground state which occurs very much lower, agreement between the two is reasonably good. Whereas in the calculations of Wildenthal et al the vector space was truncated by requiring that the number of nucleons occupying the $d_{5/2}$ orbit is greater than or equal to 10, the spectral distribution method places no

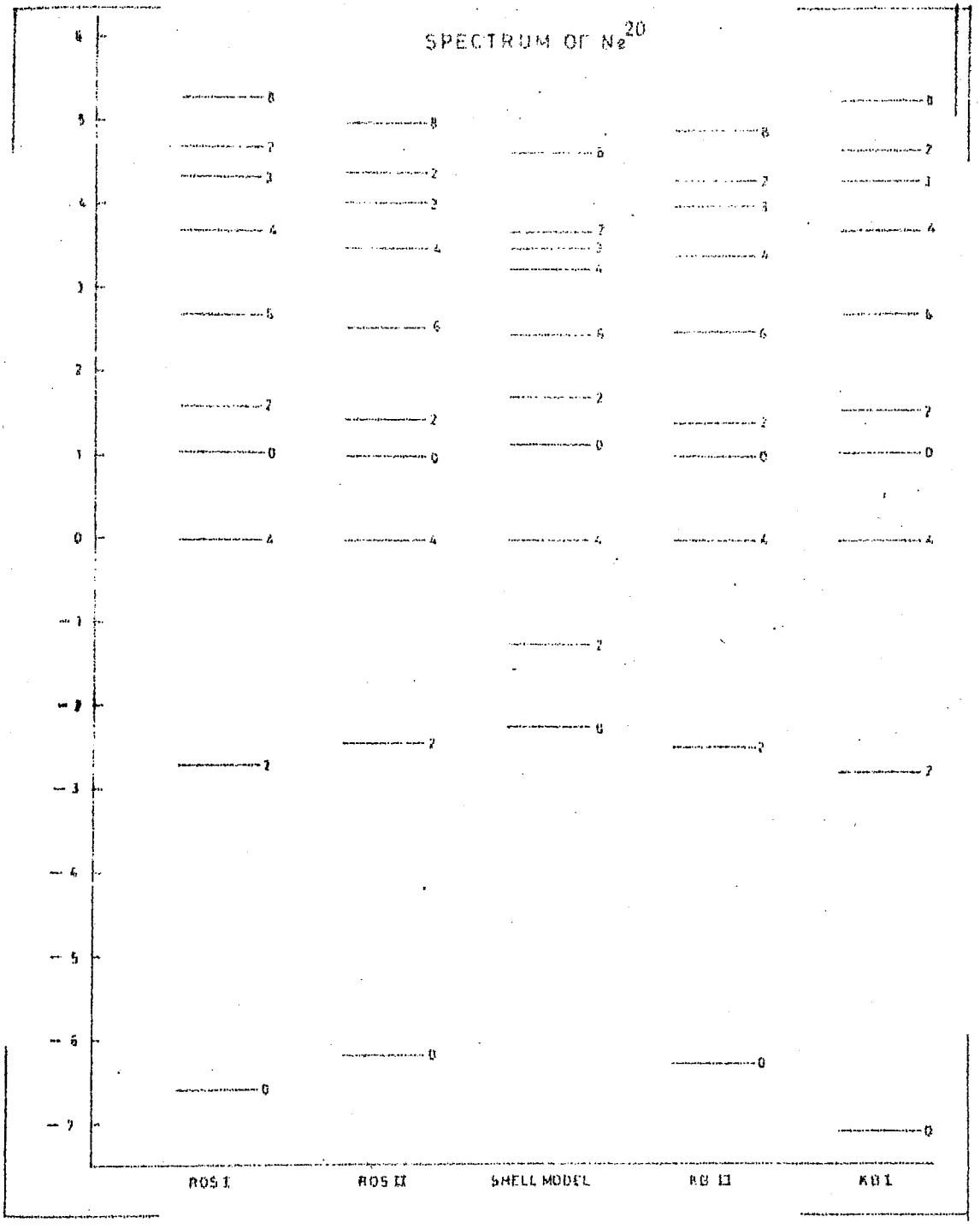

Fig. II.3 Spectrum of Ne^{20} using SU(4) TS moments compared with shell model calculations. Rosenfeld and KB (12.5) interactions were used I and II indicate that the equivalent widths were evaluated using SU(4) and (m T) widths respectively.

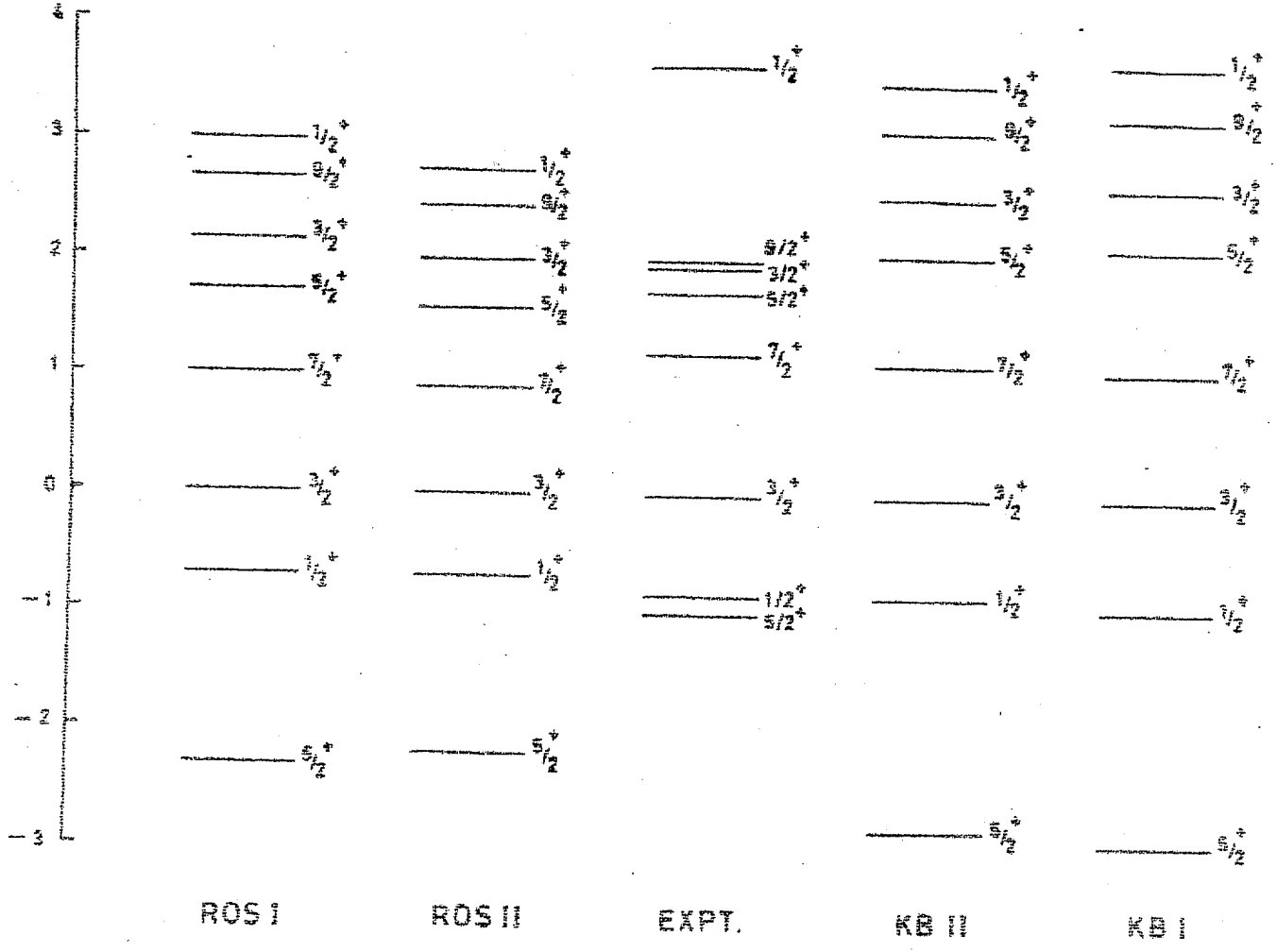

Fig. II.4 Comparison between the experimental and moment-method spectrum of Al^{27}. The $3/2^+$ level is used as reference for the comparison.

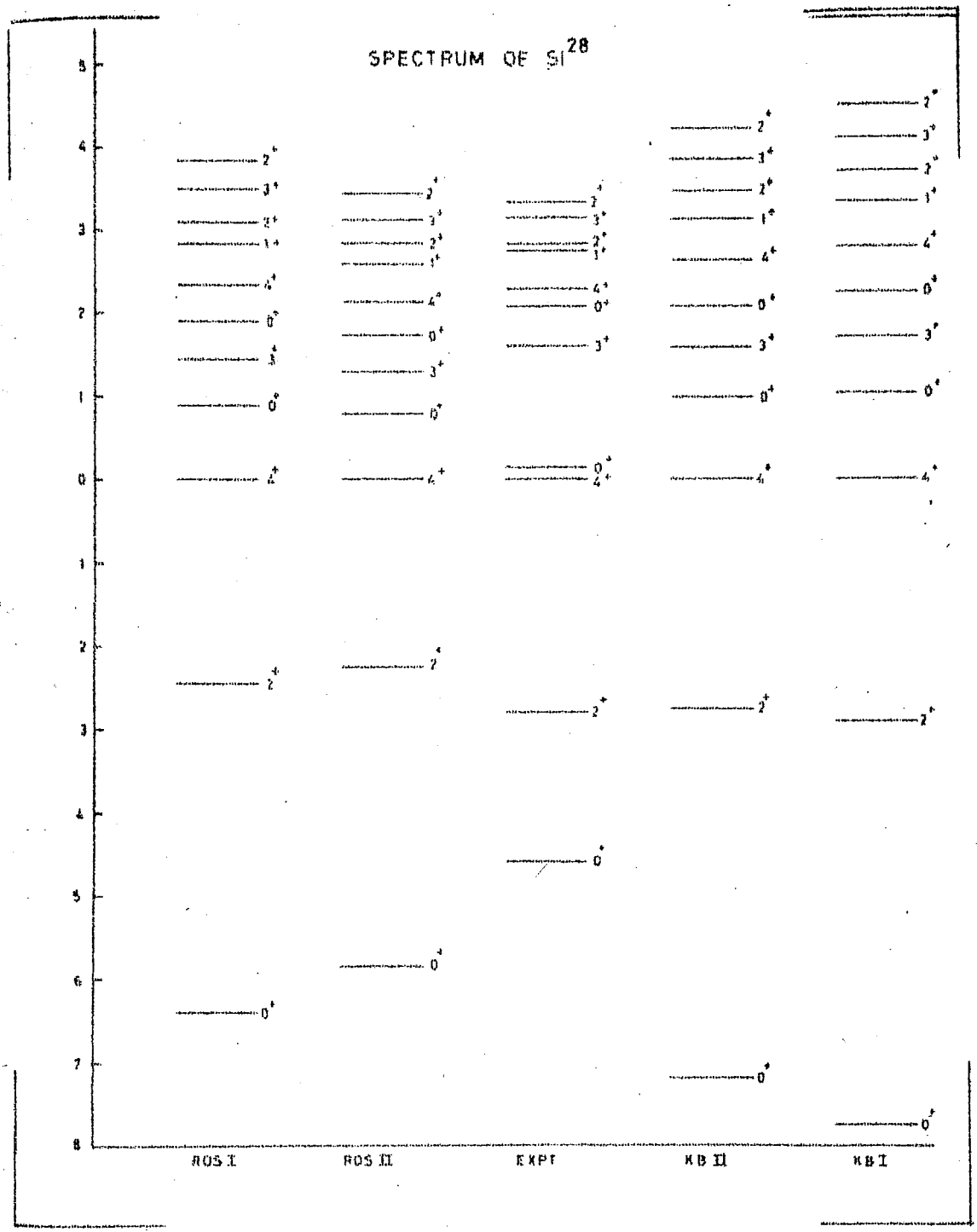

Fig. II.6 Experimental and moment method spectrum of Si^{28}.

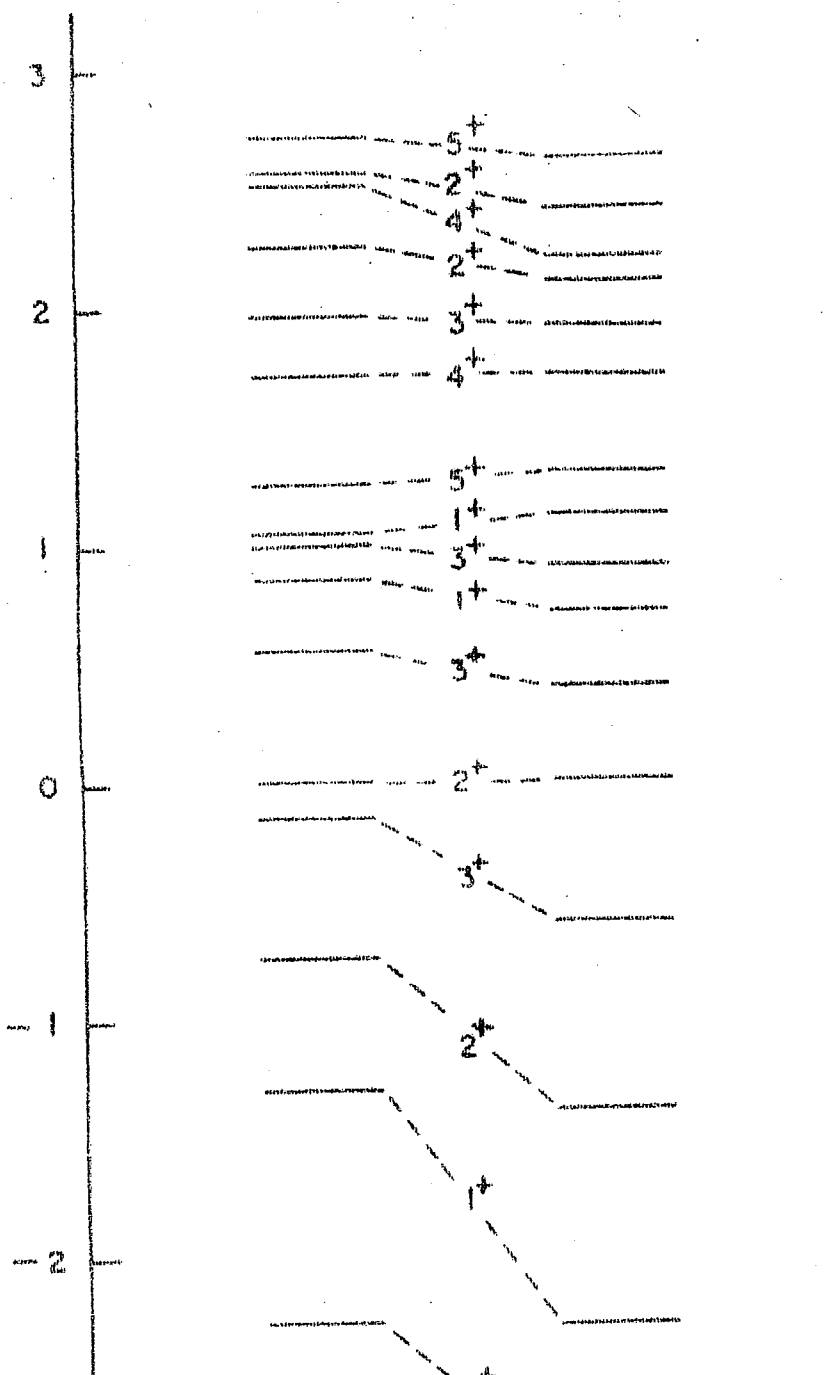

Fig. II.6 Shell model and moment-method spectrum of P^{30} using MSDI.

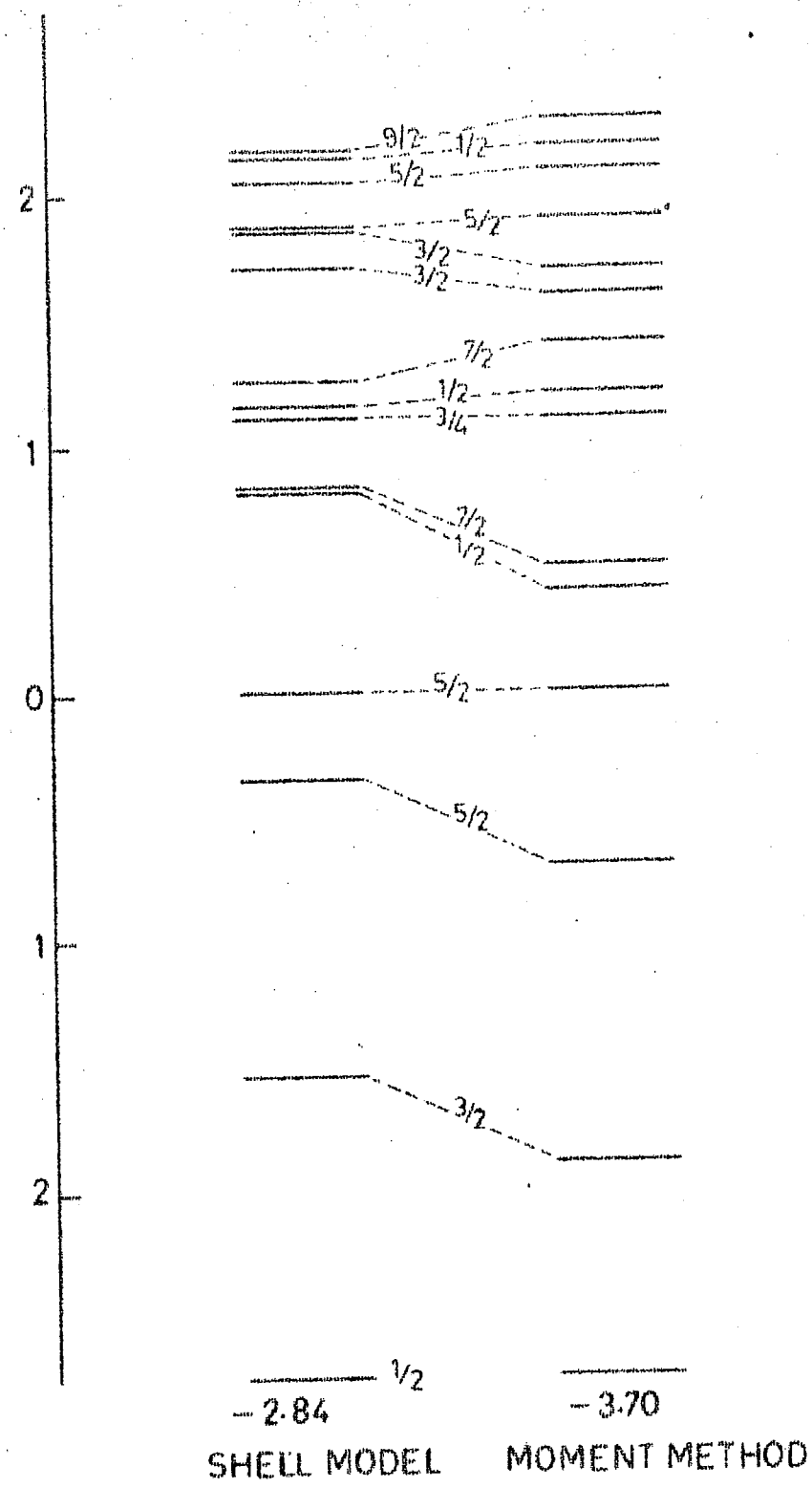

Fig. II.7 Spectrum of P^{31} using (SU(4) TS) moments compared with shell model calculations. MSDI was used.

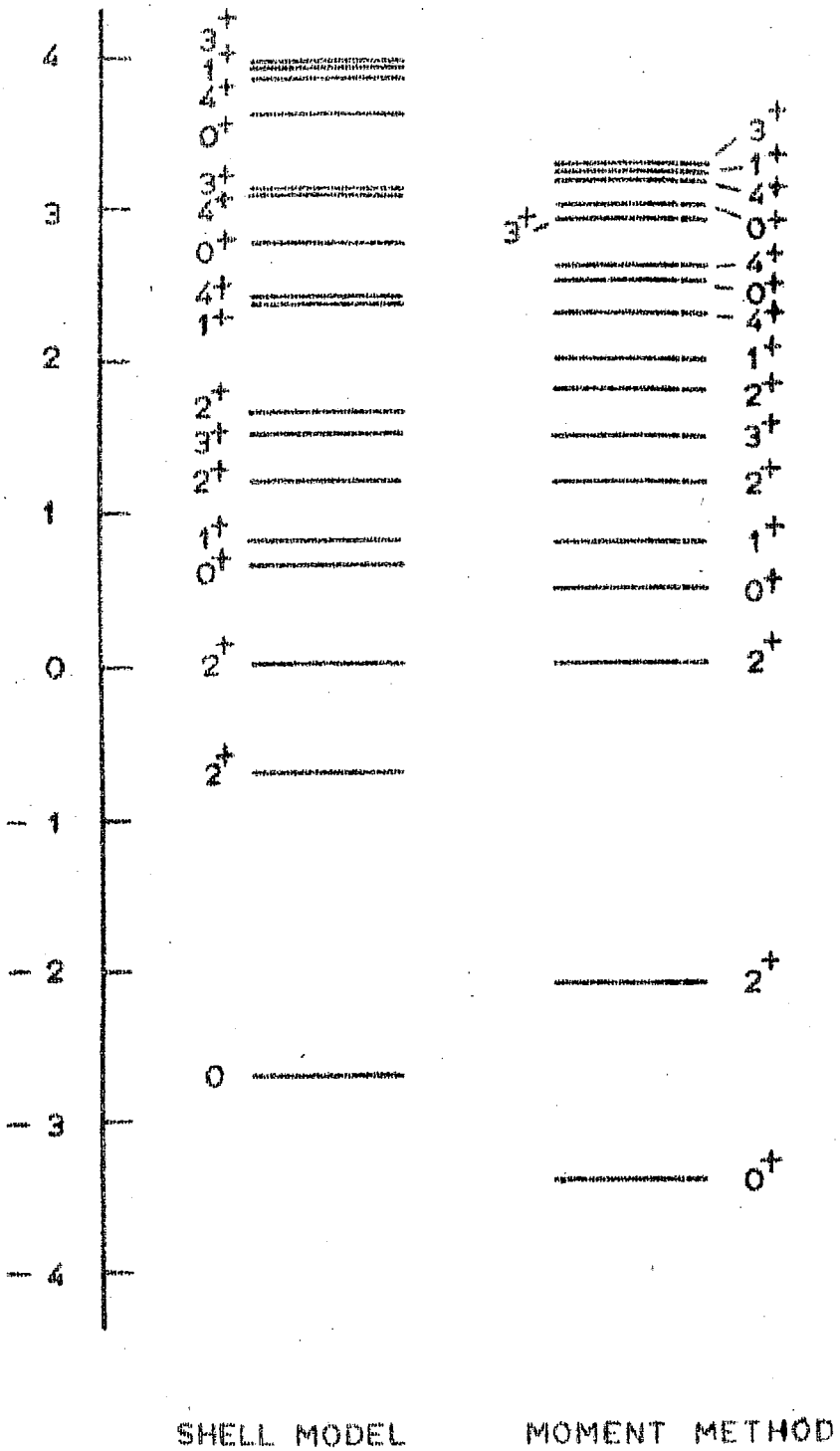

Fig. II.8 Shell model and moment method spectrum of S^{34}.

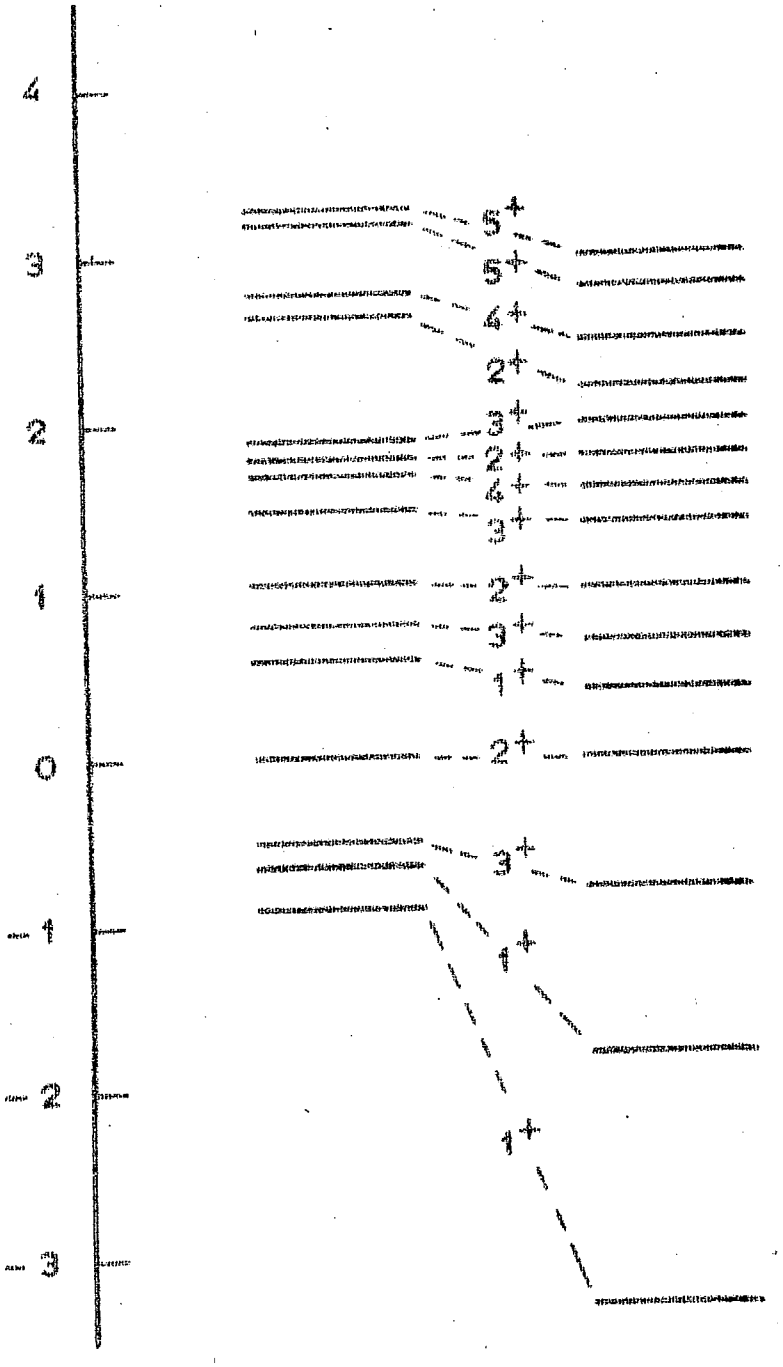

Fig. II.9 Spectrum of Cl^{34} using MSDI.

such constraint. The fact that spectra of nuclei predicted by the two methods shows very good agreement, supports the approximation made by them for determining the energy levels.

It seems therefore from the successes of the method that a large part of the spectral information is indeed contained in low moments of H.

2.5.4 'Goodness' of SU(4) symmetry

The dimensionalities, centroids and widths are used to construct a Gaussian approximation to the intensity distribution of individual representations. We therefore evaluate the intensities of various irreps at the ground state (c.f. eq.(1.2.38)). According to the arguments given earlier (c.f. sec.1.2) the relative intensities can be interpreted as mixing intensities (French and Parikh, 1971) which provide us with a measure of symmetry breaking. Intensities near the ground state for $4 \lesssim m \lesssim 20$ particles in the 2s-1d shell are given in tables (II.10) and (II.11).

We observe that the KB(12.5) interaction gives much larger mixing as compared to the Rosenfeld interaction. Apart from this, the admixture of various symmetries shows some unexpected behaviour. We see from tables (II.10) and (II.11) that for some odd-A nuclei the extent of admixture is more as compared to neighbouring even-A nuclei. Further, in several cases (e.g. A=26) the symmetry mixing alternately varies

Table (II.10)

Percentage intensities of various irreps of SU(4) at the ground state for various nuclei in the first half of the 2s-1d shell

A	T	Rosenfeld		Kuo-Brown	
		I	II	KBI	KBII
20	0	98-1	92-5-3	93-7	79-15-5
	1	99-1	96-2-2	95-3-2	90-5-3
	2	68-32	69-31	59-41	59-41
21	1/2	93-7	83-16	86-13	73-23-2-1
	3/2	68-32	62-37	64-34-2	59-38-2
	5/2	96-4	96-4	92-8	92-8
22	0	99	98	96-2	91-1-2-4
	1	67-22-9-2	56-24-14-5	56-28-12-3	45-28-16-11
	2	96-4	94-6	93-6	91-7
	3	70-30	71-29	57-42	59-40
23	1/2	84-15	76-23	71-28-1	59-36-2-2
	3/2	81-16-2	70-25-3-1	76-19-31	63-27-5-2-2
	5/2	62-37	57-42	59-40	53-46
	7/2	97-3	97-3	91-9	93-7
24	0	93-5-2	87-8-5	73-20-7	56-29-14
	1	86-12	92-3-2	88-6-4-2	80-7-7-4
	2	63-22-10-5	51-23-14-12	51-29-13-5	39-29-18-12
	3	95-5	91-7	91-8	90-9
	4	68-32	73-27	54-46	60-39
25	1/2	85-13	76-22	70-27-2	56-35-4-3
	3/2	68-30	60-36-2	61-34-2-3	51-38-3-4-3
	5/2	79-18-2	67-27-3-3	70-24-4-1	60-31-5-3
	7/2	63-27	58-42	54-45	53-46
	9/2	96-4	98-2	90-10	95-5
26	0	99	98-1	91-2-2-4	82-3-3-10
	1	61-20-16-3	51-22-19-8	46-24-20-10	34-21-24-18
	2	94-2-3	90-4-5	90-3-5	83-5-7
	3	61-21-12-6	49-21-15-14	45-28-17-10	37-28-19-14
	4	97-3	95-5	93-7	93-6
	5	67-33	83-17	55-45	69-31

Table (II.10)(contd)

A	T	Rosenfeld I	Rosenfeld II	Kuo-Brown KBI	Kuo-Brown KBII
27	1/2	83-17	73-25	65-31-2-1	51-39-4-3
	3/2	70-27-2	61-33-3-1-2	60-32-4-3-1	49-35-5-4-4
	5/2	66-32-1	58-37-2-3	57-31-3-8	49-42-3-3
	7/2	74-23-2	64-31-2-3	62-32-4-2	58-34-4-3
	9/2	68-32	65-35	56-43	61-39
	11/2	100	100	100	100
28	0	93-5-2	84-9-7	66-25-9	47-33-18
	1	95-2-1-1	90-3-3-3	84-7-4-4	74-8-7-7
	2	59-17-17-7	46-19-19-15	41-24-23-10	31-24-24-14-3
	3	95-2-2	91-4-4	90-5-4	86-6-6
	4	62-16-16-6	53-17-17-12	42-23-23-12	42-22-22-13
	5	100	100	100	100
	6	100	100	100	100

Table (II.11)

Mixing intensities for various nuclei in the second half of the 2s-1d shell using KB(12.5)

A	T	KBI	KBII
29	1/2	66-25-9	50-39-5-3
	3/2	61-31-2-3-2	51-35-4-5-3
	5/2	57-28-3-2	49-42-4-3
	7/2	60-33-4-2	57-35-5-3
	9/2	56-44	62-38
	11/2	100	100
30	0	90-4-2-2	81-12-3-3
	1	44-25-20-11	32-24-21-21
	2	88-3-6	81-5-9
	3	41-29-17-12	35-28-19-16
	4	90-10	93-6
	5	53-47	69-31
31	1/2	86-8-4	83-8-4
	3/2	58-36-2-3	48-39-3-4-2
	5/2	64-27-6-2	55-33-7-3
	7/2	51-48	51-48
	9/2	87-13	93-7
32	0	67-24-8	48-32-17
	1	85-7-4-2	76-8-9-5
	2	44-31-15-9	34-30-18-14-2
	3	88-10	88-12
	4	49-51-1	56-44
33	1/2	67-24-8	51-41-3-2
	3/2	71-21-5	59-28-6-3
	5/2	51-47	49-49
	7/2	86-14	89-11
34	0	93-1-3-3	85-3-2-9
	1	46-31-14-9	36-29-16-18
	2	90-8	88-10-1
	3	49-50-1	52-47

Table (II.11) (contd)

A	T	KBI	KBII
35	1/2	75-21-2	61-31-4-3
	3/2	56-40-2	51-44-4
	5/2	86-14	87-13
36	0	84-13-3	65-24-10
	1	90-5-4	85-7-8
	2	54-45	53-46

(either decreases or increases) with isospin. This behaviour in our calculation appears because of dimensionality factors and this feature needs to be examined with much more care. In general, one may conclude that the relative intensities involving three or four lowest lying irreps indicate large admixings, the extent of which remains the same for both low T and high T nuclei.

The external width (c.f. sec.(1.2)) would provide a more precise measure of symmetry breaking. Hecht and Draayer (1974) have evaluated SU(4) partial widths for A=25 nuclei for various modifications of the Kuo-Brown interaction and have given a simple measure of symmetry breaking. The ratio

$$\chi^2(m\underset{\sim}{f}\underset{\sim}{f}') = \frac{\sigma^2(m\underset{\sim}{f}\underset{\sim}{f}')}{\left[E_c(m\underset{\sim}{f}) - E_c(m\underset{\sim}{f}')\right]^2} \qquad (2.5.13)$$

where $\sigma(m\underset{\sim}{f}\underset{\sim}{f}')$ is the partial width, provides a measure of symmetry breaking. They find that the symmetry is violated more for realistic interactions than for the central interaction. However, it is possible that the intensities as shown in tables (II.10) and (II.11) over estimate the mixing for realistic

interactions. This may be due to the fact that ground state energies as well as the intensities may be in error because of the two moment distribution. This also needs to be investigated in detail.

In the absence of a knowledge about the partial widths one can use a different way of estimating symmetry mixing which is described here.

As mentioned earlier (c.f. eq.(1.2.12)) the total variance $\sigma^2(\underset{\sim}{mf})$ is a sum of two terms, the internal variance $\sigma^2_{int}(\underset{\sim}{mf})$ and the external variance $\sigma^2_{ext}(\underset{\sim}{mf})$. The former gives rise to spreading of states within an irrep whereas the latter leads to symmetry mixing. One now tries to determine the ground state energies and the mixing of SU(4) symmetry in terms of $\sigma^2_{int}(\underset{\sim}{mf})$ and $\sigma^2_{ext}(\underset{\sim}{mf})$. In this two step process one first determines energy of states, one from each symmetry (having the same angular momentum J and isospin T) using Ratcliff's procedure separately for each symmetry and taking $\sigma^2_{int}(\underset{\sim}{mf}) = k_1 \sigma^2(\underset{\sim}{mf})$ $(0 \leq k_1 \leq 1)$. Next these pure symmetry states are allowed to mix by i) setting up a matrix in this basis and taking for the symmetry mixing matrix elements $\sigma^2_{ext}(\underset{\sim}{mf}) = k_2 \sigma^2(\underset{\sim}{mf})$ $(k_1 + k_2 = 1)$ and ii) diagonalizing the matrix.

As an illustration consider the case of 12 particles in the 2s-1d shell, considering only the three lowest SU(4)

symmetries namely $[3333]$, $[4332]$ and $[4422]$ and using the SU(4) variances as obtained by Parikh (1973) the matrices for various values of k_1 are set up and diagonalized. Results are shown in table (II.12).

Table (II.12)

Ground state energies and symmetry mixing obtained for a decomposition of total width into an internal and external width

Interaction	Ratcliff's method		$\sigma^2_{int} = \sigma^2/4$		$\sigma^2_{int} = 9\sigma^2/10$	
	B.E.	Mixing	B.E.	Mixing	B.E.	Mixing
KB(12.5)	-140.26	47-48-5	-136.82	58-41-4	-138.8	50-49
Rosenfeld	-125.12	82-17	-122.77	78-19-2	-124.3	82-16-2

From the table one observes that as the value of k_1 is increased the ground state energy tends towards the value obtained before using $\sigma^2(mf)$. Also there are small fluctuations in the amount of symmetry mixing, the extent of mixing increasing for KB(12.5) and decreasing for Rosenfeld interaction.

2.5.5 Binding energy relationship for heavy nuclei

It has been shown by Franzini and Radicati (1963) that for a nucleus with mass number A having the SU(4) symmetry

ξ = (P, P', P'') the ground state energy in the supermultiplet scheme can be written in the form

$$E(A,\xi) = a(A) + \frac{b(A)}{2}\left[P^2 + 4P + P'^2 + 2P' + P''^2\right] \qquad (2.5.14)$$

Here a(A) and b(A) are constants which depend on the nuclear interaction and mass number. The possible values of (P,P', P'') for a given A, calculated by Wigner, have been tabulated by Wigner and Feenberg (1937) and are related to $\underset{\sim}{f} = \left[f_1 f_2 f_3 f_4\right]$ through equation (2.4.6).

The polynomial expression involving (P,P',P'') is the eigenvalue of the bilinear Casimir operator for the SU(4) symmetry ξ.

Franzini and Radicati have calculated binding energy differences between isobars using eq.(2.5.14). For each isobar, the numbers (P, P', P'') have been assigned on the basis that $P = |T_o| = 1/2\,|N-Z|$ and P' and P'' are taken to have the minimum value consistent with P and A. This corresponds to the most space symmetric state one can construct under these conditions.

With these assignments of the symmetries a ratio R involving binding energies of three isobars defined as:

$$R(T_o) = \frac{E(A,\xi(T_o)) - E(A,\xi(T_o-2))}{E(A,\xi(T_o-1)) - E(A,\xi(T_o-2))}$$

(2.5.15)

is calculated.

It can easily be checked that $R(T_o)$ is independent of A in the SU(4) scheme. The ratios $R(T_o)$ for A=2m+1, 4m and 4m+2 nuclei have been calculated by these authors using empirical values for binding energies and the agreement with theoretical values predicted by Wigner model is found to be good.

In our work, we have evaluated the ratio R using the binding energies obtained from the distribution method by choosing the appropriate isobars from tables (II.8) and (II.9). The values obtained using different interactions are compared with the theoretical SU(4) model ones R_{wig} in table (II.13). The appropriate experimental values R_{exp} are also tabulated.

For a fixed isospin T_o, the number of sets of isobars available to calculate R was small. Although the values calculated seem to be scattered around the theoretical values, agreement in general is reasonable. It should be noted that the empirical values of the ratio obtained also deviate from the theoretical

Table (II.13)

Values of the ratio R as predicted by the binding energies obtained from (SU(4)(TS)) moments for various isobars are compared with the theoretical value of R given by the Wigner model.

T_3	R_{Wigner}	R_{calc} Rosenfeld I	Rosenfeld II	KBII	R_{exp}
A=2n+1					
5/2	2.33	2.85,3.02,2.31,2.59	2.81,2.92,2.85,3.13	3.15,3.32,3.31,2.64	2.47,2.91,2.39,2.78
7/2	2.25	2.08,2.85,2.34	2.25,2.67,2.24	2.21,2.93,2.53	2.12,2.72,2.16
9/2	2.20	1.92,2.37	2.16,2.45	2.02,2.38	2.06,2.53
11/2	2.167	2.26	2.18	2.12	2.14
A=4n					
2	1.5	1.97,1.44,1.41	2.01,1.61,1.57	2.11,1.48,1.70	1.63,1.63,1.65
3	4.0	6.01,5.65	4.85,4.31	5.27,4.43	4.26,3.59
4	1.66	1.71,2.05	1.94,1.95	1.84,2.08	1.82,1.80
5	3.0	2.79	2.75	2.62	3.07
6	1.75	2.09	1.83	1.81	1.84
A=4n+2					
3	1.6	1.84,1.85	1.96,1.89	2.01,1.82	1.73,1.79
4	3.33	2.99	2.88	3.22	3.22
5	1.71	1.93	2.12	2.01	1.83

numbers. It is interesting to observe that deviation of R_{cal} and R_{exp} from R_{wig} are strongly correlated. In view of this we may conclude that the general trend supports the SU(4) model. However, the validity of the mass formula does not imply good SU(4) symmetry for the wave function. This is because the ratio R which involves differences of energies may be insensitive to symmetry mixing.

2.5.6 SU(4) Partial level densities and alpha transfer reactions

The intensities of various irreps of SU(4) in the ground state region provide a measure of symmetry mixing. One can also determine these intensities at higher excitation energies and thereby study their energy dependence. In figures (II.10 - II.11), the energy variation for 4 and 8 particles is shown.

These partial level densities of SU(4) irreps can be used in the study of alpha transfer reactions (Arima et al, 1972; Ichimura et al, 1973). In this section a very elementary account is given. For a detailed investigation, one needs the cfp's for the decomposition of a totally antisymmetric m nucleon wave function into totally antisymmetric functions for specific sets of m_1 and m_2 nucleons. Some of these coefficients for the SU(4) part of the wavefunction have been given by Hecht and Pang(1969) while those corresponding to the space part are to be investigated.

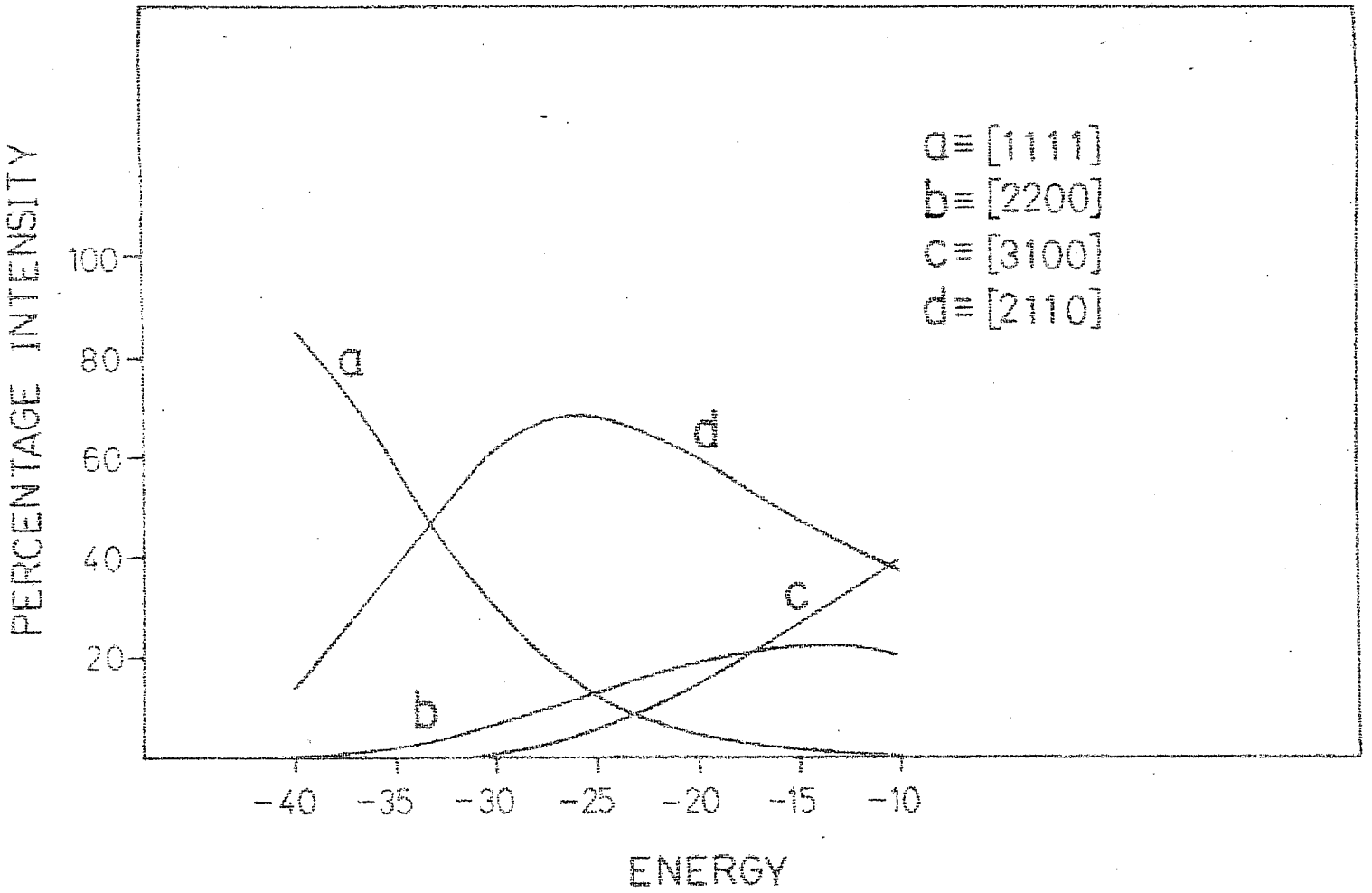

Fig. II.10 Variation of the intensities of SU(4) representations for 4 particles as a function of excitation energy.

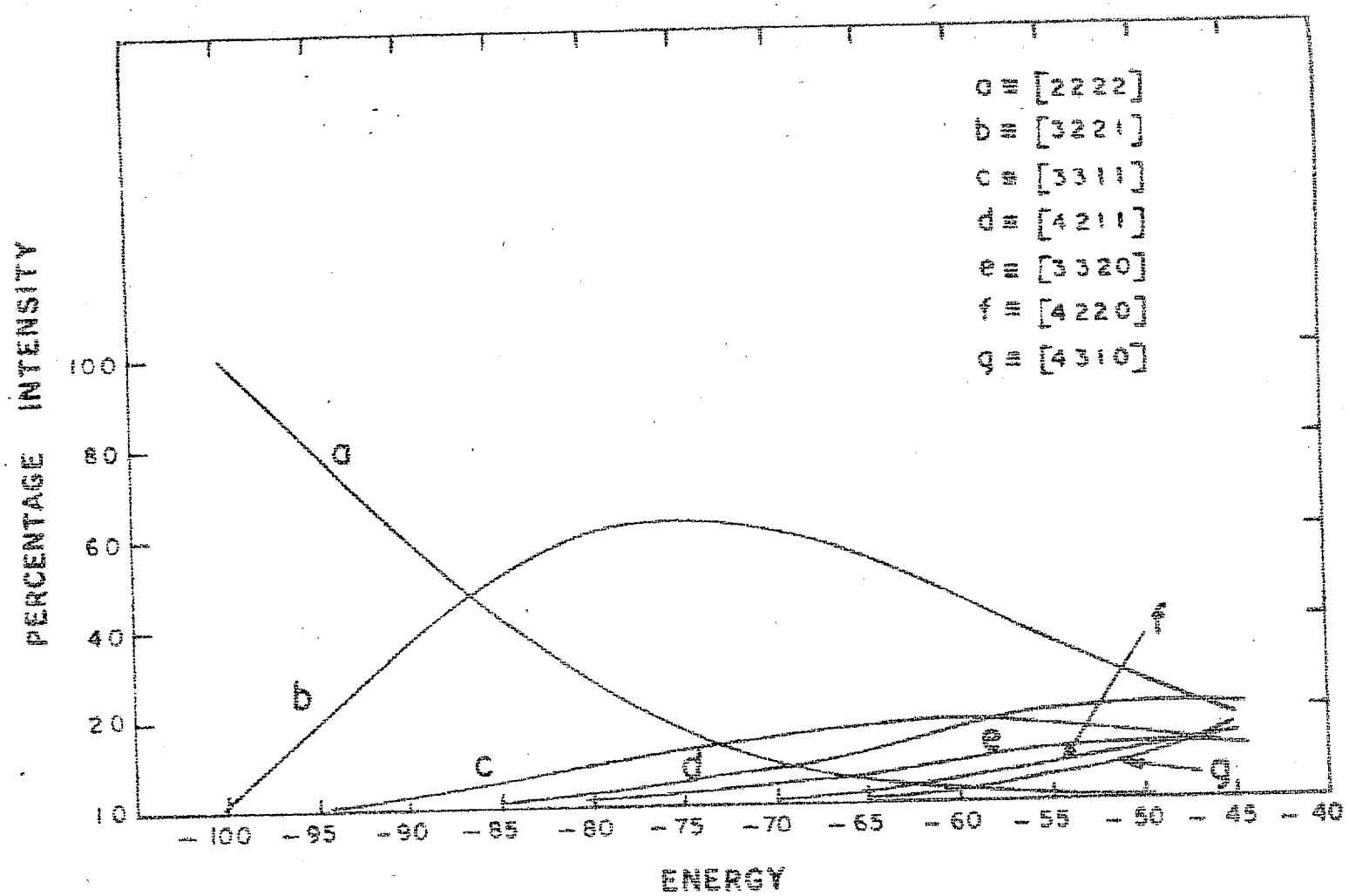

Fig. II.11 Energy variation of the intensities of SU(4) representations for 8 particles.

