# Inflationary scenario with Non-Standard Spinors

A thesis submitted in partial fulfillment of

the requirements for the degree of

**Doctor of Philosophy**

*by*


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**To**

*my family*

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# Thesis Approval

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# Abstract

Inflationary scenario has been very successful in solving various problems associated with the standard Big Bang cosmology. But the nature of the field that drives accelerated expansion (inflaton) is still unknown to us. The inflationary models with scalar fields, under the slow-roll approximations, are well studied. In contrast, inflationary scenario with spinor fields have not attracted much attention. In earlier works the 'classical' Dirac spinor field was studied as a candidate of inflaton. However, there were some issues with inflationary scenario driven by the Dirac spinor. One of the most important problem with Dirac spinor is that it produces highly scale dependent power-spectrum (with spectral index $n_s \sim 4$), which is inconsistent with the CMB observations.

Recently, one special type of spinor was proposed by Ahluwalia (2005, 2013) which is an eigenspinor of charge conjugation operator, also known as Elko. This spinor is called the Non-Standard Spinor (NSS) as it has an unusual property: $(CPT)^2 = -\mathbb{I}$. NSS field is a spin-$\frac{1}{2}$ field with mass dimension *one*, whereas the 'classical' Dirac spinor is a spin-$\frac{1}{2}$ fermion with mass dimension $\frac{3}{2}$. This new spinor field obeys the Klein-Gordon equation instead of Dirac equation. NSS can interact only through Higgs and with gravity, therefore it is dark by nature. Thus it is worth investigating the role of NSS in the unknown dark sector of the universe like: Dark matter, dark energy and inflation etc. In this thesis our focus is on the NSS driven accelerated expansion of the universe.

In the earlier NSS theories there was one major inconsistency — the equation of motion of NSS obtained from the energy-momentum tensor did not match with the equation of motion calculated using the Euler-Lagrange equation. Recently a consistent theory of NSS was developed which removed this inconsistency. In this thesis we use a consistent NSS theory to study the first order cosmological perturbation theory for NSS. The NSS Lagrangian and the energy-momentum tensor can be expressed as follows:

$$\mathcal{L} = \frac{1}{2}\bar{\lambda}\overleftarrow{\nabla}_{\mu}\nabla^{\mu}\lambda - V(\bar{\lambda}\lambda), \qquad T^{\mu\nu} = \bar{\lambda}\overleftarrow{\nabla}^{(\mu}\nabla^{\nu)}\lambda - g^{\mu\nu}\mathcal{L} + F^{\mu\nu}$$

where $\lambda$ and $\vec{\lambda}$ is the NSS and its dual. The covariant derivatives are defined as: $\vec{\lambda}\overleftarrow{\nabla}_\mu \equiv \partial_\mu \vec{\lambda} + \vec{\lambda}\Gamma_\mu$ and $\nabla_\mu \lambda \equiv \partial_\mu \lambda - \Gamma_\mu \lambda$ where, $\Gamma_\mu$ is the spin connection. In the expression of energy-momentum tensor the $F^{\mu\nu}$ term, which was absent in the earlier works, appears because of the variation of $\Gamma_\mu$ with respect to the metric (Böhmer et al., 2010). Using a simple ansatz of the perturbed NSS and its dual, $\delta\lambda = \delta\varphi\xi$, $\delta\vec{\lambda} = \delta\varphi\vec{\xi}$ where $\varphi$ is a scalar and $\xi$ is a constant spinor with the property $\vec{\xi}\xi = 1$, we have calculated components of the perturbed energy-momentum tensor. The perturbation theory for NSS becomes like a scalar field theory. However, calculation of the energy-momentum tensor shows the presence of additional terms in comparison with the standard canonical scalar field. We construct the modified Mukhanov-Sasaki equation for the NSS. Unlike scalar field case, the sound speed square is shown to be $c_s^2 \neq 1$ in general. The spectral index for the scalar perturbation is shown to give a nearly scale invariant power-spectrum which is consistent with the observation provided that $\tilde{F} \equiv \frac{\varphi^2}{8M_{pl}^2} < 10^{-4}$. With this upper bound $c_s^2 \sim 1$. Thus in case of first order perturbation theory, NSS becomes indistinguishable with the canonical scalar field theories.

