NONLINEAR WAVES IN DISPERSIVE MEDIA AND CURRENT DRIVEN INSTABILITIES IN MAGNETOPLASMAS

BY

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A THESIS

SUBMITTED FOR THE DEGREE

OF

DOCTOR OF PHILOSOPHY

OF THE

GUJARAT UNIVERSITY

JUNE 1976

PHYSICAL RESEARCH LABORATORY

AHMEDABAD 380009

INDIA

Through the haze of the pyre
glowed
a face,
In a mist of silence he lay
with a scornful delight
His sense of happiness.

That was JAYARAM
had an unpredictable sense of humor,
He died.
He would never become a memory,
To him, I dedicate.

C E R T I F I C A T E

I hereby declare that the work presented in this thesis is original and has not formed the basis for the award of any degree or diploma by any University or Institution.

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June 10, 1976

STATEMENT

In this thesis, the author has made a theoretical study of some finite amplitude low frequency long wave-length modes in dispersive and dissipative media together with a critical examination of certain beam induced collective oscillations with an emphasis on heating the plasmas. The work presented can, therefore, be broadly divided into two parts. The first part (Chapter II - V) is devoted to studying the propagation of nonlinear ion acoustic waves in collisionless plasmas and nonlinear drift waves in collisional plasmas. The second part (Chapter VI - VII) consists of studying some collective interactions induced in a plasma either by an external electron beam or by a relative drift between the two species of the plasmas. A few introductory remarks and a brief summary of the results is presented in Chapter I.

Ion acoustic solitary wave result from an exact balance between nonlinearity and dispersion. When a relatively cold component of electrons is present in an otherwise hot plasma, the strength of dispersion for the system gets reduced. Hence, in such a system, the ion acoustic solitary wave has a larger amplitude for a given width compared to one in a plasma with single electron component. If the difference between the temperatures of the two components of electrons is sufficiently large the strength of dispersion is reduced to such an extent that a solitary solution is no longer possible (Chapter II). The strength of dispersion also changes from point to point if, the plasma is inhomogeneous. Chapter V has been devoted to the study of propagation of an ion acoustic solitary wave in an inhomogeneous medium with both density and temperature inhomogeneity. For temperature gradient scalelengths much larger than the density gradient scalelengths, though the amplitude of the solitary wave is governed by the density gradients only, the velocity of the soliton increases as it propagates towards regions of increasing temperature.

Drift waves derive their importance from their causal relationship with enhanced particle losses observed in low-β plasmas. Moreover, the linear dispersion relation for the drift waves is similar to one for ion acoustic waves under certain circumstances. This motivated us to look for nonlinear steady state solutions for drift-waves in a collisional plasma. Chapter III and IV have been devoted to such studies. In general, it is found that, whenever the stabilizing ion viscosity effects are stronger than the destabilizing effects due to electron-ion collisions, there exists a stationary shock solution for the nonlinear drift waves.

Intense relativistic electron beams offer immense potentialities for heating a plasma to thermonuclear temperatures. Such a beam induces what is known as 'return current' as it enters a plasma. A new instability known as the 'return current instability' is supported by such induced currents. We have shown that, there exists a range of wavenumbers which is unstable only to return current instability and not to the usual electron-electron two stream instability. Moreover, an estimate has been made of the rate at which the return current looses energy as a result of decay of ion acoustic turbulence generated by such a current. This investigation is presented in Chapter VI.

In connection with the problem of plasma heating, the cross-field currents also play an important role, because of the anomalous resistivity they produce in a plasma. A number of electrostatic instabilities induced by such currents have been invoked as the basic mechanism for producing the observed anomalous resistivity. However, the plasma heating experiments with cross field currents often use a magnetic mirror configuration for containing the plasma. The equilibrium distribution function for such a plasma is non-Maxwellian. In Chapter VII, we have studied the effects of loss-cone and temperature anisotropy in the electron distribution function on the cross-field current driven electrostatic instabilities. We have shown that non-Maxwellian plasmas can support fast growing waves even in regions of k-space which is stable to a Maxwellian plasma.

Lost on this planet
For an unknown cause
I was
happy
To have come to an End,
To be able to apprehend
The cause to live
To be with friends,
They need no name
No overtures, no claim,
They were always there
To share
My Utopia, my shame.
In return
My gesture is silent and mute
A feeling of deep gratitude.

ACKNOWLEDGEMENTS

Words fail me when I want to express my gratitude to one person and that is my guide Professor B. Buti. Her zeal and persistence were the driving force behind completion of most of the work presented in this thesis. Many a time, it was her encouragement that used to turn my despair into hopes. I gained immensely from my close association with her for which I shall remain ever grateful to her.

It was Professor P.K. Kaw who introduced me to the field of Plasma Physics. The little discussions I had with him, evoked and sustained my interest in Physics in general and Plasma Physics in particular. Quite often, I was benefited by discussions with Professors R.K. Varma, A.C. Das and Drs. Abhijit Sen and G.S. Lakhina for which I am extremely grateful to them.

Drs. K.S. Rao and D.R. Kulkarni were always there to help me in my difficulties with numerical computations. I am thankful to them for their friendly help.

In those moments when life appeared gloomy, Dr. Anath Das and Mrs Arundhaty Das were by my side to boost up my morale. With them it was a home away from home. The memory of the little time I spent with them will remain fresh in my mind for years to come.

The preparation of the thesis in the present form has been possible only due to the neat and efficient typing of Mr. P.P. Narayanan. My special thanks are due to him.

(B.N. GOSWAMI)

CONTENTS

CHAPTER I

INTRODUCTION

The unique feature of a plasma is the rich variety of collective oscillations it can sustain. The study of these oscillations yields a wealth of information about the collective properties of a plasma. Moreover, these collective modes provide channels through which energy can be deposited in a plasma from an external source (eg. a powerful laser beam, an intense relativistic electron beam etc. etc.) in order that the plasma could be heated to a high temperature. Linear and nonlinear study of these collective modes is important as well as fascinating. It is important, because the clear understanding of these basic phenomena is a step forward towards achieving the final goal viz. the controlled fusion. It is fascinating because it offers challenging mathematical problems to be solved and requires amalgamation of knowledge from a large number of branches of physics.

The great stimulus behind the hectic research activity in the field of plasma physics in the last two decades is the goal of achieving Controlled Thermonuclear Reaction. The work presented in this thesis is not aimed at providing a 'clean' source of energy; only an attempt has been made in providing some insight into certain phenomena of nonlinear wave propagation in plasma together with a critical examination of certain plasma heating mechanisms (eg. Cross-field current induced instability heating of a plasma in a mirror machine and relativistic electron beam heating of a plasma).

Two of the most important characteristic properties of a plasma are the nonlinearity and the dispersion. A linear dispersive system is one which admits solutions of the form

$$\phi = a \cos(kx - \omega t) \quad , \qquad (1.1)$$

where the characteristic frequency of the wave $\omega = \omega(k)$, is a real function of the wave number k. The functional relation $\omega(k)$ is known as the dispersion relation. If the phase velocity of the wave $\omega(k)/k$ is not a constant but a function of k, different modes will propagate with different velocity and hence disperse. However, we will consider the wave to be dispersive only if the group velocity $v_g = \partial\omega(k)/\partial k$ defined from the Fourier integral development of Eq.(1.1), is not constant, i.e.

when $\frac{\partial^2 \omega(k)}{\partial k^2} \neq 0$.

The concept of 'group velocity' defined above is developed from an asymptotic expansion of the Fourier integral depicting superposition of several modes, similar to one given by Eq.(1.1) viz.,

$$\varphi = \int_0^\infty F(k) \cos(kx - \omega t)\, dk \quad . \qquad (1.2)$$

For nonlinear wave such a Fourier analysis may not be appropriate and it is necessary to develop the 'group velocity' concept in an independent manner. Such an approach has been developed by Whitham (1965) using 'variational' method. In general, nonlinearity modifies the dispersion in such a way that ω is no more only a function of k but also a function of the wave amplitude (Whitham 1965, 1974).

The waves in a plasma are generated either self consistantly by the particle motions or excited by externally imposed fields. The theoretical analysis is greatly simplified if the field amplitude is sufficiently small. The linear (small amplitude) theory of the plasma waves has been very well developed and the methods have been dealt in great details (Stix 1962, Krall and Trievelpiece 1973, Schmidt 1966). But, more often than not, the plasma waves are nonlinear. In equilibrium, the nonlinearity might arise due to the large amplitude nature of the waves. On the other hand if the system sustains certain unstable waves, given sufficient time, even a small amplitude initial perturbation will grow out of linear regime. The nonlinear plasma phenomena may be broadly classified into two categories - turbulent plasma phenomena and coherent plasma phenomena. By turbulent plasma phenomena, we refer to circumstances where a large number of random collective oscillations are excited by, say, a linear instability or by a similar source. That is, there are many waves present in the system, the phases of which can be considered as random in some sense. Investigations in this regard correspond to the study of average properties of a 'statistical ensamble' of systems, each evolving according to a set of basic dynamical equations. The term coherent, however, refers to circumstances where the nonlinear development of the system is followed with all due respect to the phase informations carried by the waves. While we shall touch upon only one problem of weakly turbulent plasmas (viz. the quasilinear development of the return current instability and the return current induced ion acoustic instability) in Chapter VI, the Chapters II - V are denoted to studying some steady state coherent nonlinear phenomena in plasmas.

If for small k the linear dispersion relation has the form

$$\omega = \alpha k + \beta k^3 + \cdots \tag{1.3}$$

with α and β real constants, the system of nonlinear equations can often be reduced asymptotically to an equation (Taniuti and Wei 1968, Gardner and Su 1967):

$$\frac{\partial n}{\partial \tau} + a n \frac{\partial n}{\partial \xi} + b \frac{\partial^3 n}{\partial \xi^3} = 0 \quad , \qquad (1.4)$$

where a and b are real constants. Eq.(1.4) is known as the Korteweg-deVries (K-dV) equation which was first derived to describe shallow water waves (Korteweg and deVries 1895). In plasmas, in particular, Eq.(1.4) describes the long time behaviour of nonlinear hydromagnetic waves propagating perpendicular to the magnetic field at near Alfven velocity (Gardner and Morikawa 1960) and also the weakly nonlinear propagation of ion sound disturbances propagating near the ion sound speed (Washimi and Taniuti 1966).

If we assume that n depends on τ and ξ only through $\chi = \xi - U\tau$, Eq.(1.4) can be integrated under the boundary conditions n, $\partial n/\partial \chi$, $\partial^2 n/\partial \chi^2 \to 0$ as $|\chi| \to \infty$, to give

$$n = (3U/a)\,\mathrm{Sech}^2\left[\frac{1}{2}\left(\frac{U}{b}\right)^{1/2}(\xi - U\tau)\right] . \qquad (1.5)$$

The Eq.(1.5) is known as the solitary wave solution of the K-dV equation, which shows that the pulse height, width and speed are proportional to U, $U^{-1/2}$ and U respectively. The term 'soliton' was coined by Zabusky and Kruskal (1965) for such solitary wave solutions of the K-dV equation, which they solved numerically. Gardner, Greene, Kruskal and Miura (1967) solved the initial value problem for the K-dV equation analytically with the help of what is known as 'Inverse Scattering Method'. Account of these pioneering works and the recent advances in the theory of solitons and K-dV equation can be found in a number of review articles (Jefreey and Kakutani 1972, Scott et al. 1973).

Chapters II - V have been devoted to the investigations of steady state solutions of the modified K-dV equations describing certain nonlinear physical processes. The reductive perturbation method (Taniuti and Wei 1968, Taniuti 1974), which is nothing but a perturbation scheme with proper scalings of the time and space variables , is made use of in deriving these modified K-dV equations.

It has been recently found (Jones et al. 1975) that due to the presence of even a small fraction of relatively cold electrons in an otherwise hot plasma, the phase velocity of the ion acoustic wave as well as the Debye length for the system are dominantly governed by the lower electron temperature. The strength of the dispersion, for the system, which is proportional to the square of the Debye length, gets considerably reduced in this case. In Chapter II, we have derived a K-dV equation describing the weakly nonlinear propagation of ion acoustic waves in such a system and have shown that the ion acoustic solitary wave in such a system has a larger amplitude for a given width compared to one in a plasma with single electron component. It has also been shown that, if the temperature difference between the two electron components is sufficiently large, the strength of dispersion is reduced to such an extent that a solitary solution is no longer possible.

Inhomogeneities in the physical parameters are unavoidable in most experiments. We recall that the ion acoustic solitary waves result from a balance between non-linear steepening and dispersive effects. The strength of dispersion being proportional to λ_D^2($\lambda_D^2 = T_e/4\pi n_o e^2$, where T_e is the electron temperature and n_o is the equilibrium density), the dispersion and hence the propagation charact-eristics of ion acoustic solitary waves are expected to vary as the wave propagates in an inhomogeneous medium (inhomoge-neity being in density and/or in temperature). A modified K-dV equation describing the propagation of a weakly nonlinear ion acoustic wave in an inhomogeneous plasma with both density and temperature gradients, has been derived in Chapter V. We have obtained the solitary wave solution of this equation and shown that, for temperature gradient scalelengths large compared to density gradient scalelengths, the velocity of the ion acoustic solitary wave increases as it propagates towards regions of increasing temperature. In the absence of temperature gradients, as a solitary wave moves towards a decreasing density region its amplitude (absolute magnitude) goes as $(n_o(x))^{1/2}$ and velocity goes as $(n_o(x))^{-1/2}$.

Drift waves, like ion acoustic waves are low frequency waves. The linear dispersion relation for these drift waves (Kadomtsev 1965, Krall 1968) has some similarity with that for ion acoustic waves. In particular, the linear dispersion relation for the drift waves goes over to that for ion acoustic waves as the angle between the propagation direction and the magnetic field direction becomes sufficiently small. Therefore, one expects that, at least in some cases the propagation of nonlinear drift waves can also be represented by an equation similar to the K-dV equation. In Chapter III, we have investigated the weakly nonlinear propagation of drift dissipative ion acoustic mode in the presence of ion viscosity. Drift dissipative ion acoustic mode which is a mode in an inhomogeneous collisional magnetized plasma (Kadomtsev 1965) characterised by $\omega \sim kC_s >> \Omega_i$ (ω, Ω_i, k being the wave frequency, ion cyclotron frequency and wave number respectively and C_s is the ion sound speed) is linearly unstable. From the modified K-dV equation which governs the nonlinear drift dissipative mode, we find that, when the stabilizing ion viscosity effects are strong enough to overcome the destabilizing effects due to collisions, K-dV equation allows a stationary shock solution. Numerical integration of the equation reveals that it still permits solitary solutions if the net stabilizing or destabilizing effects are not too strong. The solitons thus obtained are found to either grow or decay with time depending on whether the viscous effects are weaker or stronger compared to the collisional effects.

Having investigated the special case of drift waves in a collisional plasma in Chapter III, we have gone a step further to study the general problem of nonlinear drift waves in a collisional plasma in Chapter IV. By using reductive perturbation method, we have derived a multi-dimensional modified K-dV equation describing the propagation of nonlinear drift waves in a collisional plasma where both parallel resistivity as well as the perpendicular viscosity are important. In this case too, if the ion viscosity effects are stronger than the effects due to electron ion collisions, this equation allows a stationary shock solution. We have also investigated the dependence of the structure of the shock profile on the propagation angle. For a given set of plasma parameters it is found that the profile of the shock wave tends to change from an oscillatory one to a monotonic one as the angle between the propagation direction and the magnetic field direction increases. When the destabilizing effects due to the collisions are stronger than the stabilizing effects due to ion viscosity, a stationary solution does not exist.

One of the challenging problem of the present day Plasma Physics Research is to find a way to heat a plasma to thermonuclear temperatures. It is in this respect, that the relativistic electron beam (REB) provides an excellent tool for depositing large amount of energy in a small area via collective plasma interactions. Current technological advances have placed at our disposal intense REB with truely imposing characteristics, for example, beams are now available with electron energy upto 15 Mev, currents above 10^6A, total energy per pulse upto 3 MJ, power upto 10^{13}W, beam current density upto 5 x 10^6 A/cm^2 and pulse duration of the order of 100 ns (Yonas et al. 1973, 1974, Millar and Kuswa 1973, Korn et al. 1973). With the availability of these beams and improvements on them in sight, number of people have proposed dramatic schemes with REB (Winterberg 1972, Babykin et al. 1971, McCorkle 1975) even to compete with lasers in their inertial confinement scheme. Regarding the economics of the problem, the initial estimates by Yoshikawa (1971) were discouraging. He showed that the capital investments necessary for REB heating are higher than any other method. But more recent estimates by Winterberg (1972) and Babykin et al. (1971) are more encouraging. The authors took the self magnetic field into account and showed that it would reduce the energy needed to reach thermonuclear conditions by a factor of 100. Apart from the fascinating prospect of using a REB as a tool for heating a plasma to thermonuclear temperature, an intense REB exhibits certain very peculiar properties that are worth studying for their own sake. One class of such very interesting problems is the problems of current limitation and equilibrium stability of such beams. When an intense electron beam is injected into the vacuum, the electrostatic repulsive force due to the space charge effects exceeds the pinching force due to the self magnetic field and hence such a beam in vacuum will simply diverge. The distructive space charge electrostatic force can be removed by injecting the beam into a neutralizing background. Even the self magnetic fields, by themselves can not produce an unlimited amount of current. The current limitation in this case comes about because of the following reason.

As long as the current in the beam is small, the trajectories of the beam particles under the action of the self magnetic field remain sinusoidal. As soon as current exceeds certain critical value, known as the 'Alfven's critical current' (Alfven 1939), the trajectories of the outer-most particles in the beam, force these particles to come back to the cathode that produced them. The self magnetic field, which stands as an obstacle on the way to produce large currents, can be removed if one produces the beam in a dense plasma. The changing azimuthal self magnetic field, at the head of the beam, gives rise to an axial electric field which accelerates the plasma electrons in the opposite direction and produces what is known as the 'return current'. The azimuthal magnetic field produced by the return current cancels the self magnetic field of the beam and facilitates larger beam current to flow. Questions yet to be answered are - when will the total current neutralization takes place ? What is the exact nature of the 'return currents' ? These questions have been dealt in details by many authors (Robert and Bennett 1968, Hamer and Rostoker 1970, Lee and Sudan 1971) and we will not go into these details.

It will suffice to mention here that total current neutralization takes place with the return current flowing within the beam if the beam radius 'a' is much larger than the electromagnetic skin depth c/ω_{pe} and $n_p >> n_b$ ($n_{p,b}$ being the plasma and beam densities respectively). The neutralization of the self magnetic field of the beam allows a much larger current to flow. However, this is not a steady state. As a result of finite conductivity σ of the medium, the induced currents will eventually decay with a characteristic time $t \simeq 4\pi \sigma a^2/c^2$, the magnetic field builds up again and the energy is dissipated in the plasma.

The fact that the return current provides an additional channel through which energy from a REB can be transferred to a plasma has motivated us to examine in detail the new collective modes induced by this return current and to see what modification it produces on the already known collective modes in such a system. This is the subject of Chapter VI. Besides the usual beam-plasma electron-electron instability, the return current supports a new electron-ion two stream instability (return current instability). Although the effect of these currents on the beam-plasma e-e instability is negligible, there exists a range of wave numbers which is unstable only to return current instability and not to e-e instability. The return current has a stabilizing effect on the electromagnetic modes propagating along the direction of the external magnetic field while it has a destabilizing effect on the em waves propagating transverse to the magnetic field. As a result of the quasilinear development, in the early stages of development of the return current instability, the electrons are preferentially heated. When the electrons are sufficiently heated such that $T_e >> T_i$ and if the return current exceeds certain threshold, the return current induced ion acoustic instability sets in. The turbulence generated by this instability further heats the plasma. Assuming that the level of turbulence is limited solely due to scattering of the ion sound waves by the electrons, we have calculated the rate at which the return current will loose energy. We have shown that the fastest time scale, at which the return current looses energy, can be of the order of ω_{pi}^{-1} ; ω_{pi} being the ion plasma frequency (Goswami et al. 1974).

In connection with the problem of plasma heating, cross field currents play a special role because of the anomalous resistivity they are known to produce in a plasma. A relative drift, between two species of a plasma across an externally applied magnetic field, has been realized in a number of experimental situations. The mechanisms by which such a relative drift is produced can be different for different experiments. For example, Ioffe et al. (1961) and Alexeff et al. (1970, 1971) produce such a drift with the help of a radial d.c. electric field. Such an electric field in combination with the axial magnetic field produces an $\underset{\sim}{E} \times \underset{\sim}{B}$ drift in the azimuthal direction. Both electrons and ions tend to rotate under the influence of this drift. However, the ions, which are much heavier, experience a much stronger centrifugal force. The difference in the centrifugal forces result in a relative drift between the two species in the azimuthal direction. On the other hand in the experiment of Babykin et al. (1964) such a drift was produced by the gradient in the wave field of a magnetosonic wave propagating across the magnetic field. The magnetosonic wave was produced by setting up an oscillating magnetic field, superimposed on the confining magnetic field.

Modified two-stream instability, current-driven ion acoustic instability are some of the electrostatic instabilities induced by a cross field current. A good deal of theoretical work has been done on the study of these instabilities (Mc Bride et al. 1972, Berrett et al. 1972, Lashmore Davies and Martin 1973). However, all the investigations, so far, have been carried out on the assumption that the equilibrium distribution is well represented by a Maxwellian distribution function. We note that, most of the experiments mentioned above use a magnetic mirror configuration for confining the plasma and a Maxwellian is not a realistic distribution function for such a system. We have taken a more realistic distribution function (the generalized distribution function) which includes both loss-cone and temperature anisotropy effects. The effects of loss-cone and temperature anisotropy on the above mentioned instabilities are given in Chapter VII. We find that these effects are important when $\lambda = k^2 \rho_e^2/2 \gg 1$ (ρ_e being the electron gyroradius). It is also shown that a non Maxwellian plasma can sustain fast growing waves even in regions of k-space which is stable for a Maxwellian plasma (Goswami and Buti 1975).

CHAPTER II

ION ACOUSTIC SOLITARY WAVES IN A TWO-ELECTRON TEMPERATURE PLASMA

II.1 Introduction

Plasmas with electron velocity distribution that can be represented by the superposition of two Maxwellians are not infrequent under experimental conditions. For example, hot turbulent plasmas of thermonuclear interest ~~interest~~ often have a high energy tail (Utlaut and Cohen 1971, Sipler and Biondi 1972, Kruer et al. 1970, Kruer and Dawson 1972); strong electron beam-plasma interactions also result in such distributions (Sudan 1970) and more often, ordinary hot cathode discharge plasmas also have double electron-temperature distribution (Jones et al. 1974). Though the presence of a group of electrons with lower temperature, in an otherwise hot plasma, was known to the experimental plasma physicists for quite some time (Oleson and Found 1949), the interest has only recently been spurred in studying the propagation of ion acoustic waves in such a plasma (Jones et al. 1975). This study of Jones et al. has revealed some new aspects of propagation of ion acoustic waves in such a plasma namely, when a bunch of relatively cold electrons is present in an otherwise hot plasma, the ion acoustic speed is dominantly governed by the lower temperature. This result has some very interesting practical implications. For example, when one wants to utilize ion acoustic waves as a diagnostic tool for determining plasma parameters (eg. calculating electron temperature from the measurement of ion acoustic speed), one has to be very careful if the plasma contains a colder electron component. Another possible interesting application suggested by the above result is the utilization of the ion acoustic waves to heat ions in a plasma by Landau damping. By introducing a small amount of relatively cold electrons, as the ion acoustic speed can be appreciably reduced, an otherwise undamped wave can also possibly be driven damped, thereby transfering energy to the plasma ions.

II.2 Linear Theory

When the physical processes that produce the two types of electrons with two different temperatures, have time scales much shorter than the relevant ion time scale, the two electron components can be treated as two fluids. Thus describing the plasma by one dimensional multispecies fluid equations together with the Poisson's equation and assuming that the first order quantities go as exp $(i(kx - \omega t))$, one obtains the linear dispersion relation (Jones et al. 1975).

$$1 = \frac{\omega_{pi}^2}{\omega^2 - k^2\alpha_i^2} + \frac{\omega_{pel}^2}{\omega^2 - k^2\alpha_{el}^2} + \frac{\omega_{peh}^2}{\omega^2 - k^2\alpha_{eh}^2}, \quad (2.1)$$

where ω and k are the characteristic frequency and wave number, and ω_p is the plasma frequency. $\alpha_{e(i)}$ is the thermal velocity for the electrons (ions). The suffixes e and i refer to electrons and ions respectively whereas l and h refer to lower and higher temperature electron components respectively. For ion acoustic waves,

$$\alpha_i \ll \omega/k \ll \alpha_{el}, \alpha_{eh}, \quad (2.2)$$

in which case, Eq.(2.1) can be simplified to give

$$\omega \simeq k C_{seff} \left[1 - \frac{1}{2} k^2 \lambda_{Deff}^2 \right], \quad (2.3)$$

where $C_{seff} = (T_{eff}/m_i)^{1/2}$ and $\lambda_{Deff}^2 = T_{eff}/4\pi n_o e^2$ with

$$T_{eff} = \frac{n_o T_{eh} T_{el}}{(n_{oh} T_{el} + n_{ol} T_{eh})} \quad (2.4)$$

Eqs.(2.3) and (2.4) show that the linear dispersion relation for the ion acoustic waves in a two-electron temperature plasma is similar to the one for a plasma with single electronic component, with the difference that the ion acoustic speed

(C_{seff}) and the Debye length (λ_{Deff}) are now characterized by the 'effective temperature' (T_{eff}). In other words, the restoring force for the ion acoustic waves, in this case, is given by an electron pressure which has to be defined in terms of an 'effective temperature' given by Eq.(2.4). From Eq.(2.4), we notice that the effective temperature is a function of both the temperatures and the fractional densities of the two components. It can also be seen from Eq.(2.4) that as the difference of temperatures between the two components increases, the effective temperature and hence the propagation characteristics of ion acoustic waves becomes dominantly governed by the lower temperature. For example, for a plasma with $T_{eh} \sim 3 T_{el}$ ($T_{el} = 1eV$) and $n_{oh} = n_{ol}$, $T_{eff} = 1.5$ eV. As the temperature of the two components becomes further apart, the relative importance of the high temperature component becomes even smaller. Consider a hypothetical case, where $T_{eh} \rightarrow \infty$ and n_{ol}/n_{oh} is finite, in which case $T_{eff} = (n_{oh}/n_{ol})\, T_{el}$. Thus, even if the cold component make up only 10 per cent of the total electron density, T_{eff} can not be greater than 10 T_{el}, no matter how hot the other 90 per cent of the electrons are. These results have been verified in an experiment by Jones et al. (1975).

II.3 Nonlinear Theory

As mentioned in Chapter I, it is well known (Washimi and Taniuti 1966, Davidson 1972) that the weakly nonlinear propagation of ion sound disturbances travelling near the ion sound speed can be described by a K-dV equation. Moreover, this equation possesses a stationary solitary wave solution which results due to an exact balance between nonlinear and dispersive effects. The strength of dispersion which is proportional to λ_D^2, will be quite different for a two electron temperature plasma compared to a plasma with single electronic component. Hence, the propagation characteristics of an ion acoustic solitary wave is expected to get modified in such a plasma (Goswami and Buti 1976). To determine quantitatively, how exactly the propagation characteristics of an ion acoustic solitary wave in such a plasma is modified, we proceed as follows.