Also, since the relative phases between components having different symmetries are not known, in this preliminary application it has been assumed that the dominant symmetries are coherent and the remaining occur with random sign.

With the assumption that the energy variation of the scattering cross section for alpha transfer depends largely on the energy dependence of these intensities (c.f. sec.1.2.4) the overlaps are calculated; first by assuming that only one symmetry is dominant and then by including others (since SU(4) symmetry is broken).

As an illustration two reactions are considered.

1. $Ne^{20} + \alpha \longrightarrow Mg^{24}$
2. $Mg^{24} + \alpha \longrightarrow Si^{28}$

and the results shown in figures (II.12) and (II.13).

One observes that when only the dominant symmetry is considered the overlap varies more or less linearly with energy. When the next symmetry is included the trend changes, the overlap increases near the ground state to a maximum value and then decreases slowly. Also the overall rate of decrease in the first case is faster as compared to the second.

2.6 Concluding Remarks

We have examined the SU(4) structure of 2s-1d shell nuclei

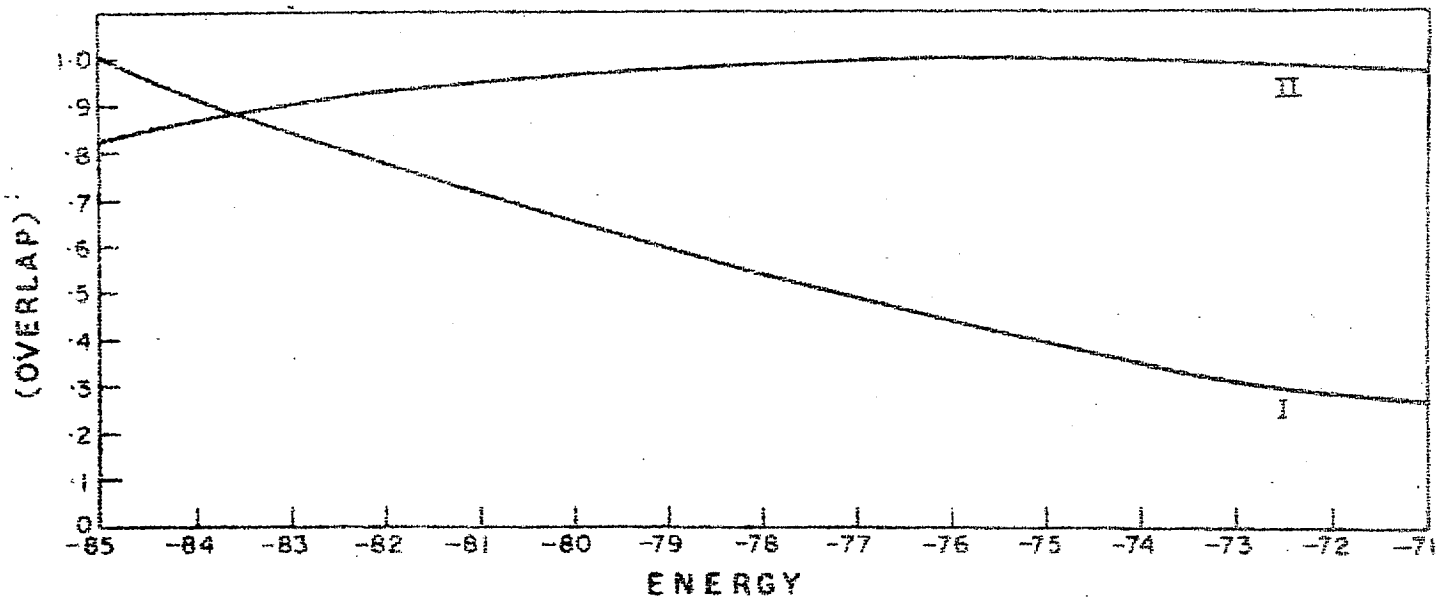

Fig. II.12 Overlap as a function of excitation energy for the reaction $Ne^{20} + \alpha \longrightarrow Mg^{24}$. (See text for details).

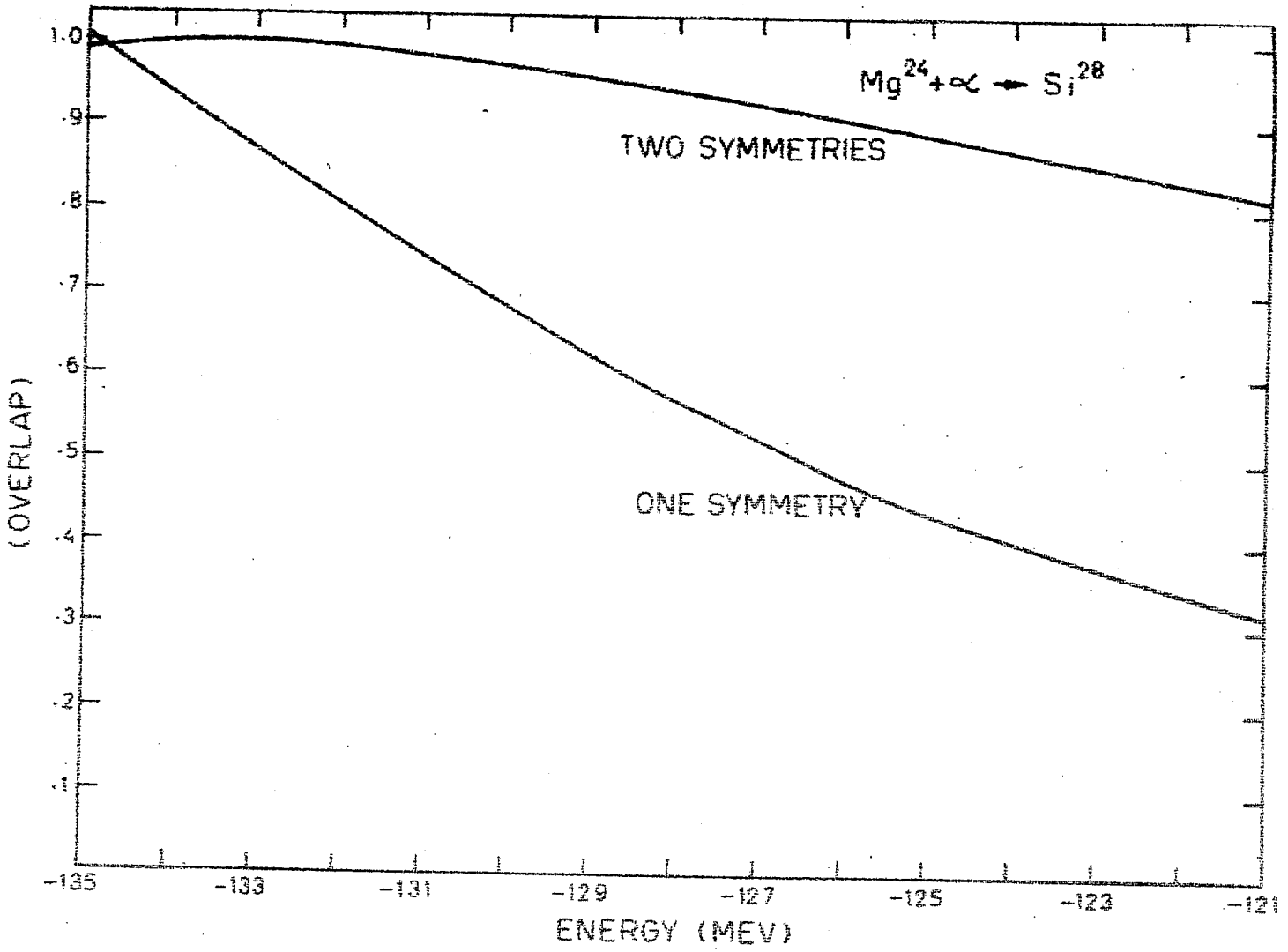

Fig. II.13 Overlap for the reaction $Mg^{24} + \alpha \longrightarrow Si^{28}$ as a function of excitation energy calculated using a) one symmetry and b) two symmetries.

in finer details using spectral distribution methods. An averaging of the SU(4) states, over fixed T and fixed S, has been carried out to obtain ground state energies and low energy spectra of several nuclei. Results of our calculations agree reasonably well with known experimental or shell model results. The reasons for disagreement observed for ground state energy have also been discussed.

Results of our calculations of relative intensities of various irreps in ground states indicate that in this shell, both for low T and high T values (for a given m), SU(4) symmetry is broken often nearly maximally. Thus in heavier nuclei, it is obvious that the symmetry cannot be violated to any greater extent. Hence it is not any more surprising to find energy systematics which are consequences of this symmetry in heavier ($A \lesssim 110$) nuclei as in the lighter ones. It appears therefore that symmetry mixing apparently does not significantly affect energy relationships, and in fact such relationships are rather successful in concealing the strong violations of the symmetry. This is not a new feature in nuclear physics where one has over and over come across "Pseudonium" nuclei (Cohen, Lawson and Soper, 1966). While such a conclusion seems reasonable, it would still be necessary to do explicit calculations for heavier nuclei to confirm the validity of this reasoning. This is done in the next chapter.

As regards the exact evaluation of the widths we just want to mention that while it has not been possible to construct a polynomial expression for $\sigma^2(\underset{\sim}{mfTS})$ we have succeeded in constructing one for $\sigma^2(\underset{\sim}{mfT})$ or $\sigma^2(\underset{\sim}{mfS})$. In these cases we are either averaging over spin states for fixed $(\underset{\sim}{mfT})$ or all isospin states for fixed $(\underset{\sim}{mfS})$. These polynomial expressions involve the operators 1, n, G_2, G_3, G_4, T^2 and one extra operator $(G_2 T^2)_{2\text{-}b}$ or $(G_2 S^2)_{2\text{-}b}$ depending upon whether we want to evaluate $\sigma^2(\underset{\sim}{mfT})$ or $\sigma^2(\underset{\sim}{mfS})$. The subscript 2-b on the operators implies that we take only the (1+2)-body parts of the new operator although it has 3 and 4-body parts as well. For physical applications we clearly want to evaluate $\sigma^2(\underset{\sim}{mfT})$. Details regarding this are mentioned in Chapter V.

CHAPTER III

SPECTRAL DISTRIBUTION STUDY OF NUCLEI IN 2p-1f SHELL

3.1 Introduction

Spectral distribution methods were used in Chapter II to study properties of nuclei in 2s-1d shell. Besides SU(4) Isospin-spin averages, spectral moments of the Hamiltonian for states defined by some other subgroups of U(N) have been evaluated by French and Ratcliff (1971) and by Chang, French and Thio (1971). They have used these moments to study properties of nuclei in 2s-1d and upper 2p-1f shells. In the application of distribution methods to nuclear energies, level densities and excitation strengths, Chang, French and Thio find that the method works very well and there is no indication of any systematic error depending upon the dimensionalities of the vector spaces which range upto 35000.

In this chapter we have studied systematics of nuclei in 2p-1f shell using these methods. There are several reasons for this. Firstly, we would be dealing with spaces which are much larger than those encountered earlier and hope to gain some insight into how these methods work in large spaces, a major concern being the accuracy of the method. Secondly, the dimensionality of vector spaces being a major constraint on conventional matrix methods, nuclei in this shell have not been extensively

studied in the shell-model. Truncation of space on the basis of physical arguments, though often quite fruitful may not always be sufficient. One then takes recourse to other approximate methods and recently deformed configuration mixing calculations (DCM) in the Hartree-Fock framework by Dhar, Kulkarni and Bhatt (1975) have been successful in describing properties of some nuclei in this shell.

It has been observed (Parikh, 1973; and Haq and Parikh, 1974) that SU(4) symmetry in light nuclei (2s-1d shell) is broken for both low and high isospin, often nearly maximally. Also previous results surprisingly indicate that symmetry mixing does not significantly affect energy relationships deduced on the assumption that the symmetry is good. In view of this it is not surprising to find energy systematics which appear to be consequences of SU(4) symmetry in heavier nuclei (Franzini and Radicati, 1963). We therefore wish to investigate the validity of the Wigner supermultiplet scheme in heavy nuclei.

In our study, we have considered a decomposition of the overall spectroscopic space (m particles in 2p-1f shell) in terms of; spherical orbit configurations ($\tilde{m}$), isospin (mT), configuration-isospin ($\tilde{m}$T) and SU(4)-isospin ($\underset{\sim}{f}$T). Centroid energies and widths for these distributions are evaluated and used to determine ground state energies, low energy spectra and fractional occupancy of single particle orbits for ground

states of nuclei. The 'goodness' of Wigner SU(4) symmetry is investigated using (fT) moments.

In Section (3.2) we briefly review how one can average over states defined by these subgroups of U(N). We first consider averaging over all m-particle states (Scalar averaging) and then deal with finer averages. This involves configuration averaging with and without isospin and SU(4) isospin-spin averages. Spectral moments are evaluated and results presented in Sec.(3.3). Application of spectral moments to study properties of nuclei is discussed in Sec.(3.4). Results are compared with shell model and other microscopic calculations where available. Concluding remarks are made in Sec.(3.5).

3.2 Averaging over states defined by U(N) and its subgroups

In Sec.(1.4) we discussed the unitary group U(N) and its subgroups. Rather than averaging over states defined by U(N) which is the simplest form of averaging, it is much more profitable to deal with subgroups of U(N) which correspond to partitions of N into subsets.

One can carry out a decomposition of U(N) via the direct-sum group $\sum_{i=1}^{\ell} U(N_i)$ where the N single particle states are partitioned into ℓ orbits with dimensionalities $N_i (i = 1, \ldots \ell)$. By assigning m_i particles to the i^{th} orbit one gets a configuration $(\tilde{m}) = (m_1, m_2, \ldots m_\ell)$ and moments of H over $(\tilde{m})$ can be

evaluated. One can also specify isospin and deal with configuration-isospin averages. By carrying out a direct-product subgroup decomposition, one can average over states defined by SU(4) symmetry or those defined by SU(4) symmetry, spin and isospin. These subgroup decompositions of U(N) have been discussed in Sec.(1.4).

In this section we consider these various types of averaging. Scalar averaging is discussed in detail and some comments regarding finer averaging are made. We show how spectral moments for these distributions can be evaluated.

3.2.1 Scalar averaging

In Sec.(1.2) we discussed how an operator with a definite particle rank propagates through the rest of the particle space. This result was then extended to propagate operators of mixed particle rank. The basic principle is to construct density operators of the system which are polynomials in the number operator n and in more complex cases involve Casimir operators of the group. For the Hamiltonian H which is a (0+1+2)-body operator and H^2 which is a (0+1+2+3+4)-body operator, one can derive expressions for centroids and widths. These have been given by French (1967).

$$E_c(m) = \frac{(N-m)(m-1)}{N} E_c(0) + \frac{(N-m)m}{(N-1)} E_c(1) + \frac{m(m-1)}{N(N-1)} E_c(N) \quad (3.2.1)$$

The centroids for m particles is written in terms of the centroids in the defining space namely 0-particle $E_c(0)$, 1-particle $E_c(1)$ and 0-hole $E_c(N)$ spaces. Normally for scalar averaging the elementary net would consist of 0-particle, 1-particle and 2-particle representations. As mentioned before this is not the most profitable way of looking at things because of the complexity in obtaining the necessary input traces. One then uses the optimum net by including hole representations in the defining space. The variance $\sigma^2(m)$ can be written as:

$$\sigma^2(m) = \frac{m(m-1)(m-2)(N-m)(N-m-1)}{(N-1)(N-2)(N-3)} \left\{ \frac{\sigma^2(N-1)}{(N-m-1)} - \right. \tag{3.2.2}$$

$$\left. \frac{(N-3)}{(m-1)} \sigma^2(1) + \frac{N-1}{2(m-2)} \sigma^2(2) \right\}$$

The input information is expressed in terms of centroids and variances for certain particle and hole representations. It is convenient to rewrite eq.(3.2.1 - 3.2.2) in terms of single particle energies and 2-body matrix elements. The Hamiltonian H can be written as:

$$H = H_0 + H_1 + H_2 \tag{3.2.3}$$

where H_0, H_1 and H_2 are the 0-body, 1-body and 2-body parts respectively. The centroid energy is given by

$$E_c(m) = \langle H \rangle^m = H_o + \binom{m}{1}\langle H_1 \rangle^1 + \binom{m}{2}\langle H_2 \rangle^2 \qquad (3.2.4)$$

where $\binom{m}{1}$ etc are binomial coefficients and $\langle H_1 \rangle^1$ and $\langle H_2 \rangle^2$ are the centroids in 1 and 2-particle spaces. In terms of the single particle energies ϵ_{ii} and 2-body matrix elements W_{ijij}, these can be written as:

$$\langle H_1 \rangle^1 = \bar{\epsilon} = \frac{1}{N}\sum_{i=1}^{N} \epsilon_{ii}$$

$$\langle H_2 \rangle^2 = \bar{W} = \binom{N}{2}^{-1} \sum_{i<j} W_{ijij} \qquad (3.2.5)$$

The final expression for centroid energy and the variance (Chang, 1970) can be written as:

$$E_c(m) = H_o + m\bar{\epsilon} + \frac{m(m-1)}{2}\bar{W} \qquad (3.2.6)$$

$$\sigma^2(m) = \frac{m(N-m)}{(N-1)}\Delta_\epsilon + \frac{2m(m-1)(N-m)}{N(N-1)(N-2)}\Delta_{\epsilon W} + \frac{m(m-1)(m-2)(N-m)}{N(N-1)(N-2)(N-3)}\Delta'_W + \frac{m(m-1)(N-m)(N-m-1)}{N(N-1)(N-2)(N-3)}\Delta_W \qquad (3.2.7)$$

Here Δ_ϵ is the variance of H_1 in single particle states defined as

$$\Delta_\epsilon = \frac{1}{N}\sum_{i=1}^{N} \epsilon_{ii}^2$$

$$\Delta_{\varepsilon W} = \sum_{i,j} \epsilon_i W_{ijij}$$

$$\Delta'_W = \sum_{ijkl} W_{ikjk} W_{iljl} - N(N-1)^2 \bar{W}^2 \qquad (3.2.8)$$

$$\Delta_W = \binom{N}{2}^{-1} \sum_{\substack{i<j \\ k<l}} (W_{ijkl})^2 - (\bar{W})^2$$

Δ_W is the variance of the H_2 in 2-particle states.

Using equations (3.2.5) and (3.2.6) one can calculate scalar eentroids and variances for any particle number m. Scalar moments have been used by Ratcliff (1971) to estimate the trend in theoretical binding energies for 2s-1d shell nuclei. Although these moments are easy to evaluate, they tell us nothing about structure of nuclei. There are many finer averages which yield information regarding nuclear structure and we discuss them next.

3.2.3 Finer averaging

By partitioning the N single particle states into orbits with dimensionalities $N_i (i = 1 \ldots . \ell)$ and distributing m_i particles in the i^{th} orbit we obtain a configuration $(\tilde{m})$. In order to evaluate moments over configuration subspaces $(\tilde{\tilde{m}})$

one can follow French and Ratcliff (1971). However, their expression for the variance $\sigma^2(\tilde{m})$ is quite involved. Chang, French and Thio (1971) have later derived an expression by considering the behaviour of operators with respect to the group $\sum U(N_i)$. This method gives a compact form for configuration centroids and variances in which each term is expressed in terms of physically significant quantities.

A natural extension of scalar (m) and configuration ($\tilde{m}$) averaging would be to deal with isospin (mT) and configuration-isospin ($\tilde{m}$T) averages. Specification of isospin is important because firstly, one achieves a finer decomposition of the space than before; and secondly, one can study nuclei with a given mass number A but having different number of neutrons (N) and protons (Z) such that N + Z = A.

The technical aspects of evaluating averages in the isospin subspace have been discussed by French (1969) and by Thio (1970). From the group theoretical point of view, as mentioned earlier, one is dealing here with the direct-product subgroup U(N/2) x U(2) of U(N) (c.f. Sec.1.4). As discussed in Sec.(1.2), the method consists in constructing an equivalent operator $\hat{H}$ for the Hamiltonian H which is a (0+1+2)-body operator. The equivalent operator $\hat{H}$ (valid only for calculating centroid energies) is written as a polynomial expression in terms of

the number operator n and isospin operator T^2.

$$\hat{H} = P_2(n) + P_o(n)\, T^2 \qquad (3.2.9)$$

The four unknown parameters in eq.(3.2.9) can be obtained in terms of averages of H ($\langle H \rangle^{mT}$) in the defining space which consists of the four representations (0,0), (1,1/2),(2,0) and (2,1). In terms of the single particle energies ϵ_{rr} and two-body matrix elements W_{rsrs}, the final expression for the centroid energy $E_c(mT)$ can be written as (Thio, 1970):

$$E_c(mT) = H_o + m\bar{\epsilon} + 1/4\,(W^o+3W^1)\binom{m}{2} + 1/2\,(W^1-W^o)\times \left\{T(T+1) - \frac{3}{4}m\right\} \qquad (3.2.10)$$

where $\bar{\epsilon} = \frac{1}{N}\sum_r N_r \epsilon_{rr}$ is the average single particle energy and

$$W^{\tau} = \frac{8}{N[N+2(-1)^{\tau}]} \sum_{\substack{rsJ \\ r\le s}} (2J+1)\, W^{J\tau}_{rsrs} \qquad (\tau = 0,1) \qquad (3.2.11)$$

Thio (1970) has derived expressions for the variance $\sigma^2(mT)$ in terms of the density operators and averages in the input space both for the 'elementary' and 'optimum' nets. For the optimum net, the input points and density operators Q(i,mT) are given in table (III.1).

Table (III.1)

Input points and density operators for isospin (mT) averaging

Input point i	Density operator Q(i,mT)
0,0	$\Big[(m-2)(m-N)\big\{(N+6)(N+1)m^2 - (N^3+7N^2-12)m + N(N^2-4)\big\} + 8m\big\{(N^2+3N-6)m - N(N^2+4N-6)\big\}T(T+1) + 16N(N^2-2)T(T+1) + 16(N-2)(N-3)T^2(T+1)^2\Big] / \big[2(N+2)N^2(N-2)\big]$
1,1/2	$\Big[(N+2)m(m-2)(N-m)\big\{(N+3)m - (N^2+2N-6)\big\} + 4m\big\{N(2N^2+9N-10) - 2m(N^2+4N-6)\big\}T(T+1) - 8(N+2)N(2N-3)T(T+1) - 16(N-2)(N-3)T^2(T+1)^2\Big] / \big[(N+2)N(N-2)(N-4)\big]$
2,0	$\big[(m+2)m - 4T(T+1)\big]\big[(N-m)(N-m+2) - 4T(T+1)\big] / \big[8N(N-2)\big]$

Table (III.1)(contd)

2,1	$\left[3(N+4)(N+2)m(m-2)(N-m)\right.$ $(N-m-2)-8(N+4)(5N-6)m$ $(N-m)T(T+1)+4(N+12)(N+2)$ $N(N-2)T(T+1)+16(3N^2-14N$ $\left.+24)T^2(T+1)^2\right] / \left[8(N+2)N(N-4)(N-6)\right]$
N,0	$\left[m(m+2)-4T(T+1)\right]\left[(m-2)\right.$ $\left\{(N+3)m-(N+2)N\right\}+4(N-3)$ $\left.T(T+1)\right] / \left[(N+2)N^2(N-2)\right]$
N-1, 1/2	$\left[m(m+2)-4T(T+1)\right]$ $\left[(N+2)(m-2)(N-m)\right.$ $\left.-4(N-2)T(T+1)\right] / \left[(N+2)N(N-2)(N-4)\right]$

Table (III.1)(contd)

$N/2, N/4$	$\Big[3(N+2)m(N-m)(N-m-2)$ $(m-2)-4(N^2+6N-12)m(N-m)$ $T(T+1)+8(N+2)N(N-2)T(T+1)$ $+16(N-2)(N-3)T^2(T+1)^2\Big] / \Big[(N+2)N^2(N-2)(N-4)\Big]$
$\frac{N-2}{2}, \frac{N-2}{4}$	$2\Big[3m(m-2)(N-m)\Big\{(3N+2)m-2$ $(N+2)(N-1)\Big\}+4m\Big\{N(2N^2$ $+11N-18)-m(3N^2+4N-12)\Big\}$ $T(T+1)-8N(N+2)(2N-3)$ $T(T+1)-16(N-2)(N-3)$ $T^2(T+1)^2\Big] / \Big[(N+2)N(N-2)(N-4)(N-6)\Big]$
$\frac{N+2}{2}, \frac{N-2}{4}$	$2\Big[m(m+2)-4T(T+1)\Big]$ $\Big[-3(m-2)(N-m)+4(N-3)$ $T(T+1)\Big] / \Big[(N+2)N(N-2)(N-4)\Big]$

Expressions for (mT) variances in the input space are given below:

(1) $\sigma^2(0,0) = 0$

(2) $\sigma^2(1, \frac{1}{2}) = (\overline{\epsilon^2} - \bar{\epsilon}^2) + \frac{1}{N} \sum_{\substack{r,s \\ r \neq s}} N_r (\epsilon_{rs})^2$

(3) $\sigma^2(2,0) = \frac{2(N+4)}{(N+2)} (\overline{\epsilon^2} - \bar{\epsilon}^2) - \frac{16}{N^2(N+2)} \sum_{\substack{r,s \\ r<s}} N_r N_s (\epsilon_{rr} - \epsilon_{ss})$

$$\times \left\{ (\epsilon_{rr} + \tilde{\epsilon}_{rr})^0 - (\epsilon_{ss} + \tilde{\epsilon}_{ss})^0 \right\} + \left\{ \overline{w^2}^0 - (w^0)^2 \right\}$$

$$+ \frac{1}{N} \sum_{\substack{r,s \\ r \neq s}} N_r \left\{ 2\epsilon_{rs}^2 + \frac{16}{(N+2)} \epsilon_{rs} \sum_u \epsilon^0(r,s;u) \right\}$$

(4) $\sigma^2(2,1) = \frac{2(N-4)}{(N-2)} (\overline{\epsilon^2} - \bar{\epsilon}^2) - \frac{16}{3N^2(N-2)} \sum_{\substack{r,s \\ r<s}} N_r N_s (\epsilon_{rr} - \epsilon_{ss})$

$$\times \left\{ (\epsilon_{rr} + \tilde{\epsilon}_{rr})^1 - (\epsilon_{ss} + \tilde{\epsilon}_{ss})^1 \right\} + \left\{ \overline{w^2}^1 - (w^1)^2 \right\}$$

$$+ \frac{1}{N} \sum_{\substack{r,s \\ r \neq s}} N_r \left\{ 2 \epsilon_{rs}^2 + \frac{16}{3(N-2)} \epsilon_{rs} \sum_u \epsilon^1(r,s;u) \right\}$$

(5) $\sigma^2(N,0) = 0$

(6) $\sigma^2(N-1,\frac{1}{2}) = (\overline{\tilde{\epsilon}^2} - \overline{\tilde{\epsilon}}^2) + \frac{1}{N} \sum_{\substack{rs \\ r \neq s}} N_r (\tilde{\epsilon}_{rs})^2$

(7) $\sigma^2(\frac{N}{2}, \frac{N}{4}) = 0$

(8) $\sigma^2(\frac{N-2}{2}, \frac{N-2}{4}) = (\overline{\tilde{\epsilon}^2(\alpha)} - \overline{\tilde{\epsilon}(\alpha)}^2) + \frac{1}{N} \sum_{\substack{rs \\ r \neq s}} N_r (\tilde{\epsilon}_{rs}(\alpha))^2$

(9) $\sigma^2(\frac{N+2}{2}, \frac{N-2}{4}) = (\overline{\tilde{\epsilon}^2(\beta)} - \overline{\tilde{\epsilon}(\beta)}^2) + \frac{1}{N} \sum_{\substack{rs \\ r \neq s}} N_r (\tilde{\epsilon}_{rs}(\beta))^2$

where the various average energies and variances are expressible in terms of the isospin average energies and induced splitting energies defined as follows:

$$\tilde{\epsilon}_{rs}(\alpha) = -\epsilon_{rs} - \frac{2}{3}\sum_{t}\epsilon^{1}(r,s;t)$$

$$\tilde{\epsilon}_{rs}(\beta) = +\ _{rs} + \sum_{t,\tau}[\tau]^{-1}\ \epsilon^{\tau}(rs;t)$$

$$W^{\tau} = \frac{8}{N(N+2(-1)^{\tau})}\sum_{\substack{rs;J \\ r\leq s}}[J]\ W^{J\tau}_{rsrs}$$

$$\overline{(W)^{2}}^{\tau} = \frac{8}{N(N+2(-1)^{\tau})}\sum_{\substack{rstu:J \\ r\leq s,\ t\leq u}}[J]\,(W^{J\tau}_{rstu})^{2}$$

$$\overline{x} = \frac{1}{N}\sum_{r} N_r\ X_{rr}$$

$$\overline{x^{2}} = \frac{1}{N}\sum_{r} N_r\,(X_{rr})^{2}$$

with $x = \epsilon, \tilde{\epsilon}$ or $\tilde{\epsilon}(\alpha), \tilde{\epsilon}(\beta)$ as appropriate.

Finally we write

$$\sigma^2(mT) = \sum_{i=1}^{q} Q(i,mT)\ \sigma^2(i) \qquad (3.2.12)$$

The dimensionality of an (mT) subspace is given by

$$d(mT) = \frac{2(2T+1)^2}{(N+2)} \binom{\Omega+1}{\frac{m}{2}-T} \binom{\Omega+1}{\frac{m}{2}+T+1} \qquad (3.2.13)$$

where $\Omega = N/2$.

It is obvious that (mT) formalism will not be convenient where protons and neutrons are filling different shell model orbits. One then needs to evaluate configuration-isospin averages where one considers the subgroup decomposition as:

$$U(N) \supset \sum_{i=1}^{l} U(N_{i/2}) \times U(2)$$

In this case one can proceed to determine the density operators for the propagation of traces of H and H^2 calculated in the defining space. However, complications arise because one is now dealing with a direct-product as well as a direct-sum subgroup decomposition. One way of avoiding this complexity is to use a p-n formalism in which the proton and neutron orbits are considered separately. The p-n formalism for evaluating (mT) averages is to proceed via an analogous chain of groups,

$$U(N) \supset \left\{ U(N/2)_p + U(N/2)_n \right\} \supset \left\{ \sum U(N_i/2)_p + \sum U(N_i/2)_n \right\}$$

In this case the forms derived for configuration averaging are applicable and isospin is finally recovered by the elementary subtraction procedure. Details of this method and expressions for the centroid energy $E_c(\tilde{m}T)$ and variance $\sigma^2(\tilde{m}T)$ have been worked out by Thio (1970).

Configuration-isospin averages are important for several reasons. An isospin subspace can be decomposed into several (mT) subspaces. Therefore, $(\tilde{m}T)$ distributions reveal finer details of the isospin subspace. As we shall see later, they enable us to calculate the occupancy of shell model orbits in the ground states of nuclei. They also tell us which configurations are significant in the low lying states of a nucleus, thereby providing a means of truncating large shell model matrices to a reasonable size.

We end this section on finer averaging with some remarks on SU(4)-isospin spin averages. We are interested (in the 2p-1f shell) in carrying out averages over states belonging to irreducible representations in the following chain decomposition:

$$U(40) \supset U(10) \times U(4) \supset U(10) \times (SU_T(2) \times SU_S(2))$$

The spectral averages are evaluated for states having fixed SU(4) symmetry $\underset{\sim}{f}$, isospin T and spin S by using the expressions derived in Chapter II.

Having discussed distributions, one might be tempted to conclude that the finer the averages the better would be the distribution curves. Usually it is true that by doing a finer averaging one takes into account more information of the vector space and should therefore expect better results than otherwise. However, it should be kept in mind that the larger the dimensionality, the closer the distribution might be to a gaussian, in which case a very fine averaging might not be a wise thing to do. It therefore appears that for large spaces one should proceed to finer averages keeping in mind that the dimensionality of the subspaces is not too small. Also for the lowest and the highest lying irreps it may often be necessary to evaluate more than 2 moments. We have not done this.

In the next section we evaluate the dimensionalities, centroids and widths for the various distributions discussed here. These are then used to study properties of nuclei in 2p-1f shell.

3.3 Spectral moments in the 2p-1f shell

The spectral moments in the 2p-1f shell have been calculated using the Kuo-Brown interaction (KB) as modified by McGrory, Wildenthal and Halbert (1970). In this modified interaction labelled MWH2, some of the T=1 matrix elements of KB interaction have been replaced by the ones obtained by McGrory et al to optimize agreement with the experimental spectra of Ca isotopes. In MWH2, the centre of gravity of interaction between two nucleons in the $1f_{7/2}$ states has been lowered and the centroid energy of interaction of a nucleon in $1f_{7/2}$ state with a nucleon in $2p_{3/2}$, $2p_{1/2}$ and $1f_{5/2}$ states has been raised as compared to KB-interaction. The Ca^{41} single particle energies have been used. These are $\epsilon_{f_{7/2}} = -8.36$; $\epsilon_{f_{5/2}} = -1.86$; $\epsilon_{p_{3/2}} = -6.26$ and $\epsilon_{p_{1/2}} = -4.46$.

In table (III.2) we list the dimensionalities d(m), centroid energies $E_c(m)$ and width $\sigma(m)$ obtained by Scalar averaging for $4 \leq m \leq 36$ particles. This table gives an idea of the dimensionalities involved.

In table (III.3) we give the Isospin (mT) averages using MWH2 interaction for m = 4, 13, 20, 27 and 36 nucleuns. We observe that the relative energy separation between two adjacent centroids increases linearly with isospin and the width decreases

Table (III.2)

The dimensionality d(m), centroid energy $E_c(m)$ and width $\sigma(m)$ obtained by scalar averaging for $4 \leq m \leq 36$ particles in the fp-shell. MWH2 interaction with Ca^{41} single particle energies was used. All energies are in MeV.

m	d(m)	$E_c(m)$	$\sigma(m)$
4	91390	-24.37	6.06
5	658008	-31.29	6.92
6	3838380	-38.54	7.72
7	18643560	-46.11	8.45
8	76904685	-54.02	9.12
9	273438880	-62.25	9.74
10	847660528	-70.81	10.30
11	2311801440	-79.70	10.82
12	5586853480	-88.92	11.28
13	12033222880	-98.47	11.69
14	23206929840	-108.35	12.05
15	40225345056	-118.56	12.36
16	62852101650	-129.10	12.63
17	88732378800	-139.96	12.84
18	113380261800	-151.16	13.01
19	131282408400	-162.69	13.13
20	137846528820	-174.54	13.19
21	131282408400	-186.72	13.21
22	113380261800	-199.23	13.18
23	88732378800	-212.08	13.11
24	62852101650	-225.25	12.98
25	40225345056	-238.75	12.80
26	23206929840	-252.58	12.57

Table (III.2)(contd)

m	d(m)	E_c(m)	σ(m)
27	12033222880	-266.73	12.29
28	5586853480	-281.22	11.95
29	2311801440	-296.03	11.56
30	847660528	-311.18	11.12
31	273438880	-326.66	10.62
32	76904685	-342.46	10.06
33	18643560	-358.59	9.43
34	3838380	-375.05	8.73
35	658008	-391.85	7.95
36	91390	-408.97	7.08

Table (III.3)

Dimensionalities, centroid energies and widths for (mT) averaging using MWH2 interaction. The fifth and seventh columns refer to the values obtained for corresponding number of holes.

m	T	d(mT)	E_c (mT)		σ (mT)	
4	0	13300	-26.04	-410.63	6.32	7.36
	1	53865	-24.84	-409.42	6.04	7.08
	2	24225	-22.43	-407.02	5.45	6.49
13	1/2	2103272640	-102.08	-270.34	11.75	12.37
	3/2	4557090720	-100.28	-268.54	11.47	12.09
	5/2	3618866160	- 97.27	-265.53	11.01	11.63
	7/2	1429675520	-93.06	-261.32	10.35	10.97
	9/2	293930000	-87.64	-255.90	9.49	10.10
	11/2	29302560	-81.03	-249.29	8.40	9.00
	13/2	1085280	-73.21	-241.46	7.04	7.63
20	0	5924217936	-179.17		13.29	
	1	37026362100	-177.96		13.11	
	2	49295452500	-175.56		12.76	
	3	31549089600	-171.95		12.24	
	4	11357672256	-167.13		11.53	
	5	2385899901	-161.12		10.64	
	6	288267525	-153.90		9.54	
	7	18952500	-145.48		8.22	
	8	606900	-135.85		6.61	
	9	7581	-125.02		4.55	
	10	21	-112.99		0	

with increasing isospin.

Configuration centroids $E_c(\tilde{m})$ and widths $\sigma(\tilde{m})$ are listed in table (III.4). The centroid span for 4 particles is about 27 MeV and the widths have more or less a constant value of 3.0 MeV.

Configuration-isospin averages $(\tilde{m}T)$ for 4 particles are given in table (III.5). The general features of $(\tilde{m}T)$ distributions are similar to (mT) distributions. The centroid span for all values of isospin is nearly equal and for 4 particles is about 27 MeV. The widths decrease with increasing isospin and for a given value of T are nearly constant. For a given configuration the centroid energies increase and widths decrease with isospin.

We have seen that for a given particle number m, the widths for various configurations $\tilde{m}$ are more or less constant. This feature is also observed for $(\tilde{m}T)$ averages where for a fixed m and T, the widths are nearly equal. Again the widths corresponding to different irreps of SU(4) for fixed m have the same value (Parikh, 1973). This constancy of widths is a striking freature for the above mentioned distributions. There is at present no clear understanding of this curious fact which needs to be investigated in detail.

Table (III.4)

Configuration centroids and widths for 4 particles in the fp-shell. Configurations are ordered as $f_{7/2}, f_{5/2}, p_{3/2}, p_{1/2}$.

$\tilde{m}$	$d(\tilde{m})$	$E_c(\tilde{m})$	$\sigma(\tilde{m})$
0004	1	-21.81	3.26
0013	32	-23.99	2.82
0022	168	-26.12	2.72
0031	224	-28.19	2.70
0040	70	-30.19	2.75
0103	48	-17.65	2.88
0112	576	-19.81	2.69
0121	1344	-21.91	2.64
0130	672	-23.95	2.67
0202	396	-14.13	2.76
0211	2112	-16.27	2.67
0220	1848	-18.35	2.69
0301	880	-11.27	2.77
0310	1760	-13.38	2.77
0400	495	-9.06	2.94
1003	64	-24.28	2.90
1012	768	-26.24	2.75
1021	1792	-28.13	2.71
1030	896	-29.96	2.72
1102	1152	-20.84	2.76
1111	6144	-22.77	2.70
1120	5376	-24.64	2.71
1201	4224	-18.06	2.76
1210	8448	-19.96	2.77

Table (III.4)(contd)

$\tilde{m}$	$d(\tilde{m})$	$E_c(\tilde{m})$	$\sigma(\tilde{m})$
1300	3520	-15.92	2.91
2002	720	-27.53	2.80
2011	3840	-29.25	2.76
2020	3360	-30.91	2.77
2101	5760	-24.82	2.78
2110	11520	-26.52	2.80
2200	7920	-22.75	2.92
3001	2240	-31.55	2.81
3010	4480	-33.04	2.85
3100	6720	-29.56	2.94
4000	1820	-36.35	2.95

Table (III.5)

Spectral averages for configuration-isospin averaging for 4 particles in fp-shell

$\tilde{m}$	T	d($\tilde{m}$)	$E_c(\tilde{m}T)$	$\sigma^2(\tilde{m}T)$
4000	0	336	-37.50	9.48
3100		1008	-31.38	9.47
3010		672	-34.50	8.80
3001		336	-33.01	8.85
2200		1176	-24.59	9.42
2110		1536	-28.29	8.89
2101		768	-26.56	8.85
2020		528	-32.33	8.28
2011		512	-31.06	8.45
2002		136	-28.73	7.80
1300		560	-17.71	9.07
1210		1152	-21.69	8.56
1201		576	-19.74	8.39
1120		768	-26.31	8.45
1111		768	-24.63	8.70
1102		192	-22.38	8.03
1030		160	-31.26	8.42
1021		256	-29.89	8.42
1012		128	-27.75	8.22
1003		16	-25.16	9.83
0400		105	-10.14	8.71
0310		280	-14.61	8.35
0301		140	-12.43	8.08
0220		300	-19.61	8.02

Table (III.5)(contd)

$\tilde{m}$	T	$d(\tilde{m}T)$	$E_c(\tilde{m}T)$	$\sigma^2(\tilde{m}T)$
0211	0	288	-17.82	8.19
0202		78	-14.27	7.03
0130		120	-25.05	8.37
0121		192	-23.44	8.18
0112		96	-21.09	7.90
0103		12	-18.26	9.16
0040		20	-31.02	8.71
0031		40	-29.85	9.21
0022		36	-27.41	8.26
0013		8	-25.25	11.14
0004		1	-21.81	10.61
4000	1	378	-36.54	8.09
3100		1344	-29.98	7.59
3010		896	-33.40	7.38
3001		448	-31.92	7.21
2200		1548	-23.31	7.24
2110		2208	-27.08	6.99
2101		1104	-25.38	6.90
2020		664	-31.29	6.95
2011		736	-29.78	6.73
2002		148	-27.75	7.08
1300		720	-16.27	7.23
1210		1632	-20.48	6.90
1201		816	-18.56	6.86
1120		1056	-25.06	6.66
1111		1152	-23.39	6.46

Table (III.5)(contd)

$\tilde{m}$	T	$d(\tilde{m}T)$	$E_c(\tilde{m}T)$	$\sigma^2(\tilde{m}T)$
1102	1	240	-21.04	6.65
1030		192	-30.13	6.69
1021		352	-28.64	6.40
1012		160	-26.54	6.53
1003		16	-23.99	7.61
0400		105	-9.15	7.87
0310		360	-13.67	7.06
0301		180	-11.55	7.08
0220		366	-18.65	6.78
0211		408	-16.69	6.55
0202		81	-14.27	7.03
0130		144	-24.12	6.64
0121		264	-22.38	6.41
0112		120	-20.09	6.49
0103		12	-17.44	7.85
0040		15	-30.06	6.73
0031		43	-28.36	6.03
0022		34	-26.38	6.21
0013		8	-23.58	6.19
4000	2	70	-34.63	5.75
3100		336	-27.46	3.91
3010		224	-31.29	4.80
3001		112	-29.81	4.26
2200		420	-20.49	3.27
2110		672	-24.57	3.33
2101		336	-22.91	3.12
2020		168	-29.09	4.14

Table (III.5)(contd)

$\tilde{m}$	T	$d(\tilde{m}T)$	$E_c(\tilde{m}T)$	$\sigma^2(\tilde{m}T)$
2011	2	224	-27.36	3.31
2002		28	-25.65	5.25
1300		160	-13.72	4.02
1210		480	-18.06	3.26
1201		240	-15.21	3.37
1120		288	-22.83	3.24
1111		384	-20.92	2.69
1102		48	-19.02	4.91
1030		32	-28.05	3.62
1021		96	-26.06	2.34
1012		32	-24.10	3.40
0400		15	-7.15	6.64
0310		80	-11.74	4.87
0301		40	-9.70	5.32
0220		90	-16.77	4.00
0211		120	-14.67	3.69
0202		15	-12.59	6.32
0130		24	-22.24	3.54
0121		72	-20.07	2.47
0112		24	-17.92	3.84
0040		1	-28.14	3.82
0031		8	-25.91	1.84
0022		6	-23.69	2.39

However, the near constancy of widths is an important requirement for stability of Gaussians under addition. The representations with small dimensionalities are situated at the ends and those having larger dimensionalities in the centre. Since the widths are nearly the same, addition of these distributions leads to a gaussian whose width is almost equal to the width of an individual representation. This has been shown by French and Ratcliff (1971) for configuration distributions, by Parikh (1973) for SU(4) distributions and by Haq and Parikh (1974) for SU(4)-T distributions.

Next we consider averages of H over states defined by SU(4) symmetry $\underset{\sim}{f}$, isospin T and spin S. The centroid energies $E_c(m\underset{\sim}{f}TS)$ are determined using eq.(2.5.6) and as before the equivalent width is calculated using (mT) widths. Input centroids and variances are shown in table (III.6).

Having determined $E_c(m\underset{\sim}{f}TS)$ and $\sigma^2(m\underset{\sim}{f}TS)$ we average over the spin quantum number S to obtain $(\underset{\sim}{f}T)$ averages denoted by $E_c(m\underset{\sim}{f}T)$ and $\sigma^2(m\underset{\sim}{f}T)$. Results for 4 particles (holes) are presented in table (III.7) and for 20 particles in table (III.8).

In the next section the spectral moments are used to study structure of nuclei.

Table (III.6)

Input centroids and widths for (SU(4)TS) averaging using MWH2 interaction with Ca^{41} single particle energies.

m	$\tilde{f}$	T	S	For particles $E_c(m\tilde{f}TS)$	For particles $\sigma(m\tilde{f}TS)$	For holes $E_c(m\tilde{f}TS)$	For holes $\sigma(m\tilde{f}TS)$
1	[1]	1/2	1/2	-5.6	2.73	-462.30	3.43
2	[2]	0	0	-10.29	4.13	-442.96	5.05
		1	1	-10.79	3.83	-443.46	4.74
	[11]	0	1	-12.98	4.13	-445.65	5.05
		1	0	-12.21	3.83	-444.88	4.74

Table (III.7)

SU(4)-T moments for 4 particles (holes) in 2p-1f shell. All energies are in MeV.