In this thesis we have also studied the attractor behaviour of NSS cosmology. In inflationary and dark energy theories it is difficult to find exact initial conditions. Therefore it is important that these theories show the attractor behaviour, which will allow a wide class of solutions with different initial conditions to have similar asymptotic behaviour. The search for an attractor in case of NSS was attempted before also (see Wei, 2011). But no stable fixed points were found in the earlier attempts. In this thesis it is shown that the NSS equations can give inflationary-attractor which corresponds to 60 e-foldings. We have also demonstrated, with a new definitions of variables, that in the presence of barotropic perfect fluid the dynamical equations of the NSS can have stable fixed points. The stable fixed points can give us late-time attractor for NSS which can be useful in alleviating the cosmic coincidence problem. The stable fixed points are achieved by redefining the kinetic and potential part of NSS.

**Keywords:** Inflation, Elko, NSS, Cosmological perturbation theory, Dark energy, Cosmic coincidence problem.

# Contents

# Chapter 1

# Introduction

One of the most important problems of cosmology is to understand evolution of the universe. General relativity provides us a theoretical framework to understand the universe at different stages of evolution. Recent developments in the observational cosmology are providing us a very detailed picture of the universe by measuring various cosmological parameters with high accuracy. Whatever theoretical models that we have, must agree with the observations. According to the standard model of cosmology, the universe was created from an extremely hot and dense state of matter which after a prolonged period of expansion became the universe that we see today. This model of cosmology is also known as the Big Bang model and it is highly sucessful in explaining many observational data. Big Bang model was first proposed by Lemaître in 1927 [1]. Lemaître's model was based on the solutions of Einstein equation found by Friedmann [2, 3] describing an expanding universe with a homogeneous and isotropic matter distribution. This idea of the expanding universe was confirmed later with the observations by Hubble in 1929 [4]. A very important discovery in cosmology came in 1965 when Penzias and Wilson discovered Cosmic Microwave Background Radiation (CMBR) [5, 6]. The discovery rendered enormous observational support to the Big Bang cosmology. According to the standard model of cosmology, the matter-radiation decoupling occurred when the universe was approximately 380,000 years old (after the Big Bang). CMB photons are the photons which we see today are travelling freely after the decoupling. Hence it can reveal to us the nature of the distribution of various components of the universe at the time of decoupling. The measurement of CMBR shows nearly perfect black body nature of the photon spectrum [7] with current temperature $2.725 \pm 0.001$K [8] throughout the sky. NASA led Cosmic Background Explorers (COBE) has measured a very little inhomogeneity ($\sim 10^{-5}$) [9] in the temperature distribution of CMBR. Thus CMBR observations imply that the matter distribution in the early universe might be highly homogeneous and isotropic. Study of the observed inhomogeneity in CMBR can give us an idea about the physics at time much earlier than decoupling (see later). Another very important observation regarding the universe is its flatness. Wilkinson Microwave Anisotropy Probe (WMAP) has measured curvature parameter $\Omega_\kappa = \frac{\kappa}{(aH)^2}$ ($\kappa$ is the spatial curvature, $a$ is the scale factor and $H$ is the Hubble parameter) to be quite small [10]. WMAP seven years data, along with Baryon Acoustic Oscillation (BAO) and Hubble parameter measurement, has constrained the curvature parameter as $-0.0133 < \Omega_\kappa < 0.0084$ in 95% CL [8]. Therefore, it can be concluded that in the flat universe total energy density, in the unit of critical density $\varepsilon_{cr} = \frac{3H^2}{8\pi G}$ where $G$ is Newtonian gravitational constant, $\Omega_{\rm tot} = 1 + \Omega_\kappa \simeq 1$. From the supernovae data cosmological models based on a purely matter dominated universe ($\Omega_{\rm m} = 1$) has been ruled out in the flat universe [11]. The $\Lambda$CDM model of the universe includes a non-zero contribution of the cosmological constant $\Lambda$ in the total energy density along with the matter (baryonic matter and Cold Dark Matter (CDM)). The origin of cosmological constant is not clear to us. It is believed that non-zero vacuum energy can be behind cosmological constant. In the $\Lambda$CDM model total energy density in the flat universe can be written as $\Omega_{\rm tot} = \Omega_m + \Omega_\Lambda \simeq 1$ where, $\Omega_m$ and $\Omega_\Lambda$ are the matter density and the cosmological constant density respectively, in the unit of critical density. $\Lambda$CDM model has been very successful in explaining CMBR. From the very recent *Planck* data $\Omega_m = 0.314 \pm 0.020$, and $\Omega_\Lambda = 0.686 \pm 0.020$ in 68% CL [12]. Hubble parameter today is measured to have the value $H_0 = 67.3 \pm 1.2$. The age of the universe is calculate to be $13.817 \pm 0.048$ Gyr.