The one dimensional basic set of equations governing the system consists of the ion continuity equation, the momentum transfer equations for ions and the two types of electrons and the Poisson's equation, namely

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v_i) = 0 \quad , \qquad (2.5)$$

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} + \frac{\partial \phi}{\partial x} = 0 \quad , \qquad (2.6)$$

$$n_{el}\frac{\partial}{\partial x}\left[\frac{T_{eff}}{T_{el}}\Phi\right]-\frac{\partial n_{el}}{\partial x}=0 \quad , \qquad (2.7)$$

$$n_{eh}\frac{\partial}{\partial x}\left[\frac{T_{eff}}{T_{eh}}\Phi\right]-\frac{\partial n_{eh}}{\partial x}=0 \quad , \qquad (2.8)$$

and
$$\frac{\partial^2\Phi}{\partial x^2}=n_{eh}+n_{el}-n_i \quad . \qquad (2.9)$$

In Eqs.(2.5) - (2.9) n_i and n_e are the ion and electron densities normalized to equilibrium value n_o, v_i is the ion fluid velocity normalized to C_{seff} and Φ is the potential normalized to T_{eff}/e. Moreover, lengths are normalized to λ_{Deff} and time is normalized to ion plasma period ω_{pi}^{-1} ($\omega_{pi}^{2} = 4\pi n_o e^2/m_i$). Eqs.(2.7) and (2.8) can be immediately integrated to give

$$n_{el}=n_{ol}\exp\left[\frac{T_{eff}}{T_{el}}\Phi\right] \qquad (2.10)$$

and
$$n_{eh}=n_{oh}\exp\left[\frac{T_{eff}}{T_{eh}}\Phi\right] \quad . \qquad (2.11)$$

Now, on combining Eqs.(2.10) and (2.11) with Eq.(2.9) and on retaining terms upto Φ^2, we get

$$\frac{\partial^2\Phi}{\partial x^2}\simeq\Phi+\Delta\frac{\Phi^2}{2}-\tilde{n}_i \qquad (2.12)$$

where $\tilde{n}_i$ is the perturbed part of the ion density and

$$\Delta=\left[\frac{n_{oh}}{T_{eh}^2}+\frac{n_{ol}}{T_{el}^2}\right]T_{eff}^2=\frac{n_{oh}(T_{el}/T_{eh})^2+n_{ol}}{[n_{oh}(T_{el}/T_{eh})+n_{ol}]^2} \qquad (2.13)$$

In writing Eq.(2.12), we have made use of the charge neutrality condition i.e., $n_o = n_{oh} + n_{ol}$. Now let us introduce the stretched variables (Davidson 1972), $\zeta = \epsilon^{1/2}(x - t)$ and $\tau = \epsilon^{3/2} t$ and for weak nonlinearities, use the following perturbation expansions:

$$\hat{n}_i = \epsilon n_i^{(1)} + \epsilon^2 n_i^{(2)} + \cdots ,$$

$$\Phi = \epsilon \Phi^{(1)} + \epsilon^2 \Phi^{(2)} + \cdots ,$$

and $$v_i = \epsilon v_i^{(1)} + \epsilon^2 v_i^{(2)} + \cdots .$$

To the lowest order (i.e. $O(\epsilon^{3/2})$) Eqs.(2.5), (2.6) and (2.12) give $n_i^{(1)} = \Phi^{(1)} = v_i^{(1)}$. This means that, in the linear approximation, the propagation characteristics of the ion acoustic wave remains same as in a plasma with single electronic component, except that it is now propagating with velocity C_{seff}. To the next order (i.e. to order $\epsilon^{5/2}$) Eqs.(2.5), (2.6) and (2.12) give

$$-\frac{\partial n_i^{(2)}}{\partial \zeta} + \frac{\partial n_i^{(1)}}{\partial \tau} + \frac{\partial}{\partial \zeta}\left(n_i^{(1)} v_i^{(1)}\right) + \frac{\partial v_i^{(2)}}{\partial \zeta} = 0 , \qquad (2.14)$$

$$-\frac{\partial v_i^{(2)}}{\partial \zeta} + \frac{\partial v_i^{(1)}}{\partial \tau} + v_i^{(1)} \frac{\partial v_i^{(1)}}{\partial \zeta} + \frac{\partial \Phi^{(2)}}{\partial \zeta} = 0 , \qquad (2.15)$$

and $$\frac{\partial^2 \Phi^{(1)}}{\partial \zeta^2} = \Phi^{(2)} + \Delta \frac{[\Phi^{(1)}]^2}{2} - n_i^{(2)} . \qquad (2.16)$$

Eliminating $n_i^{(2)}$, $\Phi^{(2)}$ and $v_i^{(2)}$, we obtain

$$\frac{\partial n_i^{(1)}}{\partial \tau} + \frac{1}{2}(3-\Delta) n_1^{(1)} \frac{\partial n_i^{(1)}}{\partial \zeta} + \frac{1}{2}\frac{\partial^3 n_i^{(1)}}{\partial \zeta^3} = 0 \qquad (2.17)$$

In deriving Eq.(2.17), we have made use of the relation $n_i^{(1)} = \Phi^{(1)} = v_i^{(1)}$. Eq.(2.17) is the K-dV equation which describes the weakly nonlinear propagation of ion acoustic waves in a plasma with two types of electrons having different temperatures. Now, let us look for a solution of Eq.(2.17) such that $n_i^{(1)}$ depend on ζ and τ only through $\chi = \zeta - U\tau$. Eq.(2.17), can then be integrated w.r.t. χ under the boundary condition $n_1^{(1)}, \frac{dn_i^{(1)}}{d\chi}, \frac{d^2 n_i^{(1)}}{d\chi^2} \to 0$ as $|\chi| \to \infty$; this gives

$$\frac{d^2 n_i^{(1)}}{d\chi^2} = 2U n_1^{(1)} - \frac{3-\Delta}{2}[n_i^{(1)}]^2 . \qquad (2.18)$$

Multiplying both sides of Eq.(2.18) by $2\,dn_i^{(1)}/d\chi$ and then integrating it again we get

$$\frac{dn_i^{(1)}}{d\chi} = n_i^{(1)}\left(2U - \frac{3-\Delta}{3} n_1^{(1)}\right)^{1/2} . \qquad (2.19)$$

Finally integrating Eq.(2.19), we get

$$n_1^{(1)} = 3U\left(\frac{2}{3-\Delta}\right) \mathrm{Sech}^2\left[\left(\frac{U}{2}\right)^{1/2}(\zeta - U\tau)\right] \qquad (2.20)$$

Eq.(2.20) is the solitary wave solution of the K-dV Eq.(2.17). Alternatively, using a scaling of the space co-ordinate $\zeta^1 = 2\zeta/(3-\Delta)$, the stationary solitary wave solution of Eq.(2.17) can also be written as

$$n_i^{(1)} = 3U' \,\mathrm{Sech}^2\left[\left(\frac{3-\Delta}{2}\right)^{1/2}\left(\frac{U'}{2}\right)^{1/2}\left(\zeta - \frac{3-\Delta}{2}U'\tau\right)\right]$$

(2.21)

II.4 Discussion

In Eqs.(2.20) and (2.21), U and U' are the velocities of the soliton which in general are arbitrary but of the order of C_{seff}. For $T_{eh} = T_{el}$, from Eq.(2.13) we get, $\Delta = 1$ and Eq.(2.20) or Eq.(2.21) then represents the solitary wave solution for ion acoustic waves in a plasma with single electronic component.

The solitary solutions of Eq.(2.17) given either by Eq.(2.20) or by Eq.(2.21) are valid only for $\Delta < 3$. For $\Delta > 3$, a solitary solution to Eq.(2.17) does not exist. This result may be understood as follows:

For Δ to be ≥ 3 one needs large ratios of T_{eh}/T_{el} and n_{oh}/n_{ol}. For example, for $T_{eh}/T_{el} = 12$ and $n_{oh}/n_{ol} = 9$, $\Delta \simeq 3.47$. When T_{eh}/T_{el} is large, the effective temperature is mostly governed by T_{el} and an appreciable reduction in the strength of dispersion (which is proportional to λ^2_{Deff}) takes place. Corresponding to $\Delta \geq 3$, the strength of dispersion is so weak that a balance between the nonlinearity and dispersion can no longer take place. Hence, no solitary wave.

Referring to the work of Jones et al. (1975), we note that, in their experiment T_{eh}/T_{el} ranged from less than 2 to 5 while n_{oh}/n_{ol} ranged from about 1/6 to 3. This gives a variation of Δ from 1.033 to 1.875. For example, when $T_{eh}/T_{el} = 5$ and $n_{oh}/n_{ol} = 3$, $\Delta = 1.875$.

Eq.(2.20) shows that, for a given width, as long as the solitary solution is maintained, the amplitude increases by a factor of $2/(3-\Delta)$. In other words, for a given amplitude, the width of a solitary wave decreases by a factor of $((3-\Delta)/2)^{1/2}$ (Eq.(2.21)). When a small fraction of relatively cold electrons is present in an otherwise hot plasma, $T_{eff} < T_{eh}$ and $\Delta > 1$, which implies an increase in amplitude of a solitary wave for a given width; this can be understood as follows: As T_{eff} decreases, the strength of dispersion also decreases. Hence, a larger amplitude is necessary to produce sharper gradients so that the dispersion effects are sufficient to produce the soliton with the given width.

Lastly, we would like to emphasize that, in order to estimate the quantitative increase in the amplitude of an ion acoustic soliton for a given set of values of T_h/T_l and n_{oh}/n_{ol}, we must take into account the decrease in C_{seff} as well. The amplitude of the soliton is proportional to U which in turn is proportional to C_{seff} and thus U goes as $T_{eff}^{1/2}$. Since for $T_h/T_l > 1$, $T_{eff} < T_h$ and $\Delta > 1$; the amplitude of the soliton will be determined by the net balance between the decrease in U and the increase in the factor $2/(3-\Delta)$. Let us take a specific example: $T_h/T_l = 5$ and $n_{oh}/n_{ol} = 3$ (Jones et al. 1975) for which case, $\Delta = 1.875$ and $2/(3-\Delta) = 1.778$. Moreover, in this case $T_{eff} = 2.5\ T_l = 0.5\ T_h$; so that, $C_{seff} = 0.707\ C_s$. Hence, the amplitude of the soliton is proportional to $3\bar{U} \times (1.778 \times 0.707) \simeq 3\bar{U} \times (1.257)$, where $\bar{U}$

is now proportional to C_s. This shows that, for this set of parameters, there is a net increase of about 25 per cent in the amplitude of the soliton compared to one in a plasma with single electronic component having a temperature equal to T_h.

II.5 Conclusions

Due to the presence of a relatively cold electron component in a plasma, the ion acoustic solitary wave of a given width has a larger amplitude. When the temperature differences between two electron components is sufficiently large, the strength of dispersion is reduced to such an extent that a solitary solution is no longer possible.

CHAPTER III

NONLINEAR DRIFT DISSIPATIVE ION ACOUSTIC WAVES

III.1 Introduction

Having discussed the case of nonlinear propagation of ion acoustic waves in a collisionless plasma in the last chapter, we shall go over to study the nonlinear propagation of a different class of low-frequency waves namely, the drift waves in a collisional plasma in this and in the next chapter. In particular, in this chapter we shall investigate the weakly nonlinear propagation of the drift-dissipative ion acoustic mode in the presence of ion viscosity.

The drift waves derive their importance from their possible causal relation to the enhanced particle losses observed in low-β plasmas. In this connection, collisional drift waves are of significant importance because, they are known to have large instability growth rates ($\mathrm{Im}\,\omega \sim \mathrm{Re}\,\omega$). Earlier theoretical work on drift waves in resistive plasmas was done by Moiseev and Sagdeev (1963) and Chen (1964). Drift dissipative instabilities are observed, among other experiments, in magnetically confined alkali metal plasmas in Q-machines (Hendel et al. 1968, Ivanov et al. 1968). In the present chapter we investigate the propagation of weakly nonlinear ion acoustic wave in an inhomogeneous and strongly collisional plasma in which both the parallel resistivity and perpendicular viscosity are present. We consider the plasma in which the electrons are magnetized ($\omega < \Omega_e$) and ions are unmagnetized ($\omega >> \Omega_i$); ω ,$\Omega_{e,i}$ being the characteristic wave frequency and electron and ion cyclotron frequencies respectively. We also assume that the electron mean free path, is much smaller than the longitudinal wavelength, so that the diffusion approximation is valid for the longitudinal motion of the electrons. Such conditions do exist in a positive column with a very low neutral pressure (Kadomtsev 1965) and in certain Q-machine experiments (Buchel'nikova 1968).

Assuming the magnetic field to be in z-direction and density gradients in the negative x-direction it has been shown (Kadomtsev 1965) that the high frequency ($\omega >> \Omega_i$) ion acoustic mode ($\omega \sim k_y C_s$) in the presence of collisions is unstable for $k_y >> k_z$ (drift dissipative instability); this is a negative energy mode and the instability is due to a phase difference between the electric field and density fluctuations introduced by the collisions and due to finite

k_z. The ion-viscosity and finite-Larmor radius corrections are known to have stabilizing effects on this instability (Hendel et al. 1968, Coppi 1964).

In a collisionless plasma, the propagation of a weakly nonlinear ion acoustic wave can be described by the K-dV equation (Washimi and Taniuti 1966, Davidson 1972). Since drift dissipative ion acoustic wave has a linear dispersion relation similar to that of ion acoustic wave itself, it is interesting to examine if, at least upto certain stage of nonlinearity, this wave can also be described by a K-dV type equation. We have taken into account the ion viscosity and by using reductive perturbation method derived a set of two coupled partial differential equations describing the propagation of nonlinear drift dissipative ion acoustic mode. We have looked for a special solution and shown that if a perturbation has a long wave length sinusoidal variation along the magnetic field direction, the propagation, along a direction transverse to both magnetic field and density gradients, is governed by a modified K-dV equation. When the stabilizing effects due to ion viscosity dominates over the destabilizing effects due to the collisions, this equation allows a shock solution.

In order to explore the region of interest when the destabilizing effects due to collisions are stronger than the stabilizing effects due to ion viscosity, we have numerically solved the equation. It is seen that the equation still permits solitary wave solutions as long as the net stabilizing or destabilizing effects are not too strong (i.e. the linear decay or growth rates are small compared to the amplitude of the initial perturbation). These solitary waves are found to either grow or decay with time depending on whether the viscous effects are weaker or stronger compared to the resistive effects.

III.2 General Theory

The basic equations governing our system are the ion and the electron continuity equations, ion equation of motion, electron parallel equation of motion and the Poisson's equation, namely

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial y}(n_i v_i) = 0, \qquad (3.1)$$

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial y} + \frac{\partial \Phi}{\partial y} = \frac{\mu}{c_s \lambda_D m_i n_o n_i} \frac{\partial^2 v_i}{\partial y^2}, \qquad (3.2)$$

$$\frac{m_e}{m_i} \frac{1}{\omega_{pi} \tau_e} \frac{\partial n_e}{\partial t} + \frac{\kappa}{\Omega_e \tau_e} \frac{\partial \Phi}{\partial y} + \frac{\partial}{\partial z}\left(n_e \frac{\partial \Psi}{\partial z}\right) = 0, \qquad (3.3)$$

$$-\frac{1}{n_e} \frac{\partial n_e}{\partial z} + \frac{\partial \Phi}{\partial z} - \frac{\partial \Psi}{\partial z} = 0 \qquad (3.4)$$

and

$$\frac{\partial^2 \Phi}{\partial y^2} = n_e - n_i \qquad (3.5)$$

with

$$\frac{\partial \Psi}{\partial z} = \frac{m_e}{T_e \tau_e} v_{ez} \qquad (3.6)$$

In this set of equations, $n_{e,i}$ are the electron and ion densities, V_i is the ion velocity, $C_s = (T_e/m_i)^{1/2}$ is the sound speed and $\lambda_D = (T_e/4\pi n_o e^2)^{1/2}$ is the electron Debye length. Eqs.(3.1) - (3.5) are written in terms of normalized quantities; densities are normalized to equilibrium value n_o, lengths to Debye length, time to ion plasma period ω_{pi}^{-1} ($\omega_{pi}^2 = 4\pi n_o e^2/m_i$), potential to T_e/e and V_i to ion acoustic speed. In Eq.(3.2), $\mathcal{K} = -\frac{dn_o}{dx}/n_o$, $\Omega_e = eB_o/m_e C$, -e being the electronic charge and τ_e is the mean collision time between electrons and ions. The quantity Ψ appearing in Eq.(3.4) is a velocity potential introduced through Eq.(3.6) where V_{ez} is the z-component of electron velocity. It is to be noted that the electron perpendicular equation of motion is not written down along with Eqs.(3.1) - (3.5), but use of which has been made to write Eq.(3.3) under local approximations (Kadomtsev,1965). Since the mode under consideration is an electrostatic mode and the propagation is nearly perpendicular to the magnetic field, the motion of the ions along the direction of the magnetic field has been neglected. The term on the right hand side of Eq.(3.2) represents the viscous force with $\mu = \eta_o/3 + \eta_1$ and $\eta_o = C_o n_i T_i/\nu_{ii}$ and $\eta_1 = C_1 n_i T_i \nu_{ii}/\Omega_i^2$, C_o and C_1 being constants and T_i and ν_{ii} being ion temperature and ion-ion collision frequency respectively. The constants as obtained by Braginskii (1965) are $C_o = 0.95$ and $C_1 = 0.3$. In writing Eq.(3.2) the ion pressure term is neglected because the ion temperature T_i is assumed to be much smaller than the electron temperature T_e and the strength of the ion pressure term compared to the ion viscosity term goes as $O(\Omega_i^2/\nu_{ii} k_y C_s)$. For $k_y C_s \sim \omega >> \Omega_i$, the strength of the ion pressure term is even smaller than that of ion-viscosity term if $\nu_{ii} \gtrsim \Omega_i$.

Let us write the ion density n_i as $n_i = 1 + \tilde{n}_i$ where $\tilde{n}_i$ is the perturbed part of ion density. Now, integrating Eq.(3.4) w.r.t. z we get $n_e = \exp(\Phi - \Psi)$, the integration constant is put equal to unity in view of the fact that the equilibrium value of n_e is also equal to unity. Substituting the expression for n_e in Eq.(3.5), we get

$$\frac{\partial^2 \Phi}{\partial y^2} \simeq \Phi - \Psi + \frac{1}{2}(\Phi - \Psi)^2 - \tilde{n}_i \quad . \qquad (3.7)$$

We now introduce the stretched variables $\zeta = \epsilon^{1/2}(y-t)$ and $\tau = \epsilon^{3/2} t$. The perturbed quantities can then be expanded as

$$\tilde{n}_i = \epsilon n_i^{(1)} + \epsilon^2 n_i^{(2)} + \text{------}$$

$$\Phi = \epsilon \Phi^{(1)} + \epsilon^2 \Phi^{(2)} + \text{-----}$$

$$v_i = \epsilon v_i^{(1)} + \epsilon^2 v_i^{(2)} + \text{-----}$$

$$\Psi = \epsilon^2 \Psi^{(1)} + \epsilon^3 \Psi^{(2)} + \text{------} \quad .$$

The last expansion, namely that for Ψ is rather crucial in our theory which follows from Eq.(3.4). In the absence of collisions the electron density fluctuations are governed only by the potential fluctuations namely $n_e = \exp(\Phi)$. The collisional term $\partial\Psi/\partial z$ in Eq.(3.4) is treated as a correction to the potential fluctuations $\partial\Phi/\partial z$. Thus the term $\partial\Psi/\partial z$ is taken to be one order smaller than the term $\partial\Phi/\partial z$. Hence the above expansion for Ψ.

The smallness parameter ϵ is chosen in such a way that, to the lowest order Eq.(3.3) gives

$$\frac{\partial^2\Psi^{(1)}}{\partial z^2} = -\frac{\kappa}{\Omega_e\tau_e}\frac{\partial\Phi^{(1)}}{\partial\zeta} \tag{3.8}$$

This requirement demands that, $(\kappa/k_z^2\Omega_e\tau_e)\sim\epsilon^{1/2})$ and $(m_e/m_i)(\omega_{pi}\tau_e)^{-1}(\kappa/\Omega_e\tau_e)^{-1}\sim\epsilon$.

To the lowest order Eqs.(3.1), (3.2) and (3.7) give $n_i^{(1)} = v_i^{(1)} = \Phi^{(1)}$. To the next higher order these equations can be written as

$$-\frac{\partial n_i^{(2)}}{\partial\zeta} + \frac{\partial v_i^{(2)}}{\partial\zeta} + \frac{\partial n_i^{(1)}}{\partial\tau} + \frac{\partial}{\partial\zeta}(n_i^{(1)}v_i^{(1)}) = 0, \tag{3.9}$$

$$-\frac{\partial v_i^{(2)}}{\partial\zeta} + \frac{\partial v_i^{(1)}}{\partial\tau} + v_i^{(1)}\frac{\partial v_i^{(1)}}{\partial\zeta} + \frac{\partial\Phi^{(1)}}{\partial\zeta} = \frac{\mu}{n_0 m_i\lambda_D c_s}\frac{\partial^2 v_i^{(1)}}{\partial\zeta^2} \tag{3.10}$$

and $$\frac{\partial^2\Phi^{(1)}}{\partial\zeta^2} = \Phi^{(2)} + [\Phi^{(1)}]^2/2 - \Psi^{(1)} - n_i^{(2)}. \tag{3.11}$$

On eliminating $n_i^{(2)}$, $\Phi^{(2)}$ and $v_i^{(2)}$ and on using the relation $n_i^{(1)} = v_i^{(1)} = \Phi^{(1)}$, Eqs.(3.9) - (3.11) can be simplified to give

$$\frac{\partial n_i^{(1)}}{\partial \tau} + n_i^{(1)}\frac{\partial n_i^{(1)}}{\partial \zeta} + \frac{1}{2}\frac{\partial^3 n_i^{(1)}}{\partial \zeta^3} + \frac{1}{2}\frac{\partial \Psi^{(1)}}{\partial \zeta} - \frac{\mu}{2 n_o m_i \lambda_D c_s}\frac{\partial^2 n_i^{(1)}}{\partial \zeta^2} = 0, \quad (3.12)$$

Eqs.(3.8) and (3.12) constitute a set of two coupled differential equations describing the propagation of nonlinear drift dissipative ion acoustic mode. We now look for a special solution for this set of equations. We assume that the z-dependence of any perturbation is sinusoidal in nature (Maxworthy and Redekopp 1976), namely $n^{(1)}(z,\zeta,\tau) = \bar{n}(\zeta,\tau)$ $\mathrm{Sin} k_z z$ and $\Psi^{(1)}(z,\zeta,\tau) = \bar{\Psi}(\zeta,\tau)\ \mathrm{Sin} k_z z$. With such a prescription Eq.(3.8) can be solved for $\Psi^{(1)}$ which is then substituted in Eq.(3.12). The z-dependence from this equation is then removed by integrating over dz from 0 to π/k_z. The resulting equation for $\bar{n}$ is

$$\frac{\partial \bar{n}}{\partial \tau} + \frac{4}{\pi}\bar{n}\frac{\partial \bar{n}}{\partial \zeta} + \frac{1}{2}\frac{\partial^3 \bar{n}}{\partial \zeta^3} + (\beta' - \alpha')\frac{\partial^2 \bar{n}}{\partial \zeta^2} = 0, \quad (3.13a)$$

where $\beta' = (\kappa/2\Omega_e \tau_e k_z^2)$ and $\alpha' = (\mu/2 n_o m_i \lambda_D C_s)$. If we drop the nonlinear and viscosity terms and assume that $\bar{n} \sim \exp(i(k_\zeta \zeta - \Omega\tau))$ Eq.(3.13a) yields the linear dispersion relation for this mode (Kadomtsev 1965). If we had Fourier analysed Eq.(3.8), then instead of Eq.(3.13a) we would have obtained a nonlinear integro-differential equation which can not be solved analytically even for some special cases that we have discussed later. By making a transformation of the space co-ordinate, $\bar{\zeta} = \frac{\pi}{4}\zeta$, Eq.(3.13a) can be written as:

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$$\frac{\partial \bar{n}}{\partial \tau} + \bar{n}\frac{\partial \bar{n}}{\partial \bar{\xi}} + \delta \frac{\partial^3 \bar{n}}{\partial \bar{\xi}^3} + (\beta - \alpha)\frac{\partial^2 \bar{n}}{\partial \bar{\xi}^2} = 0, \quad (3.13)$$

where $\delta = (1/2)\,(\pi/4)^3$, $\beta = (\pi/4)^2\beta'$ and $\alpha = (\pi/4)^2\alpha'$. Before we give the numerical solution of Eq.(3.13) we will qualitatively discuss some special cases of Eq.(3.13).

i) When $\alpha = \beta = 0$, Eq.(3.13) reduces to a K-dV equation and represents the propagation of an ion acoustic wave in a collisionless plasma.

ii) In the case when $\alpha > \beta$, the stabilizing effects due to ion viscosity is strong enough to overcome the destabilizing effects due to the collisions. By retaining only the η_0 term in the expression for μ, the condition $\alpha > \beta$ can be expressed as

$$k_z > \left(\frac{\lambda_D}{L}\right)^{1/2}\left(\frac{1}{\Omega_e \tau_e}\right)^{1/2}\left(\frac{\nu_{ii}}{\lambda_D c_s}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{1/2}, \quad (3.14)$$

where L is the scale length of density gradients. In this case Eq.(3.13) becomes a 'modified K-dV equation'. It is well known that this equation possesses a stationary shock solution with either an oscillating or a monotonic profile (Shut'ko 1970, Jhonson 1970, Jeffrey and Kakutani 1972, Grad and Hu 1969).

iii) When $\beta > \alpha$, the ion viscosity effects are not strong enough to quench the instability and hence we cannot look for a steady state solution in this case.

II.3 Numerical Analysis and Discussions

In this section we shall discuss some results of numerical solution of the equation,

$$\frac{\partial n}{\partial \tau} + n\frac{\partial n}{\partial \zeta} + \delta\frac{\partial^3 n}{\partial \zeta^3} = -\gamma\frac{\partial^2 n}{\partial \zeta^2}, \qquad (3.15)$$

where $\gamma = \beta - \alpha$. Now, for $\gamma > 0$, Eq.(3.15) corresponds to the case when the destabilizing effects are stronger while $\gamma < 0$ corresponds to the case when stabilizing viscosity effects are stronger. Although in our case $\delta = 1/2$, by an appropriate scaling of n and τ, δ can be made as small as we like. This reduces the computer time necessary to obtain the solitary solutions and only necessiates a change in the normalization of the initial perturbation. For all the calculations to be presented here we have used $\delta = 5 \times 10^{-4}$. The difference equation that approximates the modified K-dV equation, i.e. Eq.(3.15), is

$$\begin{aligned} n_i^{j+1} = n_i^{j-1} &- \tfrac{1}{3}(\Delta\tau/\Delta\zeta)(n_{i+1}^j - n_{i-1}^j)(n_{i+1}^j + n_i^j + n_{i-1}^j) \\ &- \delta(\Delta\tau/\Delta\zeta^3)(n_{i+2}^j - 2n_{i+1}^j + 2n_{i-1}^j - n_{i-2}^j) \\ &- 2\gamma(\Delta\tau/\Delta\zeta^2)(n_{i+1}^j - n_i^{j+1} - n_i^{j-1} + n_{i-1}^j), \end{aligned} \qquad (3.16)$$

where $n_i^j \equiv n(i\Delta\zeta, j\Delta\tau)$ and $\Delta\zeta$ and $\Delta\tau$ are appropriate step lengths. In Eq.(3.16) we have used the DuFort and Frankel's scheme (Richtmyer and Morton 1967) to replace the second derivative term in Eq.(3.15) by appropriate differences.

Integration is performed with 200 steps in ζ . In order that the numerical solution of Eq.(3.15) does not become numerically unstable, we have calculated the amplification matrix corresponding to Eq.(3.16) and the numerically stable region is determined for the initial perturbation having a normalized amplitude which is less than or of order unity.

First, we have taken the initial conditions as $n_i^o = \text{Cos}(i\pi \Delta\zeta)$ with periodic boundary conditions and amplitude normalized to unity. In this case we have used a mesh with $\Delta\zeta = 0.01$ and $\Delta\tau = 2 \times 10^{-4}$. The integration is performed with $\gamma = 0$ and $\gamma = \pm 10^{-4}$. In all the three cases the integration is carried on till the solitons are fully developed. In table (3.1), we have shown the amplitudes of the solitons at $\tau = 0.9$ for these three cases. Decrease of amplitude when $\gamma < 0$ and increase of amplitude when $\gamma > 0$, is observed. This result can be understood as follows:

Linearly, the second derivative term in Eq.(3.15) with $\gamma \gtrless 0$, represents a growth or damping of the initial perturbation. Since $\gamma << 1$ (the amplitude of the initial perturbation being normalized to unity) this term will simply result in an increase or decrease by a small amount compared to $\gamma = 0$ case:

Next interesting case is to see the development of an initial pulse whose amplitude is comparable to γ . For this, we take $n(\zeta, 0) = 0.03 \text{ sech}^2(5\zeta)$ and observed the development of the pulse till $\tau = 0.3$ with $\gamma = \pm 0.02$. At $\tau = 0.3$,

TABLE 3.1

Amplitudes of the solitons at $\tau = 0.9$ obtained from the numerical integration of Eq.(3.15) with $n(\zeta, 0) = \mathrm{Cos}(\pi \zeta)$ and $\delta = 5 \times 10^{-4}$. In the spatial region of 200 space steps 6 solitons are observed and the serial numbers are the numbers attached to these solitons from left to right.

Sr. No.	Amplitude of the solitons		
	$\gamma = 0$	$\gamma = -1 \times 10^{-4}$	$\gamma = +1 \times 10^{-4}$
1	1.723	1.662	1.785
2	2.128	2.042	2.174
3	2.504	2.427	2.571
4	0.351	0.317	0.387
5	0.810	0.762	0.864
6	1.282	1.225	1.342

the amplitude of the initial pulse increases to 0.0547 for the case $\gamma = 0.02$ and decreases to 0.024 for $\gamma = -0.02$. This is shown in Fig.3.1. It is to be noted that the increase in amplitude when $\gamma = +0.02$ is much larger than the decrease in amplitude when $\gamma = -0.02$. From Fig.(3.1) we also observe that the increase in amplitude is associated with decrease in width and vice versa. But this increase and decrease of amplitude do not follow the amplitude width relationship for a soliton. This is because of the fact that, to start with our initial pulse itself was not a pure soliton. In Fig.3.1 we have shown the development of the initial pulse only upto $\tau = 0.3$. Beyond $\tau = 0.3$ the solution seems to be unreliable due to accumulation of round-off errors. The reason for this may be the following: The way the second derivative term is replaced by finite differences in Eq.(3.16) introduces round-off error proportional to $(\Delta\tau/\Delta\zeta)^2 (\partial^2/\partial\tau^2)n$. The rate of change of the solution being quite rapid in this case, the accumulation of round-off error is also expected to be large.