			Particles		Holes	
m	f	T	$E_c(mfT)$	$\sigma(mfT)$	$E_c(mfT)$	$\sigma(mfT)$
4	[4000]	0	-18.45	5.95	-403.05	7.04
	[3100]		-23.21	5.95	-407.80	7.04
	[2200]		-26.21	5.99	-410.82	7.08
	[2110]		-27.32	5.95	-411.93	7.04
	[1111]		-30.80	5.95	-415.41	7.04
	[4000]	1	-18.96	5.73	-403.55	6.81
	[3100]		-23.71	5.77	-408.31	6.85
	[2200]		-25.13	5.73	-409.73	6.81
	[2110]		-27.03	5.73	-411.63	6.81
	[4000]	2	-19.96	5.27	-404.55	6.34
	[3100]		-22.80	5.27	-407.40	6.34
	[2200]		-24.22	5.27	-408.83	6.34

Table (III.8)

SU(4)-Isospin averages for 20 particles in 2p-1f shell. MWH2 interaction was used and energies are in MeV. Centroids and widths of only those irreps which are dominant in the ground state region are given.

$\underset{\sim}{f}$	T	$E_c(\underset{\sim}{mfT})$	$\sigma(\underset{\sim}{mfT})$
[5555]	0	-192.32	11.82
[6554]		-188.84	11.82
[6644]		-187.74	11.84
[7544]		-184.72	11.82
[6653]		-184.72	11.82
[7553]		-183.62	11.84
[7643]		-182.52	11.89
[6554]	1	-188.55	11.69
[6644]		-186.65	11.68
[7544]		-185.23	11.71
[6653]		-185.23	11.71
[7553]		-183.11	11.70
[7643]		-181.91	11.76
[6644]	2	-185.74	11.41
[7544]		-184.32	11.41
[6653]		-184.32	11.41
[7553]		-182.90	11.41
[7643]		-180.69	11.43
[8444]		-181.48	11.41
[6662]		-181.48	11.41

Table 'III.8)(contd)

$\underset{\sim}{f}$	T	$E_c(\underset{\sim}{mfT})$	$\sigma(\underset{\sim}{mfT})$
[8543]	2	-178.55	11.48
[7652]		-178.55	11.48
[7643]	3	-179.65	11.00
[8633]		-174.27	11.02
[8543]		-177.52	11.02
[7652]		-177.52	11.02
[8552]		-175.39	11.09
[7742]		-174.27	11.02
[8642]		-172.85	11.07
[7733]	4	-174.52	10.44
[8633]		-173.10	10.44
[7742]		-173.10	10.44
[8642]		-171.68	10.46
[8732]		-167.40	10.46
[9533]		-170.26	10.44
[7751]		-170.26	10.44
[9542]		-168.84	10.52
[8651]		-168.84	10.52
[8822]	6	-158.64	8.84
[8840]		-154.38	8.84
[10622]		-154.38	8.84
[8831]		-157.22	8.84
[9722]		-157.22	8.84
[9731]		-155.80	8.87

Table (III.8)
(contd)

$\underset{\sim}{f}$	T	$E_c(\underset{\sim}{mfT})$	$\sigma(\underset{\sim}{mfT})$
[10631]	6	-152.96	8.93
[9740]		-152.96	8.93
[9911]	8	-138.11	6.38
[10811]		-136.70	6.38
[9920]		-136.70	6.38
[10820]		-135.28	6.42
[101000]	10	-112.99	0.0

3.4 Application of Spectral Moments

In this section we use the spectral moments obtained for various distributions to study properties of nuclei in the ground state region. Binding energies and low energy spectra of nuclei are determined using Ratcliff's (1971) procedure.

Configuration-isospin distributions are used to calculate fractional occupancy of shell model orbits in the ground states of nuclei. Relative contributions of various configurations are evaluated in the ground state region. The ($\underset{\sim}{fT}$) distributions are used to determine mixing of SU(4) symmetry.

3.4.1 Ground state energies and spectra

The estimate of binding energy of a nucleus from a spectral distribution was described in detail in section (1.3). Essentially, the ground state is located at an energy E_g where the distribution function F(mE) defined in eq.(1.3.3) is equal to half the degeneracy ((2J+1)(2T+1)) of the ground state. Similarly, one locates excited states of nuclei. However, for the determination of the energy spectra one needs to know the ordering (J-sequence) of the spectrum since our distributions do not correspond to a fixed angular momentum.

As mentioned earlier it is advisable to determine the ground state energy by first locating a low lying excited state

and then using the experimental energy to fix the ground state. This shows a substantial improvement in the accuracy which is not surprising because in doing so one moves away from the extreme tail of the distribution to a region where better accuracy is expected. The accuracy with which the distribution method predicts binding energies of nuclei in comparatively smaller spaces (ds-shell (Ratcliff, 1971) and three orbit fp-shell (Chang, French and Thio, 1971)) has been very good. It is expected that even in such large spaces we should be able to do equally well.

Table (III.9) shows the binding energies of nuclei as obtained by scalar and configuration moments. For a given mass number A we have to restrict to nuclei with lowest isospin, since T in this sort of averaging is not specified. A comparison with empirical energies after making Coulomb correction is made. The empirical energies were taken from the calculations of Garvey, Gerace, Jaffe and Talmi and Kelson (1969) and Coulomb correction was made by making use of the Coulomb displacement energies tabulated by Janecke (1969).

It is expected that the binding energy determined using distributions should be lower than either shell model or empirical energies. This is due to the fact that the ground state is located in the tail of the distribution where uncertainties

Table (III.9)

Binding energies of nuclei as determined from Scalar and configuration moments compared with empirical energies (B.E). Energies are in MeV with respect to Ca^{40}.

Nucleus	Scalar	Configuration	B.E.	Nucleus	Scalar	B.E.
Ti^{44}	51.02	46.56	49.16	Ni^{57}	220.93	219.16
Ti^{45}	58.42	56.57	58.60	Cu^{58}	237.86	231.70
V^{47}	85.72	82.80	84.65	Cu^{59}	246.45	244.73
Cr^{48}	103.43	99.79	101.16	Zn^{60}	264.95	259.32
Cr^{49}	113.72	110.55	111.45	Zn^{61}	273.16	269.70
Mn^{51}	142.88	137.58	137.99	Ga^{63}	297.03	295.07
Fe^{52}	158.20	154.44	154.22	Ge^{64}	312.70	309.85
Fe^{53}	169.35	165.58	164.97	Ge^{65}	320.17	320.20
Co^{55}	196.26	193.34	192.47	Se^{68}	357.45	360.44
Ni^{56}	212.32	209.42	208.66	Kr^{72}	399.63	419.43

exist (c.f. Sec.1.2.4).

In the first half of the shell, the maximum difference between energies obtained from scalar moments and empirical energies is about 5 MeV. In the second half, differences are rather large. Reasons for this would be discussed later. Energies predicted by configuration moments are closer to empirical energies which is understandable since one has done a finer averaging of the space.

In table (III.10) binding energies of nuclei in the first half of 2p-1f shell as determined by (mT), ($\tilde{m}$T) and ($\underset{\sim}{f}$T) distributions are compared with microscopic calculations like shell model (McGrory et al, 1970), deformed configuration mixing (Dhar et al, 1975) and empirical energies. For the range of nuclei from mass number A = 44 to A = 60, agreement with empirical energies is amazingly good. Except for those nuclei where an excited reference state was not available due to lack of data, agreement is within 3 MeV. All other cases differ by about 5 MeV. This makes it evident that the accuracy with which one can determine ground state energies of nuclei using distributions is indeed quite good. A comparative study of energies as obtained by various distributions shows that finer averaging (mT→$\tilde{m}$T) significantly improves the results.

A crucial test of the accuracy of the method is that it should be able to determine the binding energies of Ca

Table (III.10)

und state energies of nuclei in first half of 2p-1f shell as predicted Isospin, configuration-isospin and SU(4)-isospin distributions. In cols. nd 9 the binding energies as obtained from microscopic calculations empirical binding energies are given. MWH2 interaction with Ca^{41} single ticle energies was used. Energies are in Mev with respect to Ca^{40}. An erix denotes lack of excited reference state. (†) denotes spin is assu-. (a) refers to configuration mixing HF calculations and (b) refers to l model calculations.

Nucleus	J_o	T_o	mT	$\tilde{m}T$	$\underset{\sim}{f}T$	MC	B.E
Ti	0	0	51.05	45.42	47.98	45.44[a]	49.16
*Sc	2	1	46.78	43.62	46.56	-	42.76
Ca	0	2	40.10	39.12	40.11	38.56[b]	39.11
Ti	7/2	1/2	61.09	55.90	58.59	55.09[a]	58.60
*Sc	7/2	3/2	57.14	54.39	56.75	-	53.96
Ca	7/2	5/2	48.95	47.52	47.62	46.27[b]	46.41
V	0	1	76.83	72.26	76.03	-	71.49
Ti	0	1	73.68	70.53	73.17	66.84[a]	71.49
*Sc	4	2	67.33	64.95	66.75	-	62.95
Ca	0	3	60.94	58.93	58.38	56.70[b]	56.79
V	3/2	1/2	86.26	82.85	85.54	78.88[a]	84.65
Ti	5/2	3/2	82.57	79.88	81.86	75.69[a]	80.66
*Sc	7/2	5/2	77.21	75.17	76.44	69.07[a]	73.51
Ca	7/2	7/2	66.30	66.18	65.95	63.96[b]	63.86
Cr	0	0	102.92	98.99	101.88	93.38[a]	101.16
V	4	1	98.76	96.36	97.73	88.20[a]	94.94
Ti	0	2	95.91	93.08	94.86	-	92.34
Sc	6	3	82.97	81.56	82.18	-	81.71
Ca	0	4	76.78	76.72	75.86	73.90[b]	73.84

Table (III.10)(contd)

Nucleus	J_o	T_o	mT	$\tilde{m}T$	$\tilde{f}T$	MC	B.E
Cr	5/2	1/2	114.31	110.55	112.99	103.50[a]	111.45
V	7/2	3/2	109.90	106.57	108.58	99.37[a]	106.59
Ti	7/2	5/2	104.86	102.20	103.59	94.16[a]	100.53
*Sc	7/2	7/2	94.46	92.96	93.39	-	91.68
Ca	3/2	9/2	81.80	82.19	80.25	78.69[b]	79.04
*Mn	0	1	131.96	127.29	129.96	-	124.28
Cr	0	1	129.25	125.15	127.45	114.95[a]	124.28
V	6	2	120.30	117.07	118.66	107.90[a]	115.93
Ti	0	3	115.08	112.55	113.24	104.44[a]	111.65
Sc	5	4	100.10	98.92	99.06	-	98.04
Ca	0	6	87.54	88.10	86.95	85.87[b]	86,16
Mn	5/2	1/2	141.47	137.49	139.53	127.80[a]	137.99
Cr	7/2	3/2	138.70	134.86	136.72	-	133.54
V	7/2	5/2	131.04	127.78	129.23	118.10[a]	126.97
Ti	3/2	7/2	123.13	120.51	121.49	110.34[a]	117.92
Sc	7/2	9/2	108.03	106.90	106.93	-	105.07
*Ca	5/2	11/2	92.23	92.88	91.72	-	89.47
Fe	0	0	157.43	153.18	155.15	141.68[a]	154.22
Mn	6	1	152.82	148.88	150.65	-	148.41
Cr	0	2	150.74	146.25	148.53	134.58[a]	145.63
*V	3	3	141.86	138.23	139.71	-	134.29
Ti	0	4	131.10	127.37	129.52	117.92[a]	126.02
Sc	6	5	113.99	112.87	112.86	-	109.43
*Ca	0	6	99.87	100.61	99.33	-	95.13

Table (III.10)(contd)

m	Nucleus	J_o	T_o	mT	$\tilde{m}T$	$\tilde{f}T$	MC	BE
13	*Fe	7/2	1/2	169.94	165.49	167.34	151.62[a]	164.97
	Mn	7/2	3/2	165.27	161.14	162.83	147.91[a]	160.93
	Cr	3/2	5/2	159.29	155.25	156.88	-	153.39
	*V	7/2	7/2	149.71	146.04	147.48	-	142.82
	†Ti	5/2	9/2	136.97	134.26	135.19	-	130.81
	†Sc	7/2	11/2	120.13	119.00	119.02	-	115.53
	*Ca	1/2	13/2	102.83	103.60	102.38	-	97.48
14	Co	0	1	180.84	176.35	177.71	-	178.54
	Fe	0	1	183.82	179.12	180.92	163.39[a]	178.54
	*Mn	3	2	177.31	172.62	174.46	-	169.74
	Cr	0	3	168.77	164.45	166.07	-	162.97
	*Ti	0	5	145.88	142.18	143.99	-	137.95
	†Sc	4	6	124.46	123.40	123.47	-	118.92
	*Ca	0	7	107.24	107.92	106.86	-	102.36
15	Co	7/2	1/2	194.71	192.15	193.80	-	192.47
	Fe	3/2	3/2	192.56	187.85	189.60	-	187.92
	Mn	5/2	5/2	184.36	179.95	181.63	166.92[a]	179.89
	*Cr	3/2	7/2	177.74	173.10	174.98	-	169.38
16	Ni	0	0	212.14	207.29	208.99	192.01[a]	208.66
	Co	4	1	206.21	201.41	203.13	-	202.72
	Fe	0	2	203.50	198.63	201.44	-	198.93
	Mn	3	3	192.67	188.05	189.84	-	187.17
	Cr	0	4	184.69	180.18	182.04	-	177.96
	*Ti	0	6	155.10	152.08	153.54	-	148.26
	*Sc	5	7	130.92	133.27	130.26	-	126.95
	*Ca	0	8	111.77	112.03	111.59	-	108.41

Table (III.10)(contd)

m	Nucleus	J_o	T_o	mT	$f_{\sim}T$	BE
17	Ni	3/2	1/2	221.95	218.68	219.16
	Co	7/2	3/2	213.02	214.82	213.99
	Fe	1/2	5/2	210.99	205.85	206.47
	*Mn	5/2	7/2	203.06	200.05	196.00
	†Cr	1/2	9/2	192.06	189.38	183.05
	†Ti	3/2	13/2	156.19	154.98	151.10
	*Sc	7/2	15/2	133.71	133.23	131.55
	Ca	1/2	17/2	111.95	111.87	109.10
18	*Cu	1	0	236.81	233.32	231.70
	Ni	0	1	234.47	231.14	231.50
	Co	2	2	227.07	223.80	222.60
	Fe	0	3	221.71	216.84	216.33
	*Cr	0	5	193.47	195.92	190.70
	*Ti	0	7	161.02	160.00	156.77
	*Ca	0	9	113.46	113.43	112.70
19	Cu	3/2	1/2	246.84	242.78	244.73
	Ni	3/2	3/2	242.90	238.62	240.55
	Co	7/2	5/2	237.04	232.26	232.95
	Fe	3/2	7/2	226.73	221.44	222.93

Table (III.10) (contd)

m	Nucleus	J_o	T_o	mT	$\tilde{m}T$	$f\tilde{T}$	BE
20	*Zn	0	0	264.03	257.80	260.33	259.32
	*Cu	2	1	259.95	254.00	256.35	254.73
	Ni	0	2	255.57	249.80	253.40	251.84
	*Co	5	3	244.97	239.57	241.73	240.49
	*Fe	0	4	238.03	232.32	234.84	232.01
	*Cr	0	6	206.09	201.48	204.20	202.10
	*Ti	0	8	163.54	159.73	163.13	163.95
	*Ca	0	10	112.99	112.99	112.99	115.77

isotopes very accurately since the interaction used has been fitted to do so. Comparison with shell model calculations shows that we are indeed able to do very well. Further comments on the accuracy of distribution methods will be made later.

Results of our calculations in the second half of 2p-1f shell are presented in table (III.11).

We observe that for nuclei upto $A = 64$, agreement between binding energies is good, deviations being not more than 5 MeV. Beyond this there are large disagreements. This brings us to the question regarding the accuracy of the method and the inadequacy of the interaction. Since results in the first half of the shell indicate that the method determines ground state energies of nuclei very accurately, disagreements here could be attributed to the inadequacy of the interaction. Strong evidence of this inadequacy is the fact that whereas in general the ground state obtained using distributions should be lower than the actual energy, for nuclei with $A > 64$ we find the reverse trend. This suggests that MWH2 interaction is underbinding these nuclei. The effect of underbinding is also seen in table (III.9) where agreement for nuclei beyond Cu^{59} is good. This is due to the fact that the two effects - overbinding of the method and underbinding of interaction cancel each other. Agreement in energy may sometimes be misleading and comparisons

Table (III.11)

Ground state energies of nuclei in the second half of 2p-1f shell. Energies are in MeV with respect to Ca^{40}.

m	Nucleus	J_o	T_o	mT	fT	B.E.
21	Zn	3/2	1/2	273.69	270.46	269.70
	Cu	3/2	3/2	268.42	265.35	266.39
	Ni	3/2	5/2	260.61	257.41	259.76
	Co	7/2	7/2	251.64	249.05	249.90
	*Fe	3/2	9/2	240.75	238.20	237.68
22	Zn	0	1	284.45	281.19	282.37
	*Cu	1	2	280.18	276.84	275.33
	Ni	0	3	270.07	268.23	270.18
	†Co	4	4	258.05	255.43	256.57
23	*Ga	3/2	1/2	297.59	294.41	295.07
	*Zn	3/2	3/2	294.43	291.21	291.54
	Cu	3/2	5/2	286.43	283.48	285.98
	*Ni	1/2	7/2	278.89	275.98	277.10
	†Co	5/2	9/2	263.68	261.37	262.25
24	*Ge	0	0	311.85	308.72	309.85
	*Ga	0	1	310.98	307.72	305.37
	Zn	0	2	302.98	300.02	303.32
	*Cu	1	3	295.15	292.29	294.02
	Ni	0	4	283.21	280.82	286.90
	*Co	1	5	268.79	266.83	271.16

Table (III.11)
(contd)

m	Nucleus	J_o	T_o	mT	f_T	B.E
25	*Ge	3/2	1/2	320.77	317.93	320.20
	*Ga	3/2	3/2	317.57	314.69	317.21
	Zn	5/2	5/2	307.56	305.02	311.42
	Cu	3/2	7/2	298.69	296.40	303.88
	Ni	5/2	9/2	284.48	282.63	292.86
26	*Ge	0	1	333.75	330.92	333.08
	*Ga	0	2	328.52	325.74	326.35
	Zn	0	3	317.16	314.81	322.33
	*Cu	1	4	305.94	303.85	310.89
	*Ni	0	5	292.97	291.31	301.68
27	*Ga	3/2	5/2	332.99	330.69	337.58
	Zn	5/2	7/2	319.70	317.87	329.86
	*Cu	3/2	9/2	309.14	307.57	320.04
28	*Se	0	0	356.81	354.63	360.44
	*Ge	0	2	350.48	348.24	354.73
	*Ga	1	3	339.60	337.66	345.86
	Zn	0	4	327.00	326.58	339.76
	Cu	1	5	312.15	311.09	326.20
29	*Ge	5/2	5/2	353.75	352.05	363.37
	Ga	3/2	7/2	342.53	341.12	356.32
	Zn	1/2	9/2	331.26	330.17	346.28
	*Cu	3/2	11/2	312.43	312.03	334.67

Table (III.11)

(contd)

m	Nucleus	J_o	T_o	mT	$\tilde{f}T$	B.E.
30	*Se	0	1	385.24	384.06	–
31	*Se	5/2	3/2	381.95	380.73	394.39
	As	5/2	5/2	372.82	371.47	389.41
32	Kr	0	0	399.23	398.27	419.43
	Se	0	2	392.25	391.21	406.79
33	*As	3/2	7/2	383.52	383.20	408.57
34	*Kr	0	1	418.21	417.69	444.14
	*Br	0	2	412.10	411.56	430.38
	*Se	0	3	401.81	401.52	427.41
35	*Br	3/2	5/2	423.06	414.28	442.36
36	*Sr	0	0	439.77	439.71	-
	*Kr	0	2	431.12	431.01	467.18

should be made keeping in mind the adequacy (inadequacy) of the interaction. Unfortunately no microscopic calculations for these nuclei with this interaction are available.

Usually calculations in the second half of 2p-1f shell are done by assuming Ni^{56} as core and allowing the valence nucleons to occupy the three orbits $p_{3/2}$, $p_{1/2}$ and $f_{5/2}$. Chang, French and Thio (1971) have studied nuclei in this region using distribution methods. All nuclei are treated as $(p_{3/2}, p_{1/2} f_{5/2})^m$. They have used Kuo-Brown (KB) interaction for which shell model calculations by Wong (1970) are available. They find that for all the nuclei considered the average error in locating the ground state is about 700 keV, the largest errors arising for cases where no excited reference state was available. In table (III.12) we compare the binding energies of nuclei obtained by ($\tilde{fT}$) distributions with their calculations.

We observe that although there is reasonable agreement, energies calculated using MWH2 are higher than those obtained from KB and the difference increases with increasing mass number. This once again suggests that MWH2 is underbinding nuclei in the upper 2p-1f shell.

In the end we would like to say that the overall agreement between binding energies as predicted by distribution method and empirical energies is good. For some cases and

Table (III.12)

Ground state energies obtained from SU(4) isospin distributions in the full (fp)-shell using MWH2 interaction compared with configuration-isospin results in three orbit (fp)-shell using KB interaction. Energies are in MeV with respect to Ni^{56}.

Nucleus	$\tilde{f}T$	$\tilde{m}T$
Ni^{60}	44.4	42.8
Cu^{60}	47.4	45.9
Ni^{61}	48.4	50.9
Cu^{61}	56.4	57.2
Zn^{61}	61.5	60.3
Ni^{62}	59.2	61.8
Cu^{62}	67.8	66.3
Zn^{62}	72.2	73.2
Ni^{63}	67.0	68.8
Cu^{63}	74.5	77.5
Zn^{63}	82.2	83.7
Ga^{63}	85.4	86.9
Ni^{64}	71.8	78.6
Cu^{64}	83.2	86.0
Zn^{64}	91.0	94.8
Ga^{64}	98.7	98.9

especially for the second half of the shell there are disagreements between the two. A possible source of this is that the MWH2 interaction is underbinding $p_{3/2}$, $f_{5/2}$ and $p_{1/2}$ orbits by about an MeV or a little more. Also it may be necessary to include $g_{9/2}$ orbit in the calculations.

We next discuss spectra of nuclei. Knowing the level sequence (JT) values of low lying states of a nucleus either from experiment or microscopic calculations, one can determine the energy of these states using the distribution method. We have obtained the low energy spectra of a few nuclei and discuss them individually.

In figures (III.1) and (III.2) we compare the spectra of Ca^{48} and Ca^{50} with shell model calculations. Agreement between the two is very good. In Ca^{48}, because of certain levels lying close together there are small differences in the energies of these as predicted by the two methods. However, the levels of Ca^{50} are reproduced almost exactly. This is expected because all the levels of $(fp)^8_{T=4}$ and $(fp)^{10}_{T=5}$ are calculated and the interaction is known to be right.

In figure (III.3) the spectrum of Ti^{47} determined using $(f\tilde{T})$ and $(\tilde{m}T)$ distributions is compared with the one obtained by Dhar et al. The J = 3/2 level is matched, it being understood that levels below this should be left out in the comparison.

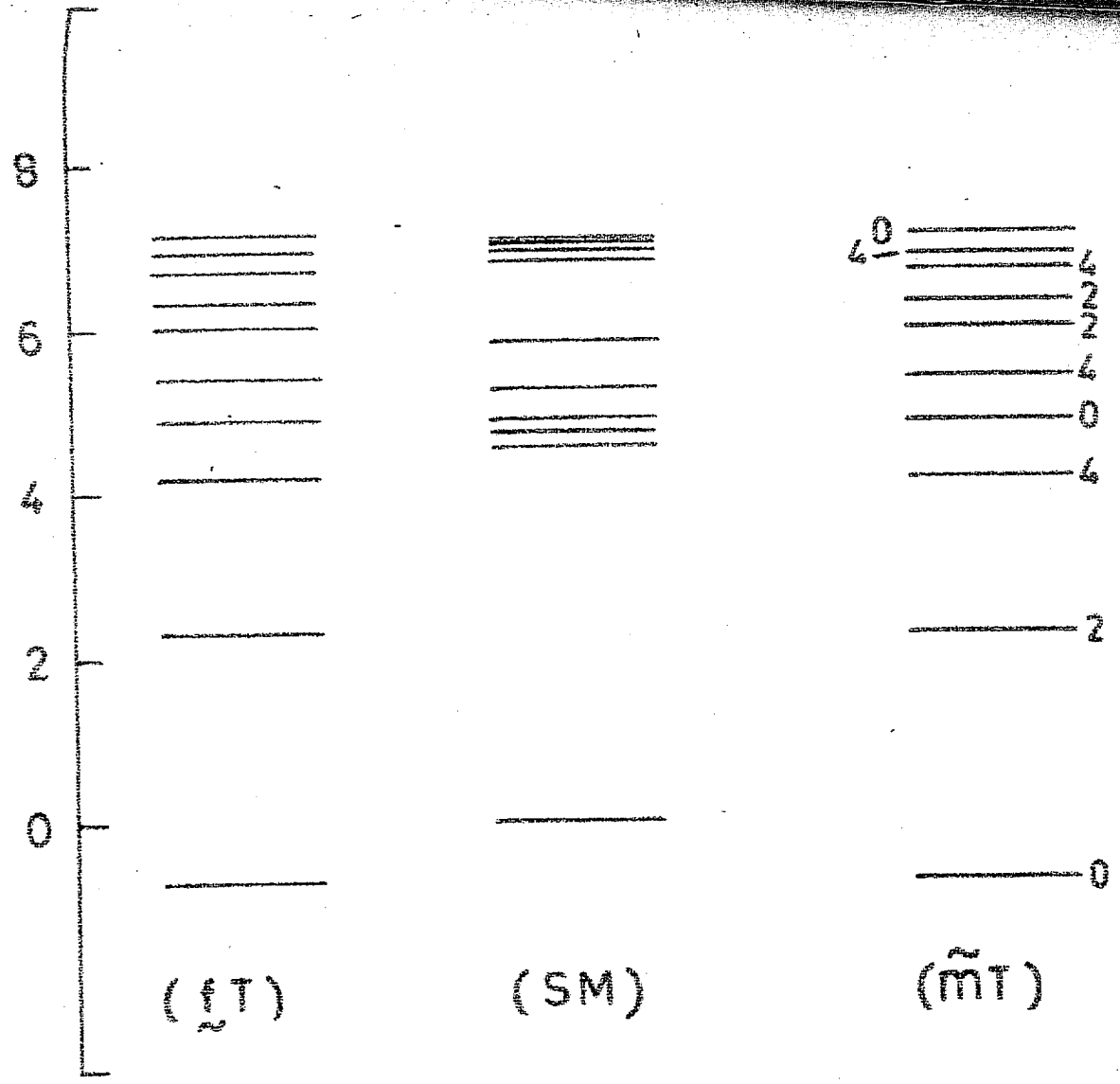

Fig. III.1 Spectrum of Ca^{48} using SU(4) - Isospin and configuration-isospin moments compared with shell model calculations.

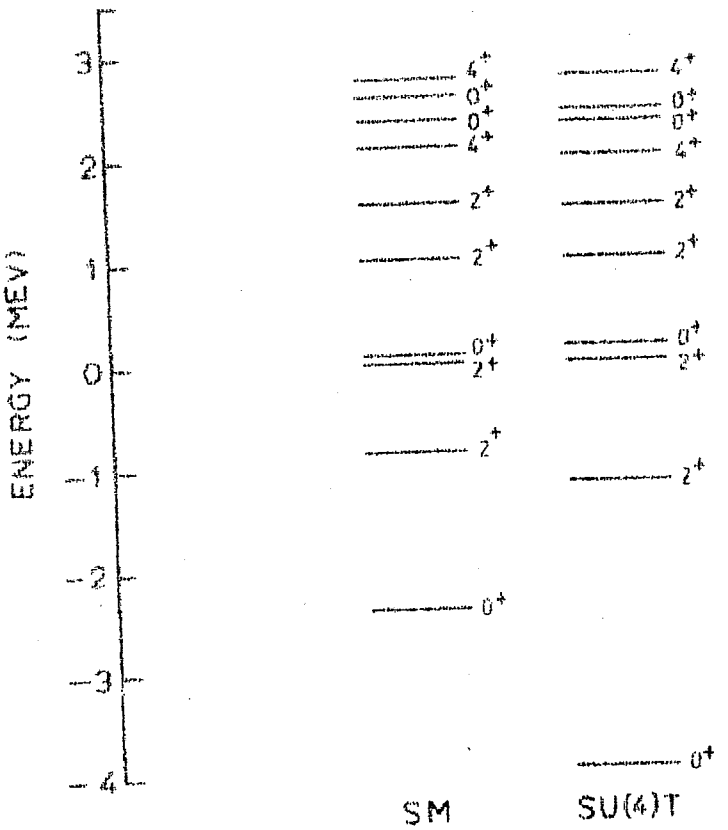

Fig. III. 2 Shell model and moment-method spectrum of Ca^{50}. The second 2^+ level is matched for comparison.

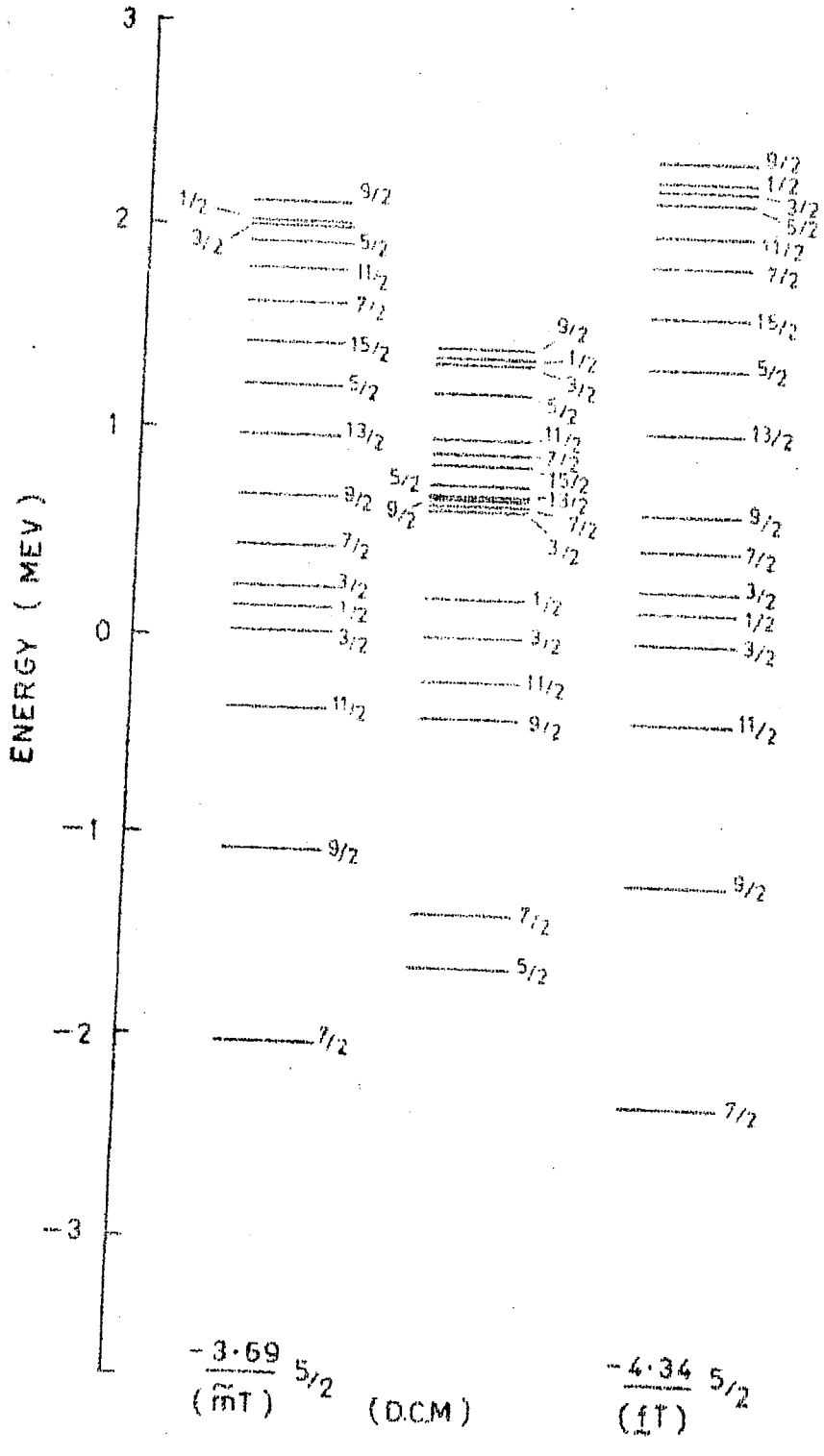

Fig. III. 3 Low-energy spectra of Ti^{47} using SU(4)-Isospin and configuration-isospin moments compared with deformed configuration mixing (DCM) calculations. The J=3/2 level is the reference level for comparison.

We observe that whereas ($f\underset{\sim}{T}$) and ($\tilde{m}T$) distributions predict almost the same energy of levels, agreement with deformed configuration mixing calculations is not very good. There are several reasons for this. The moment method will not predict very close lying levels and therefore the bunching of levels in Ti^{47} cannot be accounted for. A detailed agreement can only be expected when there is a certain smoothness in the observed spectrum. Further, we have assumed that the interaction is right and the DCM calculations give all the low-lying negative parity levels of $(f\tilde{p})^7$ configuration. If some of these are not predicted, the comparison is not very meaningful.

We conclude this section with a few general remarks. From the calculations of binding energies of nuclei we observe that although scalar averages produce results which are in reasonable agreement with empirical energies, a finer decomposition of the space improves the results significantly. Agreement between ground state energies of nuclei as obtained from distribution method and shell model calculations in small spaces, is very good. With increasing number of nucleons filling the single particle orbits we encounter larger spaces where comparisons with experimental energies is excellent. Our results for low-energy spectra of nuclei indicate that from the nature of the observed spectrum of a nucleus one can say whether it would be produced well using moment method. Disagreement for those

nuclei whose spectra show bunching of levels may thenbe regarded as a limitation of the method.

3.4.2 Configuration intensities and fractional occupancy of single particle orbits

The method of estimating the ground state energy by using a distribution which is a sum of several partial distributions for specified symmetries, quite naturally provides details about the ground state wave function in the form of relative intensities of different distributions. We use $(\tilde{m}T)$ distributions to calculate relative intensities $I(\tilde{m}T)$ of various configurations in the ground state. Details of estimating relative intensities of various distributions have been discussed in Sec. (1.2). These intensities $I(\tilde{m}T)$ are then used to calculate the fractional occupancy of each shell model orbit which is defined as

$$V_r^2 = \frac{1}{2(2j_r+1)} \sum_{\tilde{m}} m(r) I(\tilde{m}T) \qquad (3.4.1)$$

V_r^2 is simply an average measure of the number of particles in a shell model orbit in the ground state. These occupancies satisfy the relation

$$2 \sum_r V_r^2 (2j_r+1) = m \qquad (3.4.2)$$

where m is the total number of active nucleons. Results of our calculations are presented in table (III.13).

The configuration intensities $I(\tilde{m}T)$ give a rough idea as to which configurations are important in the ground state thereby providing information about which configurations need to be considered in detailed shell model calculations. The fractional occupancy result is significant since it enables one to predict the sum rule limit for single nucleon transfer strength for those cases where detailed spectroscopic calculations are formidable. Occupancies are also important parameters in the combinatorial theory of level density and in the B.C.S. treatment of pairing effects.

We remark here that to determine the configuration intensities, one should first remove those configurations which are not compatible with the ground state angular momentum. This point has been investigated by Thio (1970) and it is found that the correction does not make any significant change in the result. Further the numbers for configuration intensities should not be regarded as referring to a single state since in many cases there are rapid fluctuations from state to state. They should be regarded as average intensities over a finite energy domain which includes a few states. No such difficulty exists with the occupancies since an average is already built

Table (III.13)

Configuration intensities and fractional occupancy of single particle orbits as determined using configuration-isospin distributions. The configuration is defined as ($f_{7/2}$, $f_{5/2}$, $p_{3/2}$, $p_{1/2}$). Numbers in paranthesis are those obtained by McGrory et al(1970).

Nucleus	Configuration intensities		Occupancy of single particle orbits			
	$\tilde{m}$	I($\tilde{m}$T)	$f_{7/2}$	$f_{5/2}$	$p_{3/2}$	$p_{1/2}$
Ti^{44}	4 0 0 0	95	24.7	0.0	0.6	0.1
	3 0 1 0	4				
Sc^{44}	4 0 0 0	97	24.8	0.0	0.4	0.0
	3 0 1 0	3				
Ca^{44}	4 0 0 0	98(87)	24.9	0.0	0.2	0.0
	3 0 1 0	2				
Ti^{45}	5 0 0 0	85	30.3	0.1	1.6	0.4
	4 0 1 0	11				
Sc^{45}	5 0 0 0	92	30.7	0.0	1.0	0.1
	4 0 1 0	7				
Ca^{45}	5 0 0 0	86(88)	30.3	0.0	1.7	0.2
	4 0 1 0	13				
V^{46}	6 0 0 0	92	37.0	0.0	0.9	0.2
	5 0 1 0	7				
Ti^{46}	6 0 0 0	88	36.7	0.1	1.4	0.3
	5 0 1 0	10				

Table (III.13) (contd)

Sc^{46}	6 0 0 0 5 0 1 0	84 14	36.4	0.0	1.9	0.4
Ca^{46}	6 0 0 0 5 0 1 0	84(84) 15	36.5	0.0	2.0	0.2
Cr^{48}	8 0 0 0 7 0 1 0 7 1 0 0	69 18 5	47.8	0.5	3.1	1.2
V^{48}	8 0 0 0 7 0 1 0 7 1 0 0	66 21 5	47.5	0.4	3.6	1.4
Ti^{48}	8 0 0 0 7 0 1 0	68 23	47.7	0.2	3.7	1.2
Sc^{48}	7 0 1 0 8 0 0 0 6 0 2 0 7 0 0 1	40 29 11 8	44.3	0.4	9.0	3.4
Ca^{48}	7 0 1 0 8 0 0 0 6 0 2 0 7 0 0 1	52 28(86) 10 8	44.7	0.0	9.2	2.4
Cr^{49}	9 0 0 0 8 0 1 0 8 1 0 0	53 23 9	52.6	0.9	4.8	2.2

Table (III.13)(contd)

V^{49}	9 0 0 0	46	52.0	0.8	5.9	2.7
	8 0 1 0	28				
	8 0 0 1	7				
	8 1 0 0	7				
Ti^{49}	9 0 0 0	44	51.8	0.4	6.8	2.7
	8 0 1 0	33				
	8 0 0 1	7				
	7 0 2 0	7				
Sc^{49}	8 0 1 0	43	49.2	0.4	11.2	4.3
	9 0 0 0	18				
	7 0 2 0	16				
Ca^{49}	7 0 2 0	34	45.8	0.5	16.9	6.4
	8 0 1 0	29				
	7 0 1 1	13				
Mn^{50}	10 0 0 0	57	59.2	0.7	4.5	2.1
	9 0 1 0	23				
Cr^{50}	10 0 0 0	49	58.5	0.9	5.4	2.6
	9 0 1 0	24				
V^{50}	9 0 1 0	30	56.1	1.1	8.8	4.3
	10 0 0 0	28				
	8 0 2 0	10				
Ti^{50}	9 0 1 0	38	55.7	0.6	10.4	4.4
	10 0 0 0	22				
	8 0 2 0	14				
	9 0 0 1	9				
Sc^{50}	8 0 2 0	28	50.4	1.1	18.2	8.4
	9 0 1 0	19				
	8 0 1 1	15				
	7 0 3 0	9				
Ca^{50}	7 0 3 0	21	46.9	0.9	24.3	10.9
	8 0 1 1	18				
	7 0 2 1	15				
	8 1 1 0	7				

Table (III.13)(contd)

Mn^{51}	11 0 0 0	31	62.4	1.9	7.7	4.3
	10 0 1 0	23				
	10 1 0 0	13				
	10 0 0 1	7				
Cr^{51}	11 0 0 0	27	62.1	1.5	8.7	4.7
	10 0 1 0	27				
	10 1 0 0	10				
	9 0 2 0	9				
	10 0 0 1	8				
V^{51}	10 0 1 0	29	59.8	1.5	12.4	6.5
	11 0 0 0	12				
	9 0 2 0	15				
	10 0 0 1	9				
	9 0 1 1	9				
Ti^{51}	10 0 1 0	26	57.4	1.1	16.9	8.1
	9 0 2 0	25				
	9 0 1 1	13				
	10 0 0 1	8				
	8 0 3 0	7				
Sc^{51}	9 0 2 0	22	52.3	1.4	24.7	1.2
	8 0 3 0	18				
	8 0 2 1	16				
	9 0 1 1	15				
	9 1 1 0	8				
Ca^{51}	8 0 2 1	25	47.6	1.9	30.9	16.9
	8 0 3 0	17				
	7 0 3 1	16				
	8 1 2 0	12				
	7 0 4 0	9				
Fe^{52}	12 0 0 0	27	67.9	2.3	8.3	4.9
	11 0 1 0	21				
	11 1 0 0	15				
	10 0 2 0	8				
	11 0 0 1	7				
	10 1 0 0	6				
Mn^{52}	11 0 1 0	21	66.3	2.5	10.6	6.3
	12 0 0 0	17				
	11 1 0 0	13				
	10 0 2 0	10				
	10 1 1 0	8				
	10 0 1 1	7				

Table (III.13) (contd)

Cr^{52}	11 0 1 0	26	66.1	1.7	11.8	6.6
	12 0 0 0	14				
	10 0 2 0	13				
	11 1 0 0	9				
	10 0 1 1	8				
	11 0 0 1	8				
V^{52}	10 0 2 0	21	62.6	1.7	17.5	9.4
	11 0 1 0	19				
	10 0 1 1	13				
	10 1 1 0	7				
	9 0 3 0	7				
	9 0 2 1	7				
Ti^{52}	10 0 2 0	24	58.7	1.4	24.4	12.2
	9 0 3 0	16				
	10 0 1 1	15				
	9 0 2 1	14				
Sc^{52}	9 0 2 1	21	53.2	2.3	31.6	17.3
	9 0 3 0	15				
	8 0 3 1	15				
	8 0 4 0	8				
Ca^{52}	8 0 3 1	39	49.1	1.9	38.1	21.6
	8 0 4 0	14				
	8 1 3 0	14				
	8 0 2 2	12				
Fe^{53}	12 0 1 0	19	71.9	2.9	10.9	6.9
	13 0 0 0	15				
	12 1 0 0	14				
	11 0 2 1	9				
	11 1 1 0	8				
Mn^{53}	12 0 1 0	18	70.1	2.9	13.7	8.5
	11 0 2 0	12				
	12 1 0 0	10				
	11 0 1 1	9				
	11 1 1 0	9				
	13 0 0 0	7				

Table (III.13)(contd)

Cr^{53}	11 0 2 0	17	67.6	2.6	18.1	10.8
	12 0 1 0	14				
	11 0 1 1	12				
	11 1 1 0	10				
	10 0 2 1	8				
	10 0 3 0	7				
V^{53}	11 0 2 0	16	63.6	2.4	24.5	14.0
	10 0 2 1	14				
	10 0 3 0	13				
	11 0 1 1	12				
	11 1 1 0	8				
	10 1 2 0	7				
Ti^{53}	10 0 2 1	20	59.2	2.4	31.7	17.5
	10 0 3 0	15				
	9 0 3 1	14				
	10 1 2 0	10				
	9 0 4 0	8				
Sc^{53}	9 0 3 1	24	54.1	3.2	38.0	23.1
	9 1 3 0	14				
	9 1 2 1	11				
	9 0 2 2	9				
	9 0 4 0	8				
Ca^{53}	8 0 4 1	24	49.3	3.9	43.0	29.8
	8 1 3 1	22				
	8 1 4 0	15				
	7 0 4 2	8				
Co^{54}	13 0 1 0	13	74.6	3.8	15.1	10.0
	12 1 1 0	11				
	12 0 2 0	10				
	13 1 0 0	10				
	12 0 1 1	8				
	11 0 2 1	6				
Fe^{54}	13 0 1 0	16	76.2	3.2	13.5	8.9
	13 1 0 0	11				
	12 0 2 0	11				
	12 1 1 0	10				
	12 0 1 1	8				
	14 0 0 0	8				

Table (III.13)(contd)

Mn^{54}	12 0 2 0	13	72.7	3.4	18.5	11.9
	11 2 1 0	11				
	12 1 1 0	10				
	13 0 1 0	10				
	12 0 1 1	10				
	11 0 2 1	8				
Cr^{54}	11 0 2 1	13	68.9	3.2	24.6	15.2
	12 0 2 0	12				
	11 0 3 0	10				
	12 0 1 1	9				
	11 1 2 0	8				
	12 1 1 0	8				
Ti^{54}	10 0 3 1	26	60.4	2.7	38.9	22.4
	10 1 3 0	12				
	10 0 4 0	11				
	9 0 4 1	11				
	10 0 2 2	10				
	10 1 2 1	8				
Sc^{54}	9 1 3 1	22	54.5	4.9	43.4	30.2
	9 0 4 1	16				
	9 0 3 2	15				
	9 1 4 0	11				
	8 0 4 2	10				
Ca^{54}	8 1 4 1	42	49.9	6.1	47.4	37.1
	8 0 4 2	32				
	8 1 3 2	19				
Co^{55}	13 1 1 0	10	79.4	4.5	16.2	11.3
	14 1 0 0	9				
	14 0 1 0	9				
	13 0 2 0	9				
	13 0 1 1	7				
Fe^{55}	13 1 1 0	11	77.0	4.5	19.9	13.6
	13 0 2 0	9				
	12 0 2 1	8				
	13 0 1 1	8				
	12 1 2 0	7				

Table (III.13)(contd)

Mn^{55}	12 0 2 1	10	73.1	4.7	25.6	17.1
	12 1 2 0	9				
	12 0 3 0	7				
	13 1 1 0	7				
	13 0 2 0	6				
Cr^{55}	12 0 2 1	12	70.0	4.0	31.5	20.0
	11 0 3 1	12				
	12 1 2 0	9				
	12 0 3 0	8				
	11 1 3 0	7				
Ti^{55}	10 0 4 1	18	60.6	4.7	44.2	30.4
	10 1 3 1	18				
	10 0 3 2	15				
	10 1 4 0	11				
	9 0 4 2	9				
Sc^{55}	9 1 4 1	31	55.0	7.7	47.3	37.2
	9 1 3 2	21				
	9 0 4 2	16				
	8 1 4 2	9				
Ca^{55}	8 1 4 2	64	50.0	11.4	49.0	42.9
	8 2 4 1	26				
	8 2 3 2	8				
Ni^{56}	14 1 1 0	10	83.7	5.1	18.3	13.1
	15 1 0 0	7				
	14 0 2 0	7				
	15 0 1 0	6				
	14 0 1 1	6				
	13 1 2 0	6				
	13 0 2 1	6				
Cp^{56}	14 1 1 0	8	80.2	5.9	22.6	16.2
	13 0 2 1	7				
	13 1 2 0	7				
	13 1 1 1	6				

Table (III.13)(contd)

Fe^{56}	13 0 2 1	9	78.6	5.2	26.0	18.0
	12 1 2 1	9				
	13 1 1 1	7				
	13 0 3 0	6				
	12 1 2 1	6				
	14 1 1 0	6				
Mn^{56}	12 1 2 1	9	74.0	5.9	32.1	22.0
	12 0 3 1	9				
	12 1 3 0	7				
	13 1 2 0	7				
	13 0 2 1	6				
Cr^{56}	12 1 3 0	9	70.7	5.4	37.9	25.4
	11 1 3 1	9				
	11 0 4 1	9				
	11 0 3 2	7				
Ti^{56}	10 1 4 1	28	61.1	6.6	48.3	38.9
	10 0 4 2	23				
	10 1 3 2	19				
Sc^{56}	9 1 4 2	73	55.8	10.1	50.2	46.4
	9 2 4 1	14				
Ca^{56}	8 2 4 2	99	50.0	16.8	50.0	50.0
Zn^{60}	15 1 3 1	7	90.1	8.4	44.4	31.5
	14 1 4 1	6				
	15 1 2 2	4				
	15 0 4 1	4				
	14 1 3 2	4				
Cu^{60}	15 1 3 1	7	89.0	9.3	42.2	32.0
	14 1 4 1	6				
	14 1 3 2	5				
	15 2 2 1	4				
	15 1 2 2	4				
	14 2 3 1	4				

Table (III.13) (contd)

Ni^{60}	15 1 3 1 14 1 3 2 14 1 4 1	7 6 7	88.0	10.0	43.1	32.7
Co^{60}	14 1 4 1 14 2 3 1 14 1 3 2 13 1 4 2 13 1 5 1 13 2 4 1	8 7 7 6 5 5	84.6	11.5	45.8	35.3
Fe^{60}	13 1 4 2 14 1 3 2 14 2 3 1 14 1 4 1 13 2 4 1 13 1 5 1 13 2 3 2 12 1 5 2	12 10 9 8 7 6 6 5	82.4	11.7	48.2	38.7
Cr^{60}	12 2 4 2 11 2 5 2 12 3 4 1 12 3 3 2	61 12 11 7	73.7	18.8	50.8	47.1
Ti^{60}	10 4 4 2	99	62.5	33.3	50.1	50.0
Ca^{60} (trivial)	8 6 4 2	100	50.0	50.0	50.0	50.0

in and they can indeed be interpreted as applying to a single state.