In this thesis, we focus on the accelerated phases of expansion of the universe. The universe has undergone the phase of acceleration twice in its history. The first phase of expansion is called inflation which occurred at a very early time during the evolution of the universe [13–16]. Inflation was proposed to solve two main problems of standard expanding universe model, viz. horizon puzzle and flatness puzzle. During the inflation the universe expands nearly exponentially. Because of the rapid expansion during inflation the causally connected region blowed up exponentially [14]. This can provide an explanation for the observed correlation of temperature between two causally disconnected region around angular scales $> 1^\circ$ in the present-day sky. Thus inflationary paradigm can solve the horizon problem. The rapid expansion during inflation also makes the universe extremely flat. Indeed, CMB data are consistent with the flatness assumption. It is generally expected that CMB data can tell us about the physics at the time of the matter-radiation decoupling. But the presence of the large scale correlation (at angular scale $> 1^\circ$) can provide us insight into the physics at time scale much earlier than the matter-radiation decoupling. If the inflationary theory is correct then it can leave some imprint in the CMB spectrum on super horizon scales. Indeed the inflationary theory produces the inhomogeneity in CMB spectrum. Considering the fact that the subsequent expansion can not dissolve the inhomogeneities produced during inflation, inflationary theory should be consistent with the observation of the CMB. Observationally the power spectrum (two point correlation function) of perturbation is scale invariant in large scale. Scale dependence of the power spectrum is measured by the quantity called spectral index. Spectral index of the scalar perturbation produced during inflation ($n_s$) is constrained by *Planck*+WMAP+BAO at $n_s = 0.9643 \pm 0.0059$ [17]. Exact scale invariance of scalar perturbation, $n_s = 1$ has been ruled out by *Planck* mission. CMB can also have an imprint of the primordial gravitational waves which could have been produced by tensor perturbations during inflation. The tensor perturbations can produce B-mode polarization by Thomson scattering of the electromagnetic wave at the time of recombination [18–20]. Therefore, the detection of B-mode polarization can prove the existence of primordial gravitational wave and thereby provides evidence of the inflation. However tensor perturbations are small compared to the scalar perturbation. The ratio of power spectrum of the tensor perturbation to scalar perturbation $r$, so called tensor-to-scalar ratio, has an observational upper bound $r < 0.12$ imposed by recent *Planck* mission [17].

After the end of inflation the standard Friedmann-Lemaître-Robertson-Walker (FLRW) expansion started. The observation of type Ia Supernovae data shows that the universe is currently going through a phase of accelerated expansion [11, 21]. It has ruled out pure matter dominated universe. The acceleration can be due to cosmological constant ($\Lambda$) which contribute the most in the energy budget of the universe. This second inflationary phase started at around red shift $z \sim 1$ (the age of the universe was approximately 9 billion years old) and it's still continuing. Late time accelerated expansion is thought to be caused by the presence of the dark energy. Cosmological constant in the Einstein equation is supposed to be one of the candidate of the dark energy [22,23]. There are many models of inflation and dark energy among them single scalar field model is well studied.

In this thesis we study inflationary scenario with the non-standard spinors [24]. Spinor fields have not attracted much in the context of cosmology compared to the scalar fields. In reference [25] the authors have studied the classical homogeneous spinor fields in the context of cosmology based on the action given in (2.6.12). In case of classical Dirac spinor one can consider the spinor fields coupled to gravity by adding a term like $\xi\bar{\psi}\psi R$, where $\psi$ and $\bar{\psi}$ is the Dirac spinor and its adjoint respectively, $R$ is the Ricci scalar and $\xi$ is the non-minimal coupling constant, in the action. However, it was also shown that during evolution the bilinear $\bar{\psi}\psi$ evolves as $\propto \frac{1}{a^3}$. Therefore, during the rapid expansion in an inflationary period the the non-minimal coupling term will rapidly vanish. Thus, the effects of non-minimal coupling terms becomes insignificant during inflation.