We have also tried to examine the development of an initial pulse of sufficiently large amplitude, viz. $n(\zeta, 0) = 3.0 \operatorname{sech}^2(\zeta + 1)$, so that shock structure is produced within a reasonable amount of computer time. But in this case problems of numerical instability do not allow us to take sufficiently large values of γ (~ 3). So, we have to restrict ourselves to values as small as $|\gamma| = 10^{-4}$. At any given

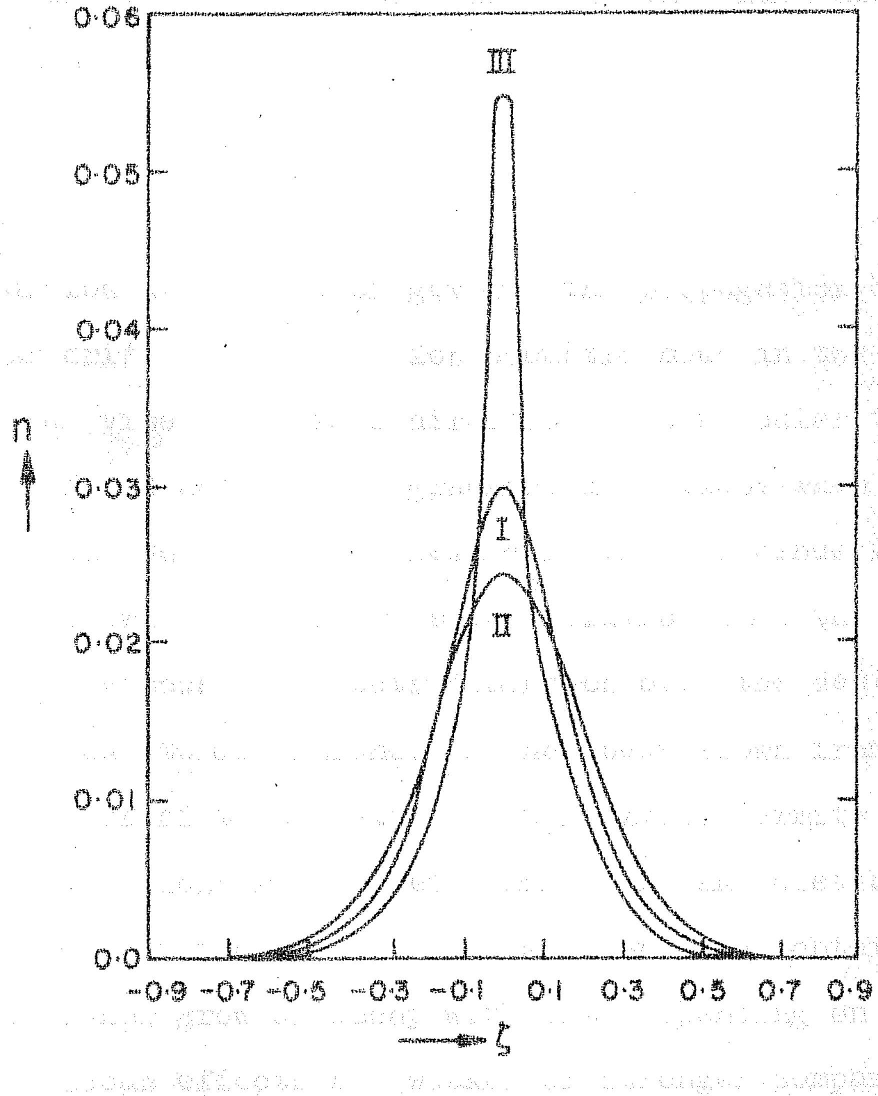


FIGURE 3.1 Time evolution of an initial pulse $n(\zeta, \tau = 0) = 0.3\ \text{Sech}^2(5\zeta)$ as given by Eq. (3.16) is shown. Curve - I is the initial pulse, Curves - II and III are $n(\zeta, \tau = 0.3)$ for $\delta = 5 \times 10^{-4}$ and $\gamma = -0.02$ and $+0.02$ respectively.

time (observed upto $\tau = 1.0$) the shock structure for these two cases (with $\gamma = \pm 10^{-4}$) are found to be almost identical (the changes in the amplitude as well as in the width are found to be less than 1 per cent).

III.4 Conclusions

A modified K-dV equation governs the propagation of the nonlinear drift dissipative ion acoustic mode in the presence of ion viscosity, in a direction perpendicular to both magnetic field and density gradient directions when the propagation along the magnetic field direction is sinusoidal. The equation allows a stationary shock solution when the stabilizing ion viscosity effects dominates over the destabilizing effects due to collisions. It has been shown from numerical solution of the equation that it still permits solitary wave solutions if the net stabilizing and destabilizing effects are not too strong. The solitons thus obtained are found to either grow or decay with time depending on whether the viscous effects are weaker or stronger compared to the resistive effects. We would like to emphasize that the actual two dimensional perturbation has to be constructed by superimposing on this solution the sinusoidal variation along the z-direction.

Even though the ion viscosity effects are not sufficient to quench the drift dissipative instability, the instability cannot make the wave grow indefinitely. Other nonlinear effects become important and finally saturates the instability. One such effect is the ion-trapping (Karatzas et al. 1975). Hence, one can modify the present theory to include this effect in order to enable one to look for a steady state solution.

CHAPTER IV

FINITE AMPLITUDE DRIFT WAVES IN A COLLISIONAL PLASMA

IV.1 Introduction

In the last chapter we considered the nonlinear propagation of a special kind of drift waves in a collisional plasma namely, the drift dissipative ion acoustic mode. This mode correspond to the case of weak magnetic field ($\omega >> \Omega_i$) and propagation almost perpendicular to the magnetic field ($k_y >> k_z$). In this chapter we wish to remove both these restrictions and consider the general case of propagation of a finite amplitude drift waves in a collisional plasma within the framework of reductive perturbation theory. The linear theory for drift waves both in collisionless plasmas (Kadomtsev 1965, Krall 1968) as well as in collisional plasmas (Kadomtsev 1965, Moiseev and Sagdeev 1963, Chen 1964), is very well established. But the attempts at developing a satisfactory nonlinear theory for drift waves in collisionless plasma or in collisional plasma have not been as successful. This is because of the fact that, the propagation of the drift waves, in general, is multi-dimensional in nature and most of the theories developed for finite amplitude waves, so far, are unidimensional.

In a collisionless plasma, the one dimensional propagation of a weakly nonlinear ion acoustic wave can be described by the Korteweg-deVries (K-dV) equation (Washimi and Taniuti 1966, Davidson 1972). The linear dispersion relation for drift waves has some characteristics similar to the one for ion acoustic waves. For example, the linear dispersion relation for drift waves reduces to that for ion acoustic waves in a magnetic field if the density gradients are sufficiently weak. Moreover, the drift waves go over to ion acoustic waves, as the angle between the direction of propagation and the magnetic field direction becomes sufficiently small. Therefore, one expects that, at least in some cases, the propagation of nonlinear drift waves can also be represented by a nonlinear equation similar to the K-dV equation. The usual reductive perturbation method for the nonlinear wave propagation in inhomogeneous media (Asano 1974) cannot be used in this case because of the multidimensional nature of the problem. However, Nozaki and Taniuti (1974) derived a multidimensional modified K-dV equation describing the weakly nonlinear propagation of drift waves in a collisionless plasma and obtained a special solution in the form of a solitary wave.

In this chapter, we have derived a modified multidimensional K-dV equation for the propagation of nonlinear drift waves in a collisional plasma in which both parallel resistivity and perpendicular viscosity are important. The method we have followed is very similar to the one used by Nozaki and Taniuti (1974). In section IV.2, we have reduced the set of nonlinear fluid equations for the drift waves in a collisional plasma to a multidimensional K-dV equation and have shown that the nonlinear term in our case is essentially same as that for ion acoustic waves but the dispersion is given by the effect of charge separation as well as ion inertia. Section IV.3 has been devoted to the analysis of this equation. When the stabilizing effects due to ion viscosity is stronger than the destabilizing effects due to electron-ion collisions, the equation allows a stationary shock solution. The structure of the shock profile depends on the propagation angle. It is found that the profile of the shock wave tends to change from an oscillatory one to a monotonic one as the angle between the propagation direction and magnetic field direction increases.

IV.2 Reduction to a Multidimensional Modified K-dV Equation

Let us consider an inhomogeneous plasma with $T_e \gg T_i$ (T_i and T_e being the ion and the electron temperature) and density gradients along the x-axis and magnetic field along z-direction. Let the magnetic field be quite strong so that in equilibrium, we can take ions to be at rest and the electrons drifting in the y-direction with a velocity

$$v_0 = -\frac{cT_e}{eB_0}\frac{dn_0}{dx}/n_0 \quad , \tag{4.1}$$

where n_0 is the equilibrium density. This drift gives rise to a current which induces a magnetic field in the z-direction. This induced magnetic field will be neglected in comparison with the applied magnetic field. The basic equations governing the system are the continuity equation and momentum transfer equation for ions and electrons respectively and the Poisson's equation, namely

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n v_x) + \frac{\partial}{\partial y}(n v_y) + \frac{\partial}{\partial z}(n v_z) = 0 \, , \tag{4.2}$$

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla}\right) v_x + \frac{e}{M}\frac{\partial \phi}{\partial x} - \Omega_i v_y + \frac{T_i}{nM}\frac{\partial n}{\partial x} + \frac{1}{nM}\left[\underset{\sim}{\nabla}\cdot\underset{\approx}{\pi}\right]_x = 0 \, , \tag{4.3}$$

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla}\right) v_y + \frac{e}{M}\frac{\partial \phi}{\partial y} + \Omega_i v_x + \frac{T_i}{nM}\frac{\partial n}{\partial y} + \frac{1}{nM}\left[\underset{\sim}{\nabla}\cdot\underset{\approx}{\pi}\right]_y = 0 \, , \tag{4.4}$$

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla}\right) v_z + \frac{e}{M}\frac{\partial \phi}{\partial z} + \frac{T_i}{nM}\frac{\partial n}{\partial z} + \frac{1}{nM}\left[\underset{\sim}{\nabla}\cdot\underset{\approx}{\pi}\right]_z = 0 \, , \tag{4.5}$$

$$\frac{m}{M}\frac{1}{C_s^2 \tau_e}\frac{\partial n_e}{\partial t} - \frac{1}{\Omega_e \tau_e}\frac{dn_0}{dx}\frac{\partial}{\partial x}\left(\frac{e\phi}{T_e}\right) - \frac{\partial}{\partial z}\left(n_e \frac{\partial \psi}{\partial z}\right) = 0 \tag{4.6}$$

$$\frac{\partial}{\partial z}\left(\frac{e\phi}{T_e}\right) - \frac{1}{n_e}\frac{\partial n_e}{\partial z} - \frac{\partial \psi}{\partial z} = 0 \tag{4.7}$$

and $$\nabla^2\phi = 4\pi e(n_e - n) \tag{4.8}$$

with $$\frac{\partial\Psi}{\partial z} = \frac{m}{T_e\tau_e}\,u_z \tag{4.9}$$

$\underset{\approx}{\Pi}$ appearing in the above equations is the stress-tensor. We shall borrow the expressions for the components of $\underset{\approx}{\Pi}$ from Braginskii's article (Braginskii 1965). For example,

$$\begin{aligned}[\underset{\sim}{\nabla}\cdot\underset{\approx}{\Pi}]_z &= \frac{\partial}{\partial x}\Pi_{zx} + \frac{\partial}{\partial y}\Pi_{zy} + \frac{\partial}{\partial z}\Pi_{zz} \\ &= -\eta_2\left[\frac{\partial^2 v_x}{\partial x\partial z} + \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_y}{\partial y\partial z} + \frac{\partial^2 v_z}{\partial y^2}\right] \\ &\quad - \eta_4\left[\frac{\partial^2 v_y}{\partial x\partial z} - \frac{\partial^2 v_x}{\partial y\partial z}\right] \\ &\quad - \frac{2\eta_0}{3}\left[2\frac{\partial^2 v_z}{\partial z^2} - \frac{\partial^2 v_x}{\partial x\partial z} - \frac{\partial^2 v_y}{\partial y\partial z}\right],\end{aligned} \tag{4.10}$$

where $\eta_0 = 0.96\, nT_i/\nu_{ii}$; $\eta_2 = 1.2\, nT_i\,\nu_{ii}/\Omega_i^2$ and $\eta_4 = nT_i/\Omega_i$. In Eqs.(4.2) – (4.10), n, n_e are the ion and electron densities respectively; v_x, v_y, v_z are the components of ion fluid velocity while u_z and C_s ($C_s^2 = T_e/M$) are the z-component of electron fluid velocity and ion acoustic speed respectively. Moreover, ϕ is the electrostatic potential while m and M are the electron and ion masses respectively. The quantity Ψ appearing in Eq.(4.7) is a velocity potential introduced through Eq.(4.9). As in Chapter III, the perpendicular equation of motion for the electrons is not explicitly written down in the above set of equation but use has been made of it in writing the electron continuity equation (Eq.(4.6)) under local approximation (Kadomtsev 1965). Thus, Eqs.(4.2) - (4.9) form a complete set. By integrating Eq.(4.7) and substituting the result in Eq.(4.8) we get

$$\nabla^2\phi = 4\pi e\left[n_0 \exp\left(\frac{e\phi}{T_e} - \Psi\right) - n\right]. \quad (4.11)$$

We now consider a drift wave of finite amplitude. Since the drift waves reduce to ion acoustic waves as the angle between the propagation direction and the magnetic field direction approaches zero, we take the same type of scalings as that for an ion acoustic wave as far as the ion density and ion motion along the magnetic field directions are concerned viz,

$$\zeta = \epsilon^{1/2}(z - c_s t) \quad , \quad \tau = \epsilon^{3/2} t$$

with

$$n = n_0 + \epsilon n^{(1)} + \epsilon^2 n^{(2)} + \cdots$$

$$v_z = \epsilon v_z^{(1)} + \epsilon^2 v_z^{(2)} + \cdots$$

$$\phi = \epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + \cdots ,$$

where ϵ is a parameter of smallness which in our system we shall define in terms of the ratio of the ion temperature to electron temperature such that $T_i/T_e = \epsilon\alpha$. Following Nozaki and Taniuti (1974), we take the order of wavelength in y-direction to be same as that in the z-direction and a slow variation in the x-direction. Hence, the stretchings used in the y and x directions can be $\eta = \epsilon^{1/2} y$ and $\xi = \epsilon x$. The stretching in the x-direction is rather crucial, which means that the density gradient approaches zero faster than the wavenumber of perturbation does. With these stretchings, v_x, v_y and Ψ can be expanded as

$$v_x = \epsilon^{3/2} v_x^{(1)} + \epsilon^{5/2} v_x^{(2)} + \cdots$$

$$v_y = \epsilon^{2} v_y^{(1)} + \epsilon^{3} v_y^{(2)} + \cdots$$

and $$\Psi = \epsilon^{2} \Psi^{(1)} + \epsilon^{3} \Psi^{(2)} + \cdots .$$

The expansions for v_x and v_y follow from Eqs.(4.3) and (4.4) while that for Ψ follows from Eq.(4.7). In the absence of collisions, Eq.(4.7) shows that the electron density fluctuation are governed only by the potential fluctuations, namely $n_e = n_o \exp(e\varphi/T_e)$. The introduction of the collisional term $\partial\Psi/\partial z$ in Eq.(4.7) is treated as a correction to the potential fluctuations. Thus the term $\frac{\partial}{\partial z}\Psi$ is taken to be one order smaller than the term $\frac{\partial}{\partial z}(e\varphi/T_e)$. Hence the above expansion for Ψ is justified.

When the electron inertia term is neglected, then to lowest order Eq.(4.6) gives

$$\frac{\partial^2 \Psi^{(1)}}{\partial \zeta^2} = - \frac{\kappa}{\Omega_e \tau_e} \frac{\partial}{\partial \eta} \left(\frac{e\varphi^{(1)}}{T_e} \right) , \qquad (4.12)$$

where $\kappa = - \frac{dn_o}{d\xi} / n_o$. In writing Eq.(4.12) it has been assumed that $\Omega_e \tau_e \sim \epsilon^{1/2}$.

Eq.(4.2) and (4.4) to order $\epsilon^{3/2}$ give

$$v_z^{(1)} = \frac{C_s}{n_0} n^{(1)} \tag{4.13}$$

and
$$v_x^{(1)} = -\frac{e}{M\Omega_i}\frac{\partial \varphi^{(1)}}{\partial \eta} . \tag{4.14}$$

Similarly $O(\epsilon)$ terms from Eq.(4.11) and $O(\epsilon^2)$ terms from Eq.(4.3) give,

$$\varphi^{(1)} = \frac{T_e}{n_0 e} n^{(1)} \tag{4.15}$$

and
$$v_y^{(1)} = -\frac{C_s}{\Omega_i}\frac{\partial v_x^{(1)}}{\partial \zeta} - \frac{\alpha \kappa C_s^2}{\Omega_i} + \frac{e}{M\Omega_i}\frac{\partial \varphi^{(1)}}{\partial \xi} . \tag{4.16}$$

We notice that to the lowest order, the stress-tensor terms in Eq.(4.3) and (4.4) do not contribute because they contain second derivation of velocity components.

For terms $O(\epsilon^{5/2})$, from Eqs.(4.1) and (4.5) and $O(\epsilon^2)$ from Eq.(4.11), we obtain

$$-C_s\frac{\partial n^{(2)}}{\partial \zeta} + n_0\frac{\partial v_z^{(2)}}{\partial \zeta} + \frac{\partial}{\partial \zeta}(n^{(1)} v_z^{(1)}) - n_0 \kappa v_x^{(1)}$$
$$+ n_0\frac{\partial v_x^{(1)}}{\partial \xi} + n_0\frac{\partial v_y^{(1)}}{\partial \eta} + \frac{\partial n^{(1)}}{\partial \tau} = 0 , \tag{4.17}$$

$$-C_s\frac{\partial v_z^{(2)}}{\partial \zeta} + \frac{e}{M}\frac{\partial \varphi^{(2)}}{\partial \zeta} + v_z^{(1)}\frac{\partial v_z^{(1)}}{\partial \zeta} + \frac{\partial v_z^{(1)}}{\partial \tau}$$
$$+(\alpha C_s^2/n_0)\frac{\partial n^{(1)}}{\partial \zeta} - \left(\eta_2\frac{\partial^2}{\partial \eta^2} + \frac{4\eta_0}{3}\frac{\partial^2}{\partial \zeta^2}\right) v_z^{(1)} = 0 \tag{4.18}$$

and $$\left(\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\zeta^2}\right)\varphi^{(1)}=4\pi e\left[-n^{(2)}+n_0\frac{e\varphi^{(2)}}{T_e}-n_0\psi^{(1)}+\frac{1}{2}n_0\frac{e^2(\varphi^{(1)})^2}{2}\right]. \quad (4.19)$$

Making use of Eqs.(4.13) - (4.16) and also eliminating $n^{(2)}$, $\varphi^{(2)}$ and $v_z^{(2)}$ from Eqs.(4.17) - (4.18), we finally get

$$\frac{\partial n^{(1)}}{\partial\tau}+\frac{1}{2}\frac{\kappa C_s^2}{\Omega_i}\frac{\partial n^{(1)}}{\partial\eta}+\frac{1}{2}\alpha C_s\frac{\partial n^{(1)}}{\partial\zeta}+\frac{C_s}{n_0}n^{(1)}\frac{\partial n^{(1)}}{\partial\zeta}$$
$$+\frac{1}{2}\frac{C_s^3}{\Omega_i^2}\frac{\partial^3 n^{(1)}}{\partial\eta^2\partial\zeta}+\frac{1}{2}C_s\lambda_D^2\left(\frac{\partial^3}{\partial\eta^2\partial\zeta}+\frac{\partial^3}{\partial\zeta^3}\right)n^{(1)}$$
$$+\frac{1}{2}n_0C_s\frac{\partial\Psi^{(1)}}{\partial\zeta}-\left(\frac{\eta_2}{2}\frac{\partial^2}{\partial\eta^2}+\frac{2\eta_0}{3}\frac{\partial^2}{\partial\zeta^2}\right)n^{(1)}=0, \quad (4.20)$$

where $\lambda_D=(T_e/4\pi n_0e^2)^{1/2}$ is the electron Debye length. Eqs. (4.12) - (4.20) constitute the set of two coupled nonlinear partial differential equations describing the propagation of a finite amplitude drift wave in a collisional plasma. We notice that Eq.(4.20) is similar to the equation obtained by Nozaki and Taniuti (1974; Eq.(23)) but it contains three additional terms. The term $\frac{1}{2}\alpha C_s\frac{\partial n^{(1)}}{\partial\zeta}$ comes due to finite ion temperature effects, the term $\frac{1}{2}n_oC_s\frac{\partial\Psi^{(1)}}{\partial\zeta}$ appears due to the parallel resistivity while the last term, comes from the ion viscosity effects. Moreover, if the viscosity, resistivity and finite ion temperature effects are neglected and then η -dependence is also neglected, Eq.(4.20) reduces to the K-dV equation for ion acoustic waves obtained by Washimi and Taniuti (1966). The effect of density gradients appear through the second term in Eq.(4.20), while the dispersion is given by the fifth and sixth terms in Eq.(4.20). The fifth term results from ion inertia while the sixth term results from charge separation effects. Differentiating Eq. (4.20) once w.r.t. ζ and making use of Eq.(4.12) we get,

$$\frac{\partial^2 n^{(1)}}{\partial\zeta\partial\tau} + \frac{1}{2}\frac{\kappa c_s^2}{\Omega_i}\frac{\partial^2 n^{(1)}}{\partial\zeta\partial\eta} + \frac{1}{2}\alpha c_s\frac{\partial^2 n^{(1)}}{\partial\zeta^2} + \frac{c_s}{n_0}\frac{\partial}{\partial\zeta}\left(n^{(1)}\frac{\partial n^{(1)}}{\partial\zeta}\right)$$
$$+\frac{1}{2}\frac{c_s^3}{\Omega_i^2}\frac{\partial^4 n^{(1)}}{\partial\eta^2\partial\zeta^2} + \frac{1}{2}c_s\lambda_D^2\left(\frac{\partial^4}{\partial\eta^2\partial\zeta^2} + \frac{\partial^4}{\partial\zeta^4}\right)n^{(1)}$$
$$-\frac{1}{2}\frac{c_s\kappa}{\Omega_e\tau_e}\frac{\partial n^{(1)}}{\partial\eta} - \left(\frac{n_2}{2}\frac{\partial^3}{\partial\eta^2\partial\zeta} + \frac{2n_0}{3}\frac{\partial^3}{\partial\zeta^3}\right)n^{(1)} = 0, \quad (4.21)$$

which is the modified K-dV equation for our system.

IV.3 Analysis of the Modified K-dV Equation

By making a Galilean transformation, $\sigma = \eta - \frac{1}{2}\frac{\kappa c_s^2}{\Omega_i}\tau$ Eq.(4.21) can be reduced to

$$\frac{\partial n^{(1)}}{\partial\zeta\partial\tau} + \frac{1}{2}\alpha c_s\frac{\partial^2 n^{(1)}}{\partial\zeta^2} + \frac{c_s}{n_0}\frac{\partial}{\partial\zeta}\left(n^{(1)}\frac{\partial n^{(1)}}{\partial\zeta}\right)$$
$$+\frac{1}{2}\frac{c_s^3}{\omega_{pi}^2}\frac{\partial^4 n^{(1)}}{\partial\zeta^4} + \frac{1}{2}\left(\frac{c_s^3}{\Omega_i^2} + \frac{c_s^3}{\omega_{pi}^2}\right)\frac{\partial^4 n^{(1)}}{\partial\zeta^2\partial\sigma^2}$$
$$-\frac{c_s\kappa}{2\Omega_e\tau_e}\frac{\partial n^{(1)}}{\partial\sigma} - \left(\frac{n_2}{2}\frac{\partial^3}{\partial\sigma^2\partial\zeta} + \frac{2n_0}{3}\frac{\partial^3}{\partial\zeta^3}\right)n^{(1)} = 0, \quad (4.22)$$

where $\omega_{pi} = (4\pi n_o e^2/M)^{1/2}$ is the ion plasma frequency. For discussing Eq.(4.22), it is more convenient to work with dimensionless quantities. For this purpose we normalize density to equilibrium value n_o, time to ion plasma period ω_{pi}^{-1} and lengths to Debye length λ_D. In terms of these normalized variables Eq.(4.22) can be rewritten as

$$\frac{\partial n}{\partial \tau \partial \zeta} + \frac{\alpha}{2}\frac{\partial^2 n}{\partial \zeta^2} + \frac{\partial}{\partial \zeta}\left(n\frac{\partial n}{\partial \zeta}\right) + \frac{1}{2}\left(\frac{\omega_{pi}^2}{\Omega_i^2}+1\right)\frac{\partial^4 n}{\partial \sigma^2 \partial \zeta^2} + \frac{1}{2}\frac{\partial^4 n}{\partial \zeta^4}$$
$$-\frac{\kappa}{2\Omega_e \tau_e}\frac{\partial n}{\partial \sigma} - \left(\frac{1.2\alpha\,\omega_{pi}\nu_{ii}}{\Omega_i^2}\frac{\partial^3}{\partial \sigma^2 \partial \zeta} + \frac{0.64\alpha\,\omega_{pi}}{\nu_{ii}}\frac{\partial^3}{\partial \zeta^3}\right)n = 0. \tag{4.23}$$

Letus now consider the propagation of the drift wave in a particular direction such that the propagation direction makes an angle θ, with the direction of the magnetic field. Therefore, the two space variables are now related through $\Lambda = \zeta \cos\theta + \sigma \sin\theta$. Writing Eq.(4.23) in terms of Λ and integrating once with respect to Λ under the boundary conditions namely, n, $\partial n/\partial \Lambda$, $\partial^2 n/\partial \Lambda^2$, $\partial^3 n/\partial \Lambda^3$ tend to zero as $\Lambda \to \pm\infty$, we get,

$$\cos\theta\frac{\partial n}{\partial \tau} + \frac{\alpha}{2}\cos^2\theta\frac{\partial n}{\partial \Lambda} + \cos^2\theta\, n\frac{\partial n}{\partial \Lambda}$$
$$+\frac{\kappa\cos\theta}{2\Omega_e\tau_e}n + \frac{\cos^2\theta}{2}\left(\sin^2\theta\frac{\omega_{pi}^2}{\Omega_i^2}+1\right)\frac{\partial^3 n}{\partial \Lambda^3}$$
$$-\left(\frac{1.2\alpha\,\omega_{pi}\nu_{ii}}{\Omega_i^2}\frac{\sin^2\theta}{\cos^2\theta} + \frac{0.64\alpha\,\omega_{pi}}{\nu_{ii}}\right)\cos^3\theta\frac{\partial^2 n}{\partial \Lambda^2} = 0. \tag{4.24}$$

It may be in order here to point out the limitations of our theory. The scalings in the x, y, z directions are not independent but related through the stretchings used. We note that k_y approaches zero as fast as k_z does but $\omega^* = k_y v_o$ approaches zero much faster. As a result, the theory is not valid in the neighbourhood of either $k_z = 0$ or $k_y = 0$, or in other words the propagation with $\theta = 0$ or $\theta = \pi/2$ cannot be investigated with the help of the present model.

Now, on dividing throughout by Cosθ, we get

$$\frac{\partial n}{\partial \tau} + \left(\frac{\alpha \cos\theta}{2} + n\cos\theta\right)\frac{\partial n}{\partial \Lambda} + \frac{\cos\theta}{2}\left(\frac{\omega_{pi}^2}{\Omega_i^2}\sin^2\theta + 1\right)\frac{\partial^3 n}{\partial \Lambda^3} - \frac{\kappa}{2\Omega_e\tau_e} n - \left(\frac{1{\cdot}2\,\alpha\,\omega_{pi}\nu_{ii}}{\Omega_i^2}\frac{\sin^2\theta}{\cos^2\theta} + \frac{0{\cdot}64\,\alpha\,\omega_{pi}}{\nu_{ii}}\right)\cos^2\theta\,\frac{\partial^2 n}{\partial \Lambda^2} = 0. \tag{4.25}$$

By introducing a change of variable $\bar{n} = n \exp(-\kappa/2\Omega_e\tau_e)$ and a transformation, $\chi = \Lambda - \frac{\alpha}{2}\cos\theta\,\tau$, Eq.(4.25) can be reduced to

$$\frac{\partial \bar{n}}{\partial \tau} + \cos\theta \exp(\delta\tau)\,\bar{n}\frac{\partial \bar{n}}{\partial \chi} + b\frac{\partial^3 \bar{n}}{\partial \chi^3} - d\frac{\partial^2 \bar{n}}{\partial \chi^2} = 0, \tag{4.26}$$

where

$$\delta = \frac{\kappa}{2\Omega_e\tau_e}\ , \quad b = \frac{\cos\theta}{2}\left(\frac{\omega_{pi}^2}{\Omega_i^2}\sin^2\theta + 1\right)$$

and $$d = \alpha\cos^2\theta\left(\frac{1{\cdot}2\,\nu_{ii}\,\omega_{pi}}{\Omega_i^2}\frac{\sin^2\theta}{\cos^2\theta} + \frac{0{\cdot}64\,\omega_{pi}}{\nu_{ii}}\right)$$

The following general conclusions can be drawn from Eq.(4.26).

i) When $\delta = d = 0$ i.e., when both viscosity and collisional effects are absent, Eq.(4.26) reduces to Eq.(26) of Nozaki and Taniuti (1974). As was shown by these authors, this equation gives the so called drift solitary wave solutions. Of course, the initial value problem of this equation has not yet been solved and one does not know how in this case a solitary wave is formed from smooth waves.

ii) When collisional effects are much weaker than the viscosity effects i.e., when we can neglect all the higher order terms in the expansion of $\exp(\delta\tau)$, Eq.(4.26) gives a stationary shock solution with either a monotonic or an oscillatory profile (Johnson 1970, Jefreey and Kakutani 1972, Dzhavakhishvili 1973). In order to see how the angle of propagation influences the profile of the shock front, we carry out the following simple minded analysis.