3.4.3 SU(4) Symmetry Mixing and Franzini-Radicati Mass Formula

Goodness of Wigner Supermultiplet Scheme is investigated using SU(4)-T moments. The intensities $I(mfT)$ of various SU(4) irreps in the ground state region provide a measure of symmetry breaking. These relative intensities $I(mfT)$ in the ground state region for $4 \leq m \leq 20$ particles are given in tables (III.14 - III.30). Results show that SU(4) symmetry is badly mixed. When the single particle energies are set to zero, the symmetry is significantly restored indicating that a large part of the breaking is due to single particle spin orbit coupling. Further the extent of mixing is the same both for low and high isospin nuclei. One may conclude that the relative intensities involving three or four low lying irreps indicate large admixing of SU(4) symmetry for ground states of 2p-1f shell nuclei.

Following Franzini and Radicati we have evaluated the ratio R defined in eq.(2.2.1) using binding energies obtained from (fT) distributions by choosing appropriate isobars from tables (III.10) and (III.11). These are compared with the values obtained from SU(4) model(R_{th}) in table (III.31). The appropriate values obtained using empirical binding energies are also given.

Table (III.14)

Mixing of SU(4) symmetry in the ground state region for nuclei in 2p-1f shell using MWH2 Interaction with and without single particle energies

Nucleus	f	I(mfT) with s.p.e.	I(mfT) s.p.e.=0
Ti^{44}	1111	42	99
	2110	35	
	2200	21	
Sc^{44}	2110	78	99
	2200	9	
	3100	13	
Ca^{44}	2200	40	96
	3100	57	4
	4000	3	
Sr^{76}	1111	40	97
	2110	35	2
	2200	23	
Kr^{76}	2200	39	86
	3100	58	14
	4000	3	

Table (III.15)

Nucleus	$\underset{\sim}{f}$	I(mfT) with s.p.e.	I(mfT) s.p.e. =0
Ti^{45}	2111	46	90
	2210	36	10
	3110	10	
	3200	8	
Sc^{45}	2210	40	77
	3110	39	23
	2300	9	
	4100	3	
Ca^{45}	3110	76	100
	3200	23	
Br^{75}	3200	75	98
	4100	25	2

Table (III.16)

Nucleus	$\underset{\sim}{f}$	I(mfT) with s.p.e.	I(mfT) s.p.e. = 0
V^{46}	2211	27	69
	3111	29	90
	2220	15	10
	3210	26	2
Ti^{46}	2211	25	69
	3111	28	19
	2220	15	10
	3210	29	2
Sc^{46}	3210	76	97
	4110	18	29
	3300	2	
	4200	4	
Ca^{46}	3300	35	91
	4200	60	9
	5100	5	
Kr^{74}	2211	23	63
	3111	27	22
	2220	14	12
	3210	31	3
Br^{74}	3210	75	97
	4110	19	3
Se^{74}	3300	34	77
	4200	61	23

Table (III.17)

Nucleus	$\underset{\sim}{f}$	I(mfT) with s.p.e.	I(mfT) s.p.e. = 0
V^{47}	2221	33	81
	3211	48	18
	3220	7	
Ti^{47}	3211	41	78
	3220	29	18
	4111	10	3
	3310	7	
	4210	13	
Sc^{47}	3310	35	67
	4210	58	33
	5110	4	
Ca^{47}	4300	74	98
	5200	25	2
As^{73}	4300	71	97
	5200	28	3
	6100	1	

Table (III.18)

Nucleus	f	I(mfT) with s.p.e.	I(mfT) s.p.e.=0
Cr^{48}	2222	26	87
	3221	38	9
	3311	28	4
	4211	3	
V^{48}	3221	57	92
	3311	10	3
	4211	17	3
	3320	9	1
	4310	4	
Ti^{48}	3311	19	55
	4211	31	24
	3320	16	12
	4220	22	8
	4310	7	
Sc^{48}	4310	70	95
	5210	27	5
Ca^{48}	4400	35	79
	5300	60	21
	6200	5	
Kr^{72}	2222	24	82
	3221	38	12
	3311	29	6
	4211	3	
Se^{72}	3311	18	50
	4211	30	27
	3320	15	14
	4220	23	9
	4310	8	
	5210	3	

Table (III.19)

Nucleus	$\tilde{f}$	I(mfT) with s.p.e.	I(mfT) s.p.e.=0
Cr^{49}	3222	32	77
	3321	39	20
	4221	12	1
	4311	12	
V^{49}	3321	29	61
	4221	41	35
	4311	10	2
	3330	3	1
	4320	10	
Ti^{49}	4311	38	70
	4320	33	23
	5211	12	4
	5220	10	
Sc^{49}	4410	32	62
	5310	61	38
Ca^{49}	5400	72	98
	6300	27	2
Se^{71}	3321	28	58
	4221	40	37
	4311	11	2
	4320	11	1
As^{71}	4311	34	66
	4320	33	26
	5211	13	5
	5220	11	3
Ge^{71}	4410	29	57
	5310	61	42
Ga^{71}	5400	67	95
	6300	31	5

Table (III.20)

Nucleus	f	I(mfT) with s.p.e.	I(mfT) s.p.e. = 0
Mn^{50}	3322	18	50
	4222	21	24
	3331	16	18
	4321	37	8
Cr^{50}	3322	17	50
	4222	21	24
	3331	16	18
	4321	38	8
V^{50}	4321	59	89
	5221	17	7
	4330	8	3
	5311	5	
	5320	5	
Ti^{50}	4411	17	50
	5311	31	26
	4420	15	13
	5320	28	11
Sc^{50}	5410	70	93
	6310	29	7
Ca^{50}	5500	32	76
	6400	61	24

Table (III.21)

Nucleus	$\underset{\sim}{f}$	I(mfT) with sp.e.	I(mfT) s.p.e. = 0
Mn^{51}	3332	26	67
	4322	44	30
	4331	11	2
	4421	12	1
	5321	5	
Cr^{51}	4322	30	61
	4331	32	30
	5222	8	4
	4421	9	2
	5321	17	2
V^{51}	4421	26	53
	5321	49	42
	4430	4	2
	5330	7	2
Ti^{51}	5411	36	69
	5420	35	24
	6311	13	4
	6320	13	3
Sc^{51}	5510	30	61
	6410	61	38
	7310	7	
Ca^{51}	6500	72	98
	7400	27	2
Ge^{69}	4421	25	52
	5321	48	43
	5330	7	3
	4430	4	2
	6221	5	
Ga^{69}	5411	33	63
	5420	34	27
	6311	14	6
	6320	15	4
Zn^{69}	5510	30	59
	6410	62	41
	7310	8	
Cu^{69}	6500	69	97
	7400	29	4

Nucleus	f	I(mfT) with s.p.e.	I(mfT) s.p.e. = 0
Fe^{52}	3333	18	71
	4332	36	18
	4422	31	10
Mn^{52}	4332	47	85
	4422	9	5
	5322	17	5
	4431	13	4
Cr^{52}	4422	14	39
	5322	23	26
	4431	17	19
	5331	26	15
	5421	10	1
V^{52}	5421	59	86
	6321	22	9
	5430	10	4
Ti^{52}	5511	16	46
	6411	30	27
	5520	15	14
	6420	31	13
Sc^{52}	6510	70	94
	7410	29	6
Ca^{52}	6600	34	78
	7500	60	22
Se^{68}	3333	18	68
	4332	36	20
	4422	31	11
Ge^{68}	4422	13	37
	5322	23	26
	4431	17	19
	5331	26	16
	5421	11	1
Ga^{68}	5421	57	86
	6321	22	9
	5430	10	5
Zn^{68}	5511	14	40
	6411	29	29
	5520	15	15
	6420	32	15
Cu^{68}	6510	69	93
	7410	29	7

Table (III.23)

Nucleus	f	I (mfT) with s.p.e	I (mfT) s.p.e = 0
Fe^{53}	4333	25	64
	4432	39	30
	5332	13	3
	5422	14	3
Mn^{53}	4432	23	53
	5332	35	38
	4441	5	3
	5422	11	3
Cr^{53}	5422	27	56
	5431	38	34
	6322	10	5
	6331	12	4
V^{53}	5521	26	51
	6421	50	43
	5530	5	2
	6430	9	3
Ti^{53}	6511	35	66
	6520	35	26
	7411	13	5
	7420	15	4
Sc^{53}	6610	31	59
	7510	61	41
Ca^{53}	7600	75	98
	8500	25	2

Table (III.24)

Nucleus	f	I (mfT) with s.p.e.	I (mfT) s.p.e. = 0
Co^{54}	4433	12	35
	5333	17	24
	4442	14	21
	5432	42	19
Fe^{54}	4433	12	35
	5333	17	24
	4442	15	21
	5432	42	19
Mn^{54}	5432	53	84
	6332	15	8
	5441	11	6
Cr^{54}	5522	11	28
	6422	22	27
	5531	17	20
	6431	34	23
Ti^{54}	6611	16	46
	7511	30	27
	6620	16	14
	7520	31	13
Sc^{54}	7610	72	93
	8510	27	7
Ca^{54}	7700	37	84
	8600	59	16

Table (III.25)

Nucleus	f	I (mfT) with s.p.e.	I (mfT) s.p.e. = 0
Co^{55}	4443	20	52
	5433	40	38
	5442	13	5
	5532	15	4
Fe^{55}	5433	24	50
	5442	31	36
	6333	7	5
	6432	20	5
	5532	10	4
Mn^{55}	5532	22	44
	6432	43	44
	7332	4	11
	6441	11	5
	5541	6	5
Cr^{55}	6522	27	53
	7422	10	50
	6531	40	36
	7431	16	6
Ti^{55}	7611	36	67
	8511	12	39
	7620	36	26
	8520	14	3
Sc^{55}	7710	33	61
	8610	61	39
Ca^{55}	8700	77	99
	9600	22	1

Table (III.26)

Nucleus	f ~	I(mfT) with s.p.e.	I(mfT) s.p.e. = 0
Ni^{56}	4444	13	48
	5443	34	30
	5533	31	20
Co^{56}	5443	41	78
	5533	9	6
	6433	17	7
	5542	15	6
	6532	10	1
Fe^{56}	5533	10	26
	6433	19	25
	5542	17	22
	6442	27	22
	6532	13	3
Mn^{56}	6532	51	79
	7432	20	11
	6541	15	8
	7441	5	2
Cr^{56}	6622	11	30
	7522	22	26
	6631	18	20
	7531	36	22
Ti^{56}	7711	17	47
	7720	17	15
	8611	29	25
	8620	31	13
Sc^{56}	8710	75	95
	9610	24	5
Ca^{56}	8300	41	93
	9700	56	7

Table (III.27)

Nucleus	f	I(mfT) with s.p.e.	I(mfT) s.p.e. = 0
Ni^{57}	5444	21	55
	5543	38	36
	6443	13	5
	6533	15	4
Co^{57}	5543	21	46
	6443	32	39
	6533	11	5
	5552	5	4
	6542	19	5
Fe^{57}	6533	23	46
	6542	39	40
	7433	8	5
	7442	13	6
Mn^{57}	6632	23	47
	7532	45	43
	6641	7	4
	7541	14	5
Cr^{57}	7622	27	54
	7631	42	36
	8522	10	4
	8531	16	5
Ti^{57}	8711	37	67
	8720	39	26
	9611	11	3
	9620	13	3
Sc^{57}	8810	38	67
	9710	59	33
Ca^{57}	9800	36	99
	10700	14	1

Table (III.28)

Nucleus	f	I(mfT) with s.p.e.	I(mfT) s.p.e. = 0
Cu^{58}	5544	48	87
	6444	4	2
	5553	4	2
	6543	33	8
	6633	7	1
Ni^{58}	5544	10	28
	6444	14	23
	5553	14	22
	6543	43	26
Co^{58}	6543	48	79
	7443	14	9
	6552	13	8
	7542	10	1
Fe^{58}	6633	10	27
	7533	20	24
	6642	18	22
	7542	34	24
Cr^{58}	7722	13	38
	8622	23	23
	7731	19	20
	8631	35	19
Ti^{58}	3811	20	52
	9711	28	22
	3820	19	15
	9720	30	11
Ca^{58}	9900	53	100
	10800	47	-

Table (III.29)

Nucleus	f	I(mfT) with s.p.e.	I(mfT) s.p.e. = 0
Cu^{59}	5554	20	53
	6544	38	37
	6553	13	5
	6643	15	4
Ni^{59}	6544	22	48
	6553	31	37
	7444	6	5
	7543	20	5
	6643	10	4
Co^{59}	6643	22	45
	7543	40	42
	6652	7	5
	7552	12	5
Fe^{59}	7633	24	50
	7642	43	40
	8542	16	6
	8533	9	4

Table (III.30)

Nucleus	f	I(mfT) with s.p.e.	I(mfT) s.p.e. = 0
Zn^{60}	5555	15	57
	6554	35	26
	6644	31	16
Cu^{60}	6554	41	78
	6644	9	6
	7544	16	7
	6653	16	7
	7643	9	1
Ni^{60}	6644	10	27
	7544	18	23
	6653	18	23
	7553	27	21
	7643	13	3
Co^{60}	7643	50	78
	8543	18	10
	7652	18	10
	8552	6	2
Fe^{60}	7733	11	31
	8633	20	22
	7742	20	22
	8642	36	22
Cr^{60}	8822	15	41
	8831	22	21
	9722	22	21
	9731	35	16
Ti^{60}	9911	26	64
	10811	24	15
	9920	24	15
	10820	25	6

Table (III.31)

Values of R as predicted by the binding energies obtained from SU(4)-Isospin moments for various isobars compared with the theoretical value of R given by the Wigner model. R_{exp} is obtained using empirical energies.

T_3	R_{th}	R_{calc}	R_{exp}
A = 2n+1			
5/2	2.33	2.47,2.13,3.67,2.32, 2.90,3.32,2.53,2.55, 3.42,3.98.	2.79,2.25,2.48,2.87 2.76,2.45,2.82,3.00 2.58,2.94
7/2	2.25	2.94,3.04,2.03,2.58, 1.83,1.65,2.70,2.05, 1.97,1.89	2.35,2.46,2.38,2.40, 2.31,2.31,2.31,2.49, 2.60,2.30
9/2	2.20	2.29,2.31,2.84,2.30, 2.95,2.60,1.80,2.00	2.43,2.14,2.24,2.24, 2.67,2.46,2.27,2.42
11/2	2.167	2.05,2.32,2.66	2.21,2.27,2.16
13/2	2.143	2.03	2.18
A = 4n			
2	1.5	1.69,1.47,1.29,1.74	1.42,1.48,1.64,1.63
3	4.0	5.42,5.16,7.86,4.96	5.09,5.08,4.10,4.93
4	1.66	1.50,2.16,1.67,1.59, 2.48,2.05	1.74,1.73,1.78,1.75, 1.77,1.69
5	3.0	2.64,1.23,2.40	3.01,3.21,3.22
6	1.75	1.81	1.86
A = 4n+2			
3	1.6	1.62,1.38,1.95,2.98	1.51,1.77,1.70,1.73
4	3.33	1.42,2.49,2.00	4.18,3.64,3.85
5	1.71	2.14	1.81
7	1.78	1.81	1.87

Although the values calculated seem to be scattered around the theoretical value as shown in figure (III.4), agreement in general is reasonable. The ratio R is very sensitive to small changes in binding energies of the isobars. Some of the points which lie far away from R_{th} correspond to those cases which contain a nucleus for which no excited reference state was available to determine the binding energy. These points should be ignored in the comparison.

However, the small spread in values of R should not be regarded as indicating the goodness of SU(4) symmetry because the ratio between the difference in energy of isobars may be insensitive to symmetry mixing. It appears that symmetry mixing does not significantly affect energy relationships. There is one very interesting feature which emerges from this study of SU(4) symmetry . From tables (III.14 - III.30) we observed from the mixing intensities that

1) For A = 4n nuclei, symmetry mixing is more for nuclei with even isospin as compared with those having odd isospin.

2) For A = 2n+1 nuclei, the extent of breaking is more or less independent of isospin.

3) For A = 4n+2 nuclei, intensities of various irreps of SU(4) indicate that the symmetry is mixed more for

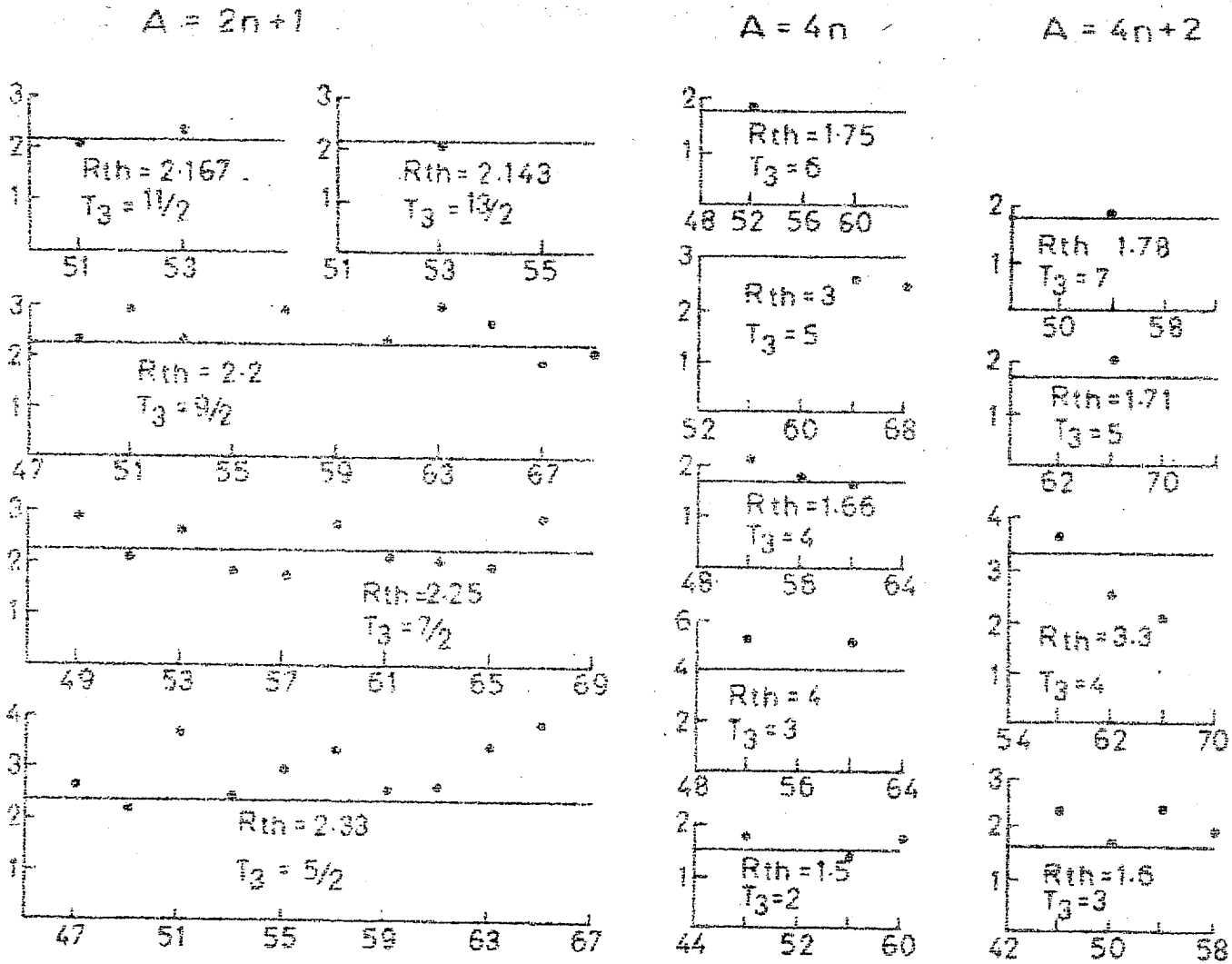

Fig. III.4 The ratio R for A = 2n+1, 4n and 4n+2 nuclei obtained using binding energies calculated from SU(4)-Isospin moments compared with the theoretical value R_{wig}.

odd T as compared to even T.

There is some sort of anticorrelation between this and the results of Franzini-Radicati. From their plot of R we observe that -

1) For A = 4n nuclei, the scatter of points as compared to R_{th} is more for odd isospins

2) For A = 2n+1 nuclei, scatter is independent of isospin

3) For A = 4n+2 nuclei, deviations from R_{th} are larger for even T as compared to odd T.

3.5 Conclusion

We have studied the structure of 2p-1f shell nuclei using the spectral distribution method. Averages of H and powers of H over states defined by the irrep of various subgroups of U(N) have been evaluated. The spectral moments for various distributions like configuration-isospin and SU(4)-Isospin have been used to study properties of nuclei from $44 \leq A \leq 76$.

Binding energies of nuclei as obtained from distributions agree remarkably well with microscopic calculations and empirical energies. This is very significant considering the fact that one is dealing with vector spaces whose dimensionalities are as large as 10^{11} states. This makes us feel confident that the accuracy in determining ground state energies is maintained

in large vector spaces and we can determine absolute energies in spaces of arbitrarily large size. One can carry out calculations by including $d_{3/2}$-hole and $g_{9/2}$-particle excitations to study nuclei where these may be important.

A systematic study of various effective interactions may also be possible using the moment method. The tremendous advantage is that whereas shell model calculations can be performed in relatively small spaces, the distribution method does not suffer from such a restriction. Our results indicate that MWH2 interaction works very well upto A = 60, after which it starts underbinding nuclei. Certain modifications need to be incorporated if one wants to use it in full 2p-1f shell space.

Spectra of nuclei as obtained from distributions agree fairly well with other calculations. It must be borne in mind that the level sequence is taken from either experiment or microscopic calculations since fixed angular momentum averages have not been evaluated. Determination of fixed J distributions will be discussed in the next chapter.

Intensities of various configurations $\tilde{m}$ obtained using ($\tilde{m}$T) averages are important parameters. These enable one to decide upon the truncation of the vector space for shell model calculations. While fractional occupancy of single particle orbits are useful in single nucleon transfer strengths and in

the theory of level densities, to get more precise information it would be necessary to calculate the occupancies of neutrons and protons respectively.

Relative intensities of SU(4) irreps in the ground state region of nuclei indicate that the symmetry in general is badly broken. Agreement of our results of the ratio R with theoretical values R_{th} as predicted by the supermultiplet scheme indicates that this ratio may be insensitive to symmetry mixing.

CHAPTER IV

FIXED ANGULAR MOMENTUM AVERAGES AND SPECTRAL DISTRIBUTION APPROACH TO DEFORMATION ENERGY CURVES

4.1 Introduction

In Chapters II and III, spectral distribution methods were used to study general properties of nuclei. Averages of the Hamiltonian H and H^2 over states defined by the irreducible representation of the group U(N) and its subgroup decompositions were evaluated. Spectral moments for configurations, configuration-isospin and SU(4)-isospin distributions were used to determine ground state energies, low energy spectra, occupancies of single particle orbits and SU(4) symmetry mixing etc. for nuclei.

In the application of distributions to the determination of low lying levels of a nucleus using Ratcliff's (1971) procedure, a knowledge of the degeneracy and therefore the angular momentum J and isospin T of each level is necessary. For even-even nuclei, one may assume the ground state spin to be 0^+ and perhaps the (0-2-4) sequence. In other cases one proceeds on the assumption that the interaction would give the observed spectrum.

Evaluation of moments of H for fixed angular momentum will enable one to determine the low-energy spectra of a nucleus

unambigously. At higher excitation energies, fixed-J moments can be used to determine the level density at a given energy and one can study the decomposition of states according to angular momentum.

In Sec.(4.2) of this chapter, fixed-J moments have been evaluated following the procedure suggested by Jacquemin (1973). The method is based on making a suitable choice of configurations and using the configuration averaging method of French (1967). These moments are used to determine spectra of some nuclei in the 2s-1d shell.

In the second part of this chapter we use spectral distribution methods to study the average deformation (quadrupole moment) of a nucleus as a function of excitation energy. Such averages may be of interest in heavy ion reaction processes where compound nuclear formation takes place. It should be emphasized that the average deformation energy curves do not describe the "response" of the nucleus to an external quadrupole field. Therefore these curves cannot be interpreted as potential energy curves for collective quadrupole motion. We do indicate later how such potential energy curves can be obtained using the spectral distribution method.

In Sec.(4.3) the method of Jacquemin (1973) is described. Centroids and variances of configurations (to be defined later) are calculated and these are then used to determine the intensities of various configurations at a fixed energy.

Average quadrupole moment for these configurations is calculated and its energy variation is studied.

In Sec.(4.4), variation of average quadrupole moment with excitation energy is studied within the frame-work of Elliott SU(3) model (1958). Some comments on the results obtained are made in Sec.(4.5).

4.2 Fixed Angular Momentum J-moments

The technical problems encountered in the evaluation of fixed-J moments cannot be handled by any straight-forward extension of the methods used for other distributions. In order to calculate averages of H and powers of H over states defined by the irrep of a group, one constructs new operators, one for each moment, in terms of the scalar operators of the group. For calculating averages over states with fixed-J, one does not have enough scalar operators.

Jacquemin (1973) has proposed a different way by which fixed-J averages can be evaluated. In this method one starts by considering a finite set N of single particle states. This space S(N) may be made up of p shells $j_1, j_2 \cdots j_p$ and for the sake of convenience it is assumed that $j_1 \geqslant j_2 \cdots \geqslant j_p$. Distribution of m particles in these N single particle states generates the space S(N,m). This space can be decomposed according to the eigenvalue K of the operator J_z giving rise to the

subspace $S(N,m,K)$. One can also decompose $S(N,m)$ according to the eigenvalue of J^2 to obtain the subspace $S(N,m,J)$. It is clear that any state in $S(N,m,J)$ is an eigenstate of J^2 with eigenvalue $J(J+1)$.

In order to define configurations the space $S(N)$ is decomposed into subspaces $S^i(N)$ $(i = 1,\ldots, \ell = 2j_i+1)$ such that every state in $S^i(N)$ is an eigenstate of J_z with eigenvalue K_i. A configuration $\tilde{m} \equiv (m_1 \ldots m_\ell)$ is obtained by distributing m_1 particles in $S^i(N)$, m_2 in $S^2(N)$ and so on such that

$$m = m_1 + m_2 + \ldots m_\ell \qquad (4.2.1)$$

Any state in m will be an eigenstate of J_z with the eigenvalue K given by

$$K = \sum_{i=1}^{\ell} m_i K_i \qquad (4.2.2)$$

Let L^i be the dimensionality of $S^i(N)$. Then

$$N = \sum_{i=1}^{\ell} L^i \qquad (4.2.3)$$

The average of an operator O in the space $S(N,m,J)$ is defined as

$$\langle O \rangle^{S(N,m,J)} = \frac{1}{d(NmJ)} \sum_{\alpha \in S(N,m,J)} \langle \alpha | O | \alpha \rangle \qquad (4.2.4)$$

where d(N,m,J) is the number of states over which the average is taken. In order to determine this average one proceeds in the following manner. In the first step the average of O over states defined by the configuration $\tilde{m}$ is evaluated. This can be written as:

$$\langle O \rangle^{\tilde{m}} = \frac{1}{d(\tilde{m})} \sum_{\beta \in \tilde{m}} \langle \beta | O | \beta \rangle \qquad (4.2.5)$$

where the dimensionality $d(\tilde{m})$ is given by

$$d(\tilde{m}) = \binom{L^1}{m_1} \cdots \binom{L^l}{m_l} \qquad (4.2.6)$$

$\binom{L}{m}$ being the binomial coefficient.

In the next step, the average of O in the subspace S(N,m,K) is determined by summing over appropriate configurations.

$$\langle O \rangle^{S(N,m,K)} = \frac{1}{d(N m K)} \sum_{\substack{\tilde{m} \\ K=\sum m_i K_i}} d(\tilde{m}) \langle O \rangle^{\tilde{m}} \qquad (4.2.7)$$

Finally, the average over S(N,m,J) is obtained by taking the linear combination of $O^{S(N,m,K)}$ as given below:

$$\langle O \rangle^{S(N,m,J)} = d^{-1}(NmJ)\left[d(NmK=J)\langle O \rangle^{S(N,m,K)} - d(NmK'=J+1)\langle O \rangle^{S(N,m,K')} \right] \qquad (4.2.8)$$

4.2.1 Evaluation of Spectral Moments with fixed-J

The centroid energy $E_c(\tilde{m})$ and variance $\sigma^2(\tilde{m})$ of a configuration $\tilde{m}$ is defined as:

$$E_c(\tilde{m}) = \frac{1}{d(\tilde{m})} \sum_{\alpha \in \tilde{m}} \langle \alpha | H | \alpha \rangle \qquad (4.2.9)$$

$$\sigma^2(\tilde{m}) = \frac{1}{d(\tilde{m})} \sum_{\alpha \in \tilde{m}} \langle \alpha | H^2 | \alpha \rangle - E_c^2(\tilde{m}) \qquad (4.2.10)$$

To determine these averages, new operators $\hat{H}$ and $\hat{H}^2$ (c.f. Sec. 1.2) are constructed with the correct maximum particle rank and which reproduce the traces of H and H^2 in the defining space. These new operators can be written as polynomial expressions in terms of the number operators $n_i (i = 1 \ldots . \ell)$ where

n_i counts the number of particles in the space $S^i(N)$.

$$\hat{H} = a_1 + \sum_{i=1}^{\ell} b_i n_i + \sum_{i\leq j=1}^{\ell} c_{ij} n_i n_j \qquad (4.2.11)$$

$$\hat{H}^2 = p_i + \sum_{i=1}^{\ell} q_i n_i + \sum_{i\leq j=1}^{\ell} r_{ij} n_i n_j + \sum_{i\leq j\leq k=1}^{\ell} s_{ijk} n_i n_j n_k + \sum_{i\leq j\leq k\leq u} t_{ijku} n_i n_j n_k n_u \qquad (4.2.12)$$

Here a,b,c,p,q,r,s and t are parameters to be determined from a knowledge of averages in the defining space. The method that we have indicated for evaluating $E_c(\tilde{m})$ and $\sigma^2(\tilde{m})$ is a "brute-force" method. Chang et al (1971) and Jacquemin (1973) use a more elegant method where they decompose H according to tensors with respect to the group $\sum U(N_i)$ and make a detailed examination of the nature of these tensor operators to simplify the evaluation of moments.

Equations (4.2.11 and 4.2.12) are used to determine configuration centroids and variances in 2s-1d shell. The space S(24) consisting of three orbits $d_{5/2}$, $d_{3/2}$ and $s_{1/2}$ is decomposed into six subspaces $S^i(24)$ ($\mathbb{K}_i = -5/2$ to $5/2$), the dimensionalities of which are $L_1 = L_6 = 2$, $L_2 = L_5 = 4$ and $L_3 = L_4 = 6$. Any configuration $\tilde{m}$ satisfies the condition $m_1 \leq 2$, $m_2 \leq 4$, $m_3 \leq 6$, $m_4 \leq 6$, $m_5 \leq 4$ and $m_6 \leq 2$.

Configurations upto 4 particles are listed in Table (IV.1).

For the calculations we have used (K+12) fp interaction (Halbert, McGrory, Wildenthal and Pandya, 1971). This interaction has been derived from Kuo's (1967) interaction by adjusting 12 parameters, namely the three single particle energies and the nine two-body matrix elements involving only the $d_{5/2}$ and $s_{1/2}$ orbits. All other two-body matrix elements are the same as in Kuo's interaction. The single particle energies are $\epsilon\, d_{5/2} = -4.24$; $\epsilon\, d_{3/2} = 1.51$ and $\epsilon\, s_{1/2} = -3.34$.

The input averages were determined by first obtaining all the states in $\tilde{m}$, calculating the determinantal energies, setting up the matrices of H and H^2 and finally taking suitable traces. These are given in Table (IV.1). These averages are then propagated by means of eq.(4.2.11) to (4.2.12) to obtain moments for all configurations belonging to a given particle number m. Results for 12 particles are given in Table (IV.2).

It should be mentioned that while determining $\sigma^2(\tilde{m})$ if one uses the elementary net consisting of configurations upto 4 particles, numerical inaccuracies creep in. The elementary net is therefore modified by replacing all 1-particle configurations by corresponding hole configurations.

Table (IV.1)

Configuration centroids and variances for $m \leq 4$ particles in 2s-1d shell. The average quadrupole moment is given in the fifth column(see text for details)

m	$\tilde{m}$	$E_c(\tilde{m})$	$\sigma^2(\tilde{m})$	$\langle Q_{20} \rangle^{\tilde{m}}$
1	000001	-4.24	0.0	-2.0
	000010	-1.36	8.26	-0.5
	000100	-2.02	6.37	1.0
	001000	-2.02	6.37	1.0
	010000	-1.36	8.26	-0.5
	100000	-4.24	0.0	-2.0
2	000002	-12.20	0.0	-4.0
	000011	-7.12	7.86	-2.5
	000020	-4.58	15.11	-1.0
	000101	-6.85	9.03	-1.0
	000110	-4.38	20.03	0.5
	000200	-5.33	15.48	2.0
	001001	-6.96	9.81	-1.0
	001010	-4.12	19.85	0.5
	001100	-5.22	19.52	2.0
	002000	-5.33	15.48	2.0
	010001	-6.71	13.03	-2.5
	010010	-3.54	24.16	-1.0
	010100	-4.12	19.85	0.5
	011000	-4.38	20.03	0.5
	020000	-4.58	15.11	-1.0
	100001	-9.73	7.96	-4.0
	100010	-6.71	13.03	-2.5
	100100	-6.96	9.81	-1.0
	101000	-6.85	9.03	-1.0
	110000	-7.12	7.86	-2.5
	200000	-12.20	0.0	-4.0

Table (IV.1)(contd)

m	$\tilde{m}$	$E_c(\tilde{m})$	$\sigma^2(\tilde{m})$
3	000012	-16.61	5.80
	000021	-11.87	13.81
	000030	-9.65	16.29
	000102	-15.40	7.19
	000111	-10.73	20.17
	000120	-8.59	29.36
	000201	-10.74	19.80
	000210	-8.68	33.34
	000300	-9.92	23.82
	001002	-15.63	11.45
	001011	-10.58	21.25
	001020	-8.07	29.80
	001101	-10.74	24.55
	001110	-8.31	37.11
	001200	-9.70	33.60
	002001	-10.97	22.10
	002010	-8.16	32.98
	002100	-9.70	33.60
	003000	-9.92	23.82
	010002	-15.79	12.43
	010020	-7.58	36.74
	010101	-10.05	26.88
	010110	-7.29	40.62
	010200	-8.16	32.98
	011001	-10.43	27.75
	011010	-7.29	40.62
	010011	-10.42	27.29

Table (IV.1) (contd)

m	$\tilde{m}$	$E_c(\tilde{m})$	$\sigma^2(\tilde{m})$
3	011100	-8.31	37.11
	012000	-8.68	33.34
	020001	-11.04	23.60
	020010	-7.58	36.74
	020100	-8.07	29.80
	021000	-8.59	29.36
	030000	-9.65	16.29
	100002	-18.94	15.89
	100011	-13.37	19.98
	100020	-11.04	23.60
	100101	-13.04	20.19
	100110	-10.43	27.75
	100200	-10.97	22.10
	101001	-13.04	20.19
	101010	-10.05	26.88
	101100	-10.74	24.55
	102000	-10.74	19.80
	110001	-13.73	19.98
	110010	-10.42	27.29
	110100	-10.58	21.25
	111000	-10.73	20.17
	120000	-11.87	13.81
	200001	-18.94	15.89
	200010	-15.79	12.43
	200100	-15.63	11.45
	201000	-15.40	7.19
	210000	-16.61	5.80

Table (IV.1) (contd)

m	$\tilde{m}$	$E_c(\tilde{m})$	$\sigma^2(\tilde{m})$
4	000022	-22.88	10.29
	000031	-18.46	12.67
	000040	-16.58	16.50
	000112	-20.81	14.15
	000121	-16.46	27.55
	000130	-14.66	30.35
	000202	-19.88	16.93
	000211	-15.62	32.93
	000220	-13.88	44.37
	000301	-15.92	28.95
	000310	-14.27	44.64
	000400	-15.80	30.10
	001012	-20.77	19.20
	001021	-16.05	29.05
	001030	-13.87	32.13
	001102	-20.00	23.29
	001111	-15.36	38.26
	001120	-13.25	48.16
	001201	-15.82	39.38
	001210	-13.79	54.41
	001300	-15.48	45.08
	002002	-20.33	25.84
	002011	-15.32	35.91
	002020	-12.84	45.41
	002101	-15.93	40.78
	002110	-13.53	53.97
	002200	-15.37	51.25
	003001	-16.26	33.48

Table (IV.1) (contd)

m	$\tilde{m}$	$E_c(\tilde{m})$	$\sigma^2(\tilde{m})$
4	003010	-13.48	44.22
	003100	-15.48	45.08
	004000	-15.80	30.10
	010012	-21.02	23.41
	010021	-15.98	37.80
	010030	-13.48	41.74
	010102	-19.72	24.03
	010111	-14.75	43.90
	010120	-12.33	55.15
	010201	-14.68	41.38
	010210	-12.33	57.39
	010300	-13.48	44.22
	011002	-20.21	28.31
	011011	-14.87	45.23
	011020	-12.06	55.78
	011101	-14.95	46.03
	011110	-12.22	61.46
	011200	-13.53	53.97
	012001	-15.43	43.76
	012010	-12.33	57.39
	012100	-13.79	54.41
	013000	-14.27	44.64
	020002	-21.23	24.93
	020011	-15.57	42.38
	020020	-12.44	54.27
	020101	-15.12	40.23
	020110	-12.06	55.78

Table (IV.1)(contd)

m	$\tilde{m}$	$E_c(\tilde{m})$	$\sigma^2(\tilde{m})$
4	020200	-12.84	45.41
	021001	-15.75	40.45
	021010	-12.33	55.15
	021100	-13.25	48.16
	022000	-13.88	44.37
	030001	-17.23	27.25
	030010	-13.48	41.74
	030100	-13.87	32.13
	031000	-14.66	30.35
	040000	-16.58	16.50
	100012	-24.47	25.22
	100021	-19.58	28.54
	100030	-17.23	27.25
	100102	-22.84	26.02
	100111	-18.03	34.99
	100120	-15.75	40.45
	100201	-17.63	33.82
	100210	-15.43	43.76
	100300	-16.26	33.48
	101002	-22.95	29.50
	101011	-17.77	35.38
	101020	-15.12	40.23
	101101	-17.52	37.19
	101110	-14.95	46.03
	101200	-15.93	40.78
	102001	-17.63	33.82
	102010	-14.68	41.38
	102100	-15.82	39.38

Table (IV.1)(contd)

m	$\tilde{m}$	$E_c(\tilde{m})$	$\sigma^{-2}(\tilde{m})$
4	103000	-15.92	28.95
	110002	-24.06	26.69
	110011	-18.54	37.84
	110020	-15.57	42.38
	110101	-17.77	35.38
	110110	-14.87	45.23
	110200	-15.32	35.91
	111001	-18.03	34.99
	111010	-14.75	43.90
	111100	-15.36	38.26
	112000	-15.62	32.93
	120001	-19.58	28.54
	120010	-15.98	37.80
	120100	-16.05	29.05
	121000	-16.46	27.55
	130000	-18.46	12.67
	200002	-29.41	31.78
	200011	-24.06	26.69
	200020	-21.23	24.93
	200101	-22.95	29.50
	200110	-20.21	28.31
	200200	-20.33	25.84
	201001	-22.84	26.02
	201010	-19.72	24.03
	201100	-20.00	23.29
	202000	-19.88	16.93
	210001	-24.47	25.22
	210010	-21.02	23.41

Table (IV.1)(contd)

m	$\tilde{m}$	$E_c(\tilde{m})$	$\sigma^2(\tilde{m})$
4	210100	-20.07	19.20
	211000	-20.81	14.15
	220000	-22.88	10.29
23	246641	-308.03	0.0
	246632	-310.93	8.40
	246542	-311.77	7.01
	245642	-311.77	7.01
	236642	-310.93	8.40
	146642	-308.03	0.0

Table (IV.2)

Configuration centroids and variances for 12 particles using K+12 fp Interaction

$\tilde{m}$	$d(\tilde{m})$	$E_c(\tilde{m})$	$\sigma^2(\tilde{m})$	$\langle Q_{20} \rangle^{\tilde{m}}$
006600	1	-105.21	78.41	12.0
015510	576	-96.17	142.85	9.0
016320	480	-93.02	145.06	7.5
016401	120	-97.62	120.61	7.5
023610	480	-93.02	145.06	7.5
024420	8100	-90.49	171.09	6.0
024501	1080	-94.36	127.01	6.0
025230	2160	-89.03	151.81	4.5
025311	5760	-92.35	149.39	4.5
026040	6	-88.63	132.20	3.0
026121	432	-91.40	133.67	3.0
026202	90	-97.70	122.23	3.0
032520	2160	-89.03	151.81	4.5
032601	120	-92.16	103.58	4.5
033330	6400	-88.19	159.76	3.0
033411	9600	-90.77	142.29	3.0
034140	360	-88.41	137.80	1.5
034221	10800	-90.44	143.13	1.5
034302	1200	-96.01	117.07	1.5
035031	192	-91.18	121.58	0.0
035112	576	-96.20	114.99	0.0
040620	6	-88.63	132.20	3.0
041430	360	-88.41	137.80	1.5
041511	288	-90.26	114.46	1.5

Table (IV.2) (contd)

$\tilde{m}$	$d(\tilde{m})$	$E_c(\tilde{m})$	$\sigma^2(\tilde{m})$	$\langle Q_{20} \rangle$
042240	225	-89.26	137.86	0.0
042321	3600	-90.55	132.82	0.0
042402	225	-95.39	102.18	0.0
043131	960	-91.91	125.05	-1.5
043212	1200	-96.19	109.36	-1.5
044022	90	-98.07	106.76	-3.0
104610	120	-97.62	120.61	7.5
105420	1080	-94.36	127.01	6.0
105501	144	-97.96	104.24	6.0
106230	120	-92.16	108.58	4.5
106311	320	-95.21	113.63	4.5
113520	5760	-92.35	149.89	4.5
113601	320	-95.21	113.63	4.5
114330	9600	-90.77	142.93	3.0
114411	14400	-93.08	142.66	3.0
115140	233	-90.26	114.46	1.5
115221	8640	-92.02	135.46	1.5
115302	960	-97.32	119.99	1.5
116031	64	-92.02	109.99	0.0
116112	192	-96.77	111.07	0.0
121620	432	-91.40	133.67	3.0
122430	10800	-90.44	148.13	1.5
122511	8640	-92.02	135.46	1.5
123240	3600	-90.55	132.82	0.0
123321	57600	-91.58	156.45	0.0
123402	3600	-96.14	121.54	0.0
124131	8640	-92.20	132.17	-1.5

Table (IV.2) (contd)

$\tilde{m}$	$d(\tilde{m})$	$E_c(\tilde{m})$	$\sigma^2(\tilde{m})$	$\langle Q_{20} \rangle^{\tilde{m}}$
124212	10800	-96.22	128.19	-1.5
125022	432	-97.35	111.00	-3.0
130530	192	-91.18	121.58	0.0
130611	64	-92.02	109.99	0.0
131340	960	-91.91	125.05	-1.5
131421	8640	-92.20	132.17	-1.5
131502	288	-96.03	103.00	-1.5
132231	14400	-93.45	136.02	-3.0
132312	9600	-96.73	119.53	-3.0
133041	320	-95.76	110.35	-4.5
133122	5760	-98.49	120.42	-4.5
140331	320	-95.76	110.35	-4.5
140412	120	-98.30	94.45	-4.5
141141	144	-98.69	104.14	-6.0
141222	1080	-100.68	103.43	-6.0
142032	120	-104.12	100.15	-7.5
202620	90	-97.70	122.28	3.0
203430	1200	-96.01	117.07	1.5
203511	960	-97.32	119.99	1.5
204240	225	-95.39	102.18	0.0
204321	3600	-96.14	121.54	0.0
204402	225	-100.44	122.36	0.0
205131	288	-96.03	103.00	-1.5
205212	360	-99.78	112.86	-1.5
206022	6	-100.18	109.79	-3.0
211530	576	-96.20	114.99	0.0
211611	192	-96.77	111.07	0.0

Table (IV.2) (contd)

$\tilde{m}$	$d(\tilde{m})$	$E_c(\tilde{m})$	$\sigma^2(\tilde{m})$	$\langle Q_{20} \rangle^{\tilde{m}\pi}$
212340	1200	-96.19	109.36	-1.5
212421	10800	-96.22	128.19	-1.5
212502	360	-99.78	112.86	-1.5
213231	9600	-96.73	119.53	-3.0
213312	6400	-99.74	118.90	-3.0
214041	120	-98.30	94.45	-4.5
214122	2160	-100.76	106.92	-4.5
220440	90	-98.07	106.76	-3.0
220521	432	-97.35	111.00	-3.0
220602	6	-100.18	109.79	-3.0
221331	5760	-98.49	120.42	-4.5
221412	2160	-100.76	106.92	-4.5
222141	1080	-100.68	103.43	-6.0
222222	8100	-102.40	115.57	-6.0
223032	480	-105.11	97.52	-7.5
230241	120	-104.12	100.15	-7.5
230322	480	-105.11	97.52	-7.5
231132	576	-108.44	96.57	-9.0
240042	1	-117.85	78.41	-12.0

Conversion of $\tilde{m}$ moments to fixed-J moments is easily accomplished using eqs.(4.2.7) and (4.2.8). Results for m = 4, 8, 9, 10, 11 and 12 are shown in Table (IV.3). One observes that the centroid energy $E_c(mJ)$ and width $\sigma(mJ)$ decrease with increasing angular momentum. This inversion of the centroid spectrum appears strange because we might normally expect the low angular momentum state to have a lower centroid energy.