For the potential of the form $V = \frac{\left(\bar{\psi}\psi\right)^n}{\left(1+\bar{\psi}\psi\right)^n}$, which could be an example for quasi de Sitter inflation, to get e-folding greater than 60 the change in the bilinear $\bar{\psi}\psi$ has to be 180 orders of magnitude. This is extremely high in contrast to the standard chaotic inflation with scalar fields. The other problems in the standard classical spinor inflation are:

1. Reheating: In case standard inflationary models with scalar fields the uni-

verse is reheated when the scalar field oscillates around the bottom of its potential and decays into particle. In contrast the equation of motion for classical spinors are first order (Dirac equation), therefore, the spinor field can not oscillate. So, the reheating can not happen with the standard mechanism.

2. Scale dependence of the power spectrum: In [25] the authors have calculated the power spectrum and spectral index. It was shown that in case of classical spinors in large scale the power spectrum is strongly scale dependent. The calculation of spectral index shows that it is blue tilted with the value $\sim 4$. Thus, the prediction of Dirac spinors are in conflict with observation.

Recently one special type of spinor is proposed by Ahluwalia and Grumiler [24, 26] which are non-standard. These are called Elko (Eigenspinoren des Ladungskonjugationsoperators). Elko is an eigen spinor of the charge conjugation operator with dual helicity. These spinors are called non standard spinors (NSS) because unlike Dirac spinors NSS have mass dimension one and $(CPT)^2 = -\mathbb{I}$. These are also called 'dark spinor' as its dominant interaction channel is via Higgs and gravity. One of the important properties of these spinors is it follows Klein-Gordon equation instead of Dirac equation. Recently there has been some interest in spinor inflation and dark energy models with NSS. In references [27, 28] authors first considered NSS as inflaton and the authors calculated the back ground equations. They have shown that the equations are similar to canonical scalar field equations with some additional terms which are the coming because of the presence of spin-connection (see in section 2.6) while considering spinors in curved space-time. In Ref. [29] the authors calculated the cosmological perturbations. However, in [30] the authors have shown that the energy momentum tensor calculated in references [27–29] may be incorrect. In the earlier works while deriving the expression of energy-momentum tensor from the action, the variation of spin-connection with respect to metric was not properly calculated. This results in the disagreement between Euler-Lagrange equations calculated directly from the action and the equation of motion calculated from the continuity equation using the energy-momentum tensor. In the same work properly considering the variation of spin-connection the corrected expression of the energy-momentum tensor is calculated and back ground equations are corrected.

In [30] it has been argued that the form of NSS considered in [24] does not have any positive definite Lorentz invariant norm for the spinor resulting negative energy ghost modes. These ghosts modes can be eliminated by introducing some special choice of projection operator. But, the presence of the projection operator in the Lagrangian can include preferred axis, hence the theory can not be Lorentz invariant. However, to study inflationary scenario one can use a specific form of the homogeneous and isotropic NSS and its dual as: $\lambda = \varphi(t)\xi$ and $\bar{\lambda} = \varphi(t)\bar{\xi}$ where, $\varphi(t)$ is a homogeneous and isotropic scalar and $\xi$ is constant spinor with the property: $\bar{\xi}\xi = 1$. With the help of this ansatz, the action and various components of the energy momentum tensor for NSS can be expressed entirely in terms of the quantity $\bar{\lambda}\lambda$ (or, $\varphi^2(t)$) and its time derivatives. As in this case $\bar{\xi}\xi > 0$, one may not have the negative energy solutions, hence the theory can be free of the ghost modes. In addition, since the action and the energy densities become functions of $\varphi(t)$ in this ansatz, the theory can be treated as Lorentz invariant. Therefore, in this thesis we use the above ansatz to study cosmology of NSS models.

In cosmology, it is useful that the dynamical equations allow to have attractor solutions during inflation. As the initial conditions are not known, the attractor behaviour can allow a wide class of solutions with different initial conditions to have similar asymptotic behaviour. In cosmology this has been done by analysing the stability of the fixed points. The attractor nature of the dynamical equations also helps us to alleviate cosmic coincidence problem associated with the dark energy. As the universe expands, the matter density and dark energy density evolve differently. The matter density falls as $\propto a^{-3}$ where as dark energy density remains almost constant. Therefore, it is not clear why at red shift $z \sim 1$ the dark energy dominated universe started. If it is because of the reason that the initial conditions were such that at red shift $z \sim 1$ the dark energy domination started, then the initial conditions should be extremely fine tuned – which is known as cosmic coincidence problem. In reference [31] the stability analysis was done for NSS in the context of dark energy. It was concluded that there do not exist any stable fixed points for NSS. In this thesis we have done the stability analysis with a set of variables different than those used in [31].