Going over to a wave frame moving with velocity U, such that $\bar{n}$ depends on χ and τ only through $\lambda \equiv \chi - U\tau$ and integrating Eq.(4.26) once w.r.t. λ under the boundary conditions $\bar{n} = \bar{n}' = \bar{n}'' = 0$ as $\lambda \to -\infty$, we obtain

$$b\frac{d^2\bar{n}}{d\lambda^2} - d\frac{d\bar{n}}{d\lambda} + \frac{\cos\theta}{2}\bar{n}^2 - U\bar{n} = 0 \qquad (4.27)$$

In the limit $\lambda \to \infty$, Eq.(4.27) gives

$$\bar{n} \simeq N = 2U/\cos\theta$$

So, in order to look for the asymptotic behaviour of Eq.(4.27) as $\lambda \to +\infty$, let us take $\bar{n} = N + \tilde{n}$. Now linearizing Eq.(4.27) we get an equation for $\tilde{n}$, namely

$$b\frac{d^2\tilde{n}}{d\lambda^2} - d\frac{d\tilde{n}}{d\lambda} + U\tilde{n} = 0 \quad . \qquad (4.28)$$

Eq.(4.28) has a solution $\tilde{n} = \exp(\gamma\lambda)$ where γ is given by

$$\gamma = -\frac{d}{2b} \pm \left(\frac{d^2}{4b^2} - \frac{U}{b}\right)^{1/2} \qquad (4.29)$$

It is clear from Eq.(4.29) that the shock wave will have a monotonic profile if $d^2/4b \geq U$ which can be rewritten as

$$2\alpha^2\left[\frac{1.2\,\nu_{ii}\,\omega_{pi}}{\Omega_i^2}\sin^2\theta + \frac{0.64\,\omega_{pi}}{\nu_{ii}}\cos^2\theta\right]^2\left[\frac{\omega_{pi}^2}{\Omega_i^2}\sin^2\theta + 1\right]^{-1} \geq U\cos\theta \qquad (4.30)$$

For all the other parameters fixed as the angle of propagation increases, it becomes increasingly easier to satisfy the above inequality and hence it becomes increasingly easier to obtain a shock wave with a monotonic profile. If the inequality (4.30) is not satisfied, the shock wave will have an oscillatory profile. This means that for a given set of plasma parameters, there exists a critical angle θ_c, which represents the angle of transition of the shock profile from an oscillatory one to a monotonic one.

iii) As for the case when the collisional growth term is sufficiently large, we cannot look for a stationary solution, for the simple reason that any initial perturbation will keep on growing because of this effect. Waves will grow to such an extent that the dispersion effects will no longer be able to balance the steepening due to nonlinearity and usual wave breaking will take place.

IV.4 Conclusions

The propagation of a small but finite amplitude drift wave in a collisional plasma can be described by means of a modified K-dV equation. When dominant effect due to the collisions is the viscous damping, the equation allows a stationary shock solution. The profile of the shock wave tends to change from oscillatory one to monotonic one as the angle between the direction of propagation and the magnetic field increases. As in the case of drift dissipative instability, if there is a growth of the wave and if the growth rate exceeds the damping rate due to the viscous effects, a stationary solution does not exist.

The initial value problem of the Eq.(4.26), even in a collisionless plasma has not been solved so far. In order to understand, how the drift-solitary wave is formed from arbitrary initial perturbations, it is essential to study the initial value problem of Eq.(4.26).

CHAPTER V

ION ACOUSTIC SOLITARY WAVES IN AN INHOMOGENEOUS PLASMA WITH NONUNIFORM TEMPERATURE

V.1 Introduction

In Chapter II, we investigated the propagation of an ion acoustic solitary wave in a medium, whose dispersion characteristics get modified by the presence of a relatively cold electron component. In this chapter we shall examine the propagation of an ion acoustic solitary wave in an inhomogeneous medium. In this case the strength of dispersion becomes a function of the space variable. As mentioned in the previous chapters, the K-dV equation describes the propagation of one dimensional weakly nonlinear ion sound disturbances in a homogeneous plasma.

In reality, however, the plasma is never strictly homogeneous. Recently Nishikawa and Kaw (1975) have derived an equation describing the propagation of a weakly nonlinear ion acoustic wave in a plasma with a density gradient. These authors have shown that the amplitude of a solitary wave (normalized to local density) decreases as it propagates towards the region of increasing density. This is because of the fact that the strength of the dispersion, which is proportional to $\lambda_D^2 = T_e/4\pi n_o e^2$, decreases as one moves towards regions of increasing density. We note that, the strength of the dispersion gets further modified if on the top of a density inhomogeneity, a temperature inhomogeneity is also present. Based on this physical ground, in this chapter, we investigate the propagation of an ion acoustic solitary wave in the presence of both density and temperature inhomogeneities.

We assume that the temperature gradient is produced by the presence of a finite thermal conductivity. Since the coefficient of thermal conductivity goes as $T^{5/2}$, (where T is the temperature) the thermal conduction is generally large for high temperature plasmas. Hence an appreciable temperature gradient is not to be expected in a high temperature plasma. Nevertheless, for not too high temperature plasmas in the presence of a magnetic field or in presence of collisions, temperature gradient can be sustained. Because of this reason we shall assume that the temperature gradient scale-length is larger than the density gradient scale-length. In Section V.2 we assume that the equilibrium density gradients are produced by a zero-order electric field E_0 and an effective gravity field. In the presence of a temperature gradient the electric field needed to sustain the density gradients is smaller than the one needed in the absence of temperature gradients. Due to the presence of the collisions which give rise to the finite thermal conductivity, one needs a smaller field to maintain the gradients. The absence of collisions simply shorts out the electric field and hence a higher field is required to maintain the gradients.

In Section V.2, using reductive perturbation method for an inhomogeneous plasma (Asano, 1974), we have derived a modified K-dV equation governing the propagation of an ion acoustic wave in the presence of both density and temperature inhomogeneities. In Section V.3, we show that a solitary wave, of a given amplitude, propagating towards increasing temperature and decreasing density has a higher velocity compared to the case when temperature gradients are absent.

Recently an experiment was performed in Physical Research Laboratory, Ahmedabad, India (John and Saxena 1976), where an ion acoustic solitary wave was launched and propagation was studied in a plasma with a density gradients. In Section V.4, we compare our results in the absence of a temperature gradient with the experimental results and show that they are in very good agreement with the experimental results.

V.2 Derivation of the Modified K-dV Equation

The basic equations governing the system are the electron and ion momentum transfer equations, ion continuity equation and Poisson's equation, namely

$$e n_e \frac{\partial \varphi}{\partial x} - \frac{\partial}{\partial x}[n_e T_e(x)] = 0 , \tag{5.1}$$

$$m_i\left[\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x}\right] + e\frac{\partial \varphi}{\partial x} = m_i g , \tag{5.2}$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}[n_i v_i] = 0 \tag{5.3}$$

and
$$\frac{\partial^2 \varphi}{\partial x^2} = 4\pi e[n_e - n_i] . \tag{5.4}$$

In Eqs.(5.1) - (5.4), φ is the potential, $n_{e,i}$ are the electron and ion densities respectively, $T_e(x)$ electron temperature, v_i is ion velocity, $-e$ is the electronic charge and g is acceleration due to the gravity field. Assuming that the scale-length for the temperature gradients is L_T, we write $T_e(x) = T_e(0)\,(1 + x/L_T)$. Thus, Eq.(5.1) shows that the equilibrium density gradients are balanced by the zero order electric field $-e\,\frac{\partial \varphi_0}{\partial x}$ and partly by the temperature gradients. If the scale-length for equilibrium density gradients is L_N, one can write $N_e(x) = N_i(x) \equiv N(x) = N(0)(1-x/L_N)$. Eq.(5.2) shows that, for ions in equilibrium the gravitational force is balanced by the zero order electric field.

We write the densities $n_{e,i} = N(x) + \Delta n_{e,i}(x,t) = N(x)(1 + \tilde{n}_{e,i}(x,t))$, where $\tilde{n}_{e,i}(x,t)$ are the perturbed electron and ion densities respectively, normalized to the local equilibrium values. It is convenient for subsequent calculations, if we write Eqs.(5.1) - (5.4) in dimensionless form. For this reason we normalize ϕ to $T_e(0)/e$, lengths to Debye length $\lambda_D(0)$ ($\lambda_D^2(0) = T_e(0)/4\pi N(0)e^2$) at $x = 0$, time to ion plasma period $\omega_{pi}^{-1}(0)$ ($\omega_{pi}^2(0) = 4\pi N(0)e^2/m_i$) and velocity to ion-sound speed $C_s(0)$ ($C_s^2(0) = T_e(0)/m_i$) at $x = 0$. In terms of these normalized quantities the equations governing the perturbed quantities can be written from Eq. (5.1) - (5.4) as

$$\frac{\partial \Phi}{\partial x} + \tilde{n}_e \frac{\partial \Phi}{\partial x} - T_e(x)\frac{\partial \tilde{n}_e}{\partial x} = 0 \ , \tag{5.5}$$

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} + \frac{\partial \Phi}{\partial x} = 0 \ , \tag{5.6}$$

$$\frac{\partial \tilde{n}_i}{\partial t} + v_i \frac{\partial \tilde{n}_i}{\partial x} + (1+\tilde{n}_i)\frac{\partial v_i}{\partial x} - \bar{\kappa}\,(1+\tilde{n}_i)v_i = 0 \tag{5.7}$$

and

$$\frac{N(0)}{N(x)}\frac{\partial^2 \Phi}{\partial x^2} = \tilde{n}_e - \tilde{n}_i \ , \tag{5.8}$$

where $\bar{\kappa} = -\dfrac{\lambda_D(0)}{N(x)}\dfrac{dN(x)}{dx}$ and $T_e(x) = 1 + \alpha\,\bar{\kappa}\,x$ with $\alpha = L_N/L_T$. We now introduce the stretched variables (Asano, 1974), $\xi = \epsilon^{1/2}(x - t)$ and $\eta = \epsilon^{3/2}x$. The smallness parameter ϵ is defined in terms of the scale-length of the density gradients such that $\bar{\kappa} = \lambda_D(0)/L_N = \epsilon^{3/2}\kappa$. We shall also assume that $\alpha = O(\epsilon)$. The perturbed quanties can now be expanded as

$$\tilde{n}_i = \epsilon n_i^{(1)} + \epsilon^2 n_i^{(2)} + \cdots\cdots$$

$$\tilde{n}_e = \epsilon n_e^{(1)} + \epsilon^2 n_e^{(2)} + \cdots\cdots$$

$$\Phi = \epsilon \Phi^{(1)} + \epsilon^2 \Phi^{(2)} + \cdots\cdots$$

and $$v_i = \epsilon v_i^{(1)} + \epsilon^2 v_i^{(2)} + \cdots\cdots .$$

To the lowest order Eqs.(5.5) - (5.8) give, $n_i^{(1)} = n_e^{(1)} = \Phi^{(1)} = v_i^{(1)}$. To the next higher order Eqs. (5.5) - (5.6) can written as

$$\frac{\partial \Phi^{(2)}}{\partial \xi} - \frac{\partial n_e^{(2)}}{\partial \xi} + n_e^{(1)} \frac{\partial \Phi^{(1)}}{\partial \xi} - \alpha\eta \frac{\partial n_e^{(1)}}{\partial \xi} = 0 , \qquad (5.9)$$

$$-\frac{\partial v_i^{(2)}}{\partial \xi} + n_i^{(1)} \frac{\partial n_i^{(1)}}{\partial \xi} + \frac{\partial n_i^{(1)}}{\partial \eta} + \frac{\partial \Phi^{(2)}}{\partial \xi} = 0 , \qquad (5.10)$$

$$-\frac{\partial n_i^{(2)}}{\partial \xi} + \frac{\partial v_i^{(2)}}{\partial \xi} + \frac{\partial n_i^{(1)}}{\partial \eta} + 2 n_i^{(1)} \frac{\partial n_i^{(1)}}{\partial \xi} - x n_i^{(1)} = 0 \qquad (5.11)$$

and $$\frac{N(0)}{N(x)} \frac{\partial^2 \Phi^{(1)}}{\partial \xi^2} = n_e^{(2)} - n_i^{(2)} \qquad (5.12)$$

By eliminating $\Phi^{(2)}$, $n_i^{(2)}$, $n_e^{(2)}$ and $v_i^{(2)}$ from Eqs.(5.9) - (5.12) one obtains

$$\frac{\partial n_i^{(1)}}{\partial \eta} + \left(n_i^{(1)} + \frac{1}{2}\alpha\eta\right) \frac{\partial n_i^{(1)}}{\partial \xi} + \frac{1}{2} f(\eta) \frac{\partial^3 n_i^{(1)}}{\partial \xi^3} - \frac{1}{2} x n_i^{(1)} = 0 \qquad (5.13)$$

where $f(\eta) = N(0)/N(x)$. In writing Eq.(5.13), use has been made of the relation $n_i^{(1)} = n_e^{(1)} = \Phi^{(1)} = v_i^{(1)}$. Eq.(5.13) is a modified K-dV equation that governs the propagation of weakly nonlinear ion acoustic waves in an inhomogeneous plasma with both density and temperature inhomogeneities. We notice that the term $\frac{1}{2}\alpha\eta\frac{\partial n_i^{(1)}}{\partial\xi}$ appears entirely due to temperature gradients. We also notice that, in terms of our stretched variables the coefficient of the dispersion term becomes a function of η only.

V.3 Steady State Solution of Eq.(5.13)

By making a co-ordinate transformation of the type $\zeta = \xi - \alpha\eta^2/4$ and $\theta = \eta$, Eq.(5.13) can be written as

$$\frac{\partial n_i^{(1)}}{\partial\theta} + n_i^{(1)}\frac{\partial n_i^{(1)}}{\partial\zeta} + \frac{1}{2}f(\theta)\frac{\partial^3 n_i^{(1)}}{\partial\zeta^3} - \frac{1}{2}\kappa n_i^{(1)} = 0 . \tag{5.14}$$

Now we introduce a change of variable $n = (f(\theta))^{-1/2}\, n_i^{(1)}$ and the following transformations:

$$\mu = [f(\theta)]^{\beta}\zeta$$

and

$$\nu = \int^{\theta}[f(\theta')]^{\gamma}d\theta'$$

For $\gamma = 1/4$ and $\beta = -1/4$, Eq.(5.14) reduces to

$$\frac{\partial n}{\partial\nu} + n\frac{\partial n}{\partial\mu} + \frac{1}{2}\frac{\partial^3 n}{\partial\mu^3} = 0 . \tag{5.15}$$

In writing Eqn.(5.15) terms of order $\beta\kappa\mu\frac{\partial}{\partial\mu}$ are neglected because of smallness of $\kappa\mu$ (Nishikawa and Kaw, 1975). Eq. (5.15) is a K-dV equation and if one assumes that n depends on μ and ν only through $\Psi = (\mu - u\nu)$, one obtains a stationary 'soliton' solution (Davidson, 1972) which is given by,

$$n = 3u \, \text{Sech}^2\left[\left(\frac{u}{2}\right)^{1/2}(\mu - u\nu)\right] , \qquad (5.16)$$

where u is the velocity of the soliton in $(\mu ; \nu)$ plane. Since $f(\theta) = \frac{N(0)}{N(x)} \approx (1 + \kappa\theta) \approx \exp(\kappa\theta)$, Eq.(5.16) can be rewritten as

$$n_i^{(1)} = 3u \exp\left(\frac{\kappa\eta}{2}\right) \text{Sech}^2\left[\left(\frac{u}{2}\right)^{1/2} \exp\left(-\frac{\kappa\eta}{4}\right) \cdot \left\{\xi - \left(\frac{\alpha\eta^2}{4} + \frac{4u}{\kappa}\exp\left(\frac{\kappa\eta}{2}\right)\right)\right\}\right] . \qquad (5.17)$$

Thus, the absolute density perturbation is given as

$$\Delta n_i = 3uN(x) \exp\left(\frac{\kappa\eta}{2}\right) \text{Sech}^2\left[\left(\frac{u}{2}\right)^{1/2} \exp\left(\frac{\kappa\eta}{4}\right) \cdot \left\{\xi - \left(\frac{\alpha\eta^2}{4} + \frac{4u}{\kappa}\exp\left(\frac{\kappa\eta}{2}\right)\right)\right\}\right] . \qquad (5.18)$$

V.4 Conclusions and Comparison with Experiment

Since $N(x)$ goes as $\exp(-\kappa\eta)$, we can derive the following conclusions from Eq.(5.18),

i) In the absence of temperature gradients, as a solitary wave moves towards decreasing density regions, its amplitude goes as $(N(x))^{1/2}$ and velocity goes as $(N(x))^{-1/2}$. The decrease in amplitude is associated with appropriate increase in width. Recently John and Saxena (1976) carried out an experiment to study the propagation of ion acoustic solitary waves in a plasma with a density gradient but with uniform temperature. Their results are shown in Fig.5.1. The variation of the amplitude and the width of the solitary wave as it propagates along decreasing (or increasing) density follow

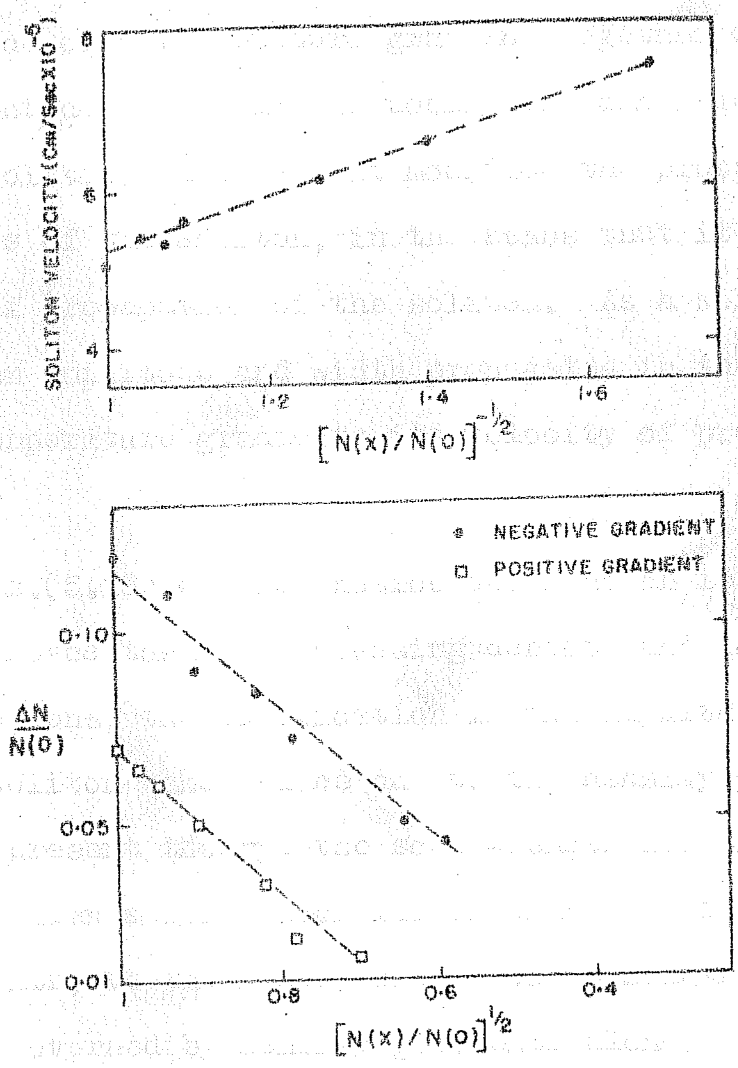


FIGURE 5.1 Variation of soliton amplitude and velocity as a solitary wave propagate along a density gradient is shown (Reproduced with permission from P.I. John and Y.C. Saxena, (1976)).

very closely the variations predicted by the theory.

ii) The presence of a temperature gradient together with a density gradient does not further modify the amplitude and the width of the soliton. However, it modifies the propagation characteristics of the soliton, in the sense that it modifies the velocity of propagation of the soliton. As a solitary wave of a given amplitude and width propagates in the direction of positive temperature gradients its velocity of propagation increases.

From Eqs.(5.18) we also notice that, as an ion acoustic solitary wave moves towards decreasing density and increasing temperature regions, the modification in the amplitude and width of the soliton takes place due to the density gradients only. In the present theory, the scale-length for the temperature gradients being much larger than that for the density gradients, to the lowest order, the strength of dispersion is governed by density gradients alone. This is the reason why the amplitude and the width of the soliton are affected only by the density gradients.

CHAPTER VI

RETURN CURRENT INSTABILITY AND ITS EFFECTS ON BEAM-PLASMA SYSTEM

VI.1 Introduction

In Chapter I, we discussed the importance of relativistic electron beams (REB) with special reference to plasma heating. We also presented a discussion on some of the basic problems connected with the production of such beams wherein, we explained how the return current is produced. The great promise, the relativistic electron beams offer, to heat a plasma to thermonuclear temperature has initiated a large number of plasma heating experiments (Altynstev et al. 1971, Miller and Kuswa 1973, Kribel et al. 1973, Ekdahl et al. 1974, Goldenbaum et al. 1974, Kapetankos and Hammer 1973, Brashitov et al. 1973). In this chapter we shall consider a relativistic beam-plasma-return current system and study certain new collective oscillations introduced due to the very existence of the return current. Moreover, we shall study the effect of the return current on some known collective modes which exist in such a system.

The processes, that can be used to deposit energy in a plasma from an intense REB, can broadly be divided into two classes. First, these beams can be used for heating solid pellets surrounded by shells of heavy material in which beam deposits energy by classical processes. This process is not very efficient and numerical calculations of Rudakov and Samarsky (1973) have shown that, in order to reach thermonuclear conditions, a beam energy of 1 - 3 MJ is required to be delivered in a time less than 10n sec. The most essential requirement of the pellet fusion scheme with REB is that, the beam be focused to the size of the pellet 1-10 mm, with current density reaching a level of $10^8 - 10^9$ A/cm^2. The situation is not very discouraging as the experiments are not lagging too far behind. Vitkovitsky et al. (1973) and Yonas et al. (1973) have achieved current densities $\lesssim 10^7$ A/cm^2, by focusing the beam within the diode. The other method of transferring energy from the REB to the plasma takes advantage of the collective oscillations generated by such beams in a plasma. Of these, basically only two are important - the e-e two stream instability and the return current instability. The efficiency of transfer of energy via these collective modes is expected to be considerably large, as the high power and energy density of the beams produce very large level of collective fluctuations.

A great deal of theoretical work has been done on the study of the e-e two-stream instability (Fainberg et al. 1969, Rudakov 1970, Breizman and Ryutov 1971, Thode and Sudan 1973, Toefer and Poukey 1973). Fainberg et al. (1969) showed that the quasilinear development of these modes produces a spread in the beam transverse momentum which ultimately stabilizes these modes. In fact, high current beams are always associated with a transverse momentum spread, which comes about because of the very nature of production of these beams. Rudakov (1970) pointed out that the spread in the transverse momentum of the beam is associated with a spread in the parallel velocity. If δE and $\langle \theta^2 \rangle$ are the energy spread and mean square angular scatter associated with a given beam distribution, the spread in the parallel velocity, $\Delta v_{\parallel}$ is given by

$$\Delta v_{\parallel}/c \simeq 0.5\langle\theta^2\rangle + \gamma_o^{-3}\delta E$$

where $\gamma_o = (1 - v^2/c^2)^{-1/2}$ is the relativistic factor. When $\Delta v_{\parallel}/c \ll \gamma_o^{-1} (n_b/n_p)^{1/3}$, n_b and n_p being the beam and plasma densities respectively, the beam is said to be cold or monoenergetic. On the other hand, when $0.5 \langle \theta^2 \rangle \gg \gamma_o^{-3} \delta E$ and $\Delta v_{\parallel}/c > \gamma_o^{-1} (n_b/n_p)^{1/3}$, the beam is said to be scattered. The important consequence of the presence of a spread in the parallel velocity of the beam is that, it leads to the stabilization of the unstable 'hydrodynamic' phase of the longitudinal modes. As, the initial assumptions used for the dominant nonlinear mechanism by different authors are widely different, the interaction length calculated by these authors also vary widely.

A number of computer simulation experiments have also been carried out in an attempt to clarify the situation. Computer simulations of Sudan (1973) shows that a strong interaction takes place for a beam with narrow spread in $\vartheta_{\parallel}$. Moreover he finds that the dominant nonlinear effects are the wave saturation by beam trapping and transfer of energy from the primary spectrum to low frequency beat waves via parametric instability or nonlinear Landau damping when the primary spectrum is broadened.

Estimates of plasma heating by the return current instability have also been attempted by some authors (Guillory and Benford 1972, Lovelace and Sudan 1971). The important questions that have to be answered are the rate at which the beam delivers energy to the plasma and the partitioning of the energy between the plasma particles and the waves. In Section VI.3.1C we have considered the quasilinear development of the return current instability and have shown that the rate at which the electrons are heated is larger than the rate at which the ions are heated. Hence, in the initial stages of the development of the instability the electrons will be preferentially heated and when the return current satisfies the condition $V_r > (C_s/2)^{1/2}\ \alpha_{\parallel e}$ (where V_r is the velocity of the return current electrons, C_s is the ion sound speed and $\alpha_{\parallel e}$ is the parallel thermal velocity of the plasma electron), the return current driven ion acoustic instability will set in. The ion acoustic turbulence decays via scattering of ion sound waves by electrons and estimate has been made of the rate at which the return current looses energy as a result of the decay of this turbulence.

VI.2 Dispersion Relation

Let us consider a hot plasma in a uniform magnetic field $\underline{B}_o$ which we take along the z-axis. A relativistic beam of electrons of radius a is streaming with velocity $\underset{\sim}{U}$ through the plasma along the direction of $\underline{B}_o$. The beam density, n_b, is taken to be much smaller than the background plasma density, n_p, through which the beam is moving. We further assume that the plasma is hot but nonrelativistic; this restricts the analysis to temperatures such that $KT << mc^2$ (m being the electron rest mass).

As the beam propagates through the plasma the changing self magnetic field towards the head of the beam induces a back current in the plasma which eventually neutralizes the beam current. The effect of such a return-current can be taken into account in an indirect way through the inclusion of a return velocity V_r, similar to the usual drift velocity, in the equilibrium distribution function of the plasma electrons. This return velocity is related to the beam velocity U through the current neutralization relation, viz $J_p^o + J_b^o = 0$, where J_p^o and J_b^o are the unperturbed plasma and beam current densities, respectively. On assuming that the return current is mainly due to the electrons, the return velocity is simply given by

$$V_r = -(n_b/n_p)\,\underset{\sim}{U} \quad . \tag{6.1}$$

Consequently for the plasma particles we can take the following equilibrium distribution function:

$$f_{oj}^{p} = \frac{1}{\pi^{3/2}\alpha_{\parallel j}\alpha_{\perp j}^2} \exp\left[-v_\perp^2/\alpha_{\perp j}^2 - (v_\parallel - V_{rj})^2/\alpha_{\parallel j}^2\right] \tag{6.2}$$

where $\alpha_{\parallel,\perp j} = (2KT_{\parallel,\perp}/m)_j^{1/2}$ are the parallel (perpendicular) thermal velocities for the j^{th} species. The subscript j labels the plasma species, i.e., j = e for electrons and i for ions. Morevoer $V_{ri} = 0$ and $V_{re} \equiv V_r$, as given by Eq.(6.1). For the beam, we choose a delta-type distribution function, namely

$$f_{oe}^{b} = \frac{1}{\pi p_\perp}\,\delta(p_\perp)\,\delta(p_\parallel - P_o) \quad . \tag{6.3}$$

The superscripts p and b are used to distinguish the plasma and the beam parameters.

For small perturbations, the motion of the charged particles moving with relativistic speeds is governed by the linearized Vlasov equation. Following the procedure outlined by Buti (1963) and Montgomery and Tidman (1964), and on using the full set of Maxwell equations, we arrive at the following dispersion relation

$$|\underset{\approx}{R}| = 0 \quad , \tag{6.4}$$

where

$$\underset{\approx}{R} = (c^2k^2 - \omega^2)\underset{\approx}{I} - c^2\underset{\sim}{k}\underset{\sim}{k} + \underset{\approx}{\sigma} \qquad (6.5)$$

with

$$\underset{\approx}{\sigma} = \mp \sum_{\beta=p,b} \sum_{j} \omega_{\beta j}^2 \frac{i\omega}{\Omega_j} \int d\underset{\sim}{p}\, \underset{\sim}{p} \int^{\varphi}_{\mp\infty} d\varphi' G(\varphi') \cdot \left[\frac{\partial f_{oj}^{\beta}}{\partial \underset{\sim}{p}} + \frac{\underset{\sim}{k} \times \underset{\sim}{p} \times (\partial f_{oj}^{\beta}/\partial p)}{m_j \omega \gamma_j} \right], \qquad (6.6)$$

$$\ln G(\varphi') = \pm \frac{i}{m_j \Omega_j} \left[(k_{\parallel} p_{\parallel} - m_j \omega \gamma_j)(\varphi - \varphi') + k_{\perp} p_{\perp} (\sin\varphi - \sin\varphi') \right]. \qquad (6.7)$$

$\omega_{\beta j} = (4\pi n_j e_j^2/m_j)^{1/2}$ and $\Omega_j = (e_j B_o/m_j c\, \gamma_j)$ are the plasma and cyclotron frequencies of the j^{th} species. The label β appearing in Eq.(6.6) implies that the summation is over the beam as well as the plasma parameters. Moreover, the upper sign in Eq.(6.6) corresponds to the ion and the lower one corresponds to the electrons. The rest of the symbols have their usual meaning and are defined in the above mentioned references (Buti 1963, Montgomery and Tidman 1964).