4.2.2 Spectra of States with fixed-J

Having determined fixed-J moments, the low lying states of fixed-J in a nucleus are located using Ratcliff's procedure. Since isospin T for the distribution is not specified, one has to restrict to nuclei with $T = T_z = 1/2\ (N-Z)$.

In Ratcliff's procedure since one is trying to determine a set of discrete energy levels from a continuous distribution, it is expected that one would obtain only the most probable position. The fluctuation of a shell model state from this most probable position is defined as:

$$\Delta E_i = E_i(SM) - E_i\ (JT)$$

and this fluctuation is upto one local level spacing (Laugheed and Wong, 1975). Therefore in comparison one can only expect the level position to be accurate utmost to the fluctuation limit.

Table (IV.3)

Centroids and variances for fixed-J moments in the 2s-1d shell using (K+12) fp interaction. Energies are in MeV with respect to O^{16}.

m	J	d(mJ)	E_c(mJ)	σ^2(mJ)
4	0	114	-13.21	9.01
	1	253	-13.16	7.82
	2	359	-13.71	7.77
	3	327	-14.02	7.16
	4	269	-14.67	6.74
	5	156	-15.64	5.94
	6	80	-16.90	5.47
	7	26	-19.61	3.72
	8	6	-22.89	3.21
8	0	3697	-44.18	11.91
	1	9928	-44.26	11.61
	2	14328	-44.52	11.55
	3	15640	-44.89	11.25
	4	14636	-45.40	10.99
	5	11677	-46.06	10.49
	6	8255	-46.91	9.96
	7	5004	-47.89	9.22
	8	2652	-49.08	8.47
	9	1151	-50.38	7.51
	10	413	-51.98	6.53
	11	101	-53.73	5.26
	12	15	-56.60	3.94
9	1/2	11630	-54.63	12.26
	3/2	20654	-54.77	12.17

Table (IV.3)(contd)

9	5/2	25746	-55.09	11.99
	7/2	26178	-55.52	11.70
	9/2	23068	-56.07	11.37
	11 /2	17704	-56.79	10.85
	13 /2	12032	-57.63	10.28
	15 /2	7102	-58.61	9.54
	17 /2	3654	-59.74	8.76
	19 /2	1550	-61.01	7.79
	21/ 2	546	-62.43	6.85
	23/ 2	130	-64.28	5.49
	25/ 2	20	-66.35	4.32
10	0	8629	-66.05	12.85
	1	23575	-66.16	12.60
	2	34272	-66.41	12.53
	3	38236	-66.76	12.26
	4	36624	-67.24	11.98
	5	30368	-67.83	11.53
	6	22449	-68.55	11.06
	7	14523	-69.37	10.41
	8	8312	-70.36	9.70
	9	4048	-71.41	8.84
	10	1681	-72.68	7.96
	11	539	-74.02	6.82
	12	129	-75.82	5.70
	13	15	-77.37	4.04
11	1/2	20452	-78.65	12.97
	3/2	36716	-78.84	12.84
	5/2	46092	-79.12	12.67
	7/2	47556	-79.52	12.37

Table (IV.3)(contd)

11	9/2	42652	-80.02	12.03
	11/2	33620	-80.63	11.57
	13/2	23598	-81.34	11.05
	15/2	14546	-82.17	10.38
	17/2	7906	-83.08	9.68
	19/2	3634	-84.14	8.78
	21/2	1408	-85.34	7.86
	23/2	412	-86.73	6.70
	25/2	88	-88.30	5.53
	27/2	6	-89.68	3.01
12	0	11434	-92.25	13.28
	1	31332	-92.38	12.99
	2	45640	-92.61	12.89
	3	51267	-92.92	12.61
	4	49441	-93.35	12.32
	5	41456	-93.85	11.89
	6	31064	-94.46	11.44
	7	20461	-95.14	10.85
	8	11975	-95.94	10.22
	9	6027	-96.79	9.44
	10	2597	-97.89	8.58
	11	837	-99.02	7.49
	12	237	-100.47	6.45
	13	35	-101.88	4.75
	14	1	-103.28	0

In fig.(IV.1) we compare the low lying levels of a fixed-J for some cases with shell model and experimental observations. The spacings between the first few levels of fixed-J match well with the experimental difference in energies. This shows that given the energy of the first level for any J, one can predict accurately the energies of other levels with the same J. It should be mentioned that the ordering of levels as determined from fixed-J averages may not be the same as the observed level sequence.

Evaluation of moments having fixed-J and T would yield more precise information. Lougheed and Wong (1975) have obtained fixed-JT averages and used them to determine the level densities and level positions for nuclei in 2s-1d shell. They find that there is a pronounced systematic energy difference between shell model and fixed-JT level positions measured in local energy units. Large deviations occur in a region where the level density is high and therefore the local spacing unit is very small in terms of energy. In general the low lying level positions from JT averaging compare favourably with those given by shell model.

4.3 Average Deformation Energy Curves

In this section the average deformation of a nucleus as a function of excitation energy is investigated using the

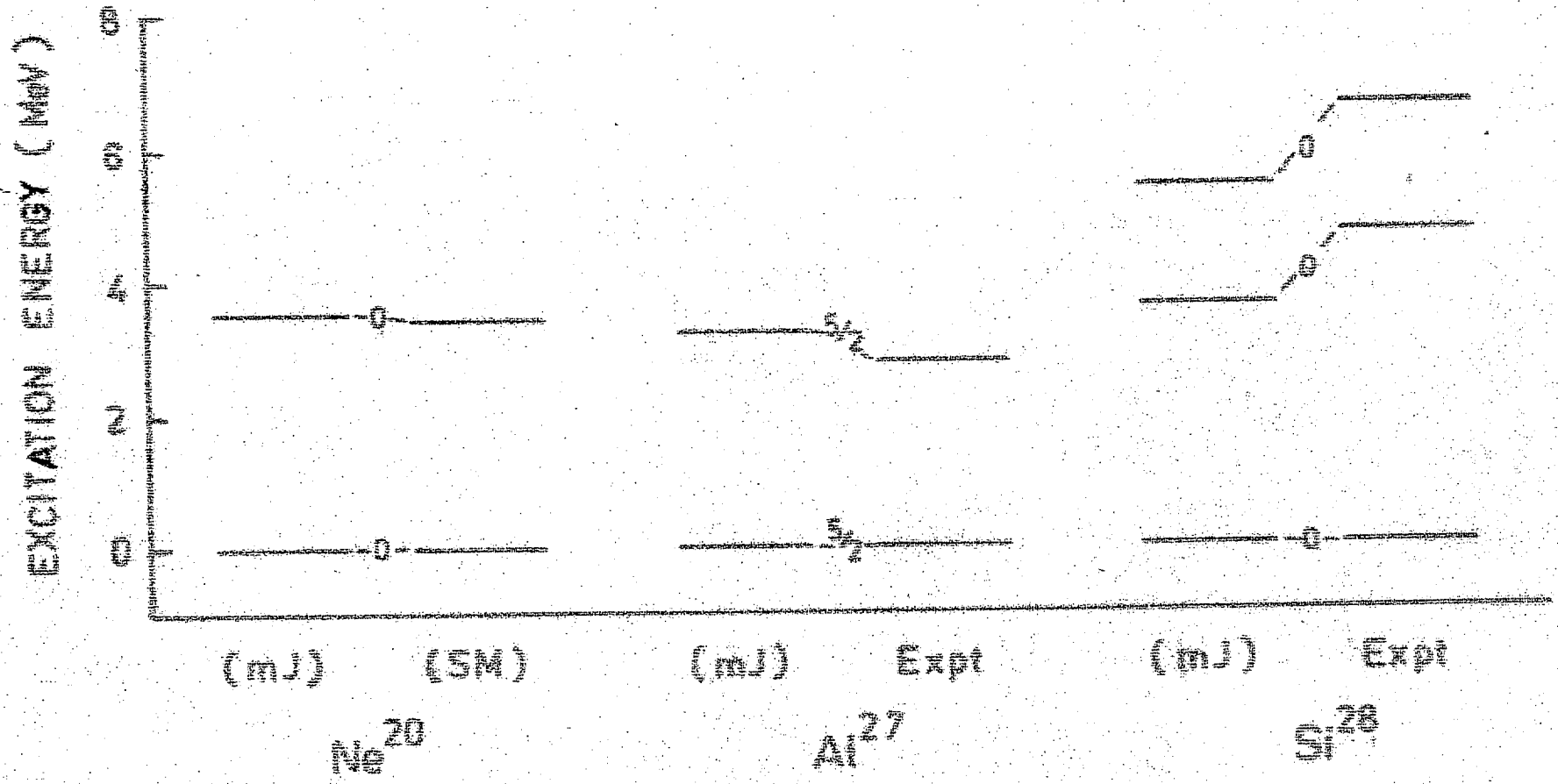

Fig. IV.1 First few levels of a given J as obtained from fixed-J moments compared with shell model results and experiment.

spectral distribution method.

According to the arguments of French (1971), the energy variation of an operator O can be written as:

$$\langle O \rangle_E = \sum_{\underset{\sim}{\alpha}} I_E(\underset{\sim}{\alpha}) \langle O \rangle^{\underset{\sim}{\alpha}} \qquad (4.3.1)$$

where $I_E(\underset{\sim}{\alpha})$ represents the normalized intensity of the representation at energy E. The assumption in writing eq.(4.3.1) is that the operator O must not connect two different representations and further, the expectation value of O does not vary strongly over the width of the representations.

For average deformation energy curves, one starts by determining the average quadrupole moment for the configuration $\tilde{m}$. This is achieved quite simply because $Q_{20} = \sqrt{\frac{16\pi}{5}} r^2 Y_{20}$ is a 1-body operator and its propagation is simple. For any configuration $\tilde{m}$ in 2s-1d shell, the average quadrupole moment can be written as:

$$\langle Q_{20} \rangle^{\tilde{m}} = -2m_1 - m_2/2 + m_3 + m_4 - m_5/2 - 2m_6 \qquad (4.3.2)$$

The energy variation is given by,

$$\langle Q_{20} \rangle_E^m = \sum_{\tilde{m}} I_E(m\tilde{m}) \langle Q_{20} \rangle^{\tilde{m}} \qquad (4.3.3)$$

where $I_E(m\,\tilde{m})$ is the relative intensity of configuration $\tilde{m}$ at energy E. These intensities are determined by first defining a frequency function f(mE) which is a sum of the individual frequency functions f ($m\tilde{m}E$) (c.f. Sec.1.2).

$$f(mE) = \sum_{\tilde{m}} f(m\tilde{m}E) = \sum_{\tilde{m}} \frac{d(\tilde{m})}{\sqrt{2\pi}\,\sigma(\tilde{m})} \exp\left[-\frac{(E - E_c(\tilde{m}))^2}{2\sigma^2(\tilde{m})}\right] \tag{4.3.4}$$

The functions $f(m\tilde{m}E)$ for m = 4 and $K(\langle J_z \rangle) = 0$ are shown in figure (IV.2). A distribution function F(mE) is then defined as:

$$F(mE) = \int_{-\infty}^{E} f(mE')\,dE' \tag{4.3.5}$$

The relative contributions $I_E(m\tilde{m})$ of each configuration $\tilde{m}$ to F(mE) is given by

$$I_E(m\tilde{m}) = \frac{1}{F(mE)} \int_{-\infty}^{E} f(m\tilde{m}E')\,dE' \tag{4.3.6}$$

The quantity $I_E(m\tilde{m})$ corresponds to the relative contributions of various configurations $\tilde{m}$ to the total area under the curve at energy E and is called mixing intensity.

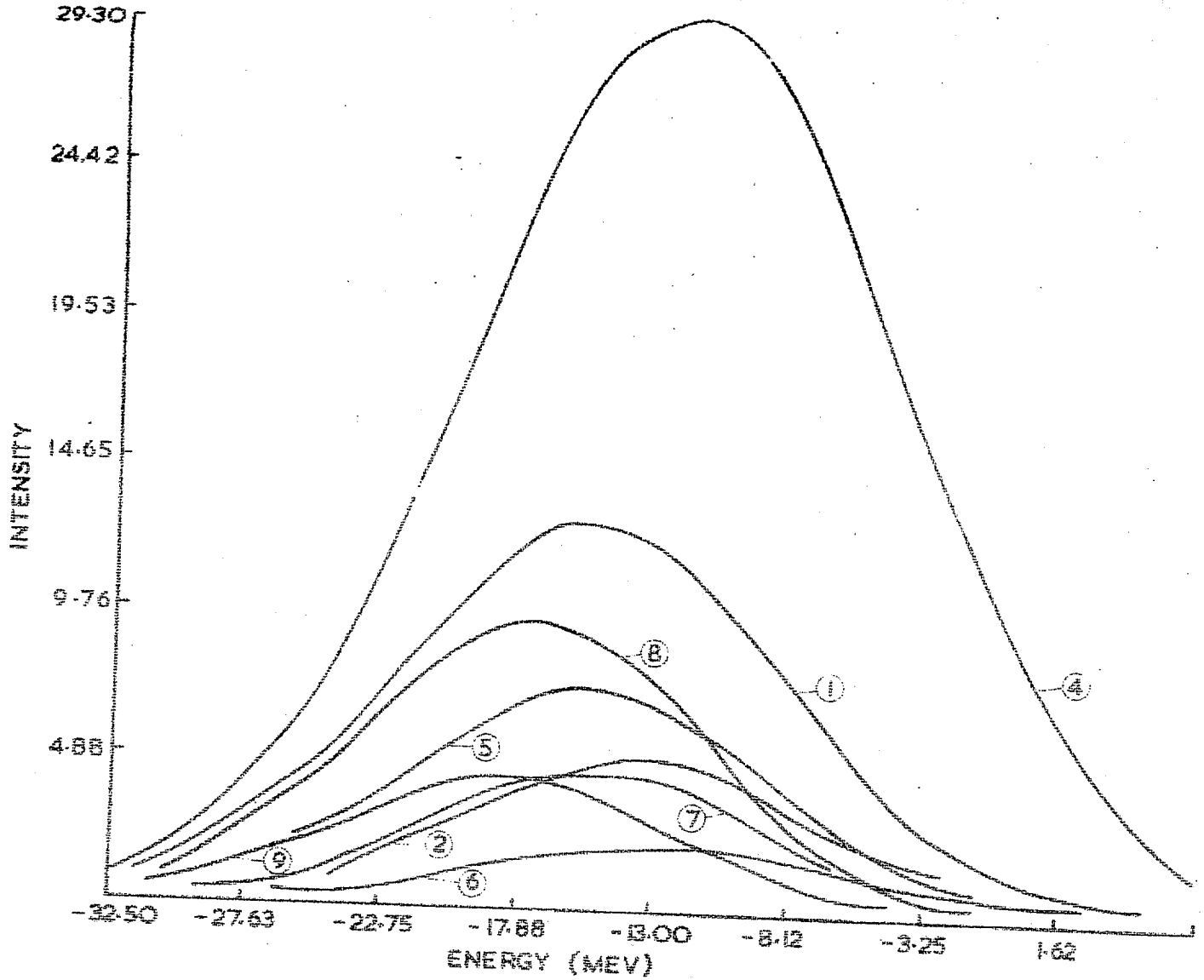

Fig. IV.2 Frequency functions for m=4 and $\langle J_z \rangle$ = 0 using K + 12fp interaction in 2s-1d shell.

Using equations (4.3.3) and (4.3.6) the energy variation of average quadrupole moment is obtained. In eq.(4.3.3) summation is restricted to all $\tilde{m}$ having a fixed K value. Results are shown in fig.(IV.3 - IV.8). For m = 4, the deformation first increases, then shows a dip and finally saturates. For m = 8, 9, 10 and 11, the general characteristics are the same, deformation first decreasing with excitation energy, then increasing and finally becoming constant. For m = 12, the trend is different and the average quadrupole moment after a small increase, decreases almost linearly with excitation energy.

In order to obtain the "response" of a nucleus to an external quadrupole field we can follow a procedure which is similar to the one used in constrained Hartree-Fock method (Jacquemin and Parikh, 1975).

We define an operator $H(\lambda)$

$$H(\lambda) = H + \lambda Q_{20}$$

where H is the usual nuclear Hamiltonian, Q_{20} is the "external" quadrupole field and λ a Lagrange multiplier.

Using the method described earlier in this section we can evaluate $\langle H(\lambda) \rangle^{\tilde{m}}$ and $\langle H^2(\lambda) \rangle^{\tilde{m}}$ and obtain $E_c(\tilde{m};\lambda)$ and $\sigma^2(\tilde{m};\lambda)$. Further by Ratcliff's procedure we can locate the lowest energy state E_g' of the operator $H(\lambda)$ and also

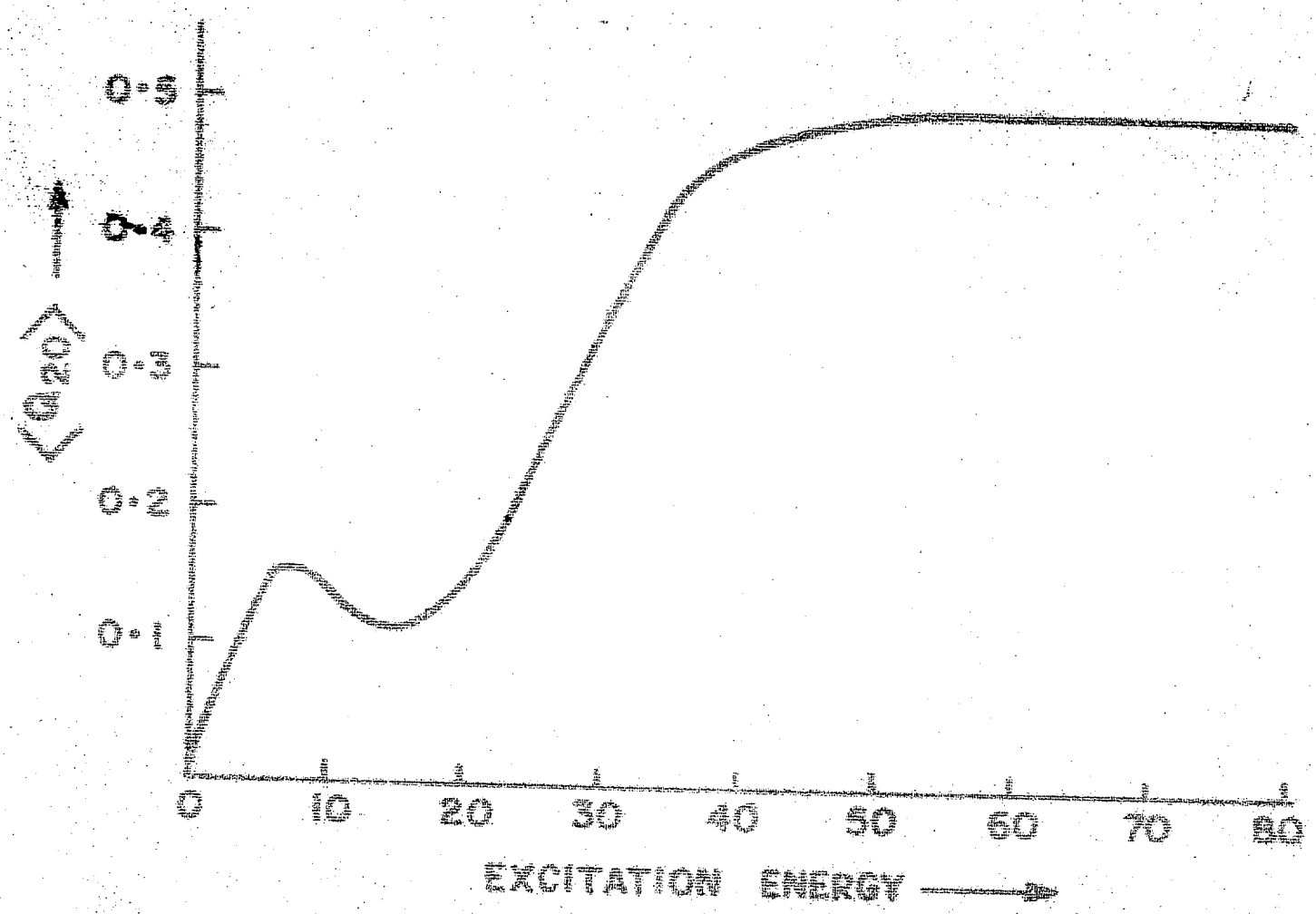

Fig. IV.3 Average quadrupole moment for 4 particles (K=0) as a function of excitation energy. (see text for details)

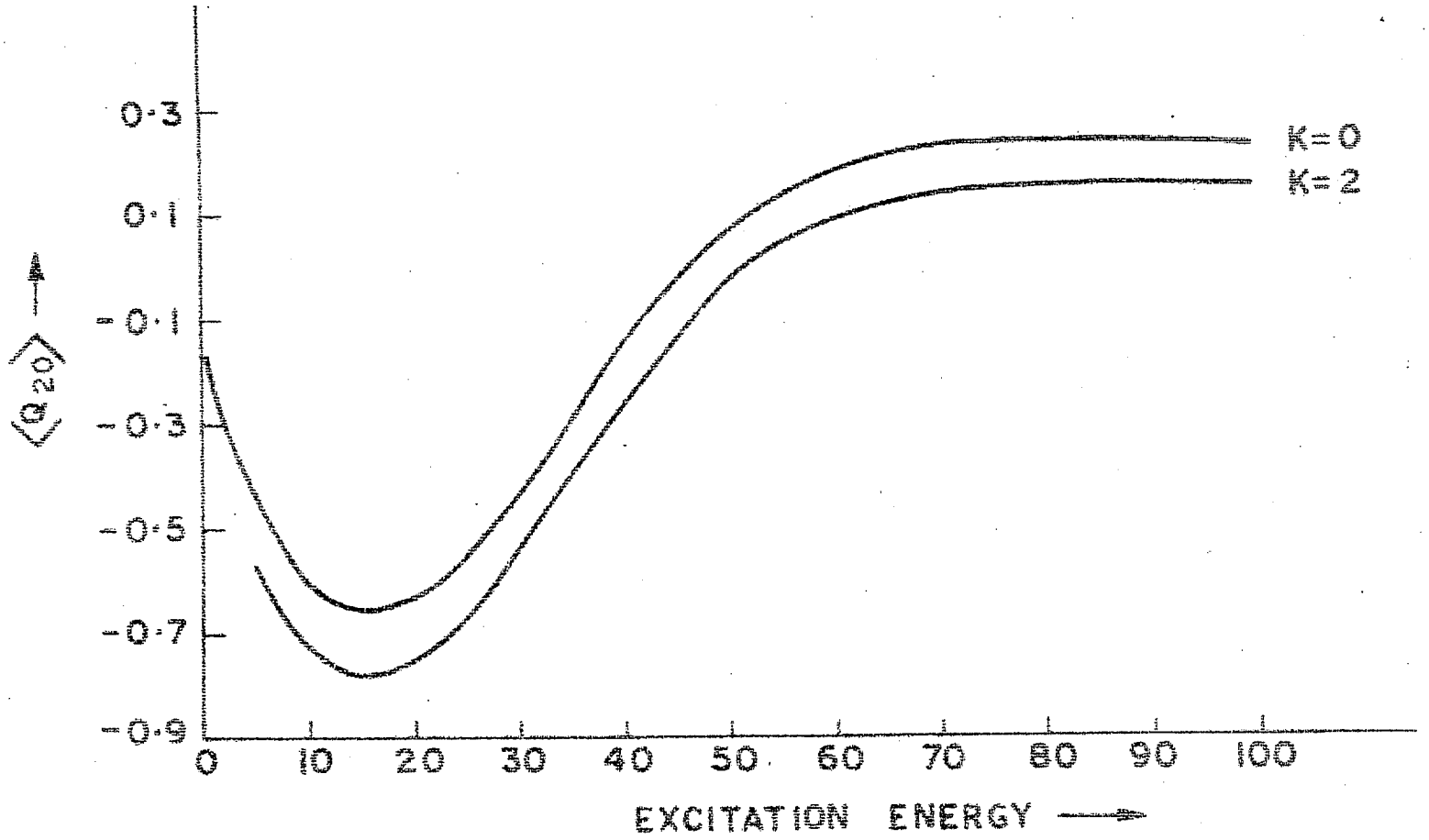

Fig. IV.4 Average deformation energy curve for m=8 (K=0 and 2) in 2s-1d shell.

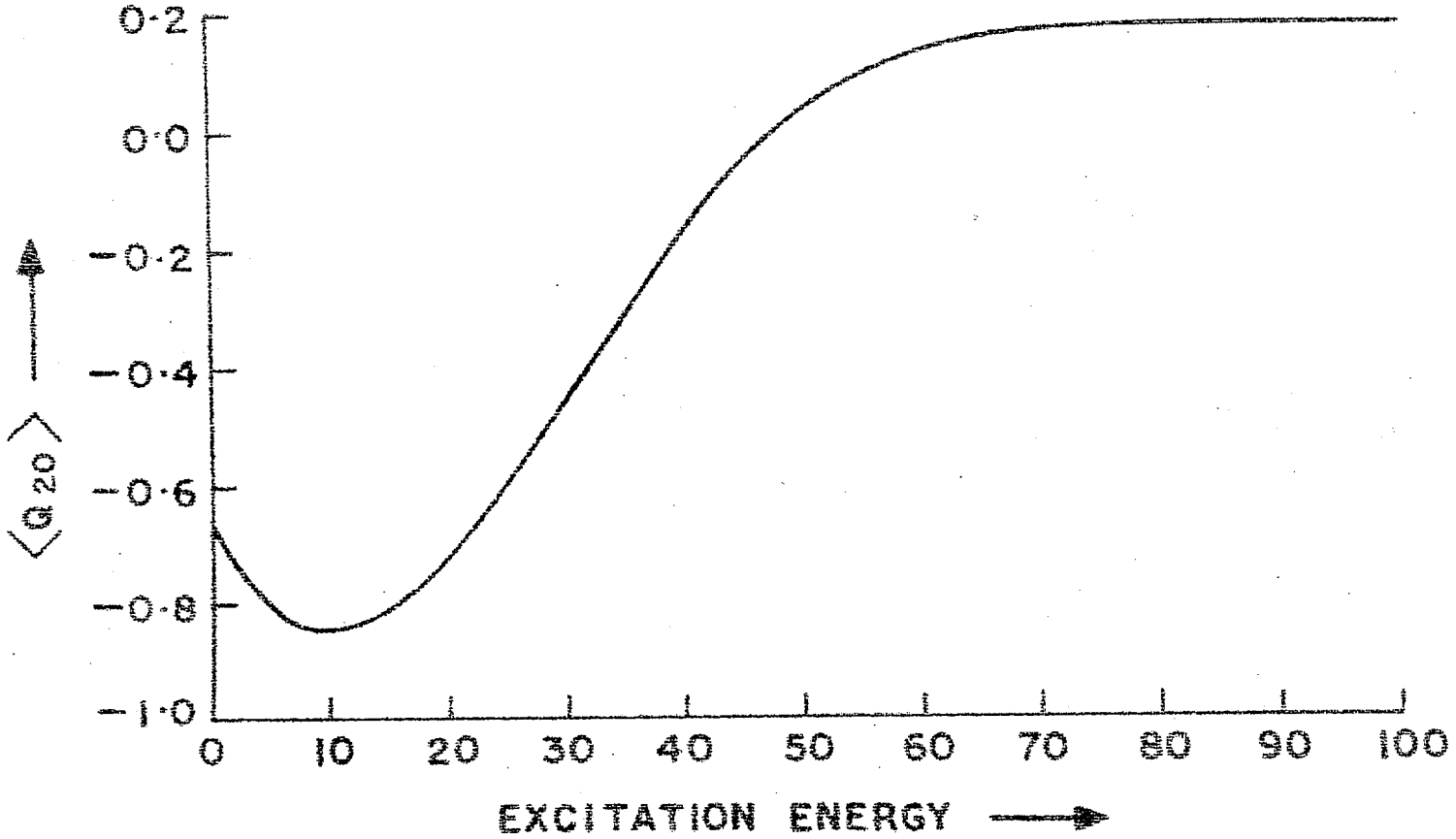

Fig. IV.5 Average deformation energy curve for m = 9 ($K=\frac{1}{2}$) particles.

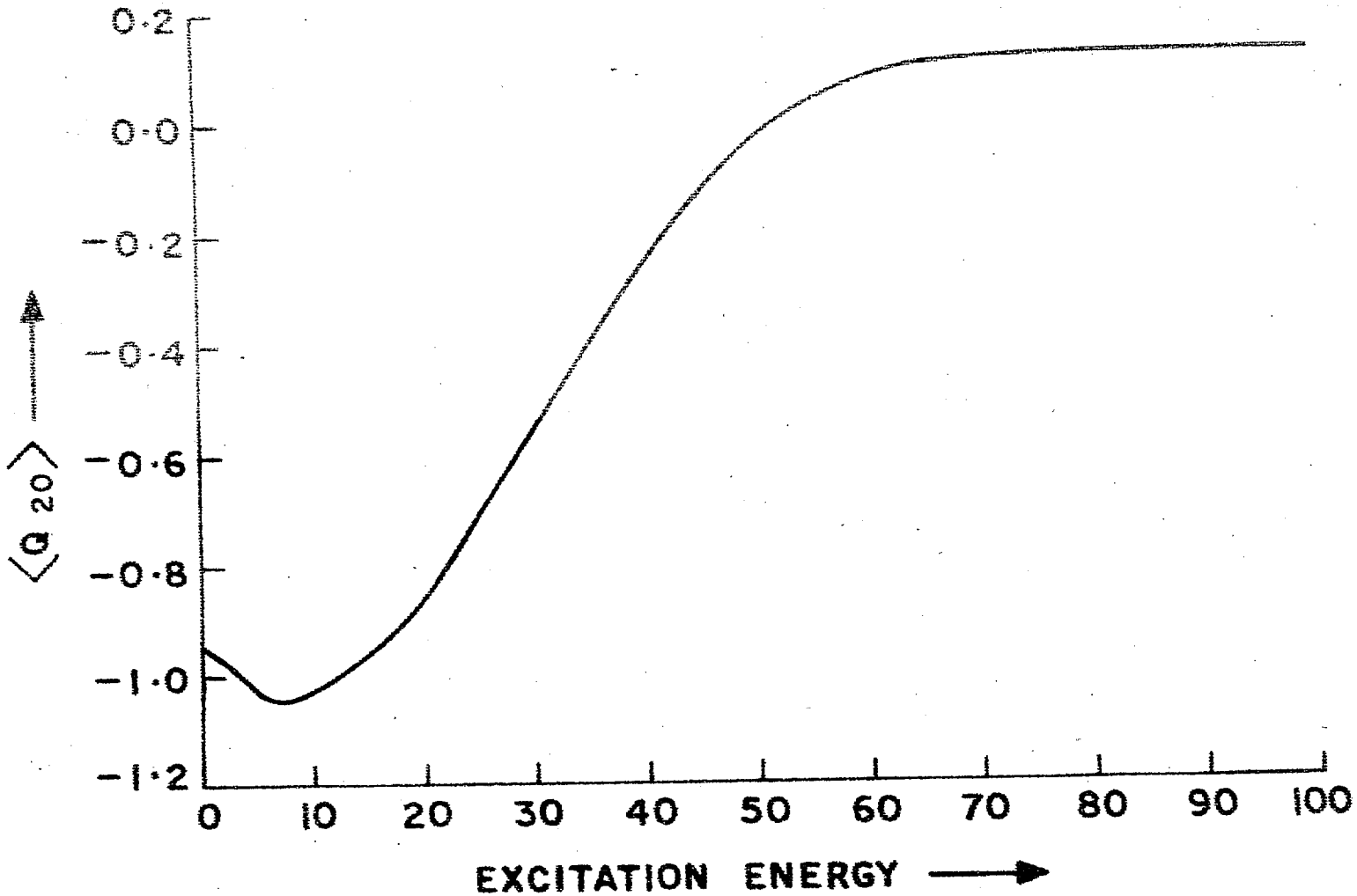

Fig. IV.6 Average quadrupole moment for m=10 (K=0) as a function of excitation energy.

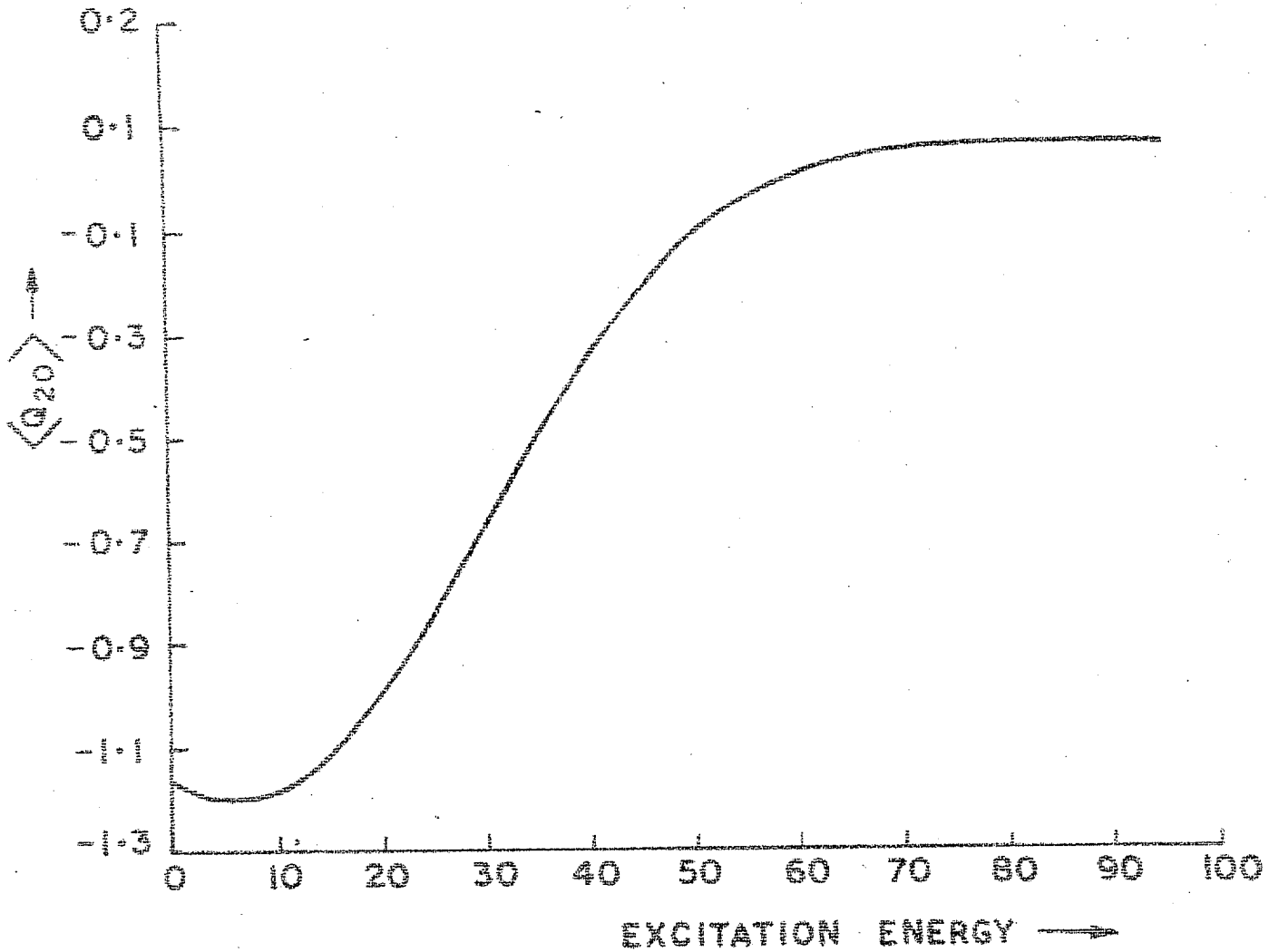

Fig. IV.7 Energy VS average deformation curve for 11 particles ($K=\frac{1}{2}$) in 2s-1d shell.

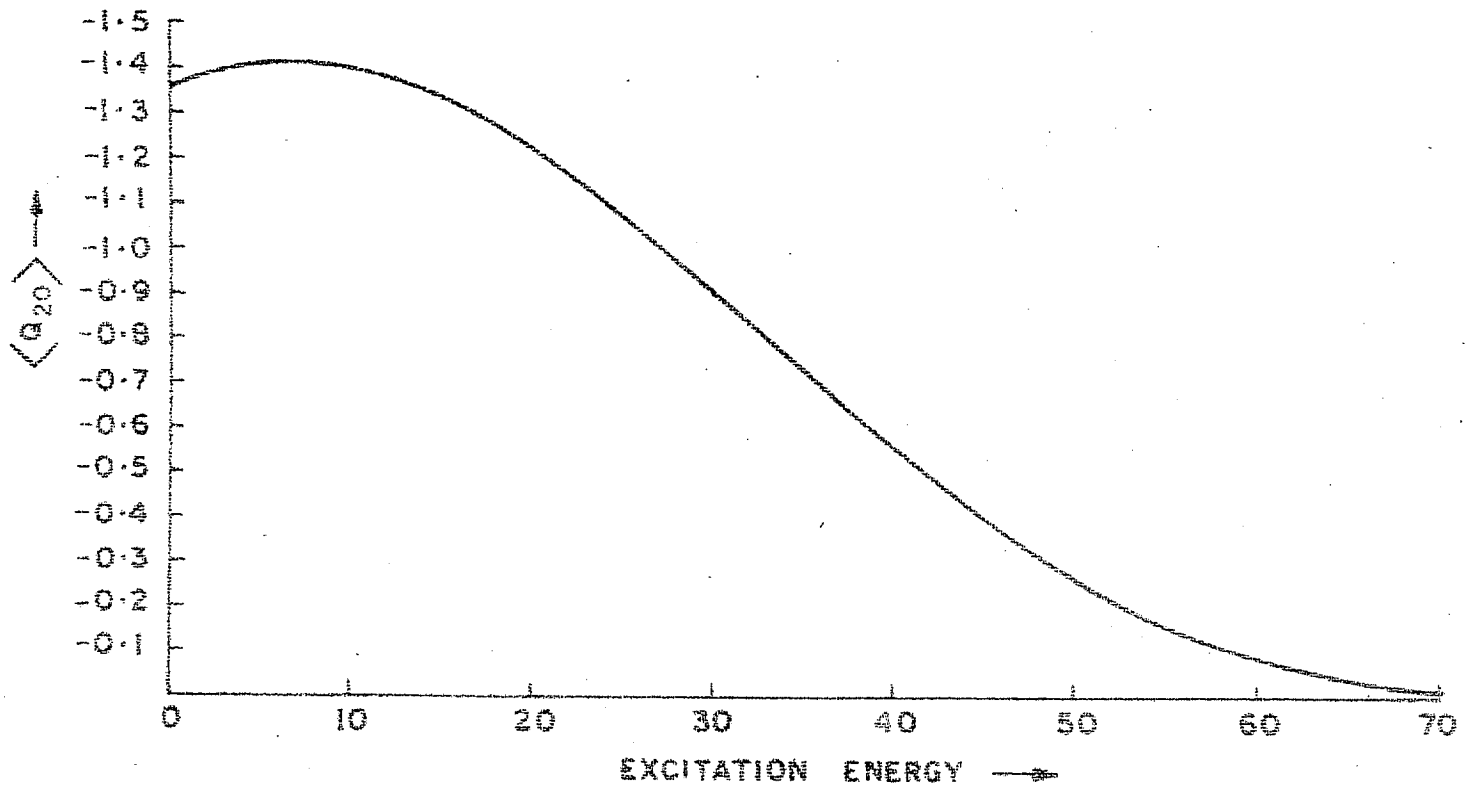

Fig.IV. 8 **Variation of the average quadrupole moment as a function of excitation energy for m=12 (K=0).**

evaluate the expectation value $\epsilon(\lambda)$ in this state. It is simple then to obtain the "true" energy $E_g(\lambda)$ of this ground state. We have

$$E_g(\lambda) = E'_g - \lambda\epsilon(\lambda)$$

Thus by taking different values of λ we can obtain $E_g(\lambda) \rightarrow \epsilon(\lambda)$ curve which may then be compared with the constrained HF results. It would be **very** interesting to carry out such calculations and study the differences (if any) with the HF results.

4.4 Energy vs Average Deformation using Elliott SU(3) Scheme

One can also study the variation of average quadrupole moment within the SU(3) frame-work. Matrix elements of Q_{20} between SU(3) states $(\lambda\mu\ KLM)$ are given by Elliott (1958).

$$\langle(\lambda\mu)KLM|Q_{20}|(\lambda\mu)KLM\rangle$$

$$= \frac{\{3K^2 - L(L+1)\}\{3M^2 - L(L+1)\}}{L(L+1)(2L-1)(2L+3)} \begin{bmatrix} \mu + 2\lambda + 3 \\ -\lambda - 2\mu - 3 \end{bmatrix} \qquad (4.4.1)$$

where the upper value in the box bracket applies to the case $\lambda \geq \mu$ and the lower value for $\lambda < \mu$. For a given $(\lambda\mu)$ the average quadrupole moment for a fixed value of M is determined.

The energy variation can be obtained as:

$$\langle Q_{20} \rangle_E^{mM} = \sum_{(\lambda\mu)} I_E\left[(\lambda\mu) M\right] \langle Q_{20} \rangle^{(\lambda\mu)M} \qquad (4.4.2)$$

The relative intensities $I_E\left[(\lambda\mu)\ M\right]$ can be determined using the distribution method. For the present an approximate way of determining these has been used. Parikh (1972) has applied the distribution method to the group SU(3) and determined the dimensionalities, centroids and variances of SU(3) representations $(\lambda\mu)$. Using these centroids and variances but changing the dimensionalities appropriately (since one is here restricting to a fixed M), the intensities $I_E\left[(\lambda\mu)M\right]$ have been evaluated.

Results for m = 4, 8 and 12 using KB Interation (1966) are shown in figures (IV.9 - IV.11). Whereas for m = 4 and 8 the curves are of a similar nature, the m = 12 curve shows a different trend.

4.5 Conclusion

In the first part of this chapter fixed angular momentum averages were evaluated and used to determine spectra of nuclei.

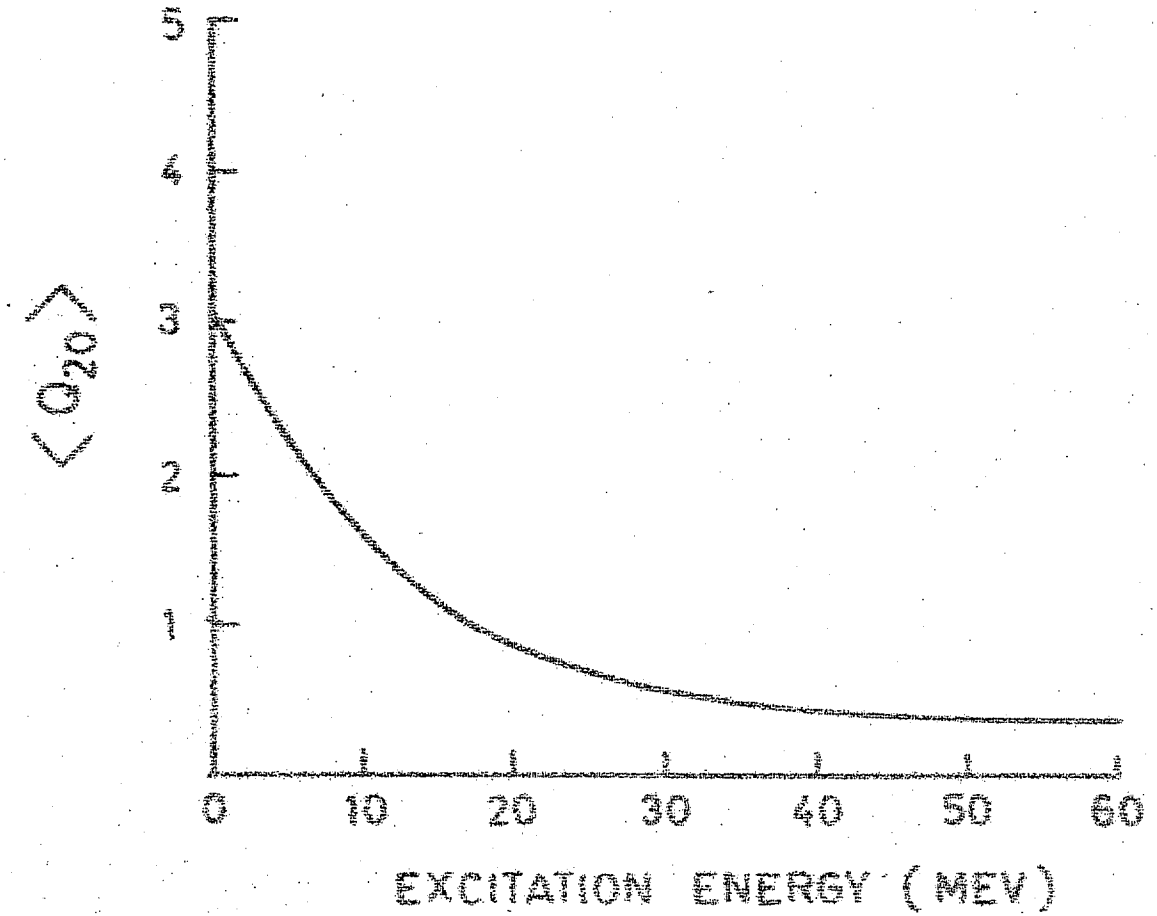

Fig. IV.9 Average deformation energy curve for m=4 using Elliott's SU(3) scheme (see text for details)

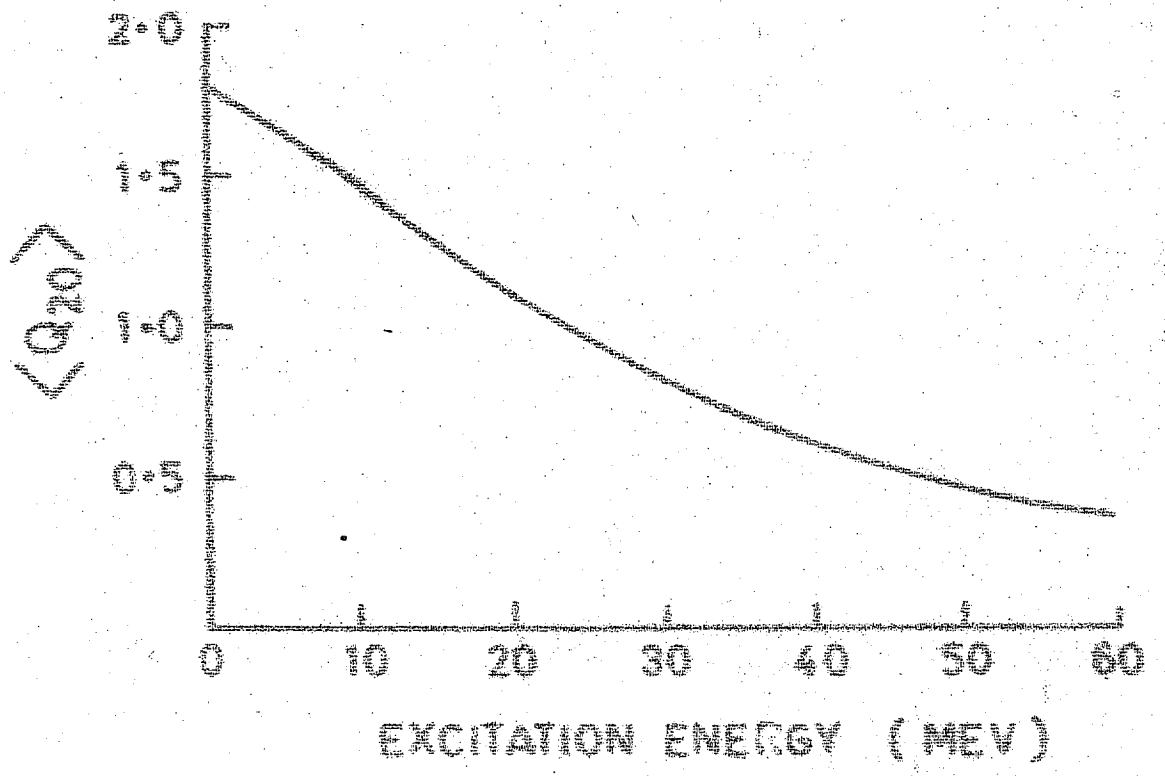

Fig. IV. 10 Average quadrupole moment as a function of excitation energy in SU(3) scheme for n=3.

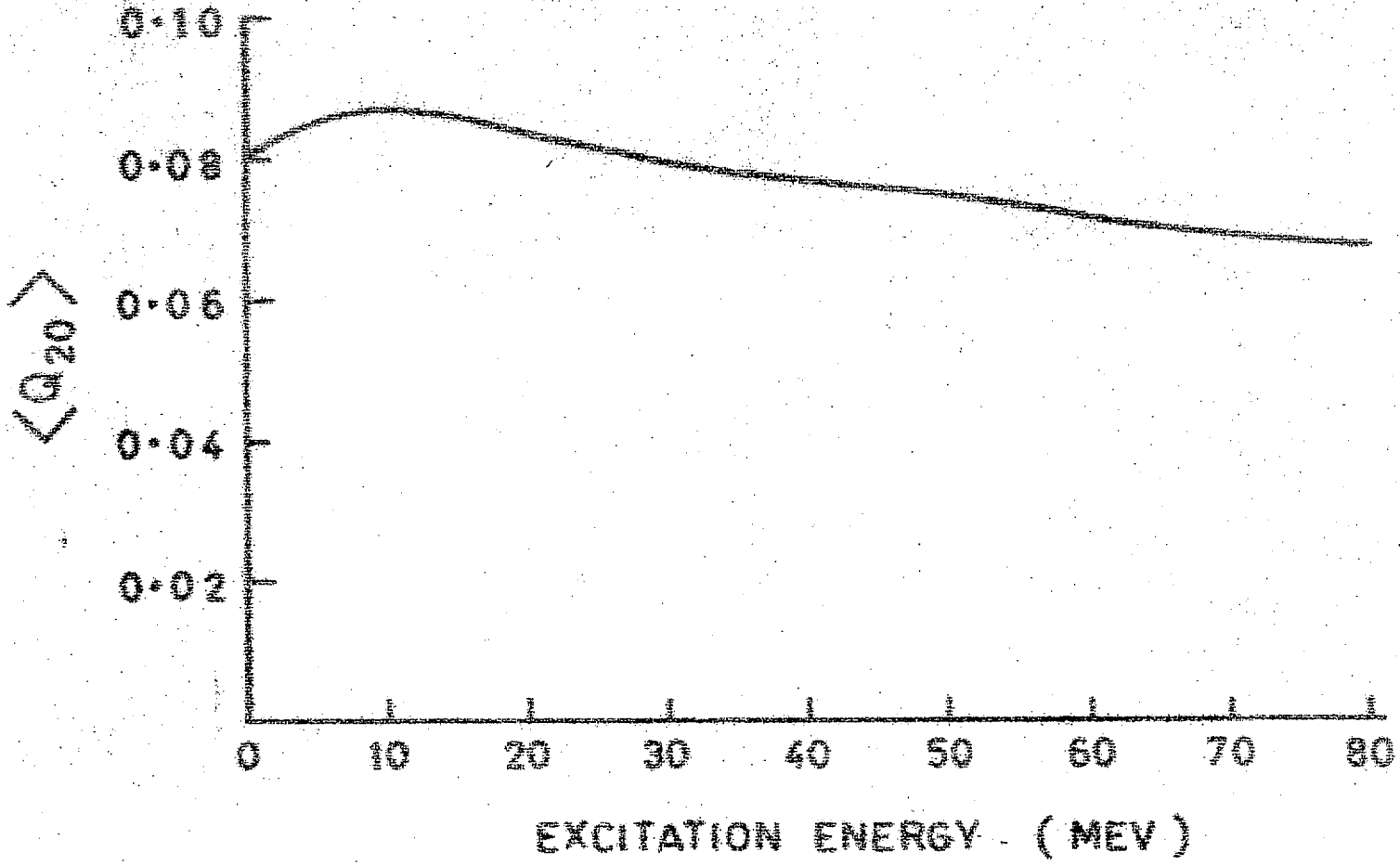

Fig. IV. 11 Average quadrupole moment as a function of excitation energy in SU(3) scheme for 12 particles in 2s-1d shell.

Results show that the level sequence as obtained using fixed-J distributions is different from the observed spectrum. However, the method estimates the spacings of levels of same J quite accurately. In order to do better one needs to evaluate fixed J and T moments. It has been shown by Lougheed and Wong (1975) using fixed-JT averages that the low-lying energy levels can be calculated with good accuracy and that the level densities are very close to the exact shell model results.

In the second part of this chapter, we have obtained the "average" deformation energy curves. A way of evaluating deformation energy curves has been described and using this method one would be able to study how the deformation of a nucleus would change as it is excited.

CHAPTER V

FORMAL ASPECTS REGARDING GROUP AVERAGING

5.1 Introduction

The spectral distribution methods have provided a way for the study of nuclear structure in large vector spaces. Since these methods deal with the averages of H and powers of H over states defined by the irrep of a group, one can under certain reasonable physical assumptions study the goodness of the symmetry defined by the relevant group. Even in cases where the validity of symmetry is approximate (no pure symmetry states which are intermingled in the low energy region), the distribution method provides information regarding general properties of nuclei.

It is clear then, that these methods are going to be very useful for a systematic study of nuclear symmetries. Therefore, it is essential that one should be able to use these methods for as many groups as possible.

In the distribution method, since the average of H and powers of H over an irrep $\underset{\sim}{\alpha}$ of a group involves a sum with equal weight for all states in $\underset{\sim}{\alpha}$, only the group scalar part of the appropriate power of H would contribute (c.f. Sec. 1.2). The evaluation of averages is simple when there are enough scalar operators of the group to reproduce the moments

of H. Explicitly then, for the operators H and H^2 which have particle rank $\leq$ 2 and $\leq$ 4 respectively, these being completely specified in the (0-2) and (0-4) particle spaces (defining space) respectively, one tries to construct equivalent operators $\hat{H}$ and $\hat{H}^2$ with the correct particle rank in terms of the scalar operators of the group. If these operators reproduce the average of H and H^2 in the defining space, they will do so in all representations. $\hat{H}$ and $\hat{H}^2$ then serve to propagate information from the defining space to many particle spaces. For the equivalence to be faithful the number of irreps in the defining space should be equal to the number of independent scalar operators.

For those cases where the number of scalar operators are not sufficient one needs extension of the averaging procedure. In this chapter a method by which averages in such cases can be determined is discussed. We will be discussing here only the formal aspects of averaging and no calculations for nuclei are given. As an application of the method, three specific cases are treated and expressions for centroid energy and variance derived.

The methods used here are quite general and could be easily extended when similar cases of averaging over other groups are encountered.

5.2 SU(4)-Isospin Averaging

For evaluating the average of H over states defined by SU(4) symmetry $\underset{\sim}{f}$ and isospin T, one observes that there are six representations in the defining space (upto 2 particles) and only five scalar operators. These are 1, n, n^2, G_2 and T^2 where n, G_2 and T^2 are the number operator, the bilinear Casimir operator of SU(4) and the isospin operator respectively. In order to construct an equivalent operator $\hat{H}$ of H, a search for other scalar operators is made. By defining a mixed operator $\mathcal{G}=(G_2T^2)_{2-b}$ where the subscript 2-b indicates that only the (0+1+2) body part of G_2T^2 should be considered, $\hat{H}$ can be written as a polynomial expression in terms of these operators. Explicitly, one can write

$$\hat{H} = P_2(n) + P_o(n)\, G_2 + P_o(n)\, T^2 + P_o(n)\mathcal{G} \qquad (5.2.1)$$

where $P_k(n)$ is a polynomial of degree k in the number operator n. Similarly the equivalent operator for H^2 can be written as,

$$\begin{aligned}\hat{H}^2 = {} & P_4(n)+P_2(n)\, G_2+P_1(n)\, G_3 + P_o(n)\, G_4 \\ & + P_o(n)\, G_2^2 + P_2(n)\, T^2+P_o(n)\, T^4+P_2(n)\mathcal{G} \\ & + P_o(n)\, \mathcal{G}^2 + P_o(n)\, G\mathcal{G} + P_o(n)\, T^2\mathcal{G}\end{aligned} \qquad (5.2.2)$$

where G_3 and G_4 are the cubic and quartic Casimir operators of SU(4). If we know the centroids for all irreps with $m \leq 2$ and variances for all irreps with $m \leq 4$ and the eigenvalues of the various operators we can solve eq.(5.2.1 - 5.2.2) for the unknown parameters. The expression for the centroid then becomes

$$E_c(m \underset{\sim}{f} T) \equiv \langle H \rangle^{m \underset{\sim}{f} T} = \langle \hat{H} \rangle^{m \underset{\sim}{f} T}$$

$$= \frac{1}{2}\left[m^2-3m+2\right]E_c(0\underset{\sim}{0}0) - m(m-2)E_c(1\underset{\sim}{1}1/2)$$

$$+ \frac{1}{8}\left[-13m + \frac{7m^2}{2} + 6T(T+1) + 2\langle G_2\rangle - \langle \ell_y\rangle\right]E_c(2\underset{\sim}{2}0)$$

$$+ \frac{1}{8}\left[3m - \frac{3m^2}{2} - 6T(T+1) + \langle \ell_y\rangle\right]E_c(2\underset{\sim}{2}1)$$

$$+ \frac{1}{8}\left[15m - \frac{5m^2}{2} - 10T(T+1) - 2\langle G_2\rangle + \langle \ell_y\rangle\right]E_c(2\underset{\sim}{1}0)$$

$$+ \frac{1}{8}\left[-9m + \frac{9m^2}{2} + 10T(T+1) - \langle \ell_y\rangle\right]E_c(2\underset{\sim}{1}1) \qquad (5.2.3)$$

where $E_c(0\underset{\sim}{0}0)$, $E_c(1\underset{\sim}{1}1/2)$ .... etc are the input centroids.

Equation (5.2.3) can be rewritten as

$$E_c(m\underset{\sim}{f}T) = \sum_{i=1}^{6} Q(i, f\underset{\sim}{T})\, E_c(i) \qquad (5.2.4)$$

where i = 1,...., 6 denotes the six input representations for

the centroid energies. The quantity $Q(i,\underset{\sim}{f}T)$ has the property that if T is taken to be any one of the representation (j) in the defining space, then

$$Q(i,j) = \delta_{ij} \qquad (i,j = 1,\ldots\ldots 6) \qquad (5.2.5)$$

Using eq.(5.2.2) the variance $\sigma^2(m\underset{\sim}{f}T)$ is given by the following expression:

$$\begin{aligned}
\sigma^2(m\underset{\sim}{f}T) = &\left[1 - \frac{25m}{12} + \frac{35m^2}{24} - \frac{5m^3}{12} + \frac{m^4}{24}\right]\sigma^2(0\,\underset{\sim}{0}\,0) \\
&+ \left[4m - \frac{13}{3}m^2 + \frac{3}{2}m^3 - \frac{1}{6}m^4\right]\sigma^2(1\,\underset{\sim}{1}\,1/2) \\
&+ \left[-\frac{39}{4}m + \frac{133}{16}m^2 - \frac{75}{32}m^3 + \frac{7}{32}m^4 + \frac{3}{2}\langle G_2\rangle\right. \\
&\quad - \frac{7}{8}m\langle G_2\rangle + \frac{1}{8}m^2\langle G_2\rangle + \frac{9}{2}\langle T^2\rangle - \frac{21}{8}m\langle T^2\rangle \\
&\quad + \frac{3}{8}m^2\langle T^2\rangle - \frac{3}{4}\langle \ell\rangle + \frac{7}{16}m\langle \ell\rangle \\
&\quad \left. - \frac{1}{16}m^2\langle \ell\rangle\right]\sigma^2(2\,\underset{\sim}{2}\,0) \\
&+ \left[\frac{9}{4}m - \frac{39}{16}m^2 + \frac{27}{32}m^3 - \frac{3}{32}m^4 - \frac{9}{2}\langle T^2\rangle\right. \\
&\quad \left. + \frac{21}{8}m\langle T^2\rangle - \frac{3}{8}m^2\langle T^2\rangle + \frac{3}{4}\langle \ell\rangle\right.
\end{aligned}$$

$$-\frac{7}{16}m\langle y\rangle+\frac{1}{16}m^2\langle y\rangle\Big]\sigma^2(2\underset{\sim}{2}1)$$

$$+\Big[\frac{45}{4}m-\frac{135}{16}m^2+\frac{65}{32}m^3-\frac{5}{32}m^4-\frac{3}{2}\langle G_2\rangle+\frac{7}{8}m\langle G_2\rangle$$

$$-\frac{1}{8}m^2\langle G_2\rangle-\frac{15}{2}\langle T^2\rangle+\frac{35}{8}m\langle T^2\rangle-\frac{5}{8}m^2\langle T^2\rangle$$

$$+\frac{3}{4}\langle y\rangle-\frac{7}{16}m\langle y\rangle+\frac{1}{16}m^2\langle y\rangle\Big]\sigma^2(2\underset{\sim}{1}0)$$

$$+\Big[-\frac{27}{4}m+\frac{117}{16}m^2-\frac{81}{32}m^3+\frac{9}{32}m^4+\frac{15}{2}\langle T^2\rangle-\frac{35}{8}m\langle T^2\rangle$$

$$+\frac{5}{8}m^2\langle T^2\rangle-\frac{3}{4}\langle y\rangle+\frac{7}{16}m\langle y\rangle-\frac{1}{16}m^2\langle y\rangle\Big]\sigma^2(2\underset{\sim}{1}1)$$

$$+\Big[\frac{164}{15}m-\frac{476}{45}m^2+\frac{199}{60}m^3-\frac{61}{180}m^4-\frac{82}{45}\langle G_2\rangle$$

$$+\frac{107}{90}m\langle G_2\rangle-\frac{11}{60}m^2\langle G_2\rangle+\frac{4}{90}\langle G_3\rangle-\frac{1}{90}m\langle G_3\rangle$$

$$-\frac{16}{3}\langle T^2\rangle+4m\langle T^2\rangle-\frac{2}{3}m^2\langle T^2\rangle+\frac{4}{5}\langle y\rangle$$

$$-\frac{3}{5}m\langle y\rangle+\frac{1}{10}m^2\langle y\rangle\Big]\sigma^2(3\underset{\sim}{3}1/2)$$

$$+\Big[-\frac{192}{45}m+\frac{316}{45}m^2-\frac{41}{15}m^3+\frac{14}{45}m^4-\frac{28}{45}\langle G_2\rangle$$

$$-\frac{11}{45}m\langle G_2\rangle+\frac{1}{10}m^2\langle G_2\rangle+\frac{8}{45}\langle G_3\rangle-\frac{4}{90}m\langle G_3\rangle$$

$$+\frac{16}{3}\langle T^2\rangle-4m\langle T^2\rangle+\frac{2}{3}m^2\langle T^2\rangle-\frac{4}{5}\langle y\rangle$$

$$+\frac{3}{5}m\langle y\rangle-\frac{1}{10}m^2\langle y\rangle\Big]\sigma^2(3\underset{\sim}{3}3\,1/2)$$

$$+\Big[-\frac{68}{5}m+\frac{471}{45}m^2-\frac{29}{10}m^3+\frac{17}{60}m^4+\frac{44}{15}\langle G_2\rangle-\frac{17}{15}m\langle G_2\rangle$$

$$+\frac{1}{10}m^2\langle G_2\rangle-\frac{12}{45}\langle G_3\rangle+\frac{1}{15}m\langle G_3\rangle+8\langle T^2\rangle$$

$$-6m\langle T^2\rangle+m^2\langle T^2\rangle-\frac{4}{5}\langle y\rangle+\frac{3}{5}m\langle y\rangle$$

$$-\frac{1}{10}m^2\langle y\rangle\Big]\sigma^2(3\underset{\sim}{2}1\,1/2)$$

$$+\Big[\frac{104}{15}m-\frac{436}{45}m^2+\frac{107}{30}m^3-\frac{71}{180}m^4+\frac{28}{45}\langle G_2\rangle+\frac{11}{45}m\langle G_2\rangle$$

$$-\frac{1}{10}m^2\langle G_2\rangle-\frac{8}{45}\langle G_3\rangle+\frac{2}{45}m\langle G_3\rangle-8\langle T^2\rangle+6m\langle T^2\rangle$$

$$-m^2\langle T^2\rangle+\frac{4}{5}\langle y\rangle-\frac{3}{5}m\langle y\rangle+\frac{1}{10}m^2\langle y\rangle\Big]\sigma^2(3\underset{\sim}{2}1\,3/2)$$

$$+\Big[\frac{4}{3}m+\frac{4}{9}m^2-\frac{1}{12}m^3-\frac{1}{36}m^4-\frac{10}{9}\langle G_2\rangle-\frac{1}{18}m\langle G_2\rangle$$

$$+\frac{1}{12}m^2\langle G_2\rangle+\frac{2}{9}\langle G_3\rangle-\frac{1}{18}m\langle G_3\rangle\Big]\sigma^2(3\underset{\sim}{1}11\,1/2)$$

$$+\left[-\frac{6553}{9360}m+\frac{495899}{112320}m^2-\frac{66557}{28080}m^3+\frac{3799}{11232}m^4\right.$$

$$-\frac{2263}{56160}\langle G_2\rangle+\frac{19}{576}\langle G_2^2\rangle-\frac{49471}{56160}m\langle G_2\rangle$$

$$+\frac{28679}{112320}m^2\langle G_2\rangle+\frac{1381}{28080}\langle G_3\rangle+\frac{23}{3510}m\langle G_3\rangle$$

$$-\frac{7}{1440}\langle G_4\rangle+\frac{795}{1404}\langle T^2\rangle+\frac{1977}{1404}\langle T^4\rangle$$

$$-\frac{6993}{1404}m\langle T^2\rangle+\frac{3975}{2808}m^2\langle T^2\rangle+\frac{2895}{5616}\langle G_2T^2\rangle$$

$$-\frac{318}{5616}\langle y\rangle+\frac{42}{1775}\langle y^2\rangle+\frac{7065}{11232}m\langle y\rangle$$

$$\left.-\frac{10233}{56160}m^2\langle y\rangle-\frac{1733}{28080}\langle G_2\rangle\langle y\rangle-\frac{522}{1404}\langle T^2\rangle\langle y\rangle\right]\sigma^2(440)$$

$$+\left[-\frac{42993}{14040}m-\frac{491925}{1404000}m^2+\frac{115638}{112320}m^3-\frac{8961}{44928}m^4\right.$$

$$+\frac{10677}{14040}\langle G_2\rangle-\frac{1}{48}\langle G_2^2\rangle+\frac{9213}{28080}m\langle G_2\rangle-\frac{8847}{56160}m^2\langle G_2\rangle$$

$$-\frac{714}{7020}\langle G_3\rangle-\frac{42}{14040}m\langle G_3\rangle+\frac{1}{120}\langle G_4\rangle+\frac{1827}{1404}\langle T^2\rangle$$

$$\left.-\frac{16605}{14040}\langle T^4\rangle+\frac{39555}{14040}m\langle T^2\rangle-\frac{142875}{140400}m^2\langle T^2\rangle\right.$$

$$-\frac{5625}{14040}\langle G_2T^2\rangle-\frac{288}{1404}\langle y\rangle-\frac{153}{7020}\langle y^2\rangle-\frac{4995}{14040}m\langle y\rangle$$

$$+\frac{188325}{1404000}m^2\langle y\rangle+\frac{729}{14040}\langle G_2\rangle\langle y\rangle+\frac{4545}{14040}\langle T^2\rangle\langle y\rangle\Big]\sigma^2(4\underset{\sim}{4}1)$$

$$+\Big[\frac{2211}{1404}m-\frac{4690375}{1404000}m^2+\frac{178975}{140400}m^3-\frac{6077}{44928}m^4$$

$$+\frac{55375}{140400}\langle G_2\rangle-\frac{1}{144}\langle G_2^2\rangle+\frac{444625}{1404000}m\langle G_2\rangle$$

$$-\frac{1201875}{14040000}m^2\langle G_2\rangle-\frac{1895}{14040}\langle G_3\rangle+\frac{145}{14040}m\langle G_3\rangle$$

$$+\frac{1}{144}\langle G_4\rangle-\frac{2622}{1404}\langle T^2\rangle-\frac{3165}{14040}\langle T^4\rangle$$

$$+\frac{30375}{14040}m\langle T^2\rangle-\frac{55875}{140400}m^2\langle T^2\rangle-\frac{16125}{140400}\langle G_2T^2\rangle$$

$$+\frac{3675}{14040}\langle y\rangle-\frac{48}{22464}\langle y^2\rangle-\frac{383625}{1404000}m\langle y\rangle$$

$$+\frac{675}{14040}m^2\langle y\rangle+\frac{1575}{140400}\langle G_2\rangle\langle y\rangle+\frac{675}{14040}\langle T^2\rangle\langle y\rangle\Big]\sigma^2(4\underset{\sim}{4}2)$$

$$+\left[\frac{279465}{140400}m-\frac{275317}{22464}m^2+\frac{928105}{140400}m^3-\frac{105701}{112320}m^4\right.$$

$$+\frac{112825}{1404000}\langle G_2\rangle-\frac{1225}{14400}\langle G_2^2\rangle+\frac{3368275}{1404000}m\langle G_2\rangle$$

$$-\frac{9893625}{14040000}m^2\langle G_2\rangle-\frac{16085}{140400}\langle G_3\rangle-\frac{197}{14040}m\langle G_3\rangle$$

$$+\frac{13}{1440}\langle G_4\rangle-\frac{2253}{1404}\langle T^2\rangle-\frac{6450}{1404}\langle T^4\rangle+\frac{21375}{1404}m\langle T^2\rangle$$

$$-\frac{6042}{1404}m^2\langle T^2\rangle-\frac{227625}{140400}\langle G_2T^2\rangle+\frac{2253}{14040}\langle y\rangle$$

$$-\frac{984}{14040}\langle y^2\rangle-\frac{2517975}{1404000}m\langle y\rangle+\frac{731025}{1404000}m^2\langle y\rangle$$

$$\left.+\frac{25335}{140400}\langle G_2\rangle\langle y\rangle+\frac{1692}{1404}\langle T^2\rangle\langle y\rangle\right]\sigma^2(4\underset{\sim}{3}10)$$

$$+\left[\frac{28791}{14040}m+\frac{207255}{14040}m^2-\frac{128163}{14040}m^3+\frac{19452}{14040}m^4\right.$$

$$-\frac{24627}{14040}\langle G_2\rangle+\frac{1}{12}\langle G_2^2\rangle-\frac{4194525}{1404000}m\langle G_2\rangle$$

$$+\frac{13872375}{14040000}m^2\langle G_2\rangle+\frac{6621}{14040}\langle G_3\rangle+\frac{267}{14040}m\langle G_3\rangle$$

$$-\frac{1}{30}\langle G_4\rangle-\frac{207}{1404}\langle T^2\rangle+\frac{4950}{702}\langle T^4\rangle-\frac{291195}{14040}m\langle T^2\rangle$$

$$+\frac{89595}{14040}m^2\langle T^2\rangle+\frac{311625}{140400}\langle G_2T^2\rangle+\frac{207}{14040}\langle y\rangle$$

$$+\frac{1494}{14040}\langle y^2\rangle+\frac{3467475}{1404000}m\langle y\rangle-\frac{1088775}{1404000}m^2\langle y\rangle$$

$$-\frac{3456}{14040}\langle G_2\rangle\langle y\rangle-\frac{2484}{1404}\langle T^2\rangle\langle y\rangle\Big]\sigma^2(4\underset{\sim}{3}1\ 1)$$

$$+\Big[-\frac{1725}{1404}m-\frac{332075}{140400}m^2+\frac{3228125}{1404000}m^3-\frac{602}{1404}m^4$$

$$-\frac{41425}{140400}\langle G_2\rangle-\frac{1}{72}\langle G_2^2\rangle+\frac{914}{1404}m\langle G_2\rangle$$

$$-\frac{354}{1404}m^2\langle G_2\rangle+\frac{113}{1404}\langle G_3\rangle-\frac{175}{35100}m\langle G_3\rangle-\frac{1}{144}\langle G_4\rangle$$

$$+\frac{2460}{1404}\langle T^2\rangle-\frac{3450}{1404}\langle T^4\rangle+\frac{77625}{14040}m\langle T^2\rangle$$

$$-\frac{29175}{14040}m^2\langle T^2\rangle-\frac{840}{1404}\langle G_2T^2\rangle-\frac{246}{1404}\langle y\rangle$$

$$-\frac{51}{1404}\langle y^2\rangle-\frac{9495}{14040}m\langle y\rangle+\frac{35775}{140400}m^2\langle y\rangle$$

$$+\frac{9225}{140400}\langle G_2\rangle\langle \bar{y}\rangle+\frac{855}{1404}\langle T^2\rangle\langle \bar{y}\rangle\Big]\sigma^2(4\tilde{3}12)$$

$$+\Big[\frac{11736}{14040}m-\frac{934275}{140400}m^2+\frac{555195}{140400}m^3-\frac{858675}{1404000}m^4$$

$$+\frac{2073}{14040}\langle G_2\rangle-\frac{1}{48}\langle G_2^2\rangle+\frac{165135}{140400}m\langle G_2\rangle$$

$$-\frac{543825}{1404000}m^2\langle G_2\rangle-\frac{1284}{14040}\langle G_3\rangle-\frac{258}{14040}m\langle G_3\rangle$$

$$+\frac{1}{120}\langle G_4\rangle-\frac{1062}{1404}\langle T^2\rangle-\frac{3690}{1404}\langle T^4\rangle$$

$$+\frac{11988}{1404}m\langle T^2\rangle-\frac{3708}{1404}m^2\langle T^2\rangle-\frac{11655}{14040}\langle G_2T^2\rangle$$

$$+\frac{1062}{14040}\langle \bar{y}\rangle-\frac{576}{14040}\langle \bar{y}^2\rangle-\frac{141615}{140400}m\langle \bar{y}\rangle$$

$$+\frac{4536}{14040}m^2\langle \bar{y}\rangle+\frac{1269}{14040}\langle G_2\rangle\langle \bar{y}\rangle$$

$$+\frac{945}{1404}\langle T^2\rangle\langle \bar{y}\rangle\Big]\sigma^2(4\tilde{2}20)$$

$$+\Big[\frac{1899}{14040}m+\frac{381}{1404}m^2-\frac{3342}{14040}m^3+\frac{582375}{14040000}m^4$$

$$-\frac{678}{14040}\langle G_2\rangle+\frac{1}{48}\langle G_2^2\rangle-\frac{214725}{1404000}m\langle G_2\rangle$$

$$+\frac{696375}{14040000}m^2\langle G_2\rangle+\frac{519}{14040}\langle G_3\rangle-\frac{57}{14040}m\langle G_3\rangle$$

$$-\frac{1}{120}\langle G_4\rangle-\frac{153}{1404}\langle T^2\rangle-\frac{135}{1404}\langle T^4\rangle-\frac{3105}{14040}m\langle T^2\rangle$$

$$+\frac{855}{14040}m^2\langle T^2\rangle+\frac{1625}{140400}\langle G_2T^2\rangle+\frac{153}{14040}\langle y\rangle$$

$$+\frac{36}{14040}\langle y^2\rangle+\frac{83025}{1404000}m\langle y\rangle-\frac{2835}{140400}m^2\langle y\rangle$$

$$-\frac{189}{14040}\langle G_2\rangle\langle y\rangle-\frac{225}{14040}\langle T^2\rangle\langle y\rangle\Big]\sigma^2(4\underset{\sim}{22}1)$$

$$+\Big[\frac{13635}{14040}m+\frac{88155}{14040}m^2-\frac{523725}{140400}m^3+\frac{8101875}{14040000}m^4$$

$$-\frac{1395}{14040}\langle G_2\rangle+\frac{1}{48}\langle G_2^2\rangle-\frac{1358625}{1404000}m\langle G_2\rangle$$

$$+\frac{4741875}{14040000}m^2\langle G_2\rangle+\frac{765}{14040}\langle G_3\rangle-\frac{75}{14040}m\langle G_3\rangle$$

$$+\frac{1215}{1404}\langle T^2\rangle+\frac{3825}{1404}\langle T^4\rangle+\frac{116775}{14040}m\langle T^2\rangle$$

$$+\frac{36225}{14040}m^2\langle T^2\rangle+\frac{100125}{140400}\langle G_2T^2\rangle-\frac{1215}{14040}\langle\mathcal{G}\rangle$$

$$+\frac{54}{1404}\langle\mathcal{G}^2\rangle+\frac{1333125}{1404000}m\langle\mathcal{G}\rangle-\frac{42525}{140400}m^2\langle\mathcal{G}\rangle$$

$$-\frac{108}{1404}\langle G_2\rangle\langle\mathcal{G}\rangle-\frac{9225}{14040}\langle T^2\rangle\langle\mathcal{G}\rangle\Big]\sigma^2(4\underset{\sim}{22}2)$$

$$+\Big[\frac{307305}{140400}m+\frac{5397777}{39440}m^2-\frac{11529525}{1404000}m^3+\frac{54798}{44928}m^4$$

$$-\frac{102675}{1404000}\langle G_2\rangle+\frac{13}{192}\langle G_2^2\rangle-\frac{368535}{140400}m\langle G_2\rangle$$

$$+\frac{1191825}{1404000}m^2\langle G_2\rangle+\frac{13245}{140400}\langle G_3\rangle+\frac{168m}{14040}\langle G_3\rangle$$

$$-\frac{1}{480}\langle G_4\rangle+\frac{70}{39}\langle T^2\rangle+\frac{2070}{351}\langle T^4\rangle$$

$$-\frac{26352}{1404}m\langle T^2\rangle+\frac{7704}{1404}m^2\langle T^2\rangle+\frac{2718}{1404}\langle G_2T^2\rangle$$

$$-\frac{7}{39}\langle\mathcal{G}\rangle+\frac{1244}{14040}\langle\mathcal{G}^2\rangle+\frac{3051}{1404}m\langle\mathcal{G}\rangle$$

$$-\frac{9288}{14040}m^2\langle\mathcal{G}\rangle-\frac{2916}{14040}\langle G_2\rangle\langle\mathcal{G}\rangle$$

$$-\frac{2052}{1404}\langle T^2\rangle\langle \bar{\psi}\psi\rangle\Big]\sigma^2(4\underset{\sim}{211}0)$$

$$+\Big[\frac{17568}{14040}m-\frac{207462}{14040}m^2+\frac{115734}{14040}m^3-\frac{16905}{14040}m^4$$

$$+\frac{14628}{14040}\langle G_2\rangle-\frac{1}{12}\langle G_2^2\rangle+\frac{39486}{14040}m\langle G_2\rangle$$

$$-\frac{12357}{14040}m^2\langle G_2\rangle-\frac{5712}{14040}\langle G_3\rangle-\frac{168}{14040}m\langle G_3\rangle$$

$$+\frac{1}{30}\langle G_4\rangle-\frac{70}{39}\langle T^2\rangle-\frac{8280}{1404}\langle T^4\rangle+\frac{26352}{1404}m\langle T^2\rangle$$

$$-\frac{7704}{1404}m^2\langle T^2\rangle-\frac{2718}{1404}\langle G_2T^2\rangle+\frac{7}{39}\langle \bar{\psi}\psi\rangle$$

$$-\frac{1224}{14040}\langle \bar{\psi}^2\psi^2\rangle-\frac{3051}{1404}m\langle \bar{\psi}\psi\rangle+\frac{9288}{14040}m^2\langle \bar{\psi}\psi\rangle$$

$$+\frac{2916}{14040}\langle G_2\rangle\langle \bar{\psi}\psi\rangle+\frac{2052}{1404}\langle T^2\rangle\langle \bar{\psi}\psi\rangle\Big]\sigma^2(4\underset{\sim}{211}1)$$

$$+\Big[\frac{1}{16}m+\frac{33}{576}m^2+\frac{5}{288}m^3+\frac{1}{576}m^4-\frac{11}{96}\langle G_2\rangle$$

$$+\frac{1}{192}\langle G_2^2\rangle-\frac{5}{72}m\langle G_2\rangle-\frac{3}{288}m^2\langle G_2\rangle+\frac{1}{16}\langle G_3\rangle$$

$$+\frac{1}{72}m\langle G_3\rangle-\frac{3}{288}\langle G_4\rangle\Big]\sigma^2(4\underset{\sim}{1111}0)$$

In order to use these equations to determine the moments one needs to know the eigenvalue of ℓ_g in any representation $\underset{\sim}{\alpha} = \underset{\sim}{f}T$. While it should be possible to write down an analytic expression for $\langle \ell_g \rangle$ in terms of $\underset{\sim}{f}$, T and S starting from first principles, a different method is indicated. The procedure used here is very specific in that it is based on the propagation method used for evaluating $(\underset{\sim}{f}, S, T)$ averages. One writes down an equivalent operator for G_2T^2 in terms of scalar operators with particle rank ≤ 2. This automatically eliminates the (3+4) body parts of G_2T^2. The operator constructed for evaluating $E_c(m\underset{\sim}{f}TS)$ in chapter II can be used to determine the expectation value of ℓ_g. Using eq.(2.4.2) we determine $\langle \ell_g \rangle$ and the results for $m \leq 4$ are given in table (V.1).

The essential point is that for cases where one does not have enough scalar operators, one can use mixed operators constructed out of the given scalar operators provided one uses the correct particle rank of these mixed operators. We would like to mention here that the method is quite general and can be used to evaluate higher moments also provided we know the eigenvalues of the "mixed" operators.

5.3 Averaging for SU(3) Group

In the study of SU(3) symmetry for 2s-1d shell using distribution methods by Parikh (1972) the notion of an equi-

Table (V.1)

Eigenvalues of ℓ_j in the representation ($\underset{\sim}{fT}$)

m	$\underset{\sim}{f}$	T	S	$\langle \ell_j \rangle$
1	[1000]	1/2	1/2	3
2	[2000]	0	0	0
		1	1	20
	[1100]	0	1	0
		1	0	12
3	[3000]	1/2	1/2	21
		3/2	3/2	51
	[2100]	1/2	1/2	15
		1/2	3/2	21
		3/2	1/2	39
	[1110]	1/2	1/2	9
4	[4000]	0	0	36
		1	1	56
		2	2	96
	[3100]	0	1	32
		1	0	44
		1	1	48
		1	2	56
		2	1	80

Table (V.1)(contd)

m	$\underset{\sim}{f}$	T	S	$\langle \psi \rangle$
	$[2200]$	0	0	24
		0	2	36
		2	0	72
		1	1	44
	$[2110]$	0	1	24
		1	0	36
		1	1	40
	$[1111]$	0	0	12

valent width was used. This is because one does not have enough scalar operators having particle rank ≤ 4 to determine the variance $\sigma^2(m(\lambda\mu))$ of m particle states belonging to the irrep $(\lambda\mu)$ of SU(3).

The scalar operators for the groups of interest are the number operator n of U(24), the bilinear $G_2(6)$, cubic $G_3(6)$ and quartic $G_4(6)$ Casimir operators of U(6), and the bilinear C_2 and cubic C_3 Casimir operators of SU(3). The total number of scalar operators with particle rank $k \leq 4$ that can be formed from these operators is 19 whereas there are 33 representations in the defining space as can be seen from table (V.2).

As before one looks for mixed operators and one possible set is given in table (V.3).

The equivalent operator for H^2 is then written as

$$\hat{H}^2 = a_1 + a_2 n^3 + \ldots\ldots a_{33} G_2^2 C_2^4 \qquad (5.3.1)$$

From a knowledge of the inputs in the defining space and the eigenvalues of the operators, the coefficients $a_i (i=1, \ldots\ldots, 33)$ can be determined. Denoting the inputs by $b_i = \sigma^2(i)$, one has to solve a system of simultaneous linear equations. Schematically this can be written as

$$Aa = b \qquad (5.3.2)$$

where A is the matrix formed out of the eigenvalues of the

Table (V.2)

Representations of SU(3) contained in an irrep $\underset{\sim}{f}$ of U(6)

$\underset{\sim}{f}$	$(\lambda\mu)$
[0]	(00)
[1]	(20)
[2]	(40)(02)
[11]	(21)
[3]	(60)(22)(00)
[21]	(41)(22)(11)
[111]	(30)(03)
[4]	(80)(42)(04)(20)
[31]	(61)(42)(23)(31)(12)(20)
[22]	(42)(31)(04)(20)
[211]	(50)(23)(31)(12)(01)
[1111]	(12)

Table (V.3)

Set of operators used in constructing a net for propagation of the variance $\sigma^2(\underset{\sim}{mf}(\lambda\omega))$

$1, n^3, n^4, G_2^2(6), G_4(6), n^3C_2, n^2G_2(6)C_2,$

$n^2G_2(6)C_2^2,\ G_3(6)C_2^2,\ n^4C_2^2,\ n\ C_2^2G_3(6),$

$n^4C_3, n^2G_2(6)\ C_3,\ G_3(6)\ C_3,\ n\ G_3(6)\ C_3,$

$n^3C_2^3,\ G_2(6)\ C_2^3,\ G_3(6)\ C_2^3,\ G_4(6)\ C_2^3,$

$G_2^2C_2^3,\ n\ C_2^4,\ n^3C_2^4,\ n^4C_2^4,\ G_2(6)\ C_2^4,$

$n\ G_2(6)\ C_2^4,\ n^2G_2(6)\ C_2^4,\ G_3(6)\ C_2^4,\ n^4\ C_2^3,$

$n^2G_2(6)\ C_2^3,\ C_2^4,\ n\ G_3(6)\ C_2^4,\ G_4(6)\ C_2^4,\ G_2^2(6)\ C_2^4.$

---

operators in the defining space, a and b are the column vectors of the unknown coefficients and the input averages respectively.

Solving eq.(5.3.2) for a and collecting terms together the variance $\sigma^2(m(\lambda\mu))$ can be written as:

$$\sigma^2(m(\lambda\mu)) = \sum_{i=1}^{33} P(i,(\lambda\mu))\sigma^2(i) \qquad (5.3.3)$$

where P(i, $(\lambda\mu)$) are the density operators.

In the expression for P(i, $(\lambda\mu)$) there are a number of mixed operators with $k > 4$. A knowledge of the eigenvalues of the (0-4) body part of these mixed operators would then enable one to determine $\sigma^2(m(\lambda\mu))$.

It should be mentioned here that the choice of operators is not unique and one could use a different set. The only condition is that these should be linearly independent.

5.4 A Projection Operator approach for evaluating averages

In this section a method of evaluating moments of operators over states defined by a chain of subgroups is given. This method makes use of the fact that - a) one can determine moments by using higher rank operators and b) the density operators act as projection operators in the defining space.

We do not discuss the method in general but give two specific applications.

5.4.1 SU(3) Averaging

As a first application we discuss averaging for the chain defined as:

$$U(6) \supset SU(3) \qquad (5.4.1)$$

One starts by constructing expression for the centroid energy and variance for U(6).

The equivalent operators can be constructed in terms of the number operator n and the Casimir operators $G_2(6)$, $G_3(6)$ and $G_4(6)$ of U(6). Denoting an irrep of U(6) by $\underset{\sim}{f}$ the final expression for the centroid $E_c(m\underset{\sim}{f})$ and $\sigma^2(m\underset{\sim}{f})$ can be written as:

$$\begin{aligned} E_c(m\underset{\sim}{f}) = {} & \frac{1}{2}(m^2-3m+2)E_c(0\underset{\sim}{0}) \\ & - m(m-2)E_c(1\underset{\sim}{1}) \\ & + \frac{1}{4}\left[m(m-7)+\langle G_2(6)\rangle\right]E_c(2\underset{\sim}{2}) \\ & + \frac{1}{4}\left[m(m+5)-\langle G_2(6)\rangle\right]E_c(2\underset{\sim}{11}) \end{aligned} \qquad (5.4.2)$$

$$\sigma^2(m\underset{\sim}{f}) = \frac{1}{24}\left[m^4 - 10m^3 + 35m^2 - 50m + 24\right]\sigma^2(0\,\underset{\sim}{0})$$

$$- \frac{m}{6}\left[m^3 - 9m^2 + 26m - 24\right]\sigma^2(1\,\underset{\sim}{1})$$

$$+ \frac{1}{8}\left[m(m^3 - 14m^2 + 61m - 84) + \langle G_2(6)\rangle(m^2 - 7m + 12)\right]\sigma^2(2\,\underset{\sim}{2})$$

$$+ \frac{1}{8}\left[m(m^3 - 2m^2 - 23m + 60) - \langle G_2(6)\rangle(m^2 - 7m + 12)\right]\sigma^2(2\,\underset{\sim}{11})$$

$$- \frac{1}{36}\left[m(m^3 - 27m^2 + 204m - 448) + \langle G_2(6)\rangle(3m^2 - 42m + 120) + 2\langle G_3(6)\rangle(m-4)\right]\sigma^2(3\,\underset{\sim}{3})$$

$$- \frac{1}{9}\left[m(m^3 - 6m^2 - 27m + 140) + 12\langle G_2(6)\rangle(m-4) - \langle G_3(6)\rangle(m-4)\right]\sigma^2(3\,\underset{\sim}{21})$$

$$- \frac{1}{36}\left[m(m^3 + 9m^2 - 12m - 160) - 3\langle G_2(6)\rangle(m^2 + 2m - 24) + 2\langle G_3(6)\rangle(m-4)\right]\sigma^2(3\,\underset{\sim}{111})$$

$$+ \frac{1}{576}\left[m(m^3 - 50m^2 + 733m - 3024) + 6\langle G_2(6)\rangle(m^2 - 30m + 191) + 3\langle G_2^2(6)\rangle + 8\langle G_3(6)\rangle(m-18) + 6\langle G_4(6)\rangle\right]\sigma^2(4\,\underset{\sim}{4})$$

$$+\frac{1}{64}\Big[m(m^3-18m^2-3m+560)+2\langle G_2(6)\rangle(m^2+4m$$
$$-131)-\langle G_2^2(6)\rangle+40\langle G_3(6)\rangle$$
$$-2\langle G_4(6)\rangle\Big]\sigma^2(4\underset{\sim}{3}1)$$

$$+\frac{1}{144}\Big[m^2(m^2-2m-35)+12m\langle G_2(6)\rangle+3\langle G_2^2(6)\rangle$$
$$-4m\langle G_3(6)\rangle\Big]\sigma^2(4\underset{\sim}{2}2)$$

$$+\frac{1}{64}\Big[m(m^3+6m^2-51m-280)-2\langle G_2(6)\rangle(m^2$$
$$-8m-83)-\langle G_2^2(6)\rangle-32\langle G_3(6)\rangle$$
$$+2\langle G_4(6)\rangle\Big]\sigma^2(4\underset{\sim}{2}11)$$

$$+\frac{1}{576}\Big[m(m^3+22m^2+157m+360)+3\langle G_2^2(6)\rangle$$
$$-6\langle G_2(6)\rangle(m^2+14m+47)$$
$$+8(m+9)\langle G_3(6)\rangle-6\langle G_4(6)\rangle\Big]\sigma^2(4\underset{\sim}{1111})$$

(54.3)

$\sigma^2(\underset{\sim}{mf})$ is written as

$$\sigma^2(\underset{\sim}{mf}) = \sum_{i=1}^{12} R(i,\underset{\sim}{f})\sigma^2(i) \qquad (5.4.4)$$

where the density operator $R(i,\underset{\sim}{f})$ have been defined in eq. (5.4.3). The variance $\sigma^2(\underset{\sim}{mf}(\lambda\mu))$ for SU(3) averaging can be written as

$$\sigma^2(m\underset{\sim}{f}(\lambda\mu)) = \sum_{i=1}^{33} S(i,\underset{\sim}{f}(\lambda\mu))\sigma^2(i)$$

where $\sigma^2(i)$ are the input variances.