The various components of $\underset{\approx}{\sigma}$ for the plasma and the beam, characterized by the distributions given by Eq.(6.2) and Eq. (6.3), respectively, are given in Appendix A. From Eqs. (6.4), (6.5) and (A-1) to (A-14) we observe that for a general $\underset{\sim}{k} = (k_{\perp}, 0, k_{\parallel})$, all the components of $\underset{\approx}{R}$ are nonvanishing, and to analyse the dispersion relation in this case is a formidable job. Instead we shall restrict ourselves to the special cases $k_{\perp} = 0$, i.e., parallel propagation or $k_{\parallel} = 0$, i.e., perpendicular propagation only.

VI.3 Parallel Propagation

In this case, on putting $k_\perp = 0$ in Eqs.(6-5)-(6-6) and (A-1) - (A-14), we find that the elements R_{xz}, R_{zx}, R_{yz} and R_{zy} vanish and the dispersion relation reduces to

$$R_{zz}(R_{xx} \pm i R_{xy}) = 0 \quad . \tag{6.8}$$

The mode corresponding to $R_{zz} = 0$ is a purely electrostatic mode, whereas $(R_{xx} \pm i R_{xy}) = 0$ correspond to the right handed and left handed circularly polarized electromagnetic modes, respectively. We shall investigate first the electrostatic mode.

VI.3.1 Electrostatic Mode and Return Current Instability

For $k_\perp = 0$, the mode $R_{zz} = 0$ leads to the following relation:

$$\frac{\omega_{be}^2}{\gamma_0^3(\omega - kU)^2} + \frac{\omega_{pe}^2}{k^2\alpha_{\parallel e}^2} Z'(\mu_e) + \frac{\omega_{pi}^2}{k^2\alpha_{\parallel i}^2} Z'(\mu_i) = 1 , \tag{6.9}$$

where $\mu_e = (\omega - kV_r)/k\alpha_{\parallel e}$, $\mu_i = \omega/k\alpha_{\parallel i}$ and $\gamma_0 = (1 - U^2/C^2)^{-1/2}$. In the absence of return currents i.e., for $V_r = 0$ Eq.(6.9) represents the usual dispersion relation for the electrostatic wave in a cold-beam-hot plasma system. On the other hand if we treat V_r as some sort of relative velocity between the plasma electrons and the ions and neglect the beam term, then Eq.(6.9) reduces to the dispersion relation for the current carrying plasmas (Stringer 1964). Hence Eq.(6.9), as one would have expected, has the characteristics of the beam plasma system or the current carrying plasmas under appropriate conditions. Since for arbitrary values of μ_e and μ_i the above equation cannot be solved analytically, we shall now consider a few cases where it is possible to extract some information regarding the stability of the electrostatic waves.

VI.3.1a. Cold Plasma $(\mu_e, \mu_i \gg 1)$.

Under this approximation, Eq.(6.9) simplifies to

$$F(\omega,k) \equiv \frac{\omega_{be}^2}{\gamma_0^3(\omega-kU)^2} + \frac{\omega_{pe}^2}{(\omega+k|V_r|)^2} + \frac{\omega_{pi}^2}{\omega^2} = 1 \,. \quad (6.10)$$

The instability occurs whenever minimum of function F becomes greater than one i.e., for $F_{min} > 1$. The case $F_{min} = 1$ defines the boundary between the stable and the unstable regions (cf. Fig.6.1).

From the schematic plot of Eq.(6.10) shown in Fig.(6.1), we observe that in the absence of return current the instability can occur for $\omega_r > 0$ which is the usual e-e type instability (Briggs 1964, Nezlin 1971). However, when $V_r \neq 0$ even the frequencies $\omega_r < 0$ can support instability. The latter instability arises because of return currents and is essentially an e-i type instability which can occur in current

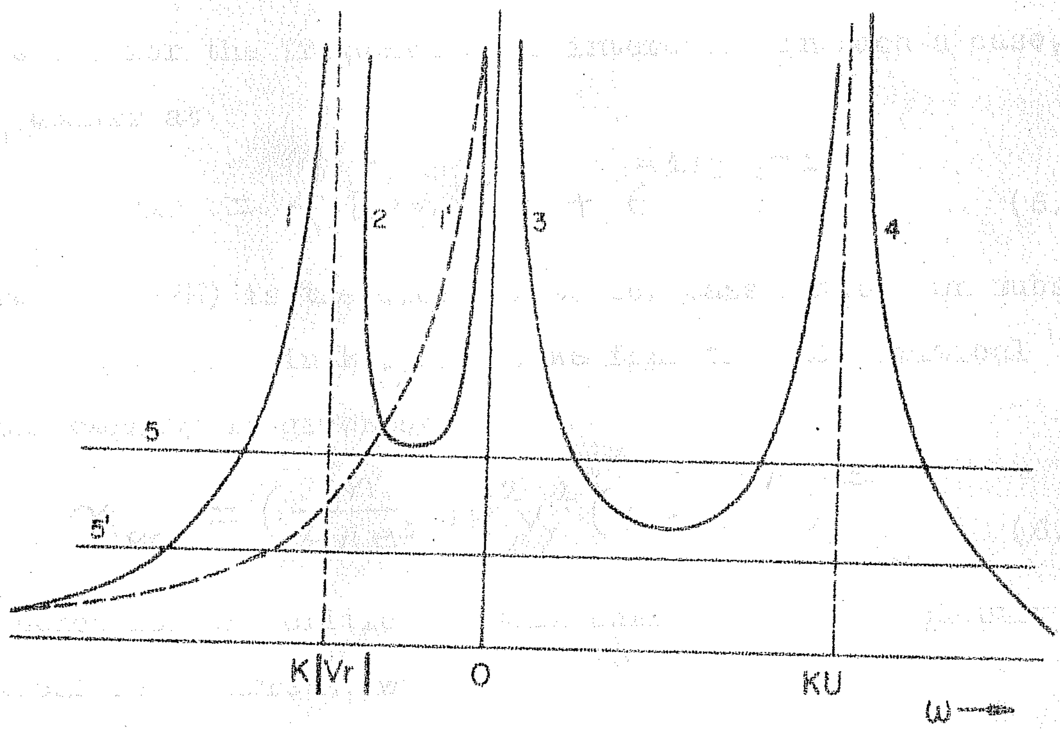


FIGURE 6.1 Schematic plot of Eq. (6.10). Curves 1 - 4 are for F (ω , k) and curves 5 and 5′ are for F (ω, k) = 1. In the absence of return currents the curves 1 and 2 merge together and form the curve 1′. Curve 5 and 2 show instability.

carrying plasmas (Nezlin 1971, Stringer 1964). We shall now discuss the conditions under which the return current instability becomes important.

From Eq.(6.10), we observe that for $\omega < 0$ and for $|\omega| \lesssim k\,|V_r|$, the beam term is approximately $\omega_{be}^2/k^2U^2\gamma_o^3$ which for relativistic velocities is << 1 and hence can be neglected for the frequencies of interest. In such a case, F_{min} occurs at

$$\omega = -k|V_r|\left(1 + \epsilon^{1/3}\right)^{-1}, \tag{6.11}$$

where $\epsilon = (m/M)$ is the electron to ion mass ratio. On substituting Eq.(6.11) in Eq.(6.10), we find that the critical plasma density is given by

$$n_{pcr} = \left(\frac{m}{4\pi e^2}\right)k^2V_r^2\left(1 + \epsilon^{1/3}\right)^{-3} \tag{6.12a}$$

and hence for the critical return current and consequently for critical beam current, we have

$$I_{cr}^b \equiv I_{cr}^R = \pi a^2 n_{pcr} V_r$$
$$= \left(\frac{ma^2}{4e}\right)k^2U^3\left(\frac{n_b}{n_p}\right)^3\left(1 + \epsilon^{1/3}\right)^{-3}. \tag{6.12b}$$

The superscript R stands for return-current.

In writing Eq.(6.12a) we have made use of Eq.(6.1). From Eqs.(6.10) and (6.11) we find that the instability exists only in the range of wave numbers $0 < k < k_{cr}^R$; where

$$k_{cr}^R = \left(\frac{n_p}{n_b}\right)^{3/2}\frac{\omega_{be}}{U}\left(1 + \epsilon^{1/3}\right)^{3/2} \tag{6.12c}$$

is the critical value of k. The maximum growth rate of this instability is given by

$$\gamma^R_{max} \sim 2^{-4/3} 3^{1/2} \epsilon^{1/3} k|V_r| \quad ; \qquad (6.12d)$$

the wave number and the frequency corresponding to the maximally gr.wing wave are given by $k^R_{max} \sim \omega_{pe}/|V_r|$ and $\omega^R_r(max) \sim 2^{-1/3} \epsilon^{1/3} \omega_{pe}$.

For $\omega_r > 0$ and $\omega \sim kU$ we can again start with Eq.(6.10) and find out critical condition for instability (beam plasma e-e instability) from the relation $F_{min} = 1$. The critical beam current necessary to excite this instability is given by

$$I_{cr} = \left(\frac{ma^2}{4e}\right) k^2 U^3 \gamma_0^3 \left[1 + (n_b/n_p)^{1/3} \gamma_0\right]^{-3} (1 + |V_r|/U)^2 . \qquad (6.13a)$$

The range of wave numbers unstable to this instability is $0 < k < k_{cr}$, where

$$k_{cr} = \frac{\omega_{be}}{U \gamma_0^{3/2}} \left[1 + (n_p \gamma_0^3/n_b)^{1/3}\right]^{3/2} (1 + |V_r|/U)^{-1} . \qquad (6.13b)$$

The maximum growth rate of this instability is given by

$$\gamma_{max} \sim 2^{-4/3} 3^{1/2} (n_b/n_p)^{1/3} \gamma_0^{-1} kU (1 + |V_r|/U), \qquad (6.13c)$$

which will occur for frequency and wave number corresponding to $\omega_r(max) \sim 2^{-4/3} \gamma_0^{-1} (n_b/n_p)^{1/3} kU (1 + |V_r|/U)$ and $k_{max} \sim \omega_{pe}/U$ respectively. From Eqs.(6.13a), (6.13b) and (6.13c), it is clear that the effect of return currents on I_{cr}, k_{cr} or γ_{max} is of the order of $|V_r|/U$ and hence negligible.

In order to ascertain the condition under which the return current instability plays a dominant role, let us compare the critical currents, critical wave numbers and growth rates of this instability with the corresponding quantities for e-e instability. From Eqs.(6.12a) - (6.12d) and Eqs.(6.13a) - (6.13c) we find that

$$I^R_{cr}/I_{cr} \sim (n_b/n_p)^2 , \tag{6.14a}$$

$$k^R_{cr}/k_{cr} \sim (n_p/n_b) \tag{6.14b}$$

and $$\gamma^R_{max}/\gamma_{max} \sim [\epsilon \gamma_0^3 (n_b/n_p)]^{1/3} \tag{6.14c}$$

From Eqs.(6.14a) - (6.14c), it is obvious that the return current instability will be important when the beam-plasma e-e instability is absent i.e., either for $I < I_{cr}$ or for wave numbers lying in the range $k_{cr} < k < k^R_{cr}$. The effect of finite temperature on the RC instability is considered in the next section.

VI.3.1b Hot Electron and Cold Ions

Under this approximation, i.e. for $\mu_e \ll 1$ and $\mu_i \gg 1$ the dispersion relation (Eq.(9)) simplifies to

$$\frac{\omega_{pi}^2}{\omega^2} - 2i\pi^{1/2}\frac{\omega_{pe}^2}{k^3\alpha_{\parallel e}^3}(\omega + k|V_r|) = 1 + \frac{2\omega_{pe}^2}{k^2\alpha_{\parallel e}^2} - \frac{\omega_{be}^2}{\gamma_0^3 k^2 U^2} . \tag{6.15}$$

Let us write $\omega = \omega_r + i\gamma$ ($\gamma > 0$ for instability) and apriori assume that $|\gamma^2| \ll |\omega_r^2|$. Then on separating the real and the imaginary parts of Eq.(6.15) and on solving for ω_r and γ, we get

$$\omega_r = \pm (\epsilon/2)^{1/2} k\alpha_{\parallel e} \chi \tag{6.16}$$

and

$$\gamma = (\pi\epsilon/8)^{1/2} k\left(|V_r| - \sqrt{\epsilon/2}\,\alpha_{\parallel e}\chi\right) , \tag{6.17}$$

with

$$\chi = \left[1 + \frac{2k^2\alpha_{\parallel e}^2}{\omega_{pe}^2} - \frac{2n_b}{n_p}\frac{\alpha_{\parallel e}^2}{\gamma_0^3 U^2}\right]^{-1/2} \approx 1 . \tag{6.18}$$

From Eq.(6.17), we notice that γ will be positive only when the return velocity exceeds a critical value i.e., if

$$|V_r| > |V_c| \approx \alpha_{\parallel e}(\epsilon/2)^{1/2} . \tag{6.19}$$

So the critical return current or the beam current is given by

$$I_{cr}^{hot} = \pi a^2 e n_p V_c = \left(\frac{m a^2}{4e}\right)(\epsilon/2)^{1/2}\omega_{pe}^2\alpha_{\parallel e} . \tag{6.20}$$

On comparing Eq.(6.20) with Eq.(6.13a), we find that

$$\frac{I_{cr}^{hot}}{I_{cr}} = \frac{\omega_{pe}^2}{k^2U^2}\left(\frac{n_p}{n_b}\frac{\alpha_{\parallel e}}{U}\right)(\epsilon/2)^{1/2} \approx O(\epsilon^{1/2}) . \tag{6.21}$$

The general dispersion relation (Eq.(6.9)) has been solved numerically; the results are shown in Figs.(6.2) and (6.3). From these figures we observe that the presence of return current forces the damped waves to grow instead. The effects of increasing (n_b/n_p) is to increase the growth rates. As $\alpha_{\parallel e}/\alpha_{\parallel i}$ is increased, the growth rates first increase with $\alpha_{\parallel e}/\alpha_{\parallel i}$ but then get saturated for larger values.

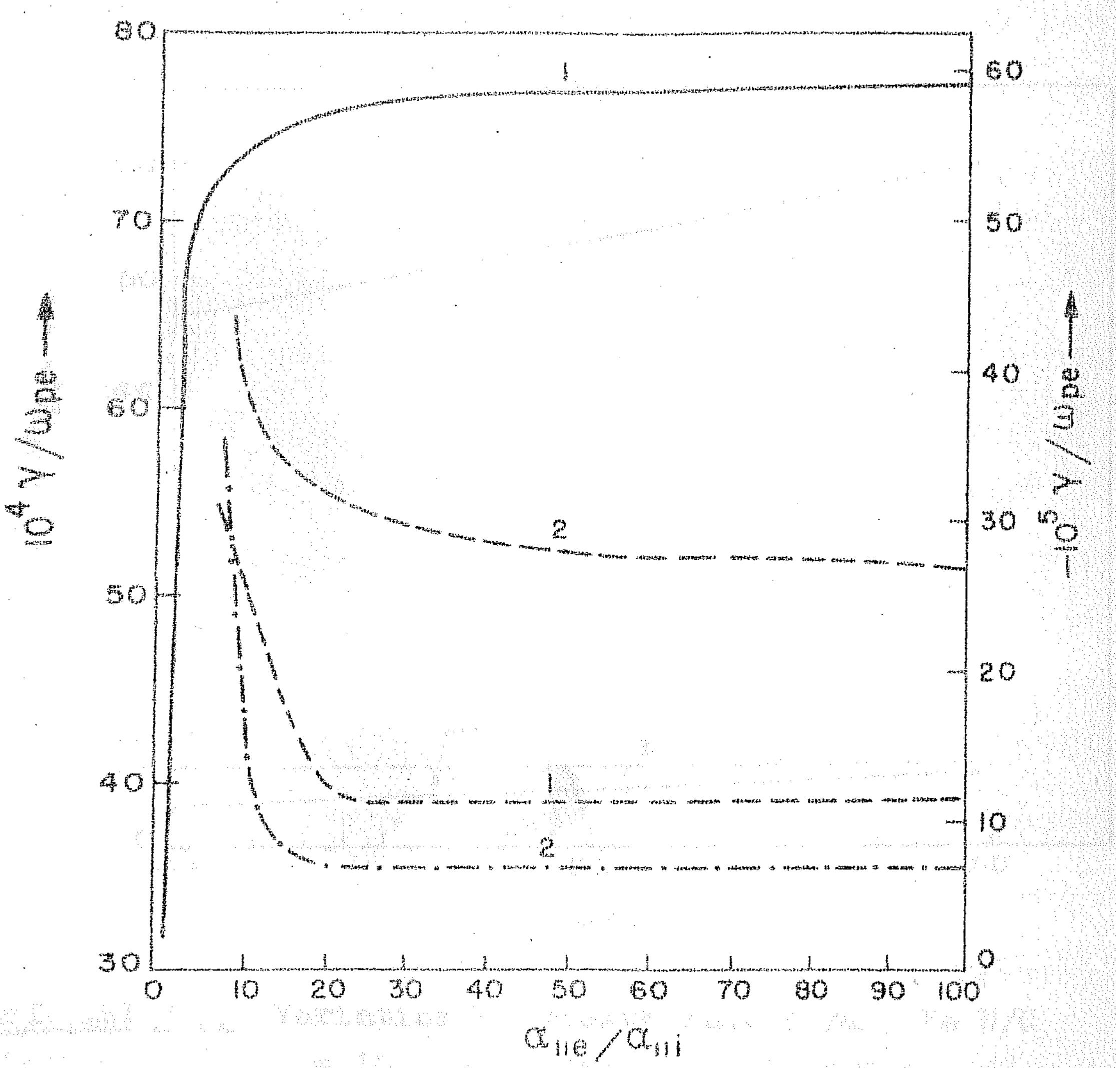


FIGURE 6.2 Variation of growth rate γ / ω_{pe} (——) and real frequency ω_r / ω_{pe} (– – –) in the presence (curve 1) and absence (curve 2) of return currents versus $\alpha_{\parallel e}/\alpha_{\parallel i}$ for $n_b/n_p = 0.05$, $U/C = 0.95$, $\omega_{pe}/k\alpha_{\parallel e} = 10$, and for $\alpha_{\parallel e}/C = 10^{-2}$. The broken curve 2 (–.–.–) denotes damping and follows the scale given on the right hand side of the figure.

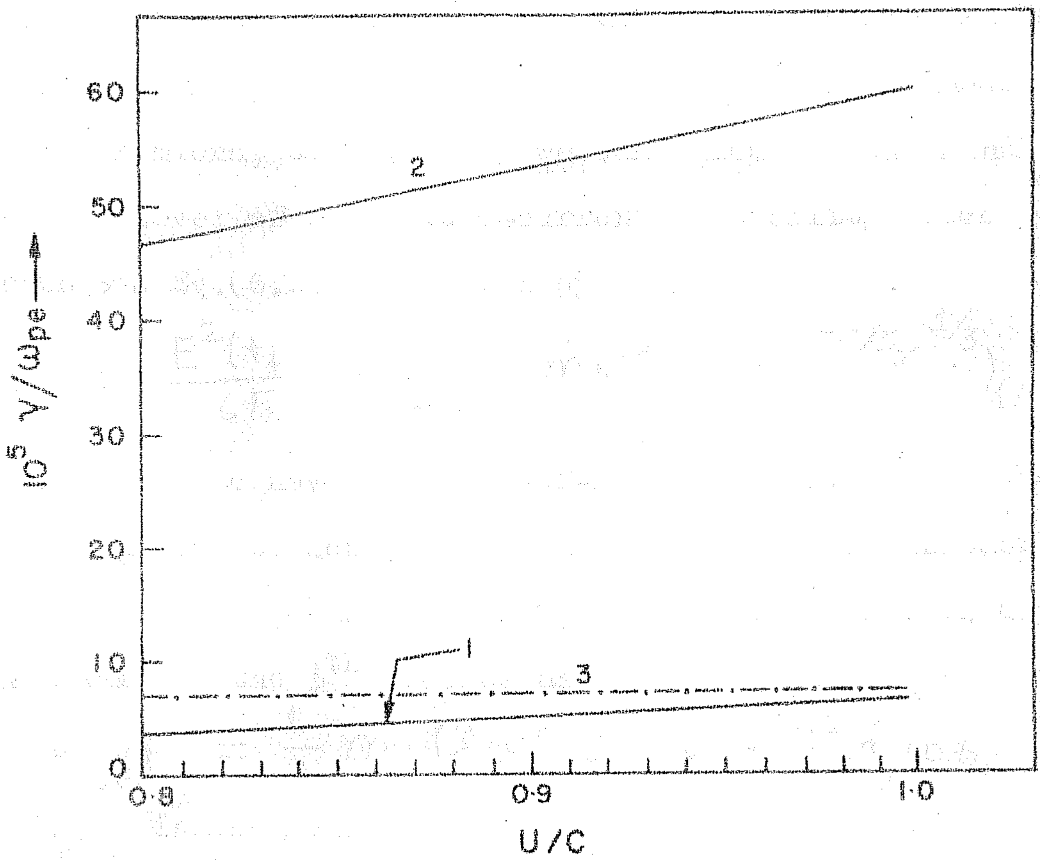


FIGURE 6.3 Variation of growth rate γ/ω_p Vs U/C for $\alpha_{\parallel e}/\alpha_{\parallel i} = 100$, $\omega_{pe}/k\alpha_{\parallel e} = 10$, and $\alpha_{\parallel e}/C = 0.1$ and for $n_b/n_p = 0.01$ and 0.05 for the curves 1 and 2 respectively. The broken curve 3, which is for the case when return currents are neglected but the other parameters are same as above, denotes damping. The damping remains unaffected by the increase of n_b/n_p. The value of the real frequency is $\omega_r/\omega_{pe} = -5.17 \times 10^{-3}$ for all the curves.

VI.3.1c Estimates of Plasma Heating

It is well known that growth of the beam plasma instability is limited by the trapping of the beam electrons in the potential of the growing wave. Following the simple model given by Drummond, et al.(1970) we can calculate the final saturation level of the return-current instability which according to Eq.(6.12d) is given by

$$W_R = \frac{E^2(t_f)}{16\pi} = \frac{1}{2} n_p m V_r^2 \left(1 - 2^{-4/3} \epsilon^{1/3}\right) \tag{6.22}$$

In order to have an order-of-magnitude estimate of how much heating can be achieved by the return current instability, we define an average kinetic energy per particle relatively to the mean for the j^{th} species as

$$K_j(t) = \frac{1}{2} m_j \int dv \left[v - V_j(t)\right]^2 f_j(v,t),$$

where $V_j(t)$ is the mean velocity defined by

$$V_j(t) = \int dv\, v\, f_j(v,t)$$

and $f_j(v,t)$ is the distribution function for the j^{th} species. The quasilinear evolution of the distribution function is governed by the equation (Drummond and Pines 1962, Vedenov et al. 1961)

$$\frac{\partial}{\partial t} f_j(v,t) = \frac{\partial}{\partial v}\left[D_j(v,t) \frac{\partial}{\partial v} f_j(v,t)\right], \tag{6.23}$$

where the diffusion co-efficient is defined by

$$D_j(v,t) = \frac{8\pi e^2}{m_j^2} \int dk\, \mathcal{E}_k(t) \frac{\gamma_k}{(\omega_k - kv)^2 + \gamma_k^2} \tag{6.24}$$

In Eq.(24) $\epsilon_k(t)$ is the spectral energy density of the wave and γ_k is the linear growth rate corresponding to the k^{th} mode. Taking appropriate velocity moment of Eq.(6.23), it can be easily shown that

$$n_j \frac{d}{dt} K_j(t) = \omega_{pj}^2 \int dk \, 2\gamma_k \mathcal{E}_k(t) \int dv f_j(v,t) \cdot \frac{[(\omega_k - kV_j)^2 + \gamma_k^2 - k^2(v - V_j)^2]}{[(\omega_k - kv)^2 + \gamma_k^2]^2} \quad (6.25)$$

In obtaining Eq.(6.25), Eq.(6.24) has been used. From an order-of-magnitude estimate of the integrand in Eq.(6.25), it can be shown (Davidson 1972a) that the electron and the ion heating rates are approximately given by,

$$n_e \frac{d}{dt} K_e(t) \simeq \frac{d}{dt} \mathcal{E}_f(t) \quad (6.26)$$

and

$$n_i \frac{d}{dt} K_i(t) \simeq \epsilon^{1/3} \frac{d}{dt} \mathcal{E}_f(t) , \quad (6.27)$$

where $E_f(t) = \int dk E_k(t)$, is the total field energy density in the unstable modes. From Eq.(6.26) and (6.27) it is clear that it is the electrons that get preferentially heated. This fact is important in the sense that after the initial development of the return-current instability, the electrons will be sufficiently heated so as to make ion-acoustic instability to take over.

Another important conclusion that can be drawn from the order-of-magnitude estimate of the integrand of Eq.(6.25) is that K_j will continue to increase until (Davidson 1972a)

$$\frac{K_j}{m_j} \simeq (\omega_k/k - V_j)^2 + (\gamma_k^R/k)^2 . \quad (6.28)$$

Thus the maximum temperature of the electrons that can be achieved by the return current instability is given by

$$T_e \simeq \frac{1}{2} m \left[(\omega_{max}/k_{max} - V_r)^2 + (\gamma^R_{max}/k_{max})^2 \right] \quad (6.29)$$

For $n_p \sim 10^{14} cm^{-3}$, $n_b/n_p \sim 0.05$ and $U/C \sim 0.99$, which corresponds to an intense beam with peak current density ~ 20 KA/cm^2, the maximum temperatures to which the electrons will be heated by the return-current instability is $\sim$ 5Kev. This is in good agreement with the estimate made by Guillory and Benford (1972).

As was mentioned earlier, the electrons will be preferentially heated in the early stages of the development of the return-current instability. When the electrons are sufficiently heated so that $T_e >> T_i$ and the return current satisfies the condition given by Eq.(6.19), the return-current induced ion acoustic instability sets in. The turbulence generated by this instability will further heat the plasma.

We shall make a rough estimate of the electron heating on the assumption that the level of turbulence (i.e., the energy in the growing wave) is limited solely due to scattering of the ion sound wave by the electrons. Following Sisonenko and Stepanov (1969) and Krall and Book (1969), we can immediately show that the result of the above mentioned process is to limit the final level of turbulent energy W, to a value given by

$$\frac{W}{n_p T_e} \sim \frac{m}{M} \frac{V_r}{C_s} , \quad (6.30)$$

where $C_s = \sqrt{T_e/M}$ is the ion sound speed.

In order to make an estimate of the electron heating, we shall first calculate the effective collision frequency ν_{eff}, for the electrons. In the resonant region of velocity space, the space-averaged distribution function for the electrons, $f_e(v,t)$ evolves according to

$$\frac{\partial f_e(\underset{\sim}{v},t)}{\partial t} = \frac{e^2}{m^2}\pi \frac{\partial}{\partial \underset{\sim}{v}} \cdot \int \frac{d^3k}{(2\pi)^3} \underset{\sim}{k}\underset{\sim}{k} |E_{\underset{\sim}{k}}|^2 \delta(\omega_{\underset{\sim}{k}} - \underset{\sim}{k}\cdot\underset{\sim}{v}) \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_e(\underset{\sim}{v},t) \tag{6.31}$$

On taking the first moment of Eq.(6.31) we obtain,

$$\frac{d}{dt}(n_e v_e) = -n_p V_e \nu_{eff} = -\frac{\pi e^2}{m^2}\int d^3\underset{\sim}{v} \int \frac{d^3k}{(2\pi)^3} \underset{\sim}{k}\underset{\sim}{k} |E_k|^2 \delta(\omega_{\underset{\sim}{k}} - \underset{\sim}{k}\cdot\underset{\sim}{v}) \frac{\partial}{\partial \underset{\sim}{v}} f_e(v,t)$$

$$= -\int \frac{d^3k}{(2\pi)^3} \underset{\sim}{k} \frac{|E_k|^2}{4\pi m} 2\gamma_{\underset{\sim}{k}} \frac{\omega_{pi}^2}{\omega_{\underset{\sim}{k}}^2} . \tag{6.32}$$

On substituting Eqs.(6.16) and (6.17) in Eq.(6.32) we get,

$$\nu_{eff} = \frac{2\pi^{1/2}\omega_{pi}^2}{4\pi m n_p \epsilon} \cdot \frac{|E_k|^2}{\langle k \rangle \alpha_{\parallel e}^3} \left[1 - \frac{c_s \langle x \rangle}{|V_r|} \right] . \tag{6.33}$$

On treating these 'effective collisions' as an isotropic Joule-heating mechanism, we can obtain the rate of heating of the electrons as

$$\frac{3}{2} n_p \frac{dT_e}{dt} \approx \eta J_p^2 , \tag{6.34}$$

where η is the resistivity defined by $\eta = 4\pi\nu_{eff}/\omega_{pe}^2$.