In order to determine $\sigma^2(\underset{\sim}{mf}(\lambda\mu))$, one observes that the density operators $S(i,\underset{\sim}{f}(\lambda\mu))$ for any representation j in the defining space should satisfy the relation

$$S(i,j) = \delta_{ij} \qquad (5.4.5)$$

One now makes uses of the Casimir operators C_2 and C_3 of SU(3) to construct S out of R. As an illustraction of the method consider $\underset{\sim}{f} = [200000]$ which contains the irreps (40) and (02) of SU(3). The eigenvalues of C_2 and C_3 in a representation $(\lambda\mu)$ is given by

$$\langle C_2 \rangle_{(\lambda\mu)} = (\lambda+\mu)(\lambda+\mu+3) - \lambda\mu \qquad (5.4.6)$$

$$\langle C_3\rangle_{(\lambda\mu)} = 2(\lambda^3-\mu^3)+3\lambda\mu(\lambda-\mu)+9(\lambda+2)(2\lambda+\mu) \qquad (5.4.7)$$

For(40) and (02), the eigenvalues of C_2 are 28 and 10 respectively. If one constructs the projection operators $P_1 = (\langle C_2\rangle - 10)/18$ and $P_2 = (28-\langle C_2\rangle)/18$ for (40) and (02) respectively, it is clear that for the representation $\underset{\sim}{\alpha} = (\underset{\sim}{2}\,(40))$, $P_1=1$ and $P_2=0$ and for $\underset{\sim}{\beta} = (\underset{\sim}{2}\,(02))$, $P_1=0$ and $P_2=1$.

Thus P_1 and P_2 satisfy the necessary requirements. In this case the projection operators could be constructed in terms of C_2 only because the eigenvalue of C_2 is sufficient to distinguish the irreps. For other cases we may need C_3. Using this method for all irreps listed in table (V.2), the final expression for $\sigma^2(\underset{\sim}{mf}(\lambda\mu))$ can be written as:

$$\sigma^2(\underset{\sim}{mf}(\lambda\mu)) = R(1)\sigma^2(0\underset{\sim}{0}\,(01)) + R(2)\sigma^2(1\underset{\sim}{1}\,(20))$$
$$+ R(3)\frac{\{\langle C_2\rangle - 10\}}{18}\sigma^2(2\underset{\sim}{2}\,(40))$$
$$+ R(3)\frac{\{28-\langle C_2\rangle\}}{18}\sigma^2(2\underset{\sim}{2}\,(02))$$
$$+ R(4)\sigma^2(2\underset{\sim}{11}\,(21))$$
$$+ R(5)\left\{\frac{\langle C_2\rangle(\langle C_2\rangle - 24)}{54\times 30}\right\}\sigma^2(3\underset{\sim}{3}\,(60))$$

$$+R(5)\left\{-\frac{\langle C_2\rangle(\langle C_2\rangle-54)}{24\times 30}\right\}\sigma^2(3\underset{\sim}{3}(22))$$

$$+R(5)\left\{\frac{(\langle C_2\rangle-54)(\langle C_2\rangle-24)}{54\times 24}\right\}\sigma^2(3\underset{\sim}{3}(00))$$

$$+R(6)\left\{\frac{(\langle C_2\rangle-24)(\langle C_2\rangle-9)}{12\times 27}\right\}\sigma^2(3\underset{\sim}{2}^{1}(41))$$

$$+R(6)\left\{-\frac{(\langle C_2\rangle-36)(\langle C_2\rangle-9)}{12\times 15}\right\}\sigma^2(3\underset{\sim}{2}^{1}(22))$$

$$+R(6)\left\{\frac{(\langle C_2\rangle-24)(\langle C_2\rangle-36)}{15\times 27}\right\}\sigma^2(3\underset{\sim}{2}^{1}(11))$$

$$+R(7)\left\{\frac{\langle C_3\rangle+48}{372}\right\}\sigma^2(3\underset{\sim}{111}(30))$$

$$+R(7)\left\{\frac{(324-\langle C_3\rangle)}{372}\right\}\sigma^2(3\underset{\sim}{111}(03))$$

$$+R(8)\left\{\frac{(\langle C_2\rangle-46)(\langle C_2\rangle-28)(\langle C_2\rangle-10)}{42\times 60\times 78}\right\}\sigma^2(4\underset{\sim}{4}(80))$$

$$+R(8)\left\{-\frac{(\langle C_2\rangle-88)(\langle C_2\rangle-28)(\langle C_2\rangle-10)}{42\times18\times36}\right\}\sigma^2(44(42))$$

$$+R(8)\left\{\frac{(\langle C_2\rangle-88)(\langle C_2\rangle-46)(\langle C_2\rangle-10)}{60\times18\times18}\right\}\sigma^2(44(04))$$

$$+R(8)\left\{-\frac{(\langle C_2\rangle-88)(\langle C_2\rangle-46)(\langle C_2\rangle-28)}{78\times36\times18}\right\}\sigma^2(44(20))$$

$$+R(9)\left\{\frac{(\langle C_2\rangle-46)(\langle C_2\rangle-34)(\langle C_2\rangle-25)(\langle C_2\rangle-16)(\langle C_2\rangle-10)}{18\times30\times39\times48\times54}\right\}\sigma^2(431(61))$$

$$+R(9)\left\{-\frac{(\langle C_2\rangle-64)(\langle C_2\rangle-34)(\langle C_2\rangle-25)(\langle C_2\rangle-16)(\langle C_2\rangle-10)}{18\times12\times21\times30\times36}\right\}\sigma^2(431(42))$$

$$+R(9)\left\{\frac{(\langle C_2\rangle-64)(\langle C_2\rangle-46)(\langle C_2\rangle-25)(\langle C_2\rangle-16)(\langle C_2\rangle-10)}{30\times12\times9\times18\times24}\right\}\sigma^2(431(23))$$

$$+R(9)\left\{-\frac{(\langle C_2\rangle-64)(\langle C_2\rangle-46)(\langle C_2\rangle-34)(\langle C_2\rangle-16)(\langle C_2\rangle-10)}{39\times21\times9\times15}\right\}\sigma^2(431(31))$$

$$+R(9)\left\{\frac{(\langle C_2\rangle-64)(\langle C_2\rangle-46)(\langle C_2\rangle-34)(\langle C_2\rangle-25)(\langle C_2\rangle-10)}{48\times30\times18\times9\times6}\right\}\sigma^2(431(12))$$

$$+R(9)\left\{-\frac{(\langle C_2\rangle-64)(\langle C_2\rangle-46)(\langle C_2\rangle-34)(\langle C_2\rangle-25)(\langle C_2\rangle-16}{54\times36\times24\times15\times6}\right\}\sigma^2(431(20))$$

$$+ R(10)\left\{\frac{(\langle c_2\rangle-25)(\langle c_2\rangle-28)(\langle c_2\rangle-40)}{21\times18\times36}\right\}\sigma^2(4\,22\,(421))$$

$$+ R(10)\left\{\frac{(\langle c_2\rangle-46)(\langle c_2\rangle-28)(\langle c_2\rangle-10)}{21\times3\times15}\right\}\sigma^2(4\,22\,(31))$$

$$+ R(10)\left\{-\frac{(\langle c_2\rangle-46)(\langle c_2\rangle-25)(\langle c_2\rangle-10)}{18\times3\times18}\right\}\sigma^2(4\,22\,(04))$$

$$+ R(10)\left\{-\frac{(\langle c_2\rangle-46)(\langle c_2\rangle-25)(\langle c_2\rangle-28)}{36\times15\times18}\right\}\sigma^2(4\,22\,(20))$$

$$+ R(11)\left\{\frac{(\langle c_2\rangle-34)(\langle c_2\rangle-25)(\langle c_2\rangle-16)(\langle c_2\rangle-4)}{6\times15\times24\times36}\right\}\sigma^2(4\,211\,(50))$$

$$+ R(11)\left\{-\frac{(\langle c_2\rangle-40)(\langle c_2\rangle-25)(\langle c_2\rangle-16)(\langle c_2\rangle-4)}{6\times9\times18\times30}\right\}\sigma^2(4\,211\,(23))$$

$$+ R(11)\left\{\frac{(\langle c_2\rangle-40)(\langle c_2\rangle-34)(\langle c_2\rangle-16)(\langle c_2\rangle-4)}{15\times9\times9\times21}\right\}\sigma^2(4\,211\,(31))$$

$$+ R(11)\left\{-\frac{(\langle c_2\rangle-40)(\langle c_2\rangle-34)(\langle c_2\rangle-25)(\langle c_2\rangle-4)}{24\times18\times9\times12}\right\}\sigma^2(4\,211\,(12))$$

$$+ R(11)\left\{\frac{(\langle c_2\rangle-40)(\langle c_2\rangle-34)(\langle c_2\rangle-25)(\langle c_2\rangle-16}{36\times30\times21\times12}\right\}\sigma^2(4\,211\,(01))$$

$$+ R(12)\,\sigma^2(4\,1111\,(12)) \qquad (5.4.8)$$

Thus one can evaluate SU(3) widths in 2s-1d shell by using the expression of eq.(5.4.8).

5.4.2 Averaging over the canonical chain of unitary groups

In this section, averaging over states defined by the canonical chain of unitary groups

$$U(N) \supset U(N-1) \supset \ldots\ldots \supset U(N-k); \quad 0 \leq k \leq N-1$$

is considered. A specific case, $U(6) \supset U(5) \supset U(4)$ is discussed in detail from which one may generalize.

The starting point is to determine the irreps in the defining space and see whether the groups in question furnish a sufficient number of scalar operators. Given an irrep $\underset{\sim}{f}$ of U(N), one can determine the representations $\underset{\sim}{f}'$ of U(N-1) contained in $\underset{\sim}{f}$ (Weyl, 1931). These should satisfy the inequality

$$f_1 \geq f_1' \geq f_2 \geq f_2' \cdots \geq f_{N-1}' \geq f_N \geq 0 \qquad (5.4.9)$$

Representations for the decomposition

$$U(6) \supset U(5) \supset U(4)$$

for $m \leq 4$ particles are listed in table (V.4).

In the next section expressions for the centroid energy and variance are derived.

Table (V.4)

Representations for $m \leq 4$ in the reduction $U(6) \supset U(5) \supset U(4)$

[f]	[g]	[h]
[1]	[1], [0]	[1], [0], [0]
[2]	[2], [1], [0]	[2], [1], [0], [1], [0], [0]
[11]	[11], [1]	[11], [1], [1], [0]
[3]	[3], [2], [1], [0]	[3], [2], [1], [0], [2], [1], [0], [1], [0], [0]
[21]	[21], [2], [11], [1]	[21], [2], [11], [1], [2], [1], [0], [11], [1], [1], [0]
[111]	[111], [11]	[111], [11], [11], [1]
[4]	[4], [3], [2], [1], [0]	[4], [3], [2], [1], [0], [3], [2], [1], [0], [2], [1], [0], [1], [0], [0]
[31]	[31], [3], [2], [1], [21], [11]	[31], [3], [2], [1], [21], [11], [3], [2], [1], [0], [2], [1], [0], [1], [0], [21], [2], [11], [1], [11], [1]
[22]	[22], [21], [2]	[22], [21], [2], [21], [2], [11], [1], [2], [1], [0]
[211]	[211], [21], [111], [11]	[211], [21], [111], [11], [21], [2], [11], [1], [111], [11], [11], [1]
[1111]	[1111], [111]	[1111], [111], [111], [11]

1. Centroid energy

a. $U(6) \supset U(5)$

The centroid energy $E_c(\underset{\sim}{mf})$ for the group U(6) is given in eq.(5.4.2). In order to determine $E_c(\underset{\sim}{mfg})$ where $\underset{\sim}{g}$ denotes the representation of U(5), we first determine the scalar operators with rank ≤ 2. Denoting by n' the number operator for U(5) such that $\langle n' \rangle = m'$ where

$$m' = \sum g_i \qquad (5.4.10)$$

and by $G_2(5)$, $G_3(5)$ and $G_4(5)$ the Casimir operators of U(5) one can use the following 8 operators:

$$1,\ n,\ n^2,\ G_2(6),\ n',\ nn',\ n'^2,\ G_2(5)$$

to construct the equivalent operator $\hat{H}$ written as

$$\hat{H} = a_1 + a_2 n + a_3 n^2 + a_4 G_2(6) + a_5 n' + a_6 nn' + a_7 n'^2 + a_8 G_2(5) \qquad (5.4.11)$$

In terms of the averages in the defining space the expression for $E_c(\underset{\sim}{mfg})$ is given by,

$$\begin{aligned} E_c(m \underset{\sim}{f} \underset{\sim}{g}) = {} & \left[1 - \frac{6}{4}m + \frac{m^2}{2}\right] E_c(0\,\underset{\sim}{0}\,\underset{\sim}{0}) + \left[2m' - mm'\right] E_c(1\,\underset{\sim}{1}\,\underset{\sim}{1}) \\ & + \left[2m - m^2 - 2m' + mm'\right] E_c(1\,\underset{\sim}{1}\,\underset{\sim}{0}) + \left[-\frac{3}{2}m' + \frac{m'}{4}\right. \\ & \left. + \frac{1}{4}\langle G_2(5)\rangle\right] E_c(2\,\underset{\sim}{2}\,\underset{\sim}{2}) \\ & + \left[-\frac{5}{4}m - \frac{m^2}{4} + \frac{1}{4}\langle G_2(6)\rangle + m' - \frac{3}{4}m'^2 - \frac{1}{4}\langle G_2(5)\rangle\right. \\ & \left. + mm'\right] E_c(2\,\underset{\sim}{2}\,\underset{\sim}{1}) \end{aligned}$$

$$+\left[-\frac{m}{2}+\frac{m^2}{2}+\frac{m'}{2}+\frac{m'^2}{2}-mm'\right]E_c(2\,\underset{\sim}{2}\,\underset{\sim}{0})$$

$$+\left[m'+\frac{m'^2}{4}-\frac{1}{4}\langle G_2(5)\rangle\right]E_c(2\,\underset{\sim}{11}\,\underset{\sim}{11})$$

$$+\left[\frac{5}{4}m+\frac{m^2}{2}-\frac{1}{4}\langle G_2(6)\rangle-m'-\frac{m'^2}{4}+\frac{1}{4}\langle G_2(5)\rangle\right]E_c(2\,\underset{\sim}{11}\,\underset{\sim}{10}) \qquad (5.4.12)$$

This expression contains no operator with particle rank > 2. A different expression for the centroid energy can be derived using the projection operator formalism. Using n' to distinguish representations of U(5) contained in $\underset{\sim}{f}$, the density operators are constructed. Rewriting eq.(5.4.2) as:

$$E_c(m\underset{\sim}{f}) = \sum_{i=1}^{4} Y(i,\underset{\sim}{f})\,E_c(i) \qquad (5.4.13)$$

the final centroid energy $E_c(mf\underset{\sim}{g})$ is given by

$$\begin{aligned}E_c(mf\underset{\sim}{g}) = {} & Y(1)E_c(0\,\underset{\sim}{0}\,\underset{\sim}{0})+Y(2)m'E_c(1\,\underset{\sim}{1}\,\underset{\sim}{1})\\ & +Y(2)(1-m')E_c(1\,\underset{\sim}{1}\,\underset{\sim}{0})+Y(3)\frac{m'(m'-1)}{2}E_c(2\,\underset{\sim}{2}\,\underset{\sim}{2})\\ & +Y(3)(2-m')m'E_c(2\,\underset{\sim}{2}\,\underset{\sim}{1})\\ & +Y(3)\frac{(m'-1)(m'-2)}{2}E_c(2\,\underset{\sim}{2}\,\underset{\sim}{0})\\ & +Y(4)(m'-1)E_c(2\,\underset{\sim}{11}\,\underset{\sim}{11})+Y(4)(2-m')E_c(2\,\underset{\sim}{11}\,\underset{\sim}{10})\end{aligned} \qquad (5.4.14)$$

This expression involves operators with rank > 2. It should be emphasised here that these expressions are not unique and one could in general derive many more by using other combinations of the given group scalars. However, when higher rank operators are included, things get complicated because one has to extract the relevant body part of these operators. In some cases this might be quite difficult.

b. U(6) $\supset$ U(5) $\supset$ U(4)

Let n'' be the number operator and $\underset{\sim}{h}$ denote any representation of U(4). Then the expectation value of n'' is given by

$$\langle n'' \rangle_{\underset{\sim}{h}} = m'' = \sum_{i=1}^{4} h_i \qquad (5.4.15)$$

In order to determine the centroid energy we first observe that the number of scalar operators with rank ≤ 2 is 13 whereas there are 14 representations in the defining space. One has therefore to include higher rank operators and then determine a linearly independent set. Rather than doing this we proceed to determine $E_c(\underset{\sim}{mfgh})$ by constructing the density operators directly using n''.

First eq.(5.4.12) is rewritten in the form

$$E_c(\underset{\sim}{m}\,\underset{\sim}{f}\,\underset{\sim}{g}) = \sum_{i=1}^{8} Z(i,\underset{\sim}{f}\,\underset{\sim}{g})\, E_c(i) \qquad (5.4.16)$$

The projection operators are determined and the final expression is:

$$E_c(m \pm g h) = Z(1) E_c(0\,0\,0\,0) + Z(2) m'' E_c(1\,1\,1\,1)$$
$$+ Z(2)(1-m'') E_c(1\,1\,1\,0) + Z(3) E_c(1\,1\,0\,0)$$
$$+ Z(4) \frac{m''(m''-1)}{2} E_c(2\,2\,2\,2) + Z(4) m''(2-m'') E_c(2\,2\,2\,1)$$
$$+ Z(4) \frac{(m''-1)(m''-2)}{2} E_c(2\,2\,2\,0) + Z(5) m'' E_c(2\,2\,1\,1)$$
$$+ Z(5)(1-m'') E_c(2\,2\,1\,0) + Z(6) E_c(2\,2\,0\,0)$$
$$+ Z(7)(m''-1) E_c(2\,1\,1\,1)$$
$$+ Z(7)(2-m'') E_c(2\,1\,1\,0)$$
$$+ Z(8) m'' E_c(2\,1\,1\,0) + Z(8)(1-m'') E_c(2\,1\,0\,0) \qquad (5.4.17)$$

By straight forward extensions one can proceed to derive expressions for the centroid energy for further decompositions.

2. Variance

Expression for the variance can be derived in a way similar to the one used for centroid energy. For the decomposition $U(6) \supset U(5)$, one has 38 representations in the defining space and only 37 scalar operators with particle rank ≤ 4. One has

therefore to include one higher rank operator and proceed to determine $\sigma^2(\underset{\sim}{mfg})$ as was done in Sec.(5.2) for the SU(4)-Isospin case.

In the projection method formalism we start with eq. (5.4.4) and construct the density operators. In this case the number operator n' is not sufficient to determine the projection operators. The bilinear Casimir operator $G_2(5)$ of U(5) is then included. The projection operators are given in table (V.5).

The final expression can be written as

$$\sigma^2(m\underset{\sim}{f}\underset{\sim}{g}) = \sum_{i=1}^{38} T(i)\sigma^2(i) \qquad (5.4.18)$$

where T(i)'s are combinations of R(i)'s and P(i)'s.

We do not derive the expression for the variance $\sigma^2(\underset{\sim\sim\sim}{mfgh})$ corresponding to the decomposition U(6) ⊃ U(5) ⊃ U(4). This can be obtained, quite simply, by an application of the projection method to eq.(5.4.18).

5.5 Conclusion

In this chapter a method of group averaging for some cases where the Casimir operators are not sufficient to reproduce

Table (V.5)

Projection operators for variance $\sigma^2(mfg)$ corresponding to the decomposition U(6) ⊃ (U5)

$\underset{\sim}{f}$	$\underset{\sim}{g}$	P
[0]	[0]	1
[1]	[1]	n'
	[0]	(1-n')
[2]	[2]	$\frac{n'(n'-1)}{2}$
	[1]	n'(2-n')
	[0]	$\frac{(n'-1)(n'-2)}{2}$
[11]	[11]	(n'-1)
	[10]	(2-n')
[3]	[3]	$\frac{n'(n'-1)(n'-2)}{6}$
	[2]	$\frac{n'(3-n')(n'-1)}{2}$
	[1]	$\frac{n'(3-n')(2-n')}{2}$
	[0]	$\frac{(1-n')(2-n')(3-n')}{6}$
[21]	[21]	$\frac{(n'-2)(n'-1)}{2}$
	[20]	$\frac{(3-n')(n'-1)(\langle G_2(5)\rangle-8)}{4}$
	[11]	$\frac{(3-n')(n'-1)(12-\langle G_2(5)\rangle)}{4}$

(Table (V.5) contd.)

	$[10]$	$\frac{(3-n')(2-n')}{2}$
$[111]$	$[111]$	$\frac{n'(n'-2)}{3}$
	$[110]$	$(3-n')$
$[4]$	$[4]$	$\frac{n'(3-n')(2-n')(n'-1)}{24}$
	$[3]$	$\frac{n'(2-n')(1-n')(4-n')}{6}$
	$[2]$	$\frac{n'(4-n')(3-n')(n'-1)}{4}$
	$[1]$	$n'(4-n')(3-n')(2-n')/6$
	$[0]$	$\frac{(4-n')(3-n')(2-n')(1-n')}{24}$
$[31]$	$[31]$	$\frac{(n'-3)(n'-2)(n'-1)}{6}$
	$[30]$	$\frac{(n'-4)(n'-2)(n'-1)(15-\langle G_2(5)\rangle)}{12}$
	$[21]$	$\frac{(n'-4)(n'-2)(n'-1)(\langle G_2(5)\rangle-21)}{12}$
	$[20]$	$\frac{(n'-4)(n'-3)(n'-1)(\langle G_2(5)\rangle-8)}{8}$
	$[11]$	$\frac{(n'-4)(n'-3)(n'-1)(12-\langle G_2(5)\rangle)}{8}$
	$[10]$	$\frac{(n'-4)(n'-3)(2-n')}{6}$

Table (V.5)(contd)

[22]	[22]	$\frac{(n'-3)(n'-2)}{2}$
	[21]	$(n'-4)(2-n')$
	[20]	$\frac{(n'-4)(n'-3)}{2}$
[211]	[211]	$\frac{(n'-3)(n'-2)}{2}$
	[210]	$\frac{(4-n')(n'-2)(\langle G_2(5)\rangle -9)}{6}$
	[111]	$\frac{(4-n')(n'-2)(15-\langle G_2(5)\rangle)}{6}$
	[110]	$\frac{(4-n')(3-n')}{2}$
[111]	[111]	$(n'-3)$
	[1110]	$(4-n')$

the averages of H and H^2 has been given. The essential point is that one can use mixed operators with correct particle rank constructed out of the given set of group scalars. Although it should be possible to derive analytical expressions for the eigenvalue of the p < k body part of rank k operator in the irrep $\underset{\sim}{\alpha}$, a different method (applicable in our case only) to evaluate this eigenvalue has been indicated.

Using this method it would be possible to apply the distribution method to many more groups of interest where one might be interested in studying the goodness of symmetries defined by these groups. It being now possible to average over a chain of subgroups, one may hope to learn more regarding normality of distributions.

CHAPTER VI

SUMMARY AND FUTURE PROSPECTS

This thesis had dealt mainly with the application of spectral distribution methods to symmetries in nuclei.

The Wigner Supermultiplet scheme was investigated for nuclei in the 2s-1d and 2p-1f shells. It is found that SU(4) symmetry in the ground state region of these nuclei is strongly mixed, the extent of breaking being more or less the same for light and heavier nuclei. In view of this, it is no longer surprising to find energy systematics which are consequences of this symmetry in heavier ($A \lesssim 110$) nuclei, than in lighter ones. It appears that symmetry mixing does not significantly affect energy relationships and in fact such relationships are rather successful in concealing the strong violations of the symmetry.

The external widths would provide in general a more precise measure of symmetry breaking. With the work of Hecht and Draayer (1974) who have given a general formula for the partial widths for the direct product subgroup U(N/k) x U(k) of the full unitary group U(N), it would be interesting to determine how the symmetry mixing results obtained here are affected.

A preliminary application of SU(4) level densities to alpha transfer reactions has been given. Much more work needs to be done in order to determine how the cross section for alpha transfer changes when one includes the dominant irreps.

SU(4)-Isospin distributions along with other distributions arising out of the decomposition of the spectroscopic space according to subgroups of U(N) have been quite successful in determining binding energies and spectra of nuclei. Results in the 2p-1f shell indicate that one can use these methods in arbitrarily large but finite spaces.

The average deformation of a nucleus as a function of excitation energy needs to be examined in greater detail. By adding an external quadrupole field, one can study the response of the nucleus to the field, and the method can be used in the study of collective motion of nuclei. Also the average moment is of interest in heavy ion reaction processes where compound nuclear formation takes place.

It has been shown how to evaluate averages for groups when the number of scalar operators are not sufficient to provide propagation laws. A projection operator approach to construct the density operators has been developed. Using the expression for variance in the case of SU(4)-isospin

averaging derived using these methods, it would be worthwhile calculating the exact widths and comparing with the equivalent width approximation. Further, it would be possible to apply the distribution methods to many more groups where one might be interested in studying the goodness of symmetries defined by these groups.

In the end we would like to mention that this is only the beginning of a systematic study of nuclear symmetries. It is hoped that such a study would reveal what goes on in complex nuclei and provide a much clearer understanding of the structure of a nucleus.

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APPENDIX

The charge-spin (TS) multiplets contained in a Wigner Supermultiplet are obtained for all possible U(4) representations for $11 \leq m \leq 20$ particles.

The structure of the Wigner supermultiplet is given by a reduction of the corresponding U(4) representation $\underset{\sim}{f} = [f_1, f_2, f_3, f_4]$ into irreps of U(2) x U(2). This reduction has been carried out by Jahn (1950), where starting from a known structure and using Littlewood's rules (1940) and adding the isospin and spin vectorially (vector addition), one can obtain the (TS) multiplets contained in a given Young shape.

As an illustration of the method, the (TS) structure for 3 particles is obtained. The charge-spin multiplets for $m \leq 2$ are given below:

$\underset{\sim}{f}$	(2T+1, 1S+1)
[1]	(22)
[2]	(11)(33)
[11]	(13)(31)

Using Littlewood's rules and coupling the isospin and spin, the representations and (TS) structure for 3 particles can be written as:

$$\underset{(11)(33)}{[2]} \times \underset{(22)}{[1]} = \underset{(22)^2(24)(42)(44)}{[3]} + [21]$$

$$\underset{(13)(31)}{[11]} \times \underset{(22)}{[1]} = \underset{(22)^2(24)(42)}{[21]} + [111]$$

Now the representation $[f_1, f_2, f_3, f_4]$ is equivalent to $[f_1, f_1-f_3, f_1-f_2, 0]$. The (TS) multiplet contained in $[111]$ is (22). Therefore the (TS) multiplets contained in $[21]$ are (22), (24) and (42). Finally we obtain

f	(2T+1, 2S+1)
$[3]$	(22)(44)
$[21]$	(22)(24)(42)
$[111]$	(22)

Jahn has tabulated the structure of all charge-spin multiplets for $m \leq 10$. Further, for some special classes of SU(4) representations where the multiplicities in (TS) values are not greater than one, Hecht and Pang (1969) have given the branching formula which gives the possible (TS) values belonging to f

In tables (1-10) the various charge-spin multiplets for all possible Young shapes for $11 \leq m \leq 20$ are enumerated. Since the multiplet (ST) for S,T occurs with (TS), in the tables (ST) is not explicitly mentioned, it being understood. Also for a given m, the (TS) structure of those f not included in the tables

can be obtained from the equivalence of the Young diagram. Besides the equivalence mentioned before one also has $[f_1,f_2,f_3,f_4] \equiv [f_1-f_4,\ f_2-f_4,\ f_3-f_4,\ 0]$. The dimensionality (dim) of $\underset{\sim}{f}$ is also listed.

Table 1

[f]	(2T+1, 2S+1)	Dm
[11]	(22)(44)(66)(88)(10,10)(12,12)	364
[101]	(22)(24)(44)(46)(66)(68)(88)(8,10)(10,10)(10,12)	780
[92]	(22)(24)(26)(44)^2(46)(48)(66)^2(68)(6,10)(88)^2(8,10)(8,12)(10,10)	1056
[911]	(22)(24)(44)(46)(66)(68)(88)(8,10)(10,10)	540
[83]	(22)(24)(26)(28)(44)^2(46)^2(48)(4,10)(66)^2(68)^2(6,10)(6,12)(88)(8,10)	1100
[821]	(22)(24)^2(26)(44)^2(46)^2(48)(66)^2(68)^2(6,10)(88)^2(8,10)	924
[74]	(22)(24)(26)(28)(2,10)(44)^2(46)^2(48)^2(4,10)(4,12)(66)^2(68)(6,10)(88)	900
[731]	(22)(24)^2(26)^2(28)(44)^3(46)^3(48)^2(4,10)(66)^3(68)^2(6,10)(88)	1000
[722]	(22)(24)(26)(44)^2(46)(48)(66)^2(68)(88)	420
[65]	(22)(24)(26)(28)(2,10)(2,12)(44)(46)(48)(4,10)(66)(68)	504
[641]	(22)(24)^2(26)^2(28)^2(2,10)(44)^3(46)^3(48)^2(4,10)(66)^2(68)	756
[632]	(22)(24)^2(26)^2(44)^3(46)^3(66)^2(28)(48)(68)	540

Table 2

[f]	(2T+1, 2S+1)	Dim
[12]	$(11)(33)(55)(77)(99)(11,11)(13,13)$	455
[111]	$(13)(33)(35)(55)(57)(77)(79)(99)(9,11)(11,11)(11,13)$	1001
[102]	$(11)(15)(33)^2(35)(37)(55)^2(57)(59)(77)^2(79)(7,11)(99)^2(9,11)(9,13)(11,11)$	1404
[93]	$(13)(17)(33)(35)^2(37)(39)(55)^2(57)^2(59)(5,11)(77)^2(79)^2(7,11)(7,13)(99)(9,11)$	1540
[84]	$(11)(15)(19)(33)^2(35)(37)^2(39)(3,11)(55)^3(57)^2(59)^2(5,11)(5,13)(77)^2(79)(7,11)(99)$	1375
[75]	$(13)(17)(1,11)(33)(35)^2(37)(39)^2(3,11)(3,13)(55)(57)^2(59)(5,11)(77)(79)$	945
[66]	$(11)(15)(19)(1,13)(33)(37)(3,11)(55)(59)(77)$	330
[1011]	$(13)(33)(35)(55)(57)(77)(79)(99)(9,11)(11,11)$	715
[921]	$(13)(15)(33)^2(35)^2(37)(55)^2(57)^2(59)(77)^2(79)^2(7,11)(99)^2(9,11)$	1280
[831]	$(13)(15)(17)(33)^2(35)^3(37)^2(39)(55)^3(57)^3(59)^2(5,11)(77)^3(79)^2(7,11)(99)$	1485
[741]	$(13)(15)(17)(19)(33)^2(35)^3(37)^3(39)^2(3,11)(55)^4(57)^3(59)^2(5,11)(77)^2(79)$	1280
[651]	$(13)(15)(17)(19)(1,11)(33)^2(35)^2(37)^2(39)^2(3,11)(55)^2(57)^2(59)(77)$	735
[822]	$(11)(15)(33)^2(35)(37)(55)^2(57)(59)(77)^2(79)(99)$	616
[732]	$(13)(15)(17)(33)^2(35)^3(37)^2(39)(55)^3(57)^3(59)(77)^2(79)$	875
[642]	$(11)(15)^2(17)(19)(33)^3(35)^3(37)^3(39)(55)^4(57)^2(59)(77)$	729
[633]	$(13)(17)(33)(35)^2(37)(55)^2(57)(77)$	300

Table 3

[f]	(2T+1, 2S+1)	Dim
[13]	$(22)(44)(66)(88)(10,10)(12,12)(14,14)$	560
[121]	$(22)(24)(44)(46)(66)(68)(88)(8,10)(10,10)(12,12)(10,12)(12,14)$	1260
[112]	$(22)(24)(26)(44)^2(46)(48)(66)^2(68)(6,10)(88)^2(8,10)(8,12)(10,10)^2(10,12)(10,14)(12,12)$	1820
[103]	$(22)(24)(26)(28)(44)^2(46)^2(48)(4,10)(66)^2(68)^2(6,10)(6,12)(88)^2(8,10)^2(8,12)(8,14)$ $(10,10)(10,12)$	2080
[94]	$(22)(24)(26)(28)(2,10)(44)^2(46)^2(48)^2(4,10)(4,12)(66)^3(68)^2(6,10)^2(6,12)(6,14)$ $(88)^2(8,10)(8,12)(10,10)$	1980
[85]	$(22)(24)(26)(28)(2,10)(2,12)(44)^2(46)^2(48)^2(4,10)^2(4,12)(4,14)(66)^2(68)^2(6,10)$ $(6,12)(88)(8,10)$	1540
[76]	$(22)(24)(26)(28)(2,10)(2,12)(2,14)(44)(46)(48)(4,10)(4,12)(66)(68)(6,10)(88)$	840
[1111]	$(22)(24)(44)(46)(66)(68)(88)(8,10)(10,10)(10,12)(12,12)$	924
[1021]	$(22)(24)^2(26)(44)^2(46)^2(48)(66)^2(68)^2(6,10)(88)^2(8,10)^2(8,12)(10,10)^2(10,12)$	1716
[931]	$(22)(24)^2(26)^2(28)(44)^3(46)^3(48)^2(4,10)(66)^3(68)^3(6,10)^2(6,12)(88)^3(8,10)^2(8,12)$ $(10,10)$	2100
[841]	$(22)(24)^2(26)^2(28)^2(2,10)(44)^3(46)^4(48)^3)(4,10)^2(4,12)(66)^4(68)^3(6,10)^2(6,12)$ $(88)^2(8,10)$	1980
[751]	$(22)(24)^2(26)^2(28)^2(2,10)^2(2,12)(44)^3(46)^3(48)^3(4,10)^2(4,12)(66)^3(68)^2(6,10)(88)$	1400
[922]	$(22)(24)(26)(44)^2(46)(48)(66)^2(68)(6,10)(88)^2(8,10)(10,10)$	864
[832]	$(22)(24)^2(26)^2(28)(44)^3(46)^3(48)^2(4,10)(66)^3(68)^3(6,10)(88)^2(8,10)$	1320
[742]	$(22)(24)^2(26)^3(28)^2(2,10)(44)^4(46)^4(48)^3(4,10)(66)^4(68)^2(6,10)(88)$	1260
[733]	$(22)(24)(26)(28)(44)^2(46)^2(48)(66)^2(68)(88)$	500

Table 4

[f]	(2T+1, 2S+1)	Dim
[14]	(11)(33)(55)(77)(99)(11,11)(13,13)(15,15)	680
[131]	(13)(33)(35)(55)(57)(77)(79)(99)(9,11)(11,11)(11,13)(13,13)(13,15)	1560
[122]	$(11)(15)(33)^2(35)(37)(55)^2(57)(59)(77)^2(79)(7,11)(99)^2(9,11)(9,13)(11,11)^2(11,13)(11,15)(13,13)$	2310
[113]	$(13)(17)(33)(37)(39)(35)^2(55)^2(57)^2(59)(5,11)(77)^2(79)^2(7,11)(7,13)(99)^2(9,11)^2(11,13)(9,15)(11,11)(11,13)$	2730
[104]	$(11)(15)(19)(33)^2(35)(37)^2(39)(3,11)(55)^3(57)^2(59)^2(5,11)(5,13)(77)^3(79)^2(7,11)^2(7,13)(7,15)(99)^2(9,11)(9,13)(11,11)$	2730
[95]	$(13)(17)(1,11)(33)(35)^2(37)(39)^2(3,11)(3,13)(55)^2(57)^3(59)^2(5,11)^2(5,13)(5,15)(77)^2(79)^2(7,11)(7,13)(99)(9,11)$	2310
[86]	$(11)(15)(19)(1,13)(33)^2(35)(37)^2(39)(3,11)^2(3,13)(3,15)(55)^2(57)(59)^2(5,11)(5,13)(77)^2(79)(7,11)(99)$	1540
[77]	(13)(17)(1,11)(1,15)(35)(39)(3,13)(57)(5,11)(79)	540
[1211]	(13)(33)(35)(55)(57)(77)(79)(99)(9,11)(11,11)(11,13)(13,13)	1170
[1121]	$(13)(15)(33)^2(35)^2(37)(55)^2(57)^2(59)(77)^2(79)^2(7,11)(99)^2(9,11)^2(9,13)(11,11)^2(11,13)$	2240
[1031]	$(13)(15)(17)(33)^2(35)^3(37)^2(39)(55)^3(57)^3(59)^2(5,11)(77)^3(79)^3(7,11)^2(7,13)(99)^3(9,11)^2(9,13)(11,11)$	2860
[941]	$(13)(15)(17)(19)(33)^2(35)^3(37)^3(39)^2(3,11)(55)^4(57)^4(59)^3(5,11)^2(5,13)(77)^4(79)^3(7,11)^2(7,13)(99)^2(9,11)$	2830
[851]	$(13)(15)(17)(19)(1,11)(33)^2(35)^3(37)^3(39)^3(3,11)^2(3,13)(55)^4(57)^4(59)^3(5,11)^2(5,13)(77)^3(79)^2(7,11)(99)$	2310
[761]	$(13)(15)(17)(19)(1,11)(1,13)(33)^2(35)^2(37)^2(39)^2(3,11)^2(3,13)(55)^2(57)^2(59)^2(5,11)(77)^2(79)$	1280
[1022]	$(11)(15)(33)^2(35)(37)(55)^2(57)(59)(77)^2(79)(7,11)(99)^2(9,11)(11,11)$	1170
[932]	$(13)(15)(17)(33)^2(35)^3(37)^2(39)(55)^3(57)^3(59)^2(5,11)(77)^3(79)^3(7,11)(99)^2(9,11)$	1890
[842]	$(11)(15)^2(17)(19)(33)^3(35)^3(37)^4(39)^2(3,11)(55)^5(57)^4(59)^3(5,11)(77)^4(79)^2(7,11)(99)$	1980

Table 4 (contd)

[f]	(2T+1, 2S+1)	Dim
[752]	$(13)(15)(17)^2(19)(1,11)(33)^2(35)^4(37)^3(39)^3(3,11)(55)^4(57)^4(59)^2(5,11)(77)^2(79)$	1470
[833]	$(13)(17)(33)(35)^2(37)(39)(55)^2(57)^2(59)(77)^2(79)(99)$	770
[743]	$(13)(15)(17)(19)(33)^2(35)^3(37)^3(39)(55)^4(57)^3(59)(77)^2(79)$	960

Table 5

[f]	(2T+1, 2S+1)	Dim
[15]	(22)(44)(66)(88)(10,10)(12,12)(14,14)(16,16)	816
[141]	(22)(24)(44)(46)(66)(68)(88)(8,10)(10,10)(10,12)(12,12)(12,14)(14,14)(14,16)	1904
[132]	$(22)(24)(26)(44)^2(46)(48)(66)^2(68)(6,10)(88)^2(8,10)(8,12)(10,10)^2(10,12)(10,14)(12,12)^2$ $(12,14)(12,16)(14,14)$	2880
[123]	$(22)(24)(26)(28)(44)^2(46)^2(48)(4,10)(66)^2(68)^2(6,10)(6,12)(88)^2(8,10)^2(8,12)(8,14)(10,10)^2$ $(10,12)^2(10,14)(10,16)(12,12)(12,14)$	3500
[114]	$(22)(24)(26)(28)(2,10)(44)^2(46)^2(48)^2(4,10)(4,12)(66)^3(68)^2(6,10)^2(6,12)(6,14)(88)^3(8,10)^2$ $(8,12)^2(8,14)(8,16)(10,10)^2(10,12)(10,14)(12,12)$	3640
[105]	$(22)(24)(26)(28)(2,10)(2,12)(44)^2(46)^2(48)^2(4,10)^2(4,12)(4,14)(66)^3(68)^3(6,10)^2(6,12)^2$ $(6,14)(6,16)(88)^2(8,10)^2(8,12)(8,14)(10,10)(10,12)$	3276
[96]	$(22)(24)(26)(28)(2,10)(2,12)(2,14)(44)^2(46)^2(48)^2(4,10)^2(4,12)^2(4,14)(66)^3(68)^2$ $(6,10)^2(6,12)(6,14)(88)^2(8,10)(8,12)(10,10)$	2464
[87]	$(22)(24)(26)(28)(2,10)(2,12)(2,14)(2,16)(44)(46)(48)(4,10)(4,12)(4,14)(66)(68)(6,10)(6,12)$ $(88)(8,10)$	1320
[1311]	$(22)(24)(44)(46)(66)(68)(88)(8,10)(10,10)(10,12)(12,12)(12,14)(14,14)$	1456
[1221]	$(22)(24)^2(26)(44)^2(46)^2(48)(66)^2(68)^2(6,10)(88)^2(8,10)^2(8,12)(10,10)^2(10,12)^2(10,14)(12,12)^2$ $(12,14)$	2860
[1131]	$(22)(24)^2(26)^2(28)(44)^3(46)^3(48)^2(4,10)(66)^3(68)^3(6,10)^2(6,12)(88)^3(8,10)^3(8,12)^2(8,14)$ $(10,10)^3(10,12)^2(10,14)(12,12)$	3780
[1041]	$(22)(24)^2(26)^2(28)^2(2,10)(44)^3(46)^4(48)^3(4,10)^2(4,12)(66)^4(68)^4(6,10)^3(6,12)^2(6,14)(88)^4$ $(8,10)^3(8,12)^2(8,14)(10,10)^2(10,12)$	4004
[951]	$(22)(24)^2(26)^2(28)^2(2,10)^2(2,12)(44)^3(46)^4(48)^4(4,10)^3(4,12)^2(4,14)(66)^5(68)^4(6,10)^3(6,12)^2$ $(6,14)(88)^3(8,10)^2(8,12)(10,10)$	3500
[861]	$(22)(24)^2(26)^2(28)^2(2,10)^2(2,12)^2(2,14)(44)^3(46)^3(48)^3(4,10)^3(4,12)^2(4,14)(66)^3(68)^3(6,10)^2$ $(6,12)(88)^2(8,10)$	2376
[1122]	$(22)(24)(26)(44)^2(46)(48)(66)^2(68)(6,10)(88)^2(8,10)(8,12)(10,10)^2(10,12)(12,12)$	1540

Table 5(Contd)

[f]	(2T+1, 2S+1)	Dim
[1032]	$(22)(24)^2(26)^2(28)(44)^3(46)^3(48)^2(4,10)(66)^3(68)^3(6,10)^2(6,12)(88)^3(8,10)^3$ $(8,12)(10,10)^2(10,12)$	2600
[942]	$(22)(24)^2(26)^3(28)^2(2,10)(44)^4(46)^4(48)^4(4,10)^2(4,12)(66)^5(68)^4(6,10)^3(6,12)$ $(88)^4(8,10)^2(8,12)(10,10)$	2916
[852]	$(22)(24)^2(26)^3(28)^3(2,10)^2(2,12)(44)^4(46)^5(48)^4(4,10)^3(4,12)(66)^5(68)^4(6,10)^2$ $(6,12)(88)^2(8,10)$	2464
[933]	$(22)(24)(26)(28)(44)^2(46)^2(48)(4,10)(66)^2(68)^2(6,10)(88)^2(8,10)(10,10)$	1120
[843]	$(22)(24)^2(26)^2(28)^2(2,10)(44)^3(46)^4(48)^3(4,10)(66)^4(68)^3(6,10)(88)^2(8,10)$	1540

Table 6

[r]	(2T+1, 2S+1)	Dim
[16]	(11)(33)(55)(77)(99)(11,11)(13,13)(15,15)(17,17)	969
[151]	(13)(33)(35)(55)(57)(77)(79)(99)(9,11)(11,11)(11,13)(13,13)(13,15)(15,15)(15,17)	2295
[142]	$(11)(15)(33)^2(35)(37)(55)^2(57)(59)(77)^2(79)(7,11)(99)^2(9,11)(9,13)(11,11)^2(11,13)$ $(11,15)(13,13)^2(13,15)(13,17)(15,15)$	3536
[133]	$(13)(17)(33)(35)^2(37)(39)(55)^2(57)^2(59)(5,11)(77)^2(79)^2(7,11)(7,13)(99)^2(9,11)^2$ $(9,13)(9,15)(11,11)^2(11,13)^2(11,15)(11,17)(13,13)(13,15)$	4400
[124]	$(11)(15)(19)(33)^2(35)(37)^2(39)(3,11)(55)^3(57)^2(59)^2(5,11)(5,13)(77)^3(79)^2(7,11)^2$ $(7,13)(7,15)(99)^3(9,11)^2(9,13)^2(9,15)(9,17)(11,11)^2(11,13)(11,15)(13,13)$	4725
[115]	$(13)(17)(1,11)(33)(35)^2(37)(39)^2(3,11)(3,13)(55)^2(57)^3(59)^2(5,11)^2(5,13)(5,15)$ $(77)^3(79)^3(7,11)^2(7,13)^2(7,15)(7,17)(99)^2(9,11)^2(9,13)(9,15)(11,11)(11,13)$	4459
[106]	$(11)(15)(19)(1,13)(33)^2(35)(37)^2(39)(3,11)^2(3,13)(3,15)(55)^3(57)^2(59)^3(5,11)^2(5,13)^2$ $(5,15)(5,17)(77)^3(79)^2(7,11)^2(7,13)(7,15)(99)^2(9,11)(9,13)(11,11)$	3640
[97]	$(13)(17)(1,11)(1,15)(33)(35)^2(37)(39)^2(3,11)(3,13)^2(3,15)(3,17)(55)(57)^2(59)(5,11)^2$ $(5,13)(5,15)(77)(79)^2(7,11)(7,13)(99)(9,11)$	2376
[88]	(11)(15)(19)(1,13)(1,17)(33)(37)(3,11)(3,15)(55)(59)(5,13)(77)(7,11)(99))	825
[1411]	(13)(33)(35)(55)(57)(77)(79)(99)(9,11)(11,11)(11,13)(13,13)(13,15)(15,15)	1785
[1321]	$(13)(15)(33)^2(35)^2(37)(55)^2(57)^2(59)(77)^2(79)^2(7,11)(99)^2(9,11)^2(9,13)(11,11)^2(11,13)^2$ $(11,15)(13,13)^2(13,15)$	3584
[1231]	$(13)(15)(17)(33)^2(35)^3(37)^2(39)(55)^3(57)^3(59)^2(5,11)(77)^3(79)^3(7,11)^2(7,13)(99)^3$ $(9,11)^3(9,13)^2(9,15)(11,11)^3(11,13)^2(11,15)(13,13)$	4875
[1141]	$(13)(15)(17)(19)(33)^2(35)^3(37)^3(39)^2(3,11)(55)^4(57)^4(59)^3(5,11)^2(5,13)(77)^4(79)^4$ $(7,11)^3(7,13)^2(7,15)(99)^4(9,11)^3(9,13)^2(9,15)(11,11)^2(11,13)$	5376
[1051]	$(13)(15)(17)(19)(1,11)(33)^2(35)^3(37)^3(39)^3(3,11)^2(3,13)(55)^4(57)^5(59)^4(5,11)^3(5,13)^2$ $(5,15)(77)^5(79)^4(7,11)^3(7,13)^2(7,15)(99)^3(9,11)^2(9,13)(11,11)$	5005
[961]	$(13)(15)(17)(19)(1,11)(1,13)(33)^2(35)^3(37)^3(39)^3(3,11)^3(3,13)^2(3,15)(55)^4(57)^4(59)^4$ $(5,11)^3(5,13)^2(5,15)(77)^4(79)^3(7,11)^2(7,13)(99)^2(9,11)$	3840