Eqs.(6.33) and (6.34) immediately give

$$\frac{dT_e}{dt} \approx \frac{2^{5/2}\pi^{1/2}}{3\lambda_D \langle k \rangle} \left(\frac{W}{nT_e} \right) \omega_{pe} \left(\frac{1}{2} m V_r^2 \right) \left[1 - \frac{c_s \langle x \rangle}{|V_r|} \right] ,$$

which can be rewritten as

$$\frac{d}{dt}(n_p T_e) \approx A\left(\frac{1}{2} m\, n_p |V_r|^2\right), \qquad (6.35)$$

where

$$A = \frac{2^{5/2}\pi^{1/2}}{3\lambda_D \langle k \rangle}\left(\frac{m}{M}\frac{|V_r|}{C_s}\right)\omega_{pe}\left[1 - \frac{C_s\langle \chi \rangle}{|V_r|}\right]. \qquad (6.36)$$

Eq.(6.35) tells us that the rate which the return-current is delivering energy to the plasma electrons is given by the quantity A. Since $\chi \sim 1$ and for the unstable modes under consideration $C_s < V_r$, the time scale of delivery of energy can be ω_{pi}^{-1} when $V_r \sim (M/m)^{1/2} C_s$ and $\langle k \rangle \lambda_D \sim 1$.

VI.3.2 Electromagnetic Modes

As mentioned earlier, two e.m. modes, of right handed and left handed circular polarization, can propagate along the direction of the magnetic field. Here we shall discuss only the right handed mode which is given by $R_{xx} + iR_{xy} = 0$. A similar treatment can be applied to the left handed mode in a straight forward manner. On using the results of Appendix A in Eq.(6.5), the dispersion relation for the right handed mode can be written as

$$-\omega^2 + c^2k^2 + \frac{\omega_{be}^2(\omega - kU)}{\gamma_0(\omega - kU + \Omega_e^r)} + \sum_j \omega_{pj}^2\left[1 + \frac{\Omega_j}{k\alpha_{\parallel j}} Z(\eta_j) + \frac{\alpha_{\perp j}^2}{2\alpha_{\parallel j}^2} Z'(\eta_j)\right] = 0, \qquad (6.37)$$

where $\Omega_e^r = \Omega_e/\gamma_0$, $\eta_e = (\omega - \Omega_e - kV_r)/k\alpha_{\parallel e}$ and $\eta_i = (\omega + \Omega_i)/k\alpha_{\parallel i}$.

For frequencies, $\omega \ll \Omega_i$, on solving Eq.(6.37), we find that the return current effects are negligible. We shall, therefore, leave out this uninteresting case and discuss here the frequency range, $\Omega_i \ll \omega \lesssim \Omega_e$, where the return current effects are expected to be important. Let us consider the following case:

VI.3.2a $\eta_e \ll 1$ and $\eta_i \sim \omega / k\alpha_{\parallel i} \gg 1$

Under these restrictions Eq.(6.37) reduces to

$$-\omega^2 + c^2k^2 + \frac{\omega_{be}^2}{\gamma_0} - (\Gamma-1)\omega_{pe}^2 + 2\omega_{pe}^2 X(\omega - \Omega_e - kV_r)/k\alpha_{\parallel e}$$
$$- i\pi^{1/2} X \omega_{pe}^2 + \omega_{pi}^2 = 0 \quad , \qquad (6.38)$$

where $\Gamma = \alpha_{\perp e}^2/\alpha_{\parallel e}^2$ and

$$X = [\Gamma(\omega - kV_r) - (\Gamma-1)\Omega_e]/k\alpha_{\parallel e} \quad . \qquad (6.39)$$

On separating Eq.(6.38) into real and imaginary parts by writing, $\omega = \omega_r + i\gamma$, and assuming that $|\gamma| \ll |\omega_r|$, we obtain

$$\omega_r = \left[(\Gamma-1) - \frac{2\gamma}{\pi^{1/2}k\alpha_{\parallel e}}\right]\frac{\Omega_e}{\Gamma} - k|V_r| \qquad (6.40)$$

and

$$\gamma \sim \frac{k\alpha_{\parallel e}}{\pi^{1/2}\Gamma}\left[(\Gamma-1)\left(1 - \frac{4\Omega_e^2}{\pi k^2\alpha_{\perp e}^2\Gamma}\right) + \left(\frac{\Gamma-1}{\Gamma}\right)^2\frac{\Omega_e^2}{\omega_{pe}^2}\left(1 - \frac{4}{\pi\Gamma}\right)\right.$$
$$\left. - \frac{2(\Gamma-1)k|V_r|\Omega_e^2}{\Gamma\Omega_e\omega_{pe}^2}\left(1 - \frac{2}{\pi\Gamma}\right) - \frac{c^2k^2}{\omega_{pe}^2}\right] . \qquad (6.41)$$

We may point out that the assumptions $\omega \lesssim \Omega_e$ and

$\eta_e = (\omega - \Omega_e + k|V_r|)/k\alpha_{\parallel e} \ll 1$ imply that $|V_r| \ll \alpha_{\parallel e}$. Therefore, Eq.(6.41) simply shows the tendency of these modes towards stabilization due to the presence of the return currents. In fact, the influence of these currents is expected to become appreciable when $|V_r| \gtrsim \alpha_{\parallel e}$, i.e., for $\eta_e \sim 1$, a case which is difficult to handle analytically. However, if we consider the case when

$$\eta_e = (\omega - \Omega_e + k|V_r|)/k\alpha_{\parallel e} \approx \frac{|V_r|}{\alpha_{\parallel e}} \gg 1$$

and $\eta_i \gg 1$ as before; from Eq.(6.38) we obtain the following dispersion relation:

$$\omega^2 = c^2k^2 + \frac{\omega_{be}^2}{\gamma_o} + \omega_{pe}^2\left(1 + \frac{\Omega_e}{k|V_r|} + \frac{\alpha_{\perp e}^2}{|V_r|^2}\right) + \omega_{pi}^2\left(1 + \frac{k^2\alpha_{\perp i}^2}{\omega^2}\right), \tag{6.42}$$

which shows no instability. Hence we conclude that the influence of the return currents on this mode is to suppress the growth rates.

VI.4 Transverse Propagation ($k_{\parallel} = 0$)

In this case we have $\underset{\sim}{k} = k\,\hat{e}_x$ and on putting $k_{\parallel} = 0$ in Eqs.(6.5), (6.6) and (A-1) – (A-14) we notice that all the elements of $\underset{\approx}{R}$ are nonvanishing and involve infinite summation over Bessel functions $I_n(\lambda)$. For any arbitrary value of λ, it is impossible to obtain some analytical results. We shall, therefore restrict ourselves to the case where $\lambda = k^2\alpha_{\perp e}^2/\Omega^2 \to \infty$, i.e. no magnetic field case and when $\lambda \ll 1$ i.e., a strong magnetic field case.

VI.4.1 Zero Magnetic Field ($B_o = 0$)

In the limit of $\Omega \to 0$ the various elements of $\underset{\approx}{R}$ are given by

$$R_{xx} = -\omega^2 + \frac{\omega_{be}^2}{\gamma_o} - \omega^2 \sum_j \frac{2\omega_{pj}^2}{k^2 \alpha_{\perp j}^2} \left[1 + \xi_j Z(\xi_j)\right], \tag{6.43}$$

$$R_{yy} = -\omega^2 + c^2k^2 + \frac{\omega_{be}^2}{\gamma_o} - \sum_j \omega_{pj}^2 \xi_j Z(\xi_j) , \tag{6.44}$$

$$R_{xz} = R_{zx} = \frac{kU\omega_{be}^2}{\gamma_o \omega} - \omega \sum_j \frac{2\omega_{pj}^2 V_r}{k\alpha_{\perp j}^2} \left[1 + \xi_j Z(\xi_j)\right], \tag{6.45}$$

$$R_{zz} = -\omega^2 + c^2k^2 + (k^2U^2\omega_{be}^2/\gamma_o\omega) + \sum_j \omega_{pj}^2 \left\{1 + \frac{\alpha_{\parallel j}^2}{\alpha_{\perp j}^2}\left(1 + \frac{2V_r^2}{\alpha_{\parallel j}^2}\right)\left[1 + \xi_j Z(\xi_j)\right]\right\} \tag{6.46}$$

and

$$R_{xy} = R_{yx} = R_{yz} = R_{zy} = 0$$

with $\xi_j = \omega / k\alpha_{\perp j}$.

The dispersion relation in this case is simply given by

$$R_{yy}\left(R_{xx}R_{zz} - R_{xz}^2\right) = 0 . \tag{6.47}$$

The mode $R_{yy} = 0$ is independent of V_r and hence we will not discuss it here. The other mode, $R_{xx}\, R_{zz} - R_{xz}^2 = 0$, is affected by the streaming velocity and we will study this in detail. The dispersion relation for this mode, on using Eqs. (6.43) - (6.46) and on neglecting ion contribution, can be written (after dropping the subscript e which is now not needed) as

$$G_1\omega^4-\left(\left\{c^2k^2+\omega_p^2-\frac{\alpha_\parallel^2}{\alpha_\perp^2}\omega_p^2[1+\xi_j Z(\xi_j)]\right\}G_1-2\omega_p^2\frac{V_r^2}{\alpha_\perp^2}\cdot[1+\xi_j Z(\xi_j)]\right)\omega^2-\frac{\omega_b^2\omega_p^2}{\gamma_0}\left\{\frac{2\tilde{U}^2}{\alpha_\perp^2}[1+\xi_j Z(\xi_j)]-1\right\}=0\,, \tag{6.48}$$

where

$$G_1=1+2\frac{\omega_p^2}{k^2\alpha_\perp^2}[1+\xi_j Z(\xi_j)]\,, \tag{6.49}$$

$$\tilde{U}^2=(U+|V_r|)^2+\alpha_\parallel^2/2\,. \tag{6.50}$$

For the case, $\xi \gg 1$ and $\omega_p^2/\omega^2 \gg 1$ Eq.(6.48) reduces to

$$\omega^2=\omega_p^2+c^2k^2\left(1+\frac{\alpha_\parallel^2+2V_r^2}{2c^2}\right)+\frac{\omega_p^2k^2}{\omega^2}\left[\left(1+\frac{3}{2}\frac{k^2\alpha_\perp^2}{\omega^2}\right)\frac{\alpha_\parallel^2}{2}+\frac{n_b\tilde{U}^2}{n_p\gamma_0}\right] \tag{6.51}$$

As $k^2\alpha_\perp^2/\omega^2 \ll 1$, we can solve Eq.(6.51) by an iterative procedure. To zeroth order Eq.(6.51) reduces to a bi-quadratic equation which yields the solution

$$\omega_0^2\simeq\frac{k^2[\alpha_\parallel^2/2+(n_b/n_p\gamma_0)\tilde{U}^2]}{1+(c^2k^2/\omega_p^2)[1-(\alpha_\parallel^2+2V_r^2)/2c^2]} \tag{6.52}$$

The other root of the bi-quadratic equation is not consistent with our assumption $\omega_p^2 \gg \omega^2$.

On using Eq.(6.52), the first order solution of Eq.(6.51) is simply given by

$$\omega^2=-\frac{k^2\{[1+\frac{3}{2}(k^2\alpha_\perp^2/\omega_0^2)](\alpha_\parallel^2/2)+(n_b/n_p\gamma_0)\tilde{U}^2\}}{1+(c^2k^2/\omega_p^2)[1-(\alpha_\parallel^2+2V_r^2)/2c^2]} \tag{6.53}$$

The effect of return currents, as seen from Eq.(6.53), is to increase the growth rate slightly.

We have solved the dispersion relation (Eq.(6.48)) numerically and the restts are given in Table 6.1. From this table, we observe at the return currents do affect the growth rates but very sghtly and that the effect of increasing $\alpha_{\perp}/\alpha_{\parallel}$ is to stabil the system; this is in agreement with Eq.(6.53). From Te 6.1 we also conclude that for low beam velocity the growtate increases with U, attains a maximum value and then reases as $U \to C$. This decrease of growth rate when $U \to C$ due to the increase of relativistic mass, $m = m_o/(1 - U^2/C^2)^{\frac{1}{2}}$. This is in agreement with the results of Lakhina and (1972).

VI.4.2 Strong Magnetic Field ($k^2\ \alpha_{\perp}^2/\Omega^2 << 1$)

Under the approxion, $\lambda << 1$ the elements of $\underset{\approx}{R}$ get simplified but all are anishing (see Appendix B) and we have to use the full diion relation

$$R_{xx}(R_{yy}R_{zz}+R_{yz}^2) - R_{xy}(R_{xy}R_{zz}+R_{yz}R_{xz}) + R_{xz}(R_{yz} - R_{yy}R_{xz}) = 0, \quad (6.54)$$

where R_{ij} are as given iendix B. However, if we assume $\omega^2/\Omega_e^2 \approx 1/\gamma_o << 1$, tms R_{xy} R_{yz} R_{xz} and R_{yy} R_{xz}^2 become much smaller than $_{zz}$ and hence can be neglected. When we do this, the disn relation (6.54) reduces to (neglecting ion contribund dropping the subscript once again)

U/C \ $\alpha_\perp/\alpha_\parallel$	1		1.0	
	$\delta = 1$	$\delta = 0$	$\delta = 1$	$\delta = 0$
Growth res γ/ω_{pe} (in units of 10^{-3})				
.4	24.04	23.99	15.11	14.99
.5	25.36	25.29	17.73	17.57
.6	26.65	26.55	19.96	19.79
.7	27.71	27.59	21.66	21.48
.8	28.28	28.16	22.54	22.35
.9	27.78	27.66	21.80	21.62
.91	27.61	27.50	21.56	21.37
.92	27.42	27.31	21.26	21.08
.93	27.18	27.07	20.89	20.72
.94	26.90	26.80	20.44	20.28
.95	26.56	26.46	19.89	19.73
.96	26.14	26.04	19.19	19.04
.97	25.61	25.52	18.28	18.13
.98	24.91	24.84	17.00	16.86
.99	23.93	23.83	14.94	14.82
.995	23.10	23.06	13.08	12.98

TABLE 6.1: Variatiorowth rate of the transverse e.m waves in the absence gnetic field for $C^2k^2/\omega_{pe}^2 = 0.1$, and $n_b/n_p = 0.01$. Tameter δ decides whether values correspond to the cas'no return current' ($\delta = 0$) or 'with return currents= 1) respectively. The waves are purely growing.

$$\omega^6 - \omega^4 c^2 k^2 \left[\left(1 + \frac{\omega_p^2}{\Omega^2}\right)(\tau^2 - \Phi) + 1 + \frac{(\tau^2 - 1)\,\omega_p^2/\Omega^2}{(1 + \omega_p^2/\Omega^2)} \right]$$
$$+ \omega^2 c^2 k^2 \left[c^2 k^2 (\tau^2 - \Phi) - \frac{\omega_b^2 U^2 (1 + \omega_p^2/\Omega^2)}{\gamma_0 c^2} - \frac{V_r^2 \omega_p^4}{c^2 \Omega^2} + \frac{(\tau^2 - \Phi)\,\omega_p^4/\Omega^2}{(1 + \omega_p^2/\Omega^2)} \right] + \left[1 + \frac{(\tau^2 - 1)\,\omega_p^2/\Omega^2}{(1 + \omega_p^2/\Omega^2)} \right] \frac{c^2 U^2 k^4 \omega_b^2}{\gamma_0} = 0 ,$$

(6.55)

where

$$\tau^2 = (1 + \omega_p^2/c^2k^2) \text{ and } \Phi = \frac{1}{2} \frac{\alpha_\parallel^2 \omega_p^2}{c^2 \Omega^2} \left(1 + 2 \frac{V_r^2}{\alpha_\parallel^2}\right). \qquad (6.56)$$

Since Eq.(6.55) is a cubic equation in ω^2 with coefficient of constant term positive, it will always give one negative root for ω^2. However, for $\omega_p^2/\Omega^2 \gtrsim 1$ we find that ω^6 term in Eq.(6.55) is $O(\omega^2/\Omega^2$ or $\omega^2/\omega_p^2)$ as compared to other terms. So, neglecting this term (which amounts to suppressing the root $\omega \sim \Omega$ or ω_p which, in any case is an invalid root) in Eq.(6.55), the resulting quadratic equation in ω^2, for the growth rate, yields

$$\gamma^2 \equiv -\omega^2 \approx \frac{(n_b/n_p\gamma_0)\, k^2 U^2}{1 + \dfrac{c^2k^2}{\omega_p^2} - \dfrac{k^2(\alpha_\parallel^2 + 2V_r^2)}{2\Omega^2} - \dfrac{V_r^2 \omega_p^2}{c^2\Omega^2}\left[1 + \dfrac{(\tau^2 - 1)\,\omega_p^2/\Omega^2}{1 + \omega_p^2/\Omega^2}\right]^{-1}}$$

(6.57)

From Eq.(6.57), we can immediately conclude that the growth rates will increase with V_r.

Once again we have solved the general dispersion relation (Eq.(6.54)) numerically. Some of the results obtained are shown in Table 6.2, which shows that the effect of return current is important when $\omega \sim \Omega_e$. For ω deviating too much from Ω_e, the effect of return current on the growth rate is very small.

A brief comment on the heating produced by these transverse e.m instabilities is in order here. Computer simulation experiments (Davidson et al., 1971, Davidson et al. 1972b) have shown that the bulk response of the plasma, such as heating, is in very good agreement with the predictions of space averaged quasilinear theory in the initial stages of the instability. The computer simulation experiments also show that the magnetic field fluctuations get saturated via magnetic trapping governed by the equation

$$\gamma_k \simeq \omega_B \tag{5.58}$$

where $\omega_B = \left| \frac{e k V_\perp B_k}{m c} \right|^{1/2}$ (B_k being the magnetic field amplitude and $V_\perp$ is the characteristic particle velocity perpendicular to the direction of propagation) is the bounce frequency of the electrons in the potential of the magnetic fluctuations. A rough estimate of rate of heating due to such an instability when $\omega \sim \Omega_e \sim \omega_{pe}$ (the region, where the effect of return currents according to Table 6.2 is most significant) was made by using quasilinear equations governing the rate of change of kinetic energy (Davidson et al. 1972b). It is found that the rate of heating achieved due to

Ω/ω_p → / U/C ↓	1.2				1.6				5.0			
	γ/ω_p (10^{-6})		ω_r/ω_p		γ/ω_p (10^{-5})		ω_r/ω_p		γ/ω_p (10^{-4})		ω_r/ω_p	
	$\delta = 1$	$\delta = 0$	$\delta = 1$	$\delta = 0$	$\delta = 1$	$\delta = 0$	$\delta = 1$	$\delta = 0$	$\delta = 1$	$\delta = 0$	$\delta = 1$	$\delta = 0$
.92	1.19	.87	1.08	1.07	4.02	4.03	1.07	1.07	3.27	3.27	1.03	1.03
.93	1.36	.99	1.08	1.07	4.30	4.31	1.07	1.07	3.28	3.28	1.03	1.03
.94	1.52	1.12	1.07	1.07	4.55	4.56	1.07	1.06	3.28	3.28	1.03	1.03
.95	1.69	1.26	1.07	1.06	4.78	4.79	1.06	1.06	3.27	3.27	1.03	1.03
.96	1.87	1.42	1.07	1.06	4.99	5.00	1.06	1.06	3.24	3.24	1.02	1.02
.97	2.07	1.58	1.07	1.06	5.19	5.19	1.06	1.06	2.99	2.99	1.02	1.02
.98	2.29	1.77	1.06	1.06	5.38	5.38	1.05	1.05	2.77	2.73	.95	.95
.99	2.55	2.00	1.06	1.06	5.57	5.57	1.05	1.05	2.90	2.90	1.04	1.04
.995	2.72	2.15	1.06	1.05	5.68	5.68	1.05	1.05	3.08	3.08	1.04	1.04
.999	2.93	2.35	1.06	1.05	5.80	5.80	1.05	1.05	3.16	3.15	1.03	1.03

TABLE 6.2: Variation of growth rate and real frequency in the presence of a magnetic field for $C^2k^2/\omega_{pe}^2 = 0.1$, $n_b/n_p = 0.1$, and $\alpha_{\parallel}/C = 10^{-2}$. The parameter δ distinguishes the 'no return currents' ($\delta = 0$) case from the 'with return current' ($\delta = 1$) case.

such an electromagnetic instability is extremely small compared to the rates obtained due to e.s. instabilities (cf. section VI.3.1c).

VI.5 Conclusion

The return currents, arising because of the motion of a relativistic beam of electrons through a nonrelativistic plasma affect the stability of the waves excited by the beam-plasma interaction in a number of ways. These currents destabilize the electrostatic waves by exciting a return current instability which requires smaller beam currents than the one required for exciting the usual beam plasma e-e instability. The electro -magnetic waves propagating along the direction of the magnetic field are stabilized by these currents. The growth rates of e.m waves propagating in the transverse direction are, however slightly increased by the presence of these currents. The return current instability can heat the plasma to Kev temperatures. Moreover, when the ion-sound turbulence generated by the return current decays via scattering of ion-sound waves by electrons the return current delivers energy to the plasma at the rate of $\sim (m/M)^{1/2} (V_r/C_s) \omega_{pi}$.

In this study the return velocities were taken to be nonrelativistic, as the analysis was restricted to beam densities much smaller than the plasma density. It would be of interest to extend the analysis for relativistic return velocities and also for the case where the plasma temperatures are high i.e. $KT \sim m c^2$.

APPENDIX A

The elements of $\underset{\approx}{\sigma}$ defined by Eq.(6.6) can be evaluated by doing the integrations as indicated by Montgomery and Tidman (1964) and Bernstein (1958). For the case of plasma characterized by the distribution function as given by Eq. (6.2) the various σ_{ij} elements turn out to have the following form:

$$\sigma_{xx}^{P} = \sum_j \omega_{pj}^2 \sum_{n=-\infty}^{\infty} \frac{n^2 I_n(\lambda_j)\, e^{-\lambda_j}}{\lambda_j} \left[1 - \frac{n\Omega_j}{k_{\parallel}\alpha_{\parallel j}} Z(\mu_{nj}) + \frac{\alpha_{\perp j}^2}{2\alpha_{\parallel j}^2} Z'(\mu_{nj}) \right], \tag{A-1}$$

$$\sigma_{xy}^{P} = -\sigma_{yx}^{P} = \sum_j \omega_{pj}^2 \sum_{n=-\infty}^{\infty} \frac{i n F_1(\lambda_j)}{\lambda_j} \left[1 - \frac{n\Omega_j}{k_{\parallel}\alpha_{\parallel j}} Z(\mu_{nj}) + \frac{\alpha_{\perp j}^2}{2\alpha_{\parallel j}^2} Z'(\mu_{nj}) \right], \tag{A-2}$$

$$\sigma_{xz}^{P} = \sigma_{zx}^{P} = \sum_j \omega_{pj}^2 \sum_{n=-\infty}^{\infty} \frac{n^2 I_n(\lambda_j)\, e^{-\lambda_j}}{2\lambda_j} \frac{k_{\perp}}{k_{\parallel}} \cdot \left[\left(\frac{\alpha_{\parallel j}^2}{\alpha_{\perp j}^2} + \frac{\omega}{n\Omega_j} - 1 \right) \frac{\alpha_{\perp j}^2}{\alpha_{\parallel j}^2} Z'(\mu_{nj}) - \frac{2 V_r}{\alpha_{\parallel j}} Z(\mu_{nj}) \right], \tag{A-3}$$

$$\sigma_{yy}^{P} = \sum_j \omega_{pj}^2 \sum_{n=-\infty}^{\infty} \Phi_1(\lambda_j) \left[1 - \frac{n\Omega_j}{k_{\parallel}\alpha_{\parallel j}} Z(\mu_{nj}) + \frac{\alpha_{\perp j}^2}{2\alpha_{\parallel j}^2} Z'(\mu_{nj}) \right], \tag{A-4}$$

$$\sigma_{yz}^{P} = -\sigma_{zy}^{P} = \sum_j \omega_{pj}^2 \sum_{n=-\infty}^{\infty} \frac{\Omega_j F_1(\lambda_j)}{i k_\perp k_\parallel \alpha_{\perp j}^2} \Big[\Big(\omega - n\Omega_j \frac{\alpha_{\perp j}^2}{\alpha_{\parallel j}^2} + n\Omega_j \Big) Z'(\mu_{nj}) - \frac{2n\Omega_j V_r}{\alpha_{\parallel j}} Z(\mu_{nj}) \Big],$$

(A-5)

$$\sigma_{zz}^{P} = -\sum_j \omega_{pj}^2 \sum_{n=-\infty}^{\infty} \frac{I_n(\lambda_j) e^{-\lambda_j}}{k_\parallel \alpha_{\parallel j}} \Big(\omega \Big[\frac{1}{2} Z''(\mu_{nj}) - \frac{V_r}{\alpha_{\parallel j}} Z'(\mu_{nj}) + Z(\mu_{nj}) \Big] + n\Omega_j \Big\{ \frac{1}{2} \Big(\frac{\alpha_{\parallel j}^2}{\alpha_{\perp j}^2} + 1 \Big) Z''(\mu_{nj}) - \frac{V_r}{\alpha_{\parallel j}} \Big(1 + \frac{\alpha_{\parallel j}^2}{\alpha_{\perp j}^2} \Big) Z'(\mu_{nj}) + \Big[\frac{\alpha_{\parallel j}^2}{\alpha_{\perp j}^2} \Big(1 + \frac{2V_r^2}{\alpha_{\parallel j}^2} \Big) + 1 \Big] Z(\mu_{nj}) \Big\} \Big),$$

(A-6)

where $\lambda_j = (k^2 \alpha_{\perp j}^2)/2\Omega_j^2$; $\mu_{nj} = \dfrac{\omega + |V_r| k_\parallel}{k_\parallel \alpha_{\parallel j}} - \dfrac{n\Omega_j}{k_\parallel \alpha_{\parallel j}}$,

$$F_1(\lambda) = \frac{\lambda}{2} \frac{d}{d\lambda} \Big\{ e^{-\lambda} [I_n(\lambda) - I_{n+1}(\lambda)] \Big\} + \frac{n}{2} e^{-\lambda} [I_n(\lambda) - I_{n+1}(\lambda)] - \frac{1}{2} I_{n+1}(\lambda) e^{-\lambda},$$

(A-7)

$$\Phi_1(\lambda) = \lambda \frac{d}{d\lambda} \Big\{ e^{-\lambda} [I_n'(\lambda) - I_n(\lambda)] \Big\} + e^{-\lambda} I_n'(\lambda),$$

and

$$Z(\mu) = \pi^{-1/2} \int_{-\infty}^{\infty} dx\, e^{-x^2}/(x-\mu)$$

is the plasma dispersion function (Fried and Conte 1961) and $I_n(\lambda)$ is the modified Bessel function of n^{th} order.

The elements of $\underset{\approx}{\sigma}$ for the beam characterized by the distribution function as given by Eq.(6.3) have been evaluated by Montgomery and Tidman (1964). These are given below for the sake of completeness:

$$\sigma_{xx}^{b} = -\omega_{be}^{2}(-\omega + k_{\parallel}U)^{2}\Lambda, \tag{A-8}$$

$$\sigma_{xy}^{b} = -\sigma_{yx}^{b} = -i\Omega_{e}^{\gamma}\omega_{be}^{2}(-\omega + k_{\parallel}U)\Lambda, \tag{A-9}$$

$$\sigma_{xz}^{b} = \sigma_{zx}^{b} = k_{\parallel}U\omega_{be}^{2}(-\omega + k_{\parallel}U)\Lambda \tag{A-10}$$

$$\sigma_{yy}^{b} = -\omega_{be}^{2}(-\omega + k_{\parallel}U)^{2}\Lambda \tag{A-11}$$

$$\sigma_{yz}^{b} = -\sigma_{zy}^{b} = -ik_{\perp}U\omega_{be}^{2}\Omega_{e}^{\gamma}\Lambda \tag{A-12}$$

$$\sigma_{zz}^{b} = -\frac{\omega\,\omega_{be}^{2}}{\Omega_{e}}\frac{\partial}{\partial p}\left[\frac{\Omega_{e}P}{\gamma_{0}(-\omega + k_{\parallel}U)}\right] - \omega_{be}^{2}k_{\perp}^{2}U^{2}\Lambda \tag{A-13}$$

where $\Omega_{e}^{\gamma} = \Omega_{e}/\gamma_{0}$

and $\Lambda = \gamma_{0}^{-1}\left[\Omega_{e}^{\gamma\,2} - (-\omega + k_{\parallel}U)^{2}\right]^{-1}$.

APPENDIX - B

Elements of R for transverse propagation in the limit $\lambda_j = k^2 \alpha_{\perp j}^2 / 2\,\Omega_j^2 \lessapprox 1$:

$$R_{xx} = -\omega^2 - \frac{\omega_{be}^2 \omega^2}{\gamma_0 (\Omega_e^{r\,2} - \omega^2)} + \sum_j \frac{\omega_{pj}^2 \omega^2}{\omega^2 - \Omega_j^2} , \qquad (B-1)$$

$$R_{xy} = -R_{yx} = \frac{i\omega\Omega_e \omega_{be}^2}{\gamma_0^2 (\Omega_e^{r\,2} - \omega^2)} + \sum_j \frac{i\omega\Omega_j \omega_{pj}^2}{\omega^2 - \Omega_j^2} , \qquad (B-2)$$

$$R_{xz} = R_{zx} = -\frac{kU\omega\omega_{be}^2}{\gamma_0 (\Omega_e^{r\,2} - \omega^2)} + \sum_j \frac{kV_r \omega\omega_{pj}^2}{\omega^2 - \Omega_j^2} , \qquad (B-3)$$

$$R_{yy} = -\omega^2 + c^2 k^2 - \frac{\omega^2 \omega_{be}^2}{\gamma_0 (\Omega_e^{r\,2} - \omega^2)} + \sum_j \frac{\omega^2 \omega_{pj}^2}{\omega^2 - \Omega_j^2} , \qquad (B-4)$$

$$R_{yz} = -R_{zy} = -\frac{ikU\Omega_e \omega_{be}^2}{\gamma_0^2 (\Omega_e^{r\,2} - \omega^2)} - \sum_j \frac{ikV_r \Omega_j \omega_{pj}^2}{\omega^2 - \Omega_j^2} , \qquad (B-5)$$

$$R_{zz} = -\omega^2 + c^2 k^2 + \frac{\omega_{be}^2}{\gamma_0^3} - \frac{k^2 U^2 \omega_{be}^2}{\gamma_0 (\Omega_e^{r\,2} - \omega^2)} + \sum_j \omega_{pj}^2 + \frac{1}{2} \sum \omega_{pj}^2 \left(1 + \frac{2V_r^2}{\alpha_{\parallel j}^2}\right) \frac{k^2 \alpha_{\parallel j}^2}{(\omega^2 - \Omega_j^2)} . \qquad (B-6)$$

CHAPTER VII

CROSS-FIELD-CURRENT-DRIVEN ELECTROSTATIC INSTABILITIES IN PLASMAS WITH GENERALIZED DISTRIBUTION FUNCTION

VII.1 Introduction

There are various kinds of plasma instabilities which can arise in beam plasma systems. The return-current instability and its impact on plasma heating was discussed in the last chapter (Chapter VI). Now, we shall go over to study another class of beam induced instabilities namely, the cross-field-current driven electrostatic instabilities, which play a special role in certain turbulent heating experiments. Cross-field-current driven electrostatic instabilities have been invoked as the basic mechanism of production of anomalous resistivity observed in a number of turbulent heating experiments (Alexeff et al. 1970, 1971; Babykin et al. 1964). In Chapter I, we described the physical mechanisms by which cross field currents were produced in a couple of experiments. However, there are other physical situations too, where cross-field currents play an important role. For example, the anomalous resistance observed in a number of collisionless shock experiments (Daughney et al. 1970, Keilhacker and Steuer 1971) is interpreted to be due to such cross field currents within the shock profile.

Cross field currents can give rise to a host of electrostatic instabilities. However, we shall concentrate on only three of them namely, the ion acoustic instability, the modified-two-stream instability and the electron cyclotron drift instability; these have rather larger groth rates ($\mathrm{Im}\,\omega > \Omega_i$) associated with them. The physical nature, of these electrostatic instabilities, allows us to classify them into two classes-'dissipative instabilities' and 'reactive instabilities' (Hasegawa 1968, Taylor and Lashmore-Davies 1970). A dissipative instability is one in which energy flows from the plasma to the wave or vice versa, depending on whether the wave carries negative or positive energy. On the other hand, a reactive instability can be considered as due to coupling between two waves carrying energies of opposite signs. Thus, a reactive instability involves more than one waves and there is no transfer of energy between the wave and the plasma. The dissipative instability involves only a small fraction of resonant particles whereas the reactive instability involves all the particles in the plasma. Therefore it is easier to stabilize a dissipative instability than a reactive one.

The cross-field-current driven ion acoustic instability is a dissipative mode. This mode has been studied by many authors (Gary 1970, Arefev 1969, Barrett 1972). The dependence of this mode on the external magnetic field has also been extensively studied (Lashmore-Davies 1973). As the angle between the direction of propagation of the mode and the magnetic field direction increases, this mode changes from a dissipative one to a reactive one.

Modified two-stream instability, on the other hand, is a reactive mode which arises as a result of coupling between the lower hybrid mode ($\omega \approx \omega_{LH}$, ω_{LH} being the lower hybrid frequency) and the Doppler shifted electron mode ($\omega \approx k_y U - (k_z/k)(m_i/m_e)^{1/2}\omega_{LH}$, where U is the relative drift between the two species of the plasma, k_z and k_y are the components of the wave vector parallel and perpendicular to the magnetic field direction respectively). For $\omega_{pe} < \Omega_e$, the Doppler shifted electron mode is nothing but the Doppler shifted electron plasma oscillations propagating almost perpendicular to the magnetic field. The modified two-stream instability has been studied quite extensively both in the linear regime (Stepanov 1965, Ashby and Paton 1967, Krall and Liewer 1971, Arefev 1969) as well as in the nonlinear regime (McBride et al. 1972). This instability derives its name from the fact that, in the fluid limit, the form of the dispersion relation is similar to the one for Bunemann two stream instability (Bunemann 1961). However, the magnetic field introduces important differences. The threshold for this instability is $U > \alpha_i$ whereas that for the Bunemann two stream instability is $U > \alpha_e$ ($\alpha_{e,i}$ being the electron and the ion thermal velocities). This instability is a nonresonant instability, the nonlinear saturation of which grossly changes the particle distribution functions. Moreover, the growth rate for this instability is comparable to the real part of the frequency. Another interesting feature that comes out of the computer simulation experiments (McBride et al. 1972) is the comparable electron and ion heating as a result of nonlinear saturation of this instability.

The electron cyclotron drift instability is also a reactive instability which results due to the coupling of an ion acoustic mode and a slow Doppler shifted Bernstein mode. This mode was discussed by Wong (1970) and Gary and Sanderson (1970). A good deal of work on the nonlinear development of this instability and a number of computer simulation experiments have also been reported recently (Forslund et al. 1970, 1971, 1972; Lamp et al. 1972). It is observed from these studies that, this instability gives rise to diffusion across the magnetic field and causes substantial electron and ion heating.

Valuable, as these investigations are, they are all carried out on the assumption that the equilibrium distribution function is well represented by a Maxwellian distribution function. However, we note that a number of cross-field current heating experiments (Alexeff et al. 1970, 1971, Babykin et al. 1964). Use of magnetic mirror configuration to confine the plasma and for such a system, we know that a Maxwellian is not a realistic distribution function. Therefore, in this chapter, we shall use a 'generalized distribution function' (Dory et al. 1965) to study the effects of loss-cone and temperature anisotropy on the above mentioned instabilities. Cross-field-current driven electrostatic instabilities being often invoked as the basic mechanism in explaining the results of the turbulent heating experiments mentioned above, it is of importance to see if the characteristics of these instabilities are significantly altered by the presence of temperature anisotropy and loss cone effects.

From our analysis, we find that, for $\lambda = k^2 \rho_e^2/2 << 1$, these effects appear as small corrections. On the other hand important modifications occure for $\lambda \sim 1$. Algebraic complexities of the dispersion relation, in the region $\lambda \sim 1$, make it difficult to draw analytic conclusions over the entire range of parameter space. For this reason, we have numerically solved the general dispersion relation over a wide range of parameter space. Results of these numerical calculations are presented in Section VII.4.

VII.2 Dispersion Relation

Let us consider the waves for which $|\omega| >> \Omega_i$ and $k \rho_i >> 1$, where ω and k are the characteristic wave frequency and wave number; Ω_i and ρ_i are the ion-cyclotron frequency and the ion gyro-radius. Under these conditions the ions are effectively unmagnetized. We choose the axis of the magnetic mirror along the z-direction.

Since the ions are essentially unmagnetized, they are governed by a Maxwellian distribution, namely

$$f_{oi} = \frac{1}{\pi^{3/2}\alpha_i^3} \exp\left[-\frac{(v_y - U)^2}{\alpha_i^2} - \frac{(v_x^2 + v_z^2)}{\alpha_i^2}\right], \qquad (7.1)$$

where $\alpha_i = (2KT_i/M)^{1/2}$ is the ion thermal velocity; K, T and M being the Boltzman constant, ion temperature and ion mass respectively. U in Eq.(7.1) is the relative streaming velocity between the two species of the plasma which we have taken to be in the y-direction. For electrons we take a generalized distribution of DGH type (Dory et al. 1965, Buti 1974, Lakhina et al. 1974), namely

$$f_{oe}^{J} = \frac{(N v_\perp)^{2J}}{\pi^{3/2}\alpha_{\perp e}^{2(J+1)}\alpha_{\parallel e} J!} \exp\left[-\frac{v_\parallel^2}{\alpha_{\parallel e}^2} - \frac{v_\perp^2}{\alpha_{\perp e}^2}\right], \qquad (7.2)$$

where J = 0, 1, 2, ... is the distribution index, $\alpha_{\parallel e}^2 = 2KT_{\parallel e}/m$ and $\alpha_{\perp e}^2 = 2(J+1)^{-1} KT_{\perp e}/m$; $T_{\perp(\parallel)e}$ and m being electron perpendicular (parallel) temperature and electron mass respectively. For J = 0, f_{oe}^J goes over to a Maxwellian distribution and for $J \to \infty$, it behaves like $\delta(v_\perp - J^{1/2}\alpha_{\perp e})$ (δ-being a Dirac delta function). Moreover f_{oe}^J is peaked about $J^{1/2}\alpha_{\perp e}$ and has a half width, $\Delta v_\perp \sim J^{-1/2}\alpha_{\perp e}$.

For small electrostatic perturbations, the perturbed distribution function for the electrons, f_{1e} is governed by the linearized Vlasov equation,

$$\frac{\partial f_{1e}}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} f_{1e} - \frac{e}{mc}\left[\underset{\sim}{v} \times \underset{\sim}{B}_o\right] \cdot \underset{\sim}{\nabla}_v f_{1e} = \frac{e}{m} \underset{\sim}{E}_1 \cdot \underset{\sim}{\nabla}_v f_{oe}^J \qquad (7.3)$$

while that for the ions, f_{1i}, is governed by

$$\frac{\partial f_{1i}}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} f_{1i} = -\frac{e}{M} \underset{\sim}{E}_1 \cdot \underset{\sim}{\nabla}_v f_{oi} \quad , \qquad (7.4)$$

where $\underline{B}_o$ is the external magnetic field. These equations are solved by following usual method of characteristics and on using these solutions in the Poisson's equation,

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E}_1 = 4\pi e\left[\int f_{1e}\, d\underset{\sim}{v} - \int f_{1i}\, d\underset{\sim}{v}\right] , \qquad (7.5)$$

we get the following dispersion relation for the electrostatic waves:

$$1 + \frac{k^2 \alpha_{\parallel e}^2}{2\omega_{pe}^2} + \frac{\omega}{k_\parallel \alpha_{\parallel e}} \sum_{n=-\infty}^{\infty} C_n^J(\lambda)\, Z\left(\frac{\omega - n\Omega_e}{k_\parallel \alpha_{\parallel e}}\right)$$

$$+ \sum_{n=-\infty}^{\infty} \left[\frac{\alpha_{\parallel e}^2}{\alpha_{\perp e}^2} D_n^J(\lambda) - C_n^J(\lambda)\right] \frac{n\Omega_e}{k_\parallel \alpha_{\parallel e}} Z\left(\frac{\omega - n\Omega_e}{k_\parallel \alpha_{\parallel e}}\right)$$

$$- \frac{T_e}{2T_i}\left(1 + \frac{\alpha_{\perp e}^2}{\alpha_{\parallel e}^2}\right)^{-1} Z'\left(\frac{\omega - \underset{\sim}{k} \cdot \underset{\sim}{U}}{k\alpha_i}\right) = 0 \qquad (7.6)$$

where ω_{pe} and Ω_e are the electron plasma frequency and electron cyclotron frequency respectively; $k_{\parallel(\perp)}$ is the component of wave vector parallel (perpendicular) to the axial magnetic field and Z is the plasma dispersion function

(Fried and Conte 1961). In obtaining Eq. (7.6), we have taken the wave propagation in the y-z plane. The functions $C_n^J(\lambda)$ and $D_n^J(\lambda)$ are defined by

$$C_n^J(\lambda) = \frac{1}{\alpha_{\perp e}^{2(J+1)} J!} \int_0^\infty dv_\perp^2 J_n^2\left(\frac{k_\perp v_\perp}{\Omega_e}\right) v_\perp^{2J} \exp[-v_\perp^2/\alpha_{\perp e}^2] \tag{7.7}$$

and

$$D_n^J(\lambda) = \frac{1}{\alpha_{\perp e}^{2J} J!} \int_0^\infty dv_\perp^2 J_n^2\left(\frac{k_\perp v_\perp}{\Omega_e}\right) \left[\frac{v_\perp^{2J}}{\alpha_{\perp e}^2} - J v_\perp^{2(J-1)}\right] \cdot \exp[-v_\perp^2/\alpha_{\perp e}^2], \tag{7.8}$$

where J_n are the Bessel functions of order n. From Eqs.(7.7) and (7.8), we immediately see that $C_n^0(\lambda) = D_n^0(\lambda) = \exp(-\lambda) I_n(\lambda)$. We can also easily show that they obey the following recurrence relations:

$$C_n^{J+1}(\lambda) = C_n^J(\lambda) + \frac{\lambda}{J+1} \frac{d}{d\lambda} C_n^J(\lambda) \tag{7.9}$$

and $$D_n^{J+1}(\lambda) = \frac{J}{J+1} D_n^J(\lambda) + \frac{\lambda}{J+1} \frac{d}{d\lambda} D_n^J(\lambda),$$

where $I_n(\lambda)$ are the Bessel functions with imaginary argument. Moreover from Eqs.(7.7) and (7.8), we note that

$$\sum_{n=-\infty}^{\infty} D_n^J(\lambda) = 1 \text{ for } J = 0$$
$$= 0 \text{ for } J \neq 0,$$

and $$\sum_{n=-\infty}^{\infty} C_n^J(\lambda) = 1 \text{ for all J's.}$$

In writing these results, we have made use of the Bessel identity, $\sum_{n=-\infty}^{\infty} J_n^2(x) = 1$. For Maxwellian distribution for electrons, i.e., for J = 0 and $\alpha_{\perp e} = \alpha_{\parallel e} \equiv \alpha_e$ Eq.(7.6) goes over to the usual dispersion relation for the crossfield current driven electrostatic instabilities (Barrett et al. 1972). Since it is not possible to solve Eq.(7.6) for ω analytically, we have to solve it numerically. However, before we do that we shall discuss some special cases.

VII.3A Stability Analysis: Low Frequency Waves

For $\omega^2 << \Omega_e^2$ and $n^2 \Omega_e^2 >> k_{\parallel}^2 \alpha_{\parallel e}^2 (n \geq 1)$
$\sum_{n=-\infty}^{\infty} C_n^J(\lambda) Z\left(\frac{\omega - n\Omega_e}{k_{\parallel}\alpha_{\parallel e}}\right) \approx C_0^J(\lambda) Z\left(\frac{\omega}{k_{\parallel}\alpha_{\parallel e}}\right)$,
since $Z(n\Omega_e/k_{\parallel}\alpha_{\parallel e}) \approx -Z(-n\Omega_e/k_{\parallel}\alpha_{\parallel e})$. Consequently Eq.(7.6) can be written as

$$2\left\{1 + k^2\lambda_D^2\left[1 + (J+1)^{-1}A\right]^{-1} - C_0^J(\lambda)\right\}$$

$$= C_0^J(\lambda)\, Z'\left(\frac{\omega}{k_{\parallel}\alpha_{\parallel e}}\right) + \frac{T_e}{T_i}\left[1 + (J+1)^{-1}\right]^{-1} Z'\left(\frac{\omega - k_{\perp}U}{k\alpha_i}\right)$$

$$-4\sum_{n=1}^{\infty}\left\{C_n^J(\lambda) - (J+1)A^{-1}D_n^J(\lambda)\right\}\frac{n\Omega_e}{k_{\parallel}\alpha_{\parallel e}} Z\left(\frac{n\Omega_e}{k_{\parallel}\alpha_{\parallel e}}\right),$$

(7.10)

where $\lambda_D = (KT_e/4\pi ne^2)^{1/2}$ is the electron Debye length and $A = T_{\perp e}/T_{\parallel e}$.

i) Modified two-stream Instability

When ions are cold ($T_i \to 0$) and $\omega^2/k_{\parallel}^2\alpha_{\parallel e}^2 \gg 1$, Eq.(7.10) yields

$$1 + \frac{\omega_{pe}^2}{\Omega_e^2}\frac{k_\perp^2}{k^2}E_0^J(\lambda) - \frac{\omega_{pe}^2 C_0^J(\lambda)\frac{k_\parallel^2}{k^2}}{\omega} - \frac{\omega_{pi}^2}{(\omega - k_\perp U)^2} = 0, \quad (7.11)$$

where $E_0^0(\lambda) = [1 - D_0^0(\lambda)]/\lambda$

and $E_0^J(\lambda) = [-D_0^J(\lambda)]/\lambda$ for $J \neq 0$ (7.12)

For J = 0, Eq.(7.11) reduces to the dispersion relation for the modified-two-stream instability obtained by McBride et al. (1972) and Lashmore Davies and Martin (1973).

For $\lambda \ll 1$; $C_0^J(\lambda) \approx D_0^J(\lambda) \approx 1$ and hence the distribution index has no effect at all. One expects significant modification only in the region $\lambda \sim 1$. In this case Eq.(7.11) can be solved only under very special circumstances. For a given $k\rho_e$(i.e., λ) near unity, let us choose the orientation of the wave vector such that

$$(k_\parallel^2/k^2)C_0^J(\lambda) = m/M \quad (7.13)$$

In this case Eq.(7.11) can be solved exactly and the solution is given by

$$\omega'^2 = \omega_{LH}'^2\left[1 + \frac{k_\perp^2 U^2}{4\omega_{LH}'^2} \pm \left(1 + \frac{k_\perp^2 U^2}{\omega_{LH}'^2}\right)^{1/2}\right], \quad (7.14)$$

where $\omega' = \omega - k_\perp U/2$ and $\omega_{LH}^2 = \omega_{pi}^2\left[1 + \frac{\omega_{pe}^2}{\Omega_e^2}E_0^J(\lambda)\right]^{-1}$.

Thus the wave having wave vector and its orientation that satisfy Eq.(7.13) becomes unstable if the relative streaming velocity satisfies the relation

$$k_\perp^2 U^2 < 8\,\omega_{LH}'^2 , \tag{7.15}$$

in which case, the growth rate is given by

$$\gamma = \omega_{LH}'\left[\left(1 + \frac{k_\perp^2 U^2}{\omega_{LH}'^2}\right)^{1/2} - \left(1 + \frac{k_\perp^2 U^2}{4\,\omega_{LH}'^2}\right)\right]^{1/2}$$

$$\approx k_\perp U/2 \qquad \text{if} \qquad k_\perp^2 U^2 \ll \omega_{LH}'^2 \tag{7.16}$$

ii) Ion Acoustic Instability

Let us consider the regime where the ions are cold, so that we can still use an asymptotic form for the function $Z'\left(\frac{\omega - k_\perp U}{k\alpha_i}\right)$, but the electrons are hot such that $T_e/T_i \gg 1$ and $(\omega^2/k_{\parallel}^2\,\alpha_{\parallel e}^2) \ll 1$, which would allow us to use a power series expansion for the function $Z(\omega^2/k_{\parallel}^2\,\alpha_{\parallel e}^2)$. Though $(\omega^2/k_{\parallel}^2\,\alpha_{\parallel e}^2) \ll 1$, we shall assume $(n^2\Omega_e^2/k_{\parallel}^2\,\alpha_{\parallel e}^2) \gg 1$ for $n \geq 1$. On writing $\omega = \omega_r + i\gamma$ and assuming a priori that $\gamma^2 \ll (\omega_r - k_\perp U)^2$, Eq.(7.10) can be solved to give

$$\omega_r = k_\perp U\left[1 - \frac{C_s}{U}\left\{1 + A(J+1)^{-1}\right\}^{-1/2}\Delta_J^{-1/2}\right] \tag{7.17}$$

and

$$\gamma = \left(\frac{\pi m}{8M}\right)^{1/2}\frac{k}{k_{\parallel}}\,C_o^J(\lambda)\,k_\perp U\,\Delta_J^{-2}\left[\Delta_J^{1/2} - \frac{C_s}{U}\right], \tag{7.18}$$

where

$$\Delta_J = C_o^J(\lambda) + \lambda(J+1)A^{-1}E_o^J(\lambda) + k^2\lambda_D^2\left[1 + A(J+1)^{-1}\right]^{-1}$$

(7.19)

and $C_s = (T_e/M)^{1/2}$ is the sound speed. It can be easily varified from Eq.(7.17) and (7.18) that the apriori assumption, $\gamma^2 \ll (\omega_r - k_\perp U)^2$, is satisfied for sufficiently large angle of orientation for the wave vector such that $(k_{\parallel}^2/k^2)(M/m) \gg 1$. For J = 0 and $\alpha_{\parallel e} = \alpha_{\perp e} = \alpha_e$, these results go over to the results obtained by Barrett et al. (1972). To see how the growth rate varies with J, we assume that the temperature anisotropy, A, is large, so that we can write $\Delta_J \approx C_o^J(\lambda)$. In this case, the ratio between the growth rates for a non-Maxwellian plasma ($J \neq 0$) and that for a Maxwellian plasma (J = 0) goes as

$$\frac{[\gamma]_{J\neq 0}}{[\gamma]_{J=0}} = \frac{C_o^o(\lambda)\left[\{C_o^J(\lambda)\}^{1/2} - C_s/U\right]}{C_o^J(\lambda)\left[\{C_o^o(\lambda)\}^{1/2} - C_s/U\right]} \qquad (7.20)$$

Since, in the region $\lambda \sim 1$, $C_o^J(\lambda)$ decreases much faster for $J \geq 1$ than for J = 0, it is clear from Eq.(7.20) that the growth rates for a non-Maxwellian plasma are higher than that for a Maxwellian Plasma.

VII.3B Stability Analysis: High Frequency Waves

i) Electron Cyclotron Instability

Since for $k_{\parallel} > 0$ the electron cyclotron waves are damped, we shall study only the case $k_{\parallel} \to 0$. Moreover, if the finite ion temperature effects are neglected, Eq.(7.6) can be simplified to obtain the dispersion relation for these high frequency ($\omega \gtrsim \Omega_e$) electron cyclotron waves; this is given by,

$$1 + \frac{k^2 \alpha_{\parallel e}^2}{2\omega_{pe}^2} = \sum_{n=-\infty}^{\infty} C_n^J(\lambda) \frac{\omega}{(\omega - n\Omega_e)} + \frac{k_\perp^2 C_s^2 (J+1) A^{-1}}{(\omega - k_\perp U)^2}$$
$$+ \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[(J+1) A^{-1} D_n^J(\lambda) - C_n^J(\lambda) \right] \frac{n\Omega_e}{(\omega - n\Omega_e)} . \quad (7.21)$$

For $\omega \approx |n|\Omega_e$ and $\gamma^2 << (\omega_r - k_\perp U)^2$, the frequency and growth rate for the n^{th} harmonic cyclotron wave are given by

$$\omega_r - k_\perp U = \pm k_\perp C_s \left\{ 1 + A(J+1)^{-1} \right\}^{-1/2}$$
$$\cdot \left[1 + k^2 \lambda_D^2 \left\{ 1 + A(J+1)^{-1} \right\}^{-1} - C_n^J(\lambda) \right]^{-1/2} , \quad (7.22)$$

and

$$\gamma^2 = -|n|\Omega_e (J+1) A^{-1} D_n^J(\lambda) (\omega_r - k_\perp U)$$
$$\cdot \left[1 + k^2 \lambda_D^2 \left\{ 1 + A(J+1)^{-1} \right\}^{-1} - C_n^J(\lambda) \right]^{-1} . \quad (7.23)$$

For $\gamma^2 > 0$, depending on the sign of D_n^J, we have to choose the appropriate root for ω_r as given by Eq.(7.22). From Eq.(7.23) it is clear that the growth rate directly depends on $D_n^J(\lambda)$ whose variation is shown in Fig.7.1. We also observe from this figure that for $n = 1$ and $k\rho_e > 1$ though $D_n^0(\lambda)$ is always positive, $D_n^J(\lambda)$ for $J \geq 1$, becomes negative after

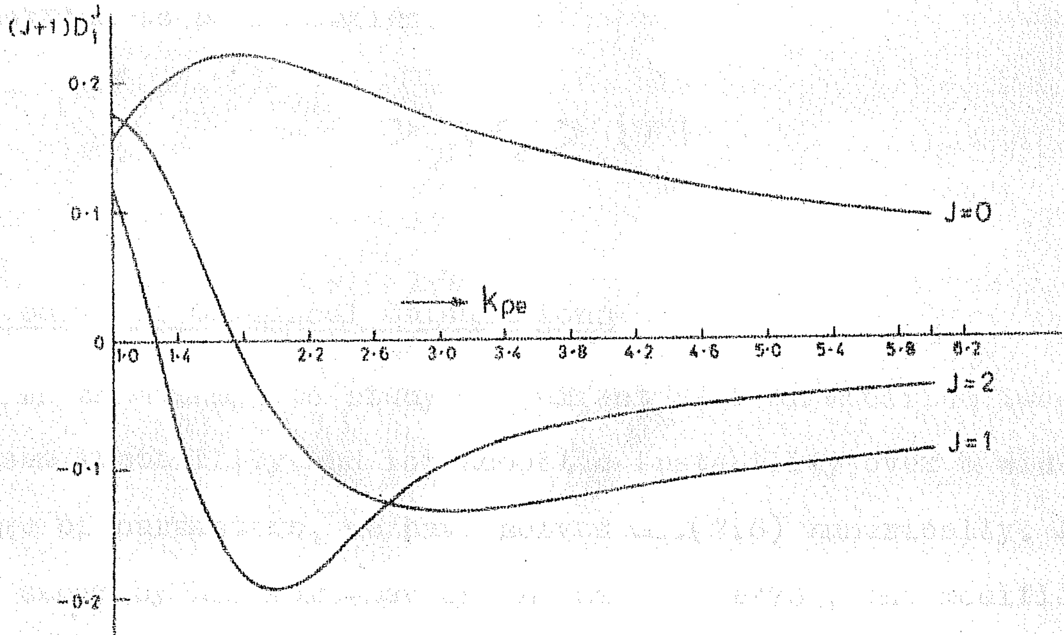


FIGURE 7.1 $(J + 1)\ D_1^J(\lambda)$ as a function of $k\rho_e$.

some critical value of $k\rho_e$. For $J = 0$, $\alpha_{\parallel e} = \alpha_{\perp e} \equiv \alpha_e$ and $\lambda > |n|$, the results given by Eqs.(7.19) and (7.20) go over to those obtained by Wong (1970).

This instability is due to the resonant coupling of a Doppler shifted ion-mode and the electron cyclotron mode. As Wong (1970) has shown that this instability needs the following condition to be satisfied.

$$\frac{\omega_{pe}}{\Omega_e} > k\rho_e > |n|$$

VII.4 Results of Numerical Calculations

In an attempt to study the effect of J on modified two-stream instability and ion acoustic instability over a wide range of parameters, we have solved Eq.(7.6) numerically. As was shown by Lashmore-Davies and Martin (1973), the modified two stream instability and ion-acoustic instability are often not separable. In fact with the variation of certain parameters (e.g. $k_{\parallel}/k$) the instability changes from one to the other. What we follow in the numerical calculations is also a combination of these two instabilities. For carrying out numerical computations, for convenience's sake, we will introduce the dimensionless variables, namely $x = \omega/\omega_{pi}$ $\bar{\theta} = (k_{\parallel}/k)(M/m)^{1/2}$, $\eta = \omega_{pe}/\Omega_e$ and $\xi = U/\alpha_i$.

The selection of the initial parameters is made in such a way that the maximum growth rate for a Maxwellian plasma (J = 0) occurs for $k\rho_e \sim 1$. In Fig.7.2 variations of frequency and growth rate with $k\rho_e$ have been shown for $J = 0, 1, 2$. It is interesting to note that, while a Maxwellian plasma ($J = 0$) is stable for $k\rho_e \geq 1.25$, a non-Maxwellian plasma ($J = 1,2$) is not only unstable but it also sustains higher growth rates for $k\rho_e > 1.25$. The growth rates for $k\rho_e > 2$ are not shown because the range $k\rho_e \lesssim 1$ is the one which is interesting from practical point of view. In the case of $J = 1$, the increase in the growth rate for $k\rho_e > 1.4$ seems to be because of the fact that in this region the root has changed from modified two stream to ion acoustic mode. Moreover an examination of Eq.(7.18) shows that the growth rate for the ion acoustic instability goes inversely as $C_o^J(\lambda)$ because $\Delta_J \sim C_o^J$ for large values of A. Since $C_o^J(\lambda)$ decreases with $k\rho_e$, there is corresponding increase in the growth rate, as depicted in Fig.7.2. From this figure, we also note that for $J \geq 1$, the growth rates are $> 0.2\ \omega_{pi}$ which for the parameters used correspond to $\gamma > \Omega_i$ and hence we conclude that in the range of k-space under consideration, the system can sustain fast growing modes.