[f]	(2T+1, 2S+1)	Dim
[871]	$(13)(15)(17)(19)(1,11)(1,13)(1,15)(33)^2(35)^2(37)^2(39)^2(3,11)^2(3,13)^2(3,15)(55)^2(57)^2(59)^2(5,11)^2(5,13)(77)^2(79)^2(7,11)(99)$	2079
[1222]	$(11)(15)(33)^2(35)(37)(55)^2(57)(59)(77)^2(79)(7,11)(99)^2(9,11)(9,13)(11,11)^2(11,13)(13,13)$	1980
[1132]	$(13)(15)(17)(33)^2(35)^3(37)^2(39)(55)^3(57)^3(59)^2(5,11)(77)^3(79)^3(7,11)^2(7,13)(99)^3(9,11)^3(9,13)(11,11)^2(11,13)$	3465
[1042]	$(11)(15)^2(17)(19)(33)^3(35)^3(37)^4(39)^2(3,11)(55)^5(57)^4(59)^4(5,11)^2(5,13)(77)^5(79)^4(7,11)^3(7,13)(99)^4(9,11)^2(9,13)(11,11)$	4095
[952]	$(13)(15)(17)^2(19)(1,11)(33)^2(35)^4(37)^4(39)^4(3,11)^2(3,13)(55)^5(57)^6(59)^4(5,11)^3(5,13)(77)^5(79)^4(7,11)^2(7,13)(99)^2(9,11)$	3780
[862]	$(11)(15)^2(17)(19)^2(1,11)(1,13)(33)^3(35)^3(37)^4(39)^3(3,11)^3(3,13)(55)^5(57)^4(59)^4(5,11)^2(5,13)(77)^4(79)^2(99)(7,11)$	2640
[1033]	$(13)(17)(33)(35)^2(37)(39)(55)^2(57)^2(59)(5,11)(77)^2(79)^2(7,11)(99)^2(9,11)(11,11)$	1560
[943]	$(13)(15)(17)(19)(33)^2(35)^3(37)^3(39)^2(3,11)(55)^4(57)^4(59)^3(5,11)(77)^4(79)^3(7,11)(99)^2(9,11)$	2304
[853]	$(13)(15)(17)^2(19)(1,11)(33)^2(35)^4(37)^4(39)^3(3,11)(55)^5(57)^5(59)^3(5,11)(77)^4(79)^2(7,11)(99)$	2156
[844]	$(11)(15)(19)(33)^2(35)(37)^2(39)(55)^3(57)^2(59)(77)^2(79)(99)$	825

Table 7

[r]	(2T+1, 2S+1)	Dim
[17]	(22)(44)(66)(88)(10,10(12,12)(14,14)(16,16)(18,18)	1140
[161]	(22)(24)(44)(46)(66)(68)(88)(8,10](10,10)(10,12)(12,12)(12,14)(14,14)(14,16)(16,16)(16,18)	2736
[152]	$(22)(24)(26)(44)^2(46)(48)(66)^2(68)(6,10)(88)^2(8,10)(8,12)(10,10)^2(10,12)(10,14)(12,12)^2$ $(12,14)(12,16)(14,14)^2(14,16)(14,18)(16,16)$	4284
[143]	$(22)(24)(26)(28)(44)^2(46)^2(48)(4,10)(66)^2(68)^2(6,10)(6,12)(88)^2(8,10)^2(3,12)(8,14)(10,10)^2$ $(10,12)^2(10,14)(10,16)(12,12)^2(12,14)^2(12,16)(12,18)(14,14)(14,16)$	5440
[134]	$(22)(24)(26)(28)(2,10)(44)^2(46)^2(48)^2(4,10)(4,12)(66)^3(68)^2(6,10)^2(6,12)(6,14)(88)^3(8,10)^2$ $(8,12)^2(8,14)(8,16)(10,10)^3(10,12)^2(10,14)^2(10,16)(10,18)(12,12)^2(12,14)(12,16)(14,14)$	6000
[125]	$(22)(24)(26)(28)(2,10)(2,12)(44)^2(46)^2(48)^2(4,10)^2(4,12)(4,14)(66)^3(68)^3(6,10)^2(6,12)^2(6,14)$ $(6,16)(88)^3(8,10)^3(8,12)^2(8,14)^2(8,16)(8,18)(10,10)^2(10,12)^2(10,14)(10,16)(12,12)(12,14)$	5880
[116]	$(22)(24)(26)(28)(2,10)(2,12)(2,14)(44)^2(46)^2(48)^2(4,10)^2(4,12)^2(4,14)(4,16)(66)^3(68)^3(6,10)^3$ $(6,12)^2(6,14)^2(6,16)(6,18)(88)^3(8,10)^2(8,12)^2(8,14)(8,16)(10,10)^2(10,12)(10,14)(12,12)$	5096
[107]	$(22)(24)(26)(28)(2,10)(2,12)(2,14)(2,16)(44)^2(46)^2(48)^2(4,10)^2(4,12)^2(4,14)^2(4,16)(4,18)$ $(66)^2(68)^2(6,10)^2(6,12)^2(6,14)(6,16)(88)^2(8,10)^2(8,12)(8,14)(10,10)(10,12)$	3744
[98]	$(22)(24)(26)(28)(2,10)(2,12)(2,14)(2,16)(2,18)(44)(46)(48)(4,10)(4,12)(4,14)(4,16)(66)$ $(68)(6,10)(6,12)(6,14)(88)(8,10)(8,12)(10,10)$	1980
[1511]	(22)(24)(44)(46)(66)(68)(88)(8,10)(10,10)(10,12)(12,12)(12,14)(14,14)(14,16)(16,16)	2160
[1421]	$(22)(24)^2(26)(44)^2(46)^2(48)(66)^2(68)^2(6,10)(88)^2(8,10)^2(8,12)(10,10)^2(10,12)^2(10,14)(12,12)^2$ $(12,14)^2(12,16)(14,14)^2(14,16)$	4420
[1331]	$(22)(24)^2(26)^2(28)(44)^3(46)^3(48)^2(4,10)(66)^3(68)^3(6,10)^2(6,12)(88)^3(8,10)^3(8,12)^2(8,14)$ $(10,10)^3(10,12)^3(10,14)^2(10,16)(12,12)^3(12,14)^2(12,16)(14,14)$	6160
[1241]	$(22)(24)^2(26)^2(28)^2(2,10)(44)^3(46)^4(48)^3(4,10)^2(4,12)(66)^4(68)^4(6,10)^3(6,12)^2(6,14)(88)^4$ $(8,10)^4(8,12)^3(8,14)^2(8,16)(10,10)^4(10,12)^3(10,14)^2(10,16)(12,12)^2(12,14)$	7020
[1151]	$(22)(24)^2(26)^2(28)^2(2,10)^2(2,12)(44)^3(46)^4(48)^4(4,10)^3(4,12)^2(4,14)(66)^5(68)^5(6,10)^4(6,12)^3$ $(6,14)^2(6,16)(88)^5(8,10)^4(8,12)^3(8,14)^2(8,16)(10,10)^3(10,12)^2(10,14)(12,12)$	6360
[1061]	$(22)(24)^2(26)^2(28)^2(2,10)^2(2,12)^2(2,14)(44)^3(46)^4(48)^4(4,10)^4(4,12)^3(4,14)^2(4,16)(66)^5(68)^5$ $(6,10)^4(6,12)^3(6,14)^2(6,16)(88)^4(8,10)^3(8,12)^2(8,14)(10,10)^2(10,12)$	5720
[971]	$(22)(24)^2(26)^2(28)^2(2,10)^2(2,12)^2(2,14)^2(2,16)(44)^3(46)^3(48)^3(4,10)^3(4,12)^3(4,14)^2(4,16)$	

Table 7 (Contd)

[f]	(2T+1, 2S+1)	Dim
	$(66)^3(68)^3(6,10)^3(6,12)^2(6,14)(88)^3(8,10)^2(8,12)(10,10)$	3780
[1322]	$(22)(24)(26)(44)^2(46)(48)(66)^2(68)(6,10)(88)^2(8,10)(8,12)(10,10)^2(10,12)(10,14)(12,12)^2$ $(12,14)(14,14)$	2496
[1232]	$(22)(24)^2(26)^2(28)(44)^3(46)^3(48)^2(4,10)(66)^3(68)^3(6,10)^2(6,12)(88)^3(8,10)^3(8,12)^2(8,14)$ $(10,10)^3(10,12)^3(10,14)(12,12)^2(12,14)$	4500
[1142]	$(22)(24)^2(26)^3(28)^2(2,10)(44)^4(46)^4(48)^4(4,10)^2(4,12)(66)^5(68)^4(6,10)^4(6,12)^2(6,14)$ $(88)^5(8,10)^4(8,12)^3(8,14)(10,10)^4(10,12)^2(10,14)(12,12)$	5544
[1052]	$(22)(24)^2(26)^3(28)^3(2,10)^2(2,12)(44)^4(46)^5(48)^5(4,10)^4(4,12)^2(4,14)(66)^6(68)^6(6,10)^4$ $(6,12)^3(6,14)(88)^5(8,10)^4(8,12)^2(8,14)(10,10)^2(10,12)$	5460
[962]	$(22)(24)^2(26)^3(28)^3(2,10)^3(2,12)^2(2,14)(44)^4(46)^5(48)^5(4,10)^4(4,12)^3(4,14)(66)^6(68)^5$ $(6,10)^4(6,12)^2(6,14)(88)^4(8,10)^2(8,12)(10,10)$	4320
[1133]	$(22)(24)(26)(28)(44)^2(46)^2(48)(4,10)(66)^2(68)^2(6,10)(6,12)(88)^2(8,10)^2(8,12)(10,10)^2$ $(10,12)(12,12)$	2100
[1043]	$(22)(24)^2(26)^2(28)^2(2,10)(44)^3(46)^4(48)^3(4,10)^2(4,12)(66)^4(68)^4(6,10)^3(6,12)(88)^4$ $(8,10)^3(8,12)(10,10)^2(10,12)$	3276
[953]	$(22)(24)^2(26)^3(28)^3(2,10)^2(2,12)(44)^4(46)^5(48)^5(4,10)^3(4,12)(66)^6(68)^5(6,10)^3(6,12)$ $(88)^4(8,10)^2(8,12)(10,10)$	3360
[944]	$(22)(24)(26)(28)(2,10)(44)^2(46)^2(48)^2(4,10)(66)^3(68)^2(6,10)(88)^2(8,10)(10,10)$	1260

Table 8

[f]	(2T+1, 2S+1)	Dim
[18]	$(11)(33)(55)(77)(99)(11,11)(13,13)(15,15)(17,17)(19,19)$	1330
[171]	$(13)(33)(35)(55)(57)(77)(79)(99)(9,11)(11,11)(11,13)(13,13)(13,15)(15,15)(15,17)(17,17)(17,19)$	3230
[162]	$(11)(15)(33)^2(35)(37)(55)^2(57)(59)(77)^2(79)(7,11)(99)^2(9,11)(9,13)(11,11)^2(11,13)(11,15)$ $(13,13)^2(13,15)(13,17)(15,15)^2(15,17)(15,19)(17,17)$	5130
[153]	$(13)(17)(33)(35)^2(37)(39)(55)^2(57)^2(59)(5,11)(77)^2(79)^2(7,11)(7,13)(99)^2(9,11)^2(9,13)(9,15)$ $(11,11)^2(11,13)^2(11,15)(11,17)(13,13)^2(13,15)^2(13,17)(13,19)(15,15)(15,17)$	6630
[144]	$(11)(15)(19)(33)^2(35)(37)^2(39)(3,11)(55)^3(57)^2(59)^2(5,11)(5,13)(77)^3(79)^2(7,11)^2(7,13)$ $(7,15)(99)^3(9,11)^2(9,13)^2(9,15)(9,17)(11,11)^3(11,13)^2(11,15)^2(11,17)(11,19)(13,13)^2(13,15)$ $(13,17)(15,15)$	7430
[135]	$(13)(17)(1,11)(33)(35)^2(37)(39)^2(3,11)(3,13)(55)^2(57)^3(59)^2(5,11)^2(5,13)(5,15)(77)^3(79)^3(7,11)^2$ $(7,13)^2(7,15)(7,17)(99)^3(9,11)^3(9,13)^2(9,15)^2(9,17)(9,19)(11,11)^2(11,13)^2(11,15)(11,17)(13,13)$ $(13,15)$	7560
[126]	$(11)(15)(19)(1,13)(33)^2(35)(37)^2(39)(3,11)^2(3,13)(3,15)(55)^3(57)^2(59)^3(5,11)^2(5,13)^2(5,15)$ $(5,17)(77)^4(79)^3(7,11)^3(7,13)^2(7,15)^2(7,17)(7,19)(99)^3(9,11)^2(9,13)^2(9,15)(9,17)(11,11)^2$ $(11,13)(11,15)(13,13)$	8360
[117]	$(13)(17)(1,11)(1,15)(33)(35)^2(37)(39)^2(3,11)(3,13)^2(3,15)(3,17)(55)^2(57)^3(59)^2(5,11)^3(5,13)$ $(5,15)^2(5,17)(5,19)(77)^2(79)^3(7,11)^2(7,13)^2(7,15)(7,17)(99)^2(9,11)^2(9,13)(9,15)(11,11)(11,13)$	5460
[108]	$(11)(15)(19)(1,13)(1,17)(33)^2(35)(37)^2(39)(3,11)^2(3,13)(3,15)^2(3,17)(3,19)(55)^2(57)(59)^2$ $(5,11)(5,13)^2(5,15)(5,17)(77)^2(79)(7,11)^2(7,13)(7,15)(99)^2(9,11)(9,13)(11,11)$	3510
[99]	$(13)(17)(1,11)(1,15)(1,19)(35)(39)(3,13)(3,17)(57)(5,11)(5,15)(79)(7,13)(9,11)$	1210
[1611]	$(13)(33)(35)(55)(57)(77)(79)(99)(9,11)(11,11)(11,13)(13,13)(13,15)(15,15)(15,17)(17,17)$	2534
[1521]	$(13)(15)(33)^2(35)^2(37)(55)^2(57)^2(59)(77)^2(79)^2(7,11)(99)^2(9,11)^2(9,13)(11,11)^2(11,13)^2$ $(11,15)(13,13)^2(13,15)^2(13,17)(15,15)^2(15,17)$	5376

Table 8 (Continued)

[f]	(2T+1, 2S+1)	Dim
[1431]	(13)(15)(17)(33)^2(35)^3(37)^2(39)(55)^3(57)^3(59)^2(5,11)(77)^3(79)^3(7,11)^2(7,13)(99)^3(9,11)^3(9,13)^2 (9,15)(11,11)^3(11,13)^3(11,15)^2(11,17)(13,13)^3(13,15)^2(13,17)(15,15)	
[1341]	(13)(15)(17)(19)(33)^2(35)^3(37)^3(39)^2(3,11)(55)^4(57)^4(59)^3(5,11)^2(5,13)(77)^4(79)^4(7,11)^3(7,13)^2 (7,15)(99)^4(9,11)^4(9,13)^3(9,15)^2(9,17)(11,11)^4(11,13)^3(11,15)^2(11,17)(13,13)^2(13,15)	7650
[1251]	(13)(15)(17)(19)(1,11)(33)^2(35)^3(37)^3(39)^3(3,11)^2(3,13)(55)^4(57)^5(59)^4(5,11)^3(5,13)^2(5,15)(77)^5 (79)^5(7,11)^4(7,13)^3(7,15)^2(7,17)(99)^5(9,11)^4(9,13)^3(9,15)^2(9,17)(11,11)^3(11,13)^2(11,15)(13,13)	8960
[1161]	(13)(15)(17)(19)(1,11)(1,13)(33)^2(35)^3(37)^3(39)^3(3,11)^3(3,13)^2(3,15)(55)^4(57)^5(59)^5(5,11)^4(5,13)^3 (5,15)^2(5,17)(77)^6(79)^5(7,11)^4(7,13)^3(7,15)^2(7,17)(99)^4(9,11)^3(9,13)^2(9,15)(11,11)^2(11,13)	9100
[1071]	(13)(15)(17)(19)(1,11)(1,13)(1,15)(33)^2(35)^3(37)^3(39)^3(3,11)^3(3,13)^3(3,15)^2(3,17)(55)^4(57)^4(59)^4 (5,11)^4(5,13)^3(5,15)^2(5,17)(77)^4(79)^4(7,11)^3(7,13)^2(7,15)(99)^3(9,11)^2(9,13)(11,11)	8064
[981]	(13)(15)(17)(19)(1,11)(1,13)(1,15)(1,17)(33)^2(35)^2(37)^2(39)^2(3,11)^2(3,13)^2(3,15)^2(3,17)(55)^2(57)^2 (59)^2(5,11)^2(5,13)^2(5,15)(77)^2(79)^2(7,11)^2(7,13)(99)^2(9,11)	6006
[1422]	(11)(15)(33)^2(35)(37)(55)^2(57)(59)(77)^2(79)(7,11)(99)^2(9,11)(9,13)(11,11)^2(11,13)(11,15)(13,13)^2 (13,15)(15,15)	3200
[1332]	(13)(15)(17)(33)^2(35)^3(37)^2(39)(55)^3(57)^3(59)^2(5,11)(77)^3(79)^3(7,11)^2(7,13)(99)^3(9,11)^3(9,13)^2 (9,15)(11,11)^3(11,13)^3(11,15)(13,13)^2(13,15)	3094
[1242]	(11)(15)^2(17)(19)(33)^3(35)^3(37)^4(39)^2(3,11)(55)^5(57)^4(59)^4(5,11)^2(5,13)(77)^5(79)^4(7,11)^4(7,13)^2 (7,15)(99)^5(9,11)^4(9,13)^3(9,15)(11,11)^4(11,13)^2(11,15)(13,13)	5720
[1152]	(13)(15)(17)^2(19)(1,11)(33)^2(35)^4(37)^4(39)^4(3,11)^2(3,13)(55)^5(57)^6(59)^5(5,11)^4(5,13)^2(5,15)(77)^6 (79)^6(7,11)^4(7,13)^3(7,15)(99)^5(9,11)^4(9,13)^2(9,15)(11,11)^2(11,13)	7590
[1062]	(11)(15)^2(17)(19)^2(1,11)(1,13)(33)^3(35)^3(37)^5(39)^4(3,11)^4(3,13)^2(3,15)(55)^6(57)^6(59)^6(5,11)^4(5,13)^3 (5,15)(77)^7(79)^5(7,11)^4(7,13)^2(7,15)(99)^4(9,11)^2(9,13)(11,11)	7546
[972]	(13)(15)(17)^2(19)(1,11)^2(1,13)(1,15)(33)^2(35)^4(37)^3(39)^4(3,11)^3(3,13)^3(3,15)(55)^4(57)^5(59)^4(5,11)^4 (5,13)^2(5,15)(77)^4(79)^4(7,11)^2(7,13)(99)^2(9,11)	6500
[1233]	(13)(17)(33)(35)^2(37)(39)(55)^2(57)^2(59)(5,11)(77)^2(79)^2(7,11)(7,13)(99)^2(9,11)^2(9,13)(11,11)^2 (11,13)(13,13)	4374
		2750

Table 8 (Continued)

[f]	(2T+1, 2S+1)	Dim
[1143]	$(13)(15)(17)(19)(33)^2(35)^3(37)^3(39)^2(3,11)(55)^4(57)^4(59)^3(5,11)^2(5,13)(77)^4(79)^4(7,11)^3(7,13)(99)^4(9,11)^3$ $(9,13)(11,11)^2(11,13)$	4480
[1053]	$(13)(15)(17)^2(19)(1,11)(33)^2(35)^4(37)^4(39)^4(3,11)^2(3,13)(55)^5(57)^6(59)^5(5,11)^3(5,13)(77)^6(79)^5(7,11)^3(7,13)$ $(99)^4(9,11)^2(9,13)(11,11)$	4914
[963]	$(13)(15)(17)^2(19)^2(1,11)(1,13)(33)^2(35)^4(37)^5(39)^4(3,11)^3(3,13)(55)^6(57)^6(59)^5(5,11)^3(5,13)(77)^6(79)^4(7,11)^2$ $(7,13)(99)^2(9,11)$	4096
[1044]	$(11)(15)(19)(33)^2(35)(37)^2(39)(3,11)(55)^3(57)^2(59)^2(5,11)(77)^3(79)^2(7,11)(99)^2(9,11)(11,11)$	1820
[954]	$(13)(15)(17)(19)(1,11)(33)^2(35)^3(37)^3(39)^3(3,11)(55)^4(57)^5(59)^3(5,11)(77)^4(79)^3(7,11)(99)^2(9,11)$	2450

Table 9

[f]	(2T+1, 2S+1)	Dim
[19]	(22)(44)(66)(88)(10,10)(12,12)(14,14)(16,16)(18,18)(20,20)	1540
[151]	(22)(24)(44)(46)(66)(68)(88)(8,10)(10,10)(10,12)(12,12)(12,14)(14,14)(14,16)(16,16)(16,18)(18,18)(18,20)	3780
[172]	(22)(24)(26)(44)^2(46)(48)(66)^2(68)(6,10)(88)^2(8,10)(8,12)(10,10)^2(10,12)(10,14)(12,12)^2(12,14)(12,16)(14,14)^2 (14,16)(14,18)(16,16)^2(16,18)(16,20)(20,20)	6080
[163]	(22)(24)(26)(28)(44)^2(46)^2(48)(4,10)(66)^2(68)^2(6,10)(6,12)(88)^2(8,10)^2(8,12)(8,14)(10,10)^2(10,12)^2(10,14)(10,16) (12,12)^2(12,14)^2(12,16)(12,18)(14,14)^2(14,16)^2(14,18)(14,20)(16,16)(16,18)	7980
[154]	(22)(24)(26)(28)(2,10)(44)^2(46)^2(48)^2(4,10)(4,12)(66)^3(68)^2(6,10)^2(6,12)(6,14)(88)^3(8,10)^2(8,12)^2(8,14)(8,16)(10,10)^3 (10,12)^2(10,14)^2(10,16)(10,18)(12,12)^3(12,14)^2(12,16)^2(12,18)(12,20)(14,14)^2(14,16)(14,18)(16,16)	9180
[145]	(22)(24)(26)(28)(2,10)(2,12)(44)^2(46)^2(48)^2(4,10)^2(4,12)(4,14)(66)^3(68)^3(6,10)^2(6,12)^2(6,14)(6,16)(88)^3(8,10)^3(8,12)^2 (8,14)^2(8,16)(8,18)(10,10)^3(10,12)^3(10,14)^2(10,16)^2(10,18)(10,20)(12,12)^2(12,14)^2(12,16)(12,18)(14,14)(14,16)	9520
[136]	(22)(24)(26)(28)(2,10)(2,12)(2,14)(44)^2(46)^2(48)^2(4,10)^2(4,12)^2(4,14)(4,16)(66)^3(68)^3(6,10)^3(6,12)^2(6,14)^2(6,16)(6,18) (88)^4(8,10)^3(8,12)^3(8,14)^2(8,16)^2(8,18)(8,20)(10,10)^3(10,12)^2(10,14)^2(10,16)(10,18)(12,12)^2(12,14)(12,14)(14,14)	8960
[137]	(22)(24)(26)(28)(2,10)(2,12)(2,14)(2,16)(44)^2(46)^2(48)^2(4,10)^2(4,12)^2(4,14)^2(4,16)(4,18)(66)^3(68)^3(6,10)^3(6,12)^3 (6,14)^2(6,16)^2(6,18)(6,20)(88)^3(8,10)^3(8,12)^2(8,14)^2(8,16)(8,18)(10,10)^2(10,12)^2(10,14)(10,16)(12,12)(12,14)	7560
[118]	(22)(24)(26)(28)(2,10)(2,12)(2,14)(2,16)(2,18)(44)^2(46)^2(48)^2(4,10)^2(4,12)^2(4,14)^2(4,16)^2(4,18)(4,20)(66)^2(68)^2(6,10)^2 (6,12)^2(6,14)^2(6,16)(6,18)(88)^2(8,10)^2(8,12)^2(8,14)(8,16)(10,10)^2(10,12)(10,14)(12,12)	5460
[109]	(22)(24)(26)(28)(2,10)(2,12)(2,14)(2,16)(2,18)(2,20)(44)(46)(48)(4,10)(4,12)(4,14)(4,16)(4,18)(66)(68)(6,10)(6,12) (6,14)(6,16)(88)(8,10)(8,12)(8,14)(10,10)(10,12)(	2860
[1711]	(22)(24)(44)(46)(66)(68)(88)(8,10)(10,10)(10,12)(12,12)(12,14)(14,14)(14,16)(16,16)(16,18)(18,18)	3060
[1621]	(22)(24)^2(26)(44)^2(46)^2(48)(66)^2(68)^2(6,10)(88)^2(8,10)^2(8,12)(10,10)^2(10,12)^2(10,14)(12,12)^2(12,14)^2(12,16)(14,14)^2 (14,16)^2(14,18)(16,16)^2(16,18)	6460
[1531]	(22)(24)^2(26)^2(28)(44)^3(46)^3(48)^2(4,10)(66)^3(68)^3(6,10)^2(6,12)(88)^3(8,10)^3(8,12)^2(8,14)(10,10)^3(10,12)^3(10,14)^2 (10,16)(12,12)^3(12,14)^3(12,16)^2(12,18)(14,14)^3(14,16)^2(14,18)(16,16)	9360
[1441]	(22)(24)^2(26)^2(28)^2(3,10)(44)^3(46)^4(48)^3(4,10)^2(4,12)(66)^4(68)^4(6,10)^3(6,12)^2(6,14)(88)^4(8,10)^4(8,12)^3(8,14)^2(8,16) (10,10)^4(10,12)^4(10,14)^3(10,16)^2(10,18)(12,12)^4(12,14)^3(12,16)^2(12,18)(14,14)^2(14,16)	11220

Table 9 (Continued)

[f]	(2T+1,2S+1)	Dim
[1351]	$(22)(24)^2(26)^2(28)^2(2,10)^2(2,12)(44)^3(46)^4(48)^4(4,10)^3(4,12)^2(4,14)(66)^5(68)^5(6,10)^4(6,12)^3(6,14)^2(6,16)(68)^5$ $(8,10)^5(8,12)^4(8,14)^3(8,16)^2(8,18)(10,10)^5(10,12)^4(10,14)^3(10,16)^2(10,18)(12,12)^3(12,14)^2(12,16)(14,14)$	11760
[1261]	$(22)(24)^2(26)^2(28)^2(2,10)^2(2,12)^2(2,14)(44)^3(46)^4(48)^4(4,10)^4(4,12)^3(4,14)^2(4,16)(66)^5(68)^6(6,10)^5(6,12)^4$ $(6,14)^3(6,16)^2(6,18)(88)^6(8,10)^5(8,12)^4(8,14)^3(8,16)^2(8,18)(10,10)^2(10,12)^3(10,14)^2(10,16)(12,12)^2(12,14)$	10920
[1171]	$(22)(24)^2(26)^2(28)^2(2,10)^2(2,12)^2(2,14)^2(2,16)(44)^3(46)^4(48)^4(4,10)^4(4,12)^4(4,14)^3(4,16)^2(4,18)(66)^5(68)^5$ $(6,10)^5(6,12)^4(6,14)^3(6,16)^2(6,18)(88)^5(8,10)^4(8,12)^3(8,14)^2(8,16)(10,10)^3(10,12)^2(10,14)(12,12)$	8820
[1081]	$(22)(24)^2(26)^2(28)^2(2,10)^2(2,12)^2(2,14)^2(2,16)^2(2,18)(44)^3(46)^3(48)^3(4,10)^3(4,12)^3(4,14)^3(4,16)^2(4,18)(66)^3$ $(68)^3(6,10)^3(6,12)^3(6,14)^2(6,16)(88)^3(8,10)^3(8,12)^2(8,14)(10,10)^2(10,12)$	5720
[1522]	$(22)(24)(26)(44)^2(46)(48)(66)^2(68)(6,10)(68)^2(8,10)(8,12)(10,10)^2(10,12)(10,14)(12,12)^2(12,16)(12,16)(14,14)^2$ $(14,16)(16,16)$	3780
[1432]	$(22)(24)^2(26)^2(28)(44)^3(46)^3(48)^2(4,10)(66)^3(68)^3(6,10)^2(6,12)(88)^3(8,10)^3(8,12)^2(8,14)(10,10)^3(10,12)^3(10,14)^2$ $(10,16)(12,12)^3(12,14)^3(12,16)(14,14)^2(14,16)$	7140
[1342]	$(22)(24)^2(26)^3(28)^2(2,10)(44)^4(46)^4(48)^4(4,10)^2(4,12)(66)^5(68)^4(6,10)^4(6,12)^2(6,14)(88)^5(8,10)^4(8,12)^4(8,14)^2$ $(8,16)(10,10)^5(10,12)^4(10,14)^3(10,16)(12,12)^4(12,14)^2(12,16)(14,14)$	9360
[1252]	$(22)(24)^2(26)^3(28)^3(2,10)^2(2,12)(44)^4(46)^5(48)^5(4,10)^4(4,12)^2(4,14)(66)^6(68)^6(6,10)^5(6,12)^4(6,14)^2(6,16)(88)^6$ $(8,10)^6(8,12)^4(8,14)^3(8,16)(10,10)^5(10,12)^4(10,14)^2(10,16)(12,12)^2(12,14)$	10080
[1162]	$(22)(24)^2(26)^3(28)^3(2,10)^2(2,12)^2(2,14)(44)^4(46)^5(48)^6(4,10)^5(4,12)^4(4,14)^2(4,16)(66)^7(68)^7(6,10)^6(6,12)^4(6,14)^3$ $(6,16)(88)^7(8,10)^5(8,12)^4(8,14)^2(8,16)(10,10)^4(10,12)^2(10,14)(12,12)$	9240
[1072]	$(22)(24)^2(26)^3(28)^3(2,10)^3(2,12)^3(2,14)^2(2,16)(44)^4(46)^5(48)^5(4,10)^5(4,12)^4(4,14)^3(4,16)(66)^6(68)^6(6,10)^5(6,12)^4$ $(6,14)^2(6,16)(88)^5(8,10)^4(8,12)^2(8,14)(10,10)^2(10,12)$	7020
[1333]	$(22)(24)(26)(28)(44)^2(46)^2(48)(4,10)(66)^2(68)^2(6,10)(6,12)(88)^2(8,10)^2(8,12)(8,14)(10,10)^2(10,12)^2(10,14)(12,12)^2$ $(12,14)(14,14)$	3520
[1243]	$(22)(24)^2(26)^2(28)^2(2,10)(44)^3(46)^4(48)^3(4,10)^2(4,12)(66)^4(68)^4(6,10)^3(6,12)^2(6,14)(88)^4(8,10)^4(8,12)^3(8,14)$ $(10,10)^4(10,12)^3(10,14)(12,12)^2(12,14)$	5940
[1153]	$(22)(24)^2(26)^3(28)^3(2,10)^2(2,12)(44)^4(46)^5(48)^5(4,10)^4(4,12)^2(4,14)(66)^6(68)^6(6,10)^5(6,12)^3(6,14)(88)^6(8,10)^5$ $(8,12)^3(8,14)(10,10)^4(10,12)^2(10,14)(12,12)$	6860
[1063]	$(22)(24)^2(26)^3(28)^4(2,10)^3(2,12)^2(2,14)(44)^4(46)^6(48)^6(4,10)^5(4,12)^3(4,14)(66)^7(68)^7(6,10)^5(6,12)^3(6,14)(88)^6$ $(8,10)^4(8,12)^2(8,14)(10,10)^2(10,12)$	6240

Table 9 (Continued)

	(2T+1, 2S+1)	Dim
[5]		
[1144]	$(22)(24)(26)(28)(2,10)(44)^2(46)^2(48)^2(4,10)(4,12)(66)^3(68)^2(6,10)^2(6,12)(88)^3(8,10)^2(8,12)(10,10)^2(10,12)(12,12)$	252
[1054]	$(22)(24)^2(26)^2(28)^2(2,10)^2(2,12)(44)^3(46)^4(48)^4(4,10)^3(4,12)(66)^5(68)^5(6,10)^3(6,12)(88)^4(8,10)^3(8,12)(10,10)^2(10,12)$	30

Table 10

[f]	(2T+1, 2S+1)	Dim
[20]	(11)(33)(55)(77)(99)(11,11)(13,13)(15,15)(17,17)(19,19)(21,21)	1771
[191]	(13)(33)(35)(55)(57)(77)(79)(99)(9,11)(11,11)(11,13)(13,13)(13,15)(15,15)(15,17)(17,17)(17,19)(19,19)(19,21)	4389
[182]	(11)(15)(33)^2(35)(37)(55)^2(57)(59)(77)^2(79)(7,11)(99)^2(9,11)(9,13)(11,11)^2(11,13)(11,15)(13,13)^2(13,15)(13,17)(15,15)^2(15,17)(15,19)(17,17)^2(17,19)(17,21)(19,19)	7140
[173]	(13)(17)(33)(35)^2(37)(39)(55)^2(57)^2(59)(5,11)(77)^2(79)^2(7,11)(7,13)(99)^2(9,11)^2(9,13)(9,15)(11,11)^2(11,13)(11,15)(11,17)(13,13)^2(13,15)^2(13,17)(13,19)(15,15)^2(15,17)^2(15,19)(15,21)(17,17)(17,19)	9500
[164]	(11)(15)(19)(33)^2(35)(37)^2(39)(3,11)(55)^3(57)^2(59)^2(5,11)(5,13)(77)^3(79)^2(7,11)^2(7,13)(7,15)(99)^3(9,11)^2(9,13)^2(9,15)(9,17)(11,11)^3(11,13)^2(11,15)^2(11,17)(11,19)(13,13)^2(13,15)^2(13,17)^2(13,19)(13,21)(15,15)^2(15,17)(15,19)(17,17)	11115
[155]	(13)(17)(1,11)(33)(35)^2(37)(39)^2(3,11)(3,13)(55)^2(57)^3(59)^2(5,11)^2(5,13)(5,15)(77)^3(79)^3(7,11)^2(7,13)^2(7,15)(7,17)(99)^3(9,11)^2(9,13)^2(9,15)^2(9,17)(9,19)(11,11)^3(11,13)^2(11,15)^2(11,17)^2(11,19)(11,21)(13,13)^2(13,15)^2(13,17)(13,19)(15,15)(15,17)	11781
[146]	(11)(15)(19)(1,13)(33)^2(35)(37)^2(39)(3,11)^2(3,13)(3,15)(55)^2(57)^2(59)^2(5,11)^2(5,13)^2(5,15)(5,17)(77)^4(79)^3(7,11)^3(7,13)^2(7,15)^2(7,17)(7,19)(99)^4(9,11)^3(9,13)^3(9,15)^2(9,17)^2(9,19)(9,21)(11,11)^3(11,13)^2(11,15)^2(11,17)(11,19)(13,13)^2(13,15)(13,17)(15,15)	11424
[137]	(13)(17)(1,11)(1,15)(33)(35)^2(37)(39)^2(3,11)(3,13)^2(3,15)(3,17)(55)^2(57)^3(59)^2(5,11)^3(5,13)^2(5,15)^2(5,17)(5,19)(77)^3(79)^4(7,11)^3(7,13)^3(7,15)^2(7,17)^2(7,19)(7,21)(99)^3(9,11)^3(9,13)^2(9,15)^2(9,17)(9,19)(11,11)^2(11,13)^2(11,15)(11,17)(13,13)(13,15)	10004
[128]	(11)(15)(19)(1,13)(1,17)(33)^2(35)(37)^2(39)(3,11)^3(3,13)(3,15)^2(3,17)(3,19)(55)^3(57)^2(59)^2(5,11)^2(5,13)^3(5,15)^2(5,17)^2(5,19)(5,21)(77)^3(79)^2(7,11)^3(7,13)^2(7,15)^2(7,17)(7,19)(99)^3(9,11)^2(9,13)^2(9,15)(9,17)(11,11)^2(11,13)(11,15)(13,13)	7871
[119]	(13)(17)(1,11)(1,15)(1,19)(33)(35)^2(37)(39)^2(3,11)(3,13)^2(3,15)(3,17)^2(3,19)(3,21)(55)(57)^2(59)(5,11)^2(5,13)(5,15)^2(5,17)(5,19)(77)(79)^3(7,11)(7,13)^2(7,15)(7,17)(99)(9,11)^2(9,13)(9,15)(11,11)(11,13)	5005
[1911]	(11)(15)(19)(1,13)(1,17)(1,21)(33)(37)(3,11)(3,15)(3,19)(55)(59)(5,13)(5,17)(77)(7,11)(7,15)(99)(9,13)(11,11)	1716
[1811]	(13)(33)(35)(55)(57)(77)(79)(99)(9,11)(11,11)(11,13)(13,13)(13,15)(15,15)(15,17)(17,17)(17,19)(19,19)	3591
[1721]	(13)(15)(33)^2(35)^2(37)(55)^2(57)^2(59)(77)^2(79)^2(7,11)(99)^2(9,11)^2(9,13)(11,11)^2(11,13)^2(11,15)(13,13)^2(13,15)^2(13,17)(15,15)^2(15,17)^2(15,19)(17,17)^2(17,19)	7680
[1631]	(13)(15)(17)(33)^2(35)^3(37)^2(39)(55)^3(57)^3(59)^2(5,11)(77)^3(79)^3(7,11)^2(7,13)(99)^3(9,11)^3(9,13)^2(9,15)(11,11)^3(11,13)^3(11,15)^2(11,17)(13,13)^3(13,15)^3(13,17)^2(13,19)(15,15)^3(15,17)^2(15,19)(17,17)	11305

(Table 10 Continued)

[f]	(2T+1, 2S+1)	Dim
[1541]	$(13)(15)(17)(19)(33)^2(35)^3(37)^3(39)^2(3,11)(55)^4(57)^4(59)^3(5,11)^2(5,13)(77)^4(79)^4(7,11)^3(7,13)^2(7,15)(99)^4(9,13)^3$ $(9,11)^4(9,15)^2(9,17)(11,11)^4(11,13)^4(11,15)^3(11,17)^2(11,19)(13,13)^4(13,15)^3(13,17)^2(13,19)(15,15)^2(15,17)$	13824
[1451]	$((13)(15)(17)(19)(1,11)(33)^2(35)^3(37)^3(39)^3(3,11)^2(3,13)(55)^4(57)^5(59)^4(5,11)^3(5,13)^2(5,15)(77)^5(79)^5(7,11)^4$ $(7,13)^3(7,15)^2(7,17)(99)^5(9,11)^5(9,13)^4(9,15)^3(9,17)^2(9,19)(11,11)^5(11,13)^4(11,15)^3(11,17)^2(11,19)(13,13)^3(13,15)^2$ $(13,17)(15,15)$	14875
[1361]	$(13)(15)(17)(19)(1,11)(1,13)(33)^2(35)^3(37)^3(39)^3(3,11)^3(3,13)^2(3,15)(55)^4(57)^5(59)^5(5,11)^4(5,13)^3(5,15)^2(5,17)$ $(77)^6(79)^6(7,11)^5(7,13)^4(7,15)^3(7,17)^2(7,19)(99)^6(9,11)^5(9,13)^4(9,15)^3(9,17)^2(9,19)(11,11)^4(11,13)^3(11,15)^2(11,17)$ $(13,13)^2(13,15)$	14336
[1271]	$(13)(15)(17)(19)(1,11)(1,13)(1,15)(33)^2(35)^3(37)^3(39)^3(3,11)^3(3,13)^3(3,15)^2(3,17)(55)^4(57)^5(59)^5(5,11)^5(5,13)^4$ $(5,15)^3(5,17)^2(5,19)(77)^6(79)^6(7,11)^5(7,13)^4(7,15)^3(7,17)^2(7,19)(99)^5(9,11)^4(9,13)^3(9,15)^2(9,17)(11,11)^3(11,13)^2$ $(11,15)(13,13)$	12285
[1181]	$(13)(15)(17)(19)(1,11)(1,13)(1,15)(1,17)(33)^2(35)^3(37)^3(39)^3(3,11)^3(3,13)^3(3,15)^3(3,17)^2(3,19)(55)^4(57)^4(59)^4$ $(5,11)^4(5,13)^4(5,15)^3(5,17)^2(5,19)(77)^4(79)^4(7,11)^4(7,13)^3(7,15)^2(7,17)(99)^4(9,11)^3(9,13)^2(9,15)(11,11)^2(11,13)$	8960
[1091]	$(13)(15)(17)(19)(1,11)(1,13)(1,15)(1,17)(1,19)(33)^2(35)^2(37)^2(39)^2(3,11)^2(3,13)^2(3,15)^2(3,17)^2(3,19)(55)^2(57)^2(59)^2$ $(5,11)^2(5,13)^2(5,15)^2(5,17)(77)^2(79)^2(7,11)^2(7,13)^2(7,15)(99)^2(9,11)^2(9,13)(11,11)$	4719
[1622]	$(11)(15)(33)^2(35)(37)(55)^2(57)(59)(77)^2(79)(7,11)(99)^2(9,11)(9,13)(11,11)^2(11,13)(11,15)(13,13)^2(13,15)(13,17)(15,15)^2$ $(15,17)(17,17)$	4560
[1532]	$(13)(15)(17)(33)^2(35)^3(37)^2(39)(55)^3(57)^3(59)^2(5,11)(77)^3(79)^3(7,11)^2(7,13)(99)^3(9,11)^3(9,13)^2(9,15)(11,11)^3(11,13)^3$ $(11,15)^2(11,17)(13,13)^3(13,15)^3(13,17)(15,15)^2(15,17)$	9775
[1442]	$(11)(15)^2(17)(19)(33)^3(35)^3(37)^4(39)^2(3,11)(55)^5(57)^4(59)^4(5,11)^2(5,13)(77)^5(79)^4(7,11)^4(7,13)^2(7,15)(99)^5(9,11)^4$ $(9,13)^4(9,15)^2(9,17)(11,11)^5(11,15)^4(11,17)^3(11,19)(13,13)^4(13,15)^2(13,17)(15,15)$	11781
[1352]	$(13)(15)(17)^2(19)(1,11)(33)^2(35)^4(37)^4(39)^4(3,11)^2(3,13)(55)^5(57)^6(59)^5(5,11)^4(5,13)^2(5,15)(77)^6(79)^6(7,11)^5(7,13)^4$ $(7,15)^2(7,17)(99)^6(9,11)^6(9,13)^4(9,15)^3(9,17)(11,11)^5(11,13)^4(11,15)^2(11,17)(13,13)^2(13,15)$	13104
[1262]	$(11)(15)^2(17)(19)^2(1,11)(1,13)(33)^3(35)^3(37)^5(39)^4(3,11)^4(3,13)^2(3,15)(55)^6(57)^6(59)^7(5,11)^5(5,13)^4(5,15)^2(5,17)$ $(77)^8(79)^7(7,11)^6(7,13)^4(7,15)^3(7,17)(99)^7(9,11)^5(9,13)^4(9,15)^2(9,17)(11,11)^4(11,13)^2(11,15)(13,13)$	12600
[1172]	$(13)(15)(17)^2(19)(1,11)^2(1,13)(1,15)(33)^2(35)^4(37)^4(39)^5(3,11)^4(3,13)^4(3,15)^2(3,17)(55)^5(57)^7(59)^6(5,11)^6(5,13)^4$ $(5,15)^3(5,17)(77)^7(79)^7(7,11)^5(7,13)^4(7,15)^2(7,17)(99)^5(9,11)^4(9,13)^2(9,15)(11,11)^2(11,13)$	10395
[1082]	$(11)(15)^2(17)(19)^2(1,11)(1,13)^2(1,15)(1,17)(33)^3(35)^3(37)^4(39)^3(3,11)^4(3,13)^3(3,15)^3(3,17)(55)^5(57)^4(59)^5(5,11)^4$ $(5,13)^4(5,15)^2(5,17)(77)^5(79)^4(7,11)^4(7,13)^2(7,15)(99)^4(9,11)^2(9,13)(11,11)$	6825

Table 10 (Continued)

[f]	(2T+1, 2S+1)	Dim
[1433]	$(13)(17)(33)(35)^2(37)(39)(55)^2(57)^2(59)(5,11)(77)^2(79)^2(7,11)(7,13)(99)^2(9,11)^2(9,13)(9,15)(11,11)^2(11,13)^2$ $(11,15)(13,13)^2(13,15)(15,15)$	4420
[1343]	$(13)(15)(17)(19)(33)^2(35)^3(37)^3(39)^2(3,11)(55)^4(57)^4(59)^3(5,11)^2(5,13)(77)^4(79)^4(7,11)^3(7,13)^2(7,15)(99)^4(9,11)^4$ $(9,13)^3(9,15)(11,11)^4(11,13)^3(11,15)(13,13)^2(13,15)$	7680
[1253]	$(13)(15)(17)^2(19)(1,11)(33)^2(35)^4(37)^4(39)^4(3,11)^2(3,13)(55)^5(57)^6(59)^5(5,11)^4(5,13)^2(5,15)(77)^6(79)^6(7,11)^5$ $(7,13)^3(7,15)(99)^6(9,11)^5(9,13)^3(9,15)(11,11)^4(11,13)^2(11,15)(13,13)$	9240
[1163]	$(13)(15)(17)^2(19)^2(1,11)(1,13)(33)^2(35)^4(37)^5(39)^5(3,11)^4(3,13)^2(3,15)(55)^6(57)^7(59)^7(5,11)^5(5,13)^3(5,15)(77)^8$ $(79)^7(7,11)^5(7,13)^3(7,15)(99)^6(9,11)^4(9,13)^2(9,15)(11,11)^2(11,13)$	8960
[1073]	$(13)(15)(17)^2(19)^2(1,11)^2(1,13)(1,15)(33)^2(35)^4(37)^5(39)^5(3,11)^4(3,13)^3(3,15)(55)^6(57)^7(59)^6(5,11)^5(5,13)^3$ $(5,15)(77)^7(79)^6(7,11)^4(7,13)^2(7,15)(99)^4(9,11)^2(9,13)(11,11)$	7020
[1244]	$(11)(15)(19)(33)^2(35)(37)^2(39)(3,11)(55)^3(57)^2(59)^2(5,11)(5,13)(77)^3(79)^2(7,11)^2(7,13)(99)^3(9,11)^2(9,13)(11,11)^2$ $(11,13)(13,13)$	3375
[1154]	$(13)(15)(17)(19)(1,11)(33)^2(35)^3(37)^3(39)^3(3,11)^2(3,13)(55)^4(57)^5(59)^4(5,11)^3(5,13)(77)^5(79)^5(7,11)^3(7,13)(99)^4$ $(9,11)^3(9,13)(11,11)^2(11,13)$	5145
[1064]	$(11)(15)^2(17)(19)^2(1,11)(1,13)(33)^3(35)^3(37)^5(39)^4(3,11)^3(3,13)(55)^6(57)^6(59)^6(5,11)^3(5,13)(77)^7(79)^5(7,11)^3$ $(7,13)(99)^4(9,11)^2(9,13)(11,11)$	5200
[1055]	$(13)(17)(1,11)(33)(35)^2(37)(39)^2(3,11)(55)^2(57)^3(59)^2(5,11)(77)^3(79)^2(7,11)(99)^2(9,11)(11,11)$	1911