In Fig.7.3, we have plotted the variation of frequency and growth rate with the angle of orientation of the wave vector for a fixed $k(k\rho_e = 1.2)$. It is observed that the growth rates are higher for higher J values when $\tilde{\theta}$ is small, but they are smaller for higher J values as $\tilde{\theta}$ increases. However, the real part of the frequency decreases with an

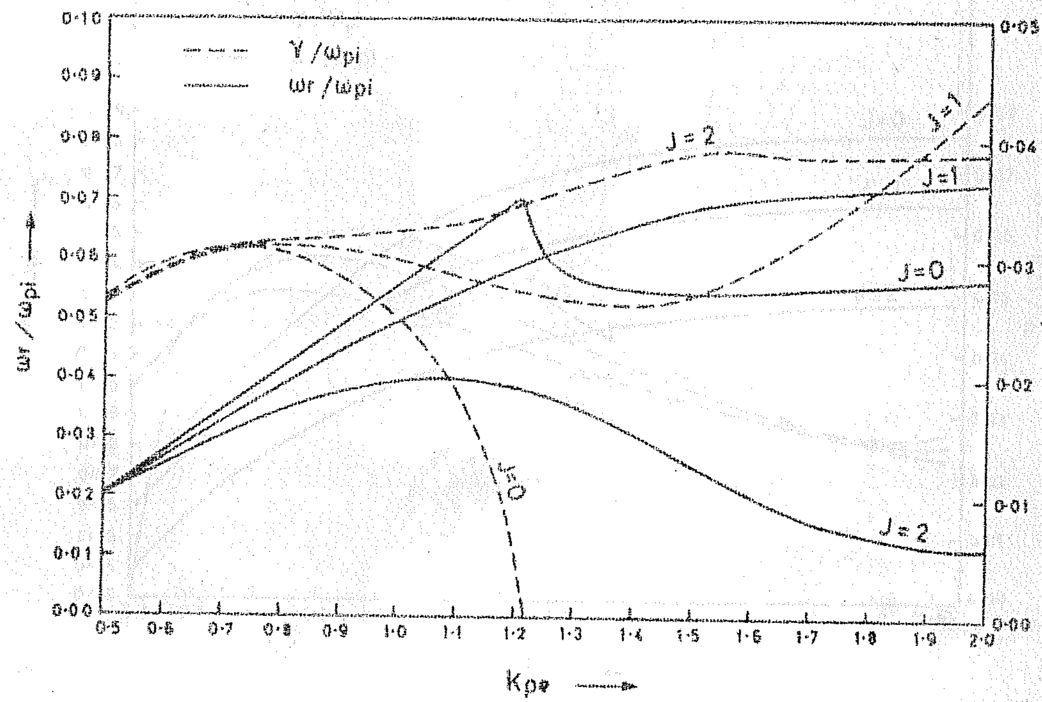


FIGURE 7.2 Variation of frequency (ω_r/ω_{pi}) and growth (γ/ω_{pi}) rate with wave number $k\rho_e$. The parameters labeling the curve is the distribution index J. Other parameters are ω_{pe}/Ω_e = 10.0, T_e/T_i = 50.0, U/α_i = 12.5, A = 10.0, θ = 0.4 and m/M = 1/1836.

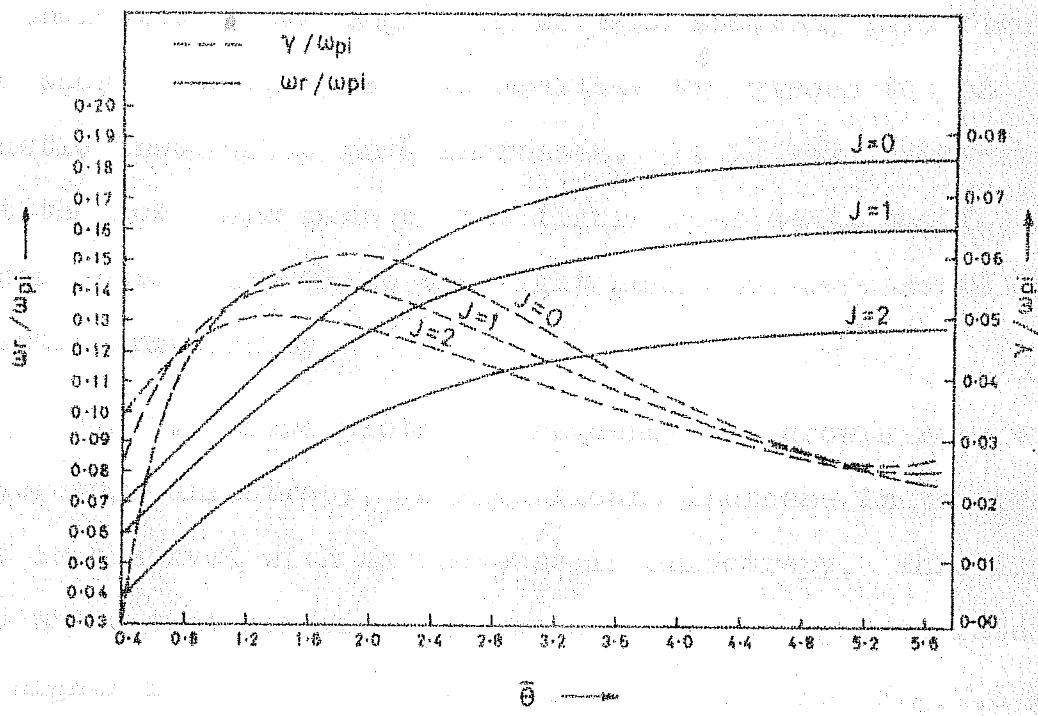
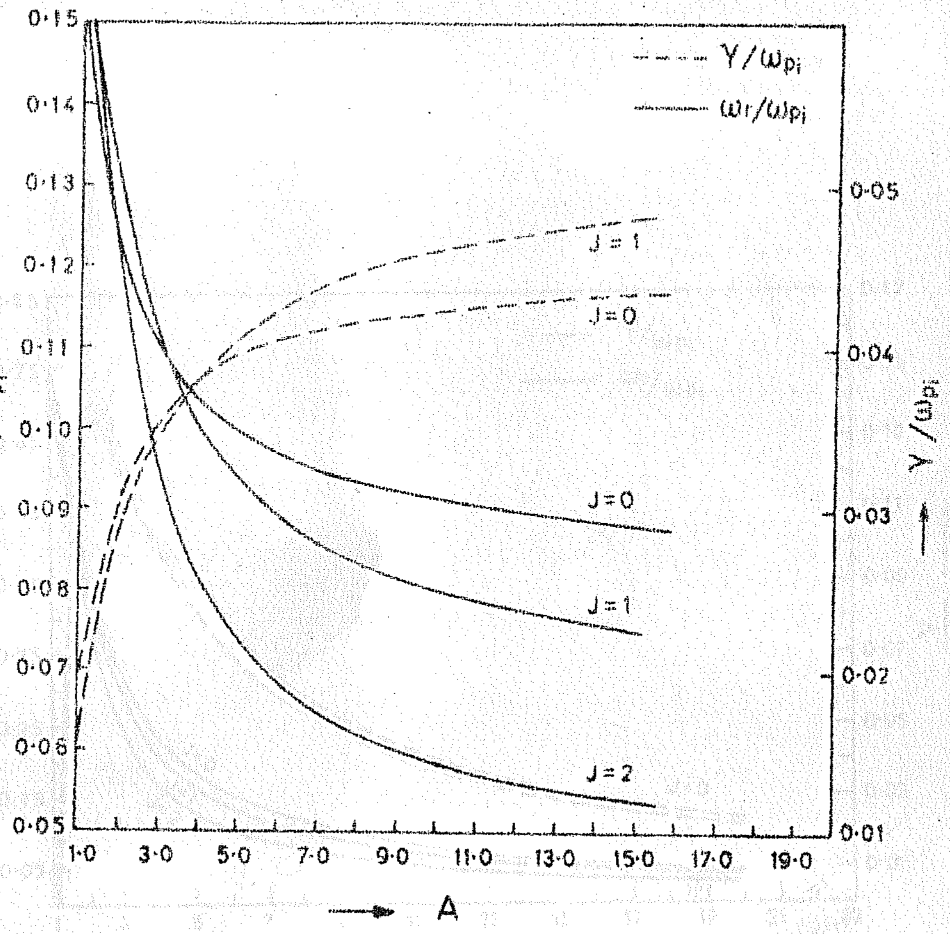


FIGURE 7.3 Plots of frequency (ω_r/ω_{pi}) and growth rate (γ/ω_{pi}) versus the angle of orientation of the wave vector ($\bar{\theta}$). The parameter labeling the curves is the distribution index J. $k\rho_e$ for these curves is 1.2 and other parameters are same as in Fig.7.2.

increase in J. It is well known that, as $(k_{\parallel}/k)$ increases, the electron Landau damping becomes stronger and stronger and finally it stabilizes the wave. The above results seem to indicate that the electron Landau damping play a more vital role for a loss-cone plasma than for a Maxwellian plasma. As was indicated in the beginning of this section, this figure also shows a transition from modified two stream to ion acoustic instability as $\bar{\theta}$ increases. It is easy to verify that the left hand side of the figure represents modified two stream instability while the right hand side represents ion-acoustic instability.

Fig.7.4 shows plots of frequency and growth rate versus temperature anisotropy. A significant increase in the growth rate is observed with an increase in anisotropy. This figure also shows that for an anisotropic plasma the growth rates are higher for $J \geq 1$ than those for $J = 0$. From Fig.7.5 it is apparent that for $\eta \geq 2$ the magnetic field has a destabilising effect on the system. Fig.7.6 shows that the threshold for T_e/T_i for instability decreases as J increases and the maximum growth rate (maximized over T_e/T_i) is higher for higher J values.

In order to see the variation with the streaming velocity we determined the maximum growth rate, which occurs at $k = k^*$, for a number of streaming velocities. These results are shown in Table 7.1. For reasons stated earlier, we have restricted ourselves to values of $k\rho_e$ only upto 3.0.



<u>FIGURE 7.4</u> Variation of frequency (ω_r/ω_{pi}) and growth rate (γ/ω_{pi}) with anisotropy parameter $A(\equiv T_{\perp e}/T_{\parallel e})$. The parameter labeling the curves is the distribution index J. For these curves $k\xi_e = 1.2$ and $\bar{\theta} = 0.8$. All other parameters are same as in Fig.7.2. The growth rates for J = 2 are not shown, because with the scales used in this figure they overlap with those for J = 1.

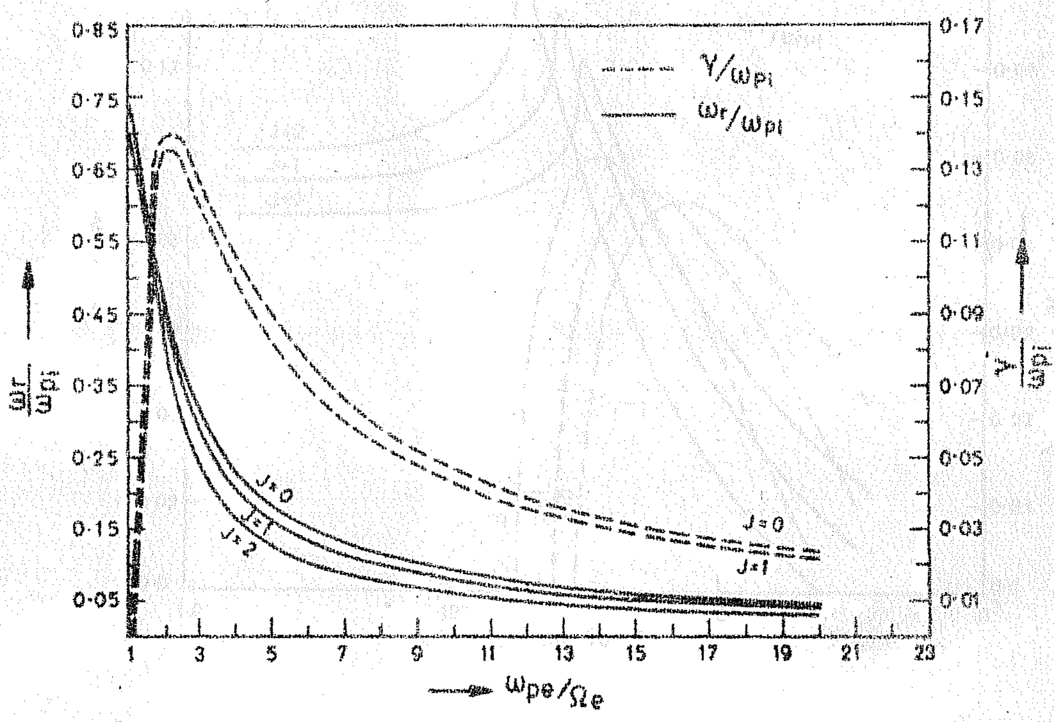


FIGURE 7.5 Variation of frequency (ω_r/ω_{pi}) and growth rate (γ/ω_{pi}) with ω_{pe}/Ω_e. The parameter labeling the curves is the distribution index J. For these curves A = 10.0 and all other parameters are same as in Fig.7.3. The growth rates for J = 2 are not shown, because with the scales used in this figure they overlap with those for J = 1.

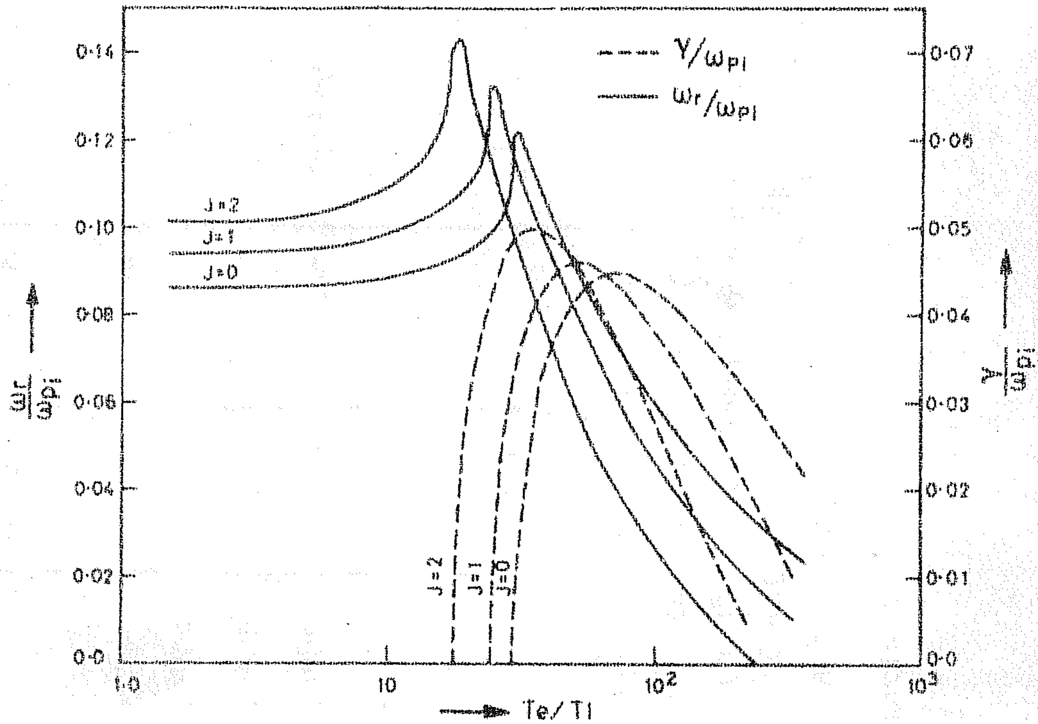


FIGURE 7.6 Plots showing the variation of frequency and growth rate (γ / ω_{pi}) with the temperature ratio (T_e/T_i). The parameter labeling the curves is the distribution index J. Other parameters are ω_{pe}/Ω_e = 10.0, A = 10.0, $k\rho_e$ = 1.2, U/α_i = 12.5 and θ = 0.8.

J \ $\bar{U}/\alpha_i$	0			1		
	$k^*\rho_e$	γ^*/ω_{pi}	ω_r^*/ω_{pi}	$k^*\rho_e$	γ^*/ω_{pi}	ω_r^*/ω_{pi}
25.0	0.45	0.0444	0.0730	0.40	0.0445	0.0663
20.0	0.55	0.0446	0.0704	0.55	0.0448	0.0714
18.0	0.60	0.0446	0.0685	0.65	0.0448	0.0746
15.0	0.75	0.0446	0.0702	0.80	0.0450	0.0729
12.0	1.0	0.0447	0.0722	>3.0	–	–
10.0	1.3	0.0449	0.0745	>3.0	–	–
5.0	>3.0	–	–	>3.0	–	–

TABLE 7.1 Maximum growth rates (γ^*/ω_{pi}) and corresponding frequency (ω_r^*/ω_{pi}) together with the value of $k\rho_e$ for which the growth rates are maximum ($k^*\rho_e$) are shown for a number of streaming velocities. The constant parameters are $T_e/T_i = 50.0$, $\omega_{pe}/\Omega_e = 10.0$, $A = 10.0$ and $\bar{\theta} = 0.8$.

– indicates the absence of maximum

We observe that γ^* as well as k^* increase as the streaming velocity decreases. Another interesting thing to be noted is that, there are certain values of streaming velocities (for example $U/\alpha_i \leq 12.0$, for the set of parameters considered in Table 7.1) for which the growth rate maximizes for $k\rho_e \sim 1.0$ in the case of $J = 0$ while it goes on increasing even upto $k\rho_e = 3.0$ in the case of $J = 1$.

VII.5 Conclusions

We have shown that the loss-cone and temperature anisotropy in the electron velocity distribution has profound effects on the modified two-stream instability and ion acoustic instability when $\lambda = k^2\rho_e^2/2 \sim 1$. We also find that, non-Maxwellian plasmas with distribution under $J \geq 1$, can sustain fast growing waves ($\mathrm{Im}\,\omega > \Omega_i$) even in regions of k-space which is stable for a Maxwellian plasma. Moreover, when $\lambda \sim 1$ the temperature anisotropy is found to have a destabilizing effect on the modified two-stream instability and on the ion acoustic instability.

In this chapter we have not attempted to ascertain quantitatively the modifications in the non-linear theory of these instabilities due to the modifications introduced by the loss-cone and the anisotropy effects; however, we would qualitatively discuss the evolution of the distribution function due to these nonlinearities. The result of numerical simulation experiments performed by McBride et al. (1972) indicate that the nonlinear stabilization of the modified two stream instability takes place due to particle trapping in the potential well. They have also shown that the electrons predominantly get heated along the direction of the magnetic field while the ions predominantly get heated along the direction perpendicular to the magnetic field, such that the final parallel temperature of the electrons is comparable to the perpendicular ion temperature. In our case the consequence of such an effect will be that, the predominant parallel heating of the electrons due to modified two-stream instability will eventually isotropize the distribution function even if the equilibrium distribution function for the electrons was sufficiently anisotropic to start with.

REFERENCES

Alexeff I, Estabrook K, Hirose A, Jones W D, Neidigh R V, Olsen J N, Scott F R, Sterling W L and Wing W R (1970) Phys. Rev. Lett. 25, 848

Alexeff I, Berry L A, Dudley J M, Estabrook K G, Hirose A, Jone W D, Neidigh R V, Olsen J N, Scott F R, Stirling W L, Winder M M, and Wing W R (1971) Plasma Physics and Controlled Nuclear Fusion Research, Vol 2, 221 (IAEA, Vienna)

Alfven H (1939) Phys. Rev. 55, 425

Altynstev A T, Es'kov A G, Zolotoviskii O A, Koroteev V I, Kurtmullaev R Kh, Masalov V D and Semenov V N (1971) (ZhETF Pis. Red 13, 139 (1971))

Arefev V I (1969) Zh tekh Fiz 39, 1973 (Sov. Phys. Tech. Phys. 14, 1487 (1970))

Asano N (1974) Suppl. Progr. Theor. Phys. No.55, P52

Ashby DETF and Paton A (1967) Plasma Phys. 9, 359

Babykin M B, Zavoiskii E K, Ivanov A A and Rudakov L I (1971) Plasma Physics and Controlled Nuclear Fusion Research (IAEA, Vienna) Vol.1, P635

Babykin M B, Gavrin P P, Zavoiskii E K, Rudakov L I, Skoriupin V and Sholin G V (1964) Sov. Phys. JETP 19, 349

Barrett P J, Fried B D, Kennel C F, Sellen J M and Taylor R J (1972) Phys. Rev. Lett. 28, 337

Bernstein I B (1958) Phys. Rev. 109, 10

Braginskii S I (1965) Reviews of Plasma Physics (Ed. Leontovich M A, Consultants Bureau, N Y) Vol.1, P205

Briggs R J (1964) Electron Stream Interactions with Plasmas (MIT Press, Cambridge, Mass.)

Brashitov Yu I A, Koidan V S, Konyukhov V V, Lagunov V M, Luk'yanov V N and Mekler K I (1973) ZhETP Pis Red. 18, 675 (JETP Lett. 18, 395 (1973))

Breizman B and Ryutov D (1971) Zh Eksp. Teor. Fis. 60, 408 (Sov. Phys. JETP 30, 220 (1971))

Buchel'nikova N S (1968) Sov. Phys. Tech. Phys. 13, 455

Buneman O (1961) J. Nucl. Energy Part C4, 11

Buti B (1963) Phys. Fluids 13, 89

Buti B (1974) Phys. Fluids 17, 258

Chen F F (1964) Phys. Fluids 7, 949

Coppi B (1964) Phys. Rev. Letts. 12, 417

Dahavakhishvili D I (1973) Sov. Phys. Tech. Phys. 17, 1438

Davidson R C, Haber I and Hammer D A (1971) Phys. Letts. 34A 235

Davidson R C (1972) Methods in Nonlinear Plasma Theory (Academic Press, New York, London) Chap.2

Davidson R C (1972a) Methods in Nonlinear Plasma Theory (Academic Press, N Y, London) Chap.11

Davidson R C, Hammer D A, Haber I and Wagner C E (1972b) Phys. Fluids 15, 317

Daughney C C, Holmes L S and Paul J W M (1970) Phys. Rev. Lett. 25, 497

Dory R A, Guest G E and Harris E G (1965) Phys. Rev. Lett. 14, 131

Drummond W E, Malmberg J H, O'Neil T M and Thompson J R (1970) Phys. Fluids 13, 2422

Drummond W E and Pines D (1962) Nuclear Fusion Suppl. Part III, 1049

Ekdahl C, Greenspan M, Kribel R E, Sethian J and Wharton C B (1974) Phys. Rev. Letts. 33, 346

Fainberg Ya B, Shapiro V D and Shevchenko V I (1969) Zh. Eksp. Teor. Fiz 57, 966 (Sov. Phys. JETP 30, 528 (1970))

Forslund D W, Morse R L and Nielson C W (1971) Phys. Rev. Letts. 27, 1424

Forslund D W, Morse R L and Nielson C W (1972) Phys. Fluids 15, 2363

Fried B D and Conte S D (1961) Plasma Dispersion Function (Academic Press, N Y, London)

Gardner C S, Greene J M, Kruskal M D and Miura R M (1967) Phys. Rev. Letts. 19, 1095

Gardner C S and Morikawa G K (1960) Rep. NYU - 9082 (Courant Inst. Math. Sci. N Y Univ. New York)

Gardner C S and Su C H (1966) Ann. Rep. Princeton Univ. Plasma Phys. Lab. Matt-Q-24

Gary S P (1970) J. Plasma Phys. 4, 758

Gary S P and Sanderson J J (1970) J. Plasma Phys. 4, 739

Goswami B N, Lakhina G S and Buti B (1974) Pramana 3, 10

Goswami B N, and Buti B (1975) Nuclear Fusion 15, 991

Goswami B N and Sinha M (1976) Pramana (in press)

Goswami B N and Buti B (1976) Phys. Lett.A (in press)

Goldenbaum G C, Dove W.G, Gerber K A and Logan B G (1974) Phys. Rev. Lett. 32, 830

Guillory J and Benford G (1972) Plasma Physics 14, 1131

Handel H W, Chu T K and Politzer P A (1968) Phys. Fluids 11, 2426

Hammer D and Rostoker N (1970) Phys. Fluids 13, 1831

Hasegawa A (1968) Phys. Rev. 169, 204

Ioffe M S, Sobolev R I, Telkoviskii V G and Yushmanov E E (1961) Sov. Phys. JETP 12, 1117

Ivanov A A, Kazakov Yu B, Alukyanchuk A N, Rusanov V D, Sobolev S S and Teikhman I (1969) JETP Letts. 9, 210

Jefreey A and Kakutani T (1972) SIAM Review 14, 582

John P I and Saxena Y C (1976) Physics Letts. A (in press)

Johnson R S (1970) J. Fluid Mech. 42, 49

Johnson R S (1972) J Fluid Mech. 81, 81

Jones W D, Lee A, Gleman S, Starr O and Doucet H J (1974) Bull. Am. Phys. Soc. 19, 863

Jones W D, Lee A, Gleman S and Doucet H J (1975) Phys. Rev. Lett. 35, 1349

Kadomtsev B B (1965) Plasma Turbulence (Academic Press, N Y, London)

Kapetankos C A and Hammer D A (1973) Appl. Phys. Lett. 23, 17

Karatzas N Th, Anastassiadis A J and Papadopoulos K (1975) Phys. Rev. Lett. 35, 30

Karpman V I (1973) Nonlinear Waves in Dispersive Media ('Nauka', Moscow)

Keilhacker M and Steuer K H (1971) Phys. Rev. Lett. 26, 694

Korn P, Sandel E and Wharton C B (1973) Phys. Rev. Lett. 31, 579

Korteweg D J and deVries G (1895) Phil Mag. 39, 422

Krall N A (1968) Advances in Plasma Physics (Ed. Simon A and Thompson W B, Interscience Publishers, N Y, London) Vol.1, P153

Krall N A and Trievelpiece A W (1973) Principles of Plasma Physics (McGraw Hill Book Co., N Y, London, Sydney)

Krall N A and Liewer P C (1971) Phys. Rev. A4, 2094

Kruer W L, Kaw P K, Dawson J M and Oberman C (1970) Phys. Rev. Lett. 24, 987

Kruer W L and Dawson J M (1972) Phys. Fluids 15, 446

Lakhina G S and Buti B (1972) Phys. Rev. 5A, 1846

Lakhina G S, Buti B and Nayar G (1974) J. Plasma Phys. 11, 461

Lamp M, Manheimer W M, McBride J B and Orens J H (1972) Phys. Fluids 15, 2356

Lashmore-Davies C N and Martin T J (1973) Nuclear Fusion 13, 193

Lee R and Sudan R N (1971) Phys. Fluids 14, 1213

Lovelace R V and Sudan R N (1971) Phys. Rev. Lett. 27, 1256

Maxworthy T and Redekopp L G (1976) Nature 260, 511

McBride J B, Ott E, Boris J P and Orens J H (1972) Phys. Fluids 15, 2367

McCorkle R A (1975) Phys. Rev. Letts. 20, 1344

Millar P A and Kuswa G W (1973) Phys. Rev. Lett. 30, 958

Mikhailovskii A B and Tsypin V S (1966) Zh. Eksp. Teor. Fiz Pis Red 3, 247 (JETP Letts. 3, 158 (1966))

Moiseev S S and Sagdeev R Z (1963) Sov. Phys. JETP. 17, 515

Montgomery D C and Tidman D A (1964) Plasma Kinetic Theory (McGraw Hill, New York)

Nezlin M V (1971) Sov. Phys. Usp. 13, 608

Nishikawa K and Kaw P K (1975) Phys. Letts. 50A, 455

Nozaki K and Taniuti T (1974) J. Phys. Soc. Japan. 36, 267

Oleson N L and Found C G (1949) J, Appl. Phys. 20, 416

Richtmyer R D and Morton K W (1967) Difference Methods for Initial Value Problems, 2nd Ed. P 176 (Interscience Publishers, N Y, London, Sydney)

Roberts T G Bennett W H (1968) Plasma Phys, 10, 381

Rudakov L I (1970) Zh. Eksp. Teor. Fiz 59, 2091 (Sov. Phys. JETP 32, 1134 (1971))

Rudakov L I and Samarsky A A (1973) Proc. VIth European Plasma Physics Conference, Moscow P487

Schmidt G (1966) Physics of High Temperature Plasmas. An Introduction (Academic Press N Y, London)

Scott A C, Chu F Y F and Mclanghlin D W (1973) Proc. IEEE 61, 1443

Shutko A V (1970) Sov. Phys. JETP 30, 248

Sipler D W and Biondi M A (1972) J. Geophys. Res. 77, 6202

Stepanov K N (1964) Zh. Tech. Fiz 34, 2146 (Sov. Phys. Tech. Phys. 9, 1653 (1965))

Stringer T E (1964) Plasma Physics 6, 267

Stix T H (1962) The Theory of Plasma Waves (McGraw Hill Book Co., N Y, London, Sydney)

Sudan R N (1973) Proc. VIth European Conference on Controlled Fusion and Plasma Physics Vol.II, P184

Taniuti T (1974) Suppl. Progr. Theor. Phys. No.55, P1

Taniuti T and Wei C C (1968) J. Phys. Soc. Japan 24, 941

Taylor J B and Lashmore-Davies C N (1970) Phys. Rev. Lett. 24, 1340

Thode L E and Sudan R N (1973) Phys. Rev. Lett. 30, 732

Toefer A and Poukey J (1973) Phys. Letts. 42A, 383

Utlaut W F and Cohen R (1971) Science 174, 245

Vedenov A A, Velikhov E P and Sagdeev R Z (1961) Nuclear Fusion 1, 82; Nuclear Fusion Suppl. Part II, 465

Vitkovisky I M, Levine L S, Mosher D and Stephanakis S J (1973) Appl. Phys. Letts. 23, 9

Washimi H and Taniuti T (1966) Phys. Rev. Letts. 17, 996

Whitham G B (1974) Linear and Nonlinear Waves (John Wiley and Sons, N Y, London, Sydney)

Whitham G B (1965) J. Fluid Mech. 22, 273

Wong H V (1970) Phys. Fluids 14, 757

Yonas G, Poukey J W, Freeman J R, Prestwich K R, Toefer A J, Clauser M J and Beckner (1973) Proc. VIth European Conference on Controlled Fusion (Moscow, USSR) Vol.1, P483

Yonas G, Poukey J W, Prestwich K R, Freeman J R, Toefer A J and Clauser M J (1974) Nuclear Fusion 14, 731