The thesis mainly focuses on the following two issues in the NSS cosmology: 1. Developing a consistent cosmological perturbation theory, 2. to study the robustness of NSS based models to produce accelerated expansion.

1. We ask the question if, unlike classical Dirac spinors, NSS driven inflation can give us scale invariant power spectrum at large scale? To answer this question we have used consistent NSS theory [30] and studied first order perturbation theory [32]. It has been shown that the scalar spectral index in case of NSS can be in the observed range provided $\varphi$ satisfies an upper bound $\frac{\varphi^2}{8M_{pl}^2} < 10^{-4}$ where $M_{pl}$ is the reduced Planck mass.

2. The other issue that has been studied in this thesis is the attractor behaviour in the NSS cosmology [33]. It is shown that inflationary attractor can exist in case of NSS. With the redefinition of the variables it is shown that the dynamical equations can behave as an attractor in the inflationary era. It is also shown that in case of late-time acceleration the dynamical equations have stable fixed points which can alleviate the cosmic coincidence problem associated with the dark energy.

**Plan of the thesis:** In Chapter-2 we give a general description of the inflationary universe. We start with the standard model of cosmology with the FLRW metric as a solution of Einstein equation. Using the approximations for the metric and the energy-momentum tensor for an ideal fluid, the Einstein equation is written in terms of the Friedmann equation and the acceleration equation. With the help of acceleration equation the condition for the accelerated expansion (negative pressure) has been established. Next, we describe motivation for inflation, namely — two puzzles associated with the standard model of cosmology: horizon puzzle and flatness puzzle. Solutions of these puzzles using inflationary scenario has been explained in this section. After this we introduce the concept of dark energy in order to explain the current acceleration of the universe. Next we study the inflationary universe scenario with a canonical single scalar field. We have explained some of the general features of the inflationary scenario with scalar field like, slow-roll, fast-roll and multi-field inflation. Then we consider the spinors in the curved space-time. We discuss the tetrad formalism which is important for incorporating spinors in the curved space-time and use it to write the Dirac equation in the curved space-time. Finally, we write the consistent set of NSS equations. It has been shown that in comparison with the canonical scalar field case, the Friedmann equation and acceleration equation in case of NSS have additional terms proportional to $\tilde{F} = \frac{\varphi^2}{8M_{pl}^2}$ when a simple ansatz on the form of NSS is used. From Friedmann equation it is shown that the condition $\tilde{F} < 1$ is necessary in order to have a real Hubble parameter $H$, i.e. $H^2 > 0$.

In Chapter-3 we focus on the first order perturbation theory of NSS based inflationary model. At first we provide discussions on the metric perturbation. Then we discuss properties of the gauge transformation and define the gauge invariant quantities using the metric perturbation. By using a simple ansatz, the NSS perturbations can be expressed in terms of scalar quantity $\delta\varphi$. So in this chapter we discuss the various aspects of the perturbation theory for single scalar field. Then we construct the Mukhanov-Sasaki equation and show that the solutions of this equation leads to nearly scale invariant power spectrum under the slow-roll paradigm. Next, we calculate the perturbed energy-momentum tensor for NSS. It is demonstrated that the pressure perturbation for NSS, in general, can be anisotropic. Using the assumption that $\tilde{F} \ll 1$, we calculate the modified Mukhanov-Sasaki equation for NSS in the linear order of $\tilde{F}$. We solve the Mukhanov-Sasaki equation for NSS as corrections to the solutions found in the single scalar field case. Finally, we show that NSS can give us a nearly scale invariant power spectrum, which is consistent with the observation if the term $\tilde{F}$ satisfies an upper bound, $\tilde{F} < 10^{-4}$.

In Chapter-4 we focus on the early universe attractor scenario with NSS. We discuss the attractor scenario in case of the canonical scalar field. We have shown that in case of NSS, the Friedmann and acceleration equation can show attractor behaviour during inflation. The attractor in this case corresponds to 60 e-folds, which is necessary for successful inflationary scenario, when $\tilde{F} < 10^{-4}$. Then we find the fixed points for the dynamical equations of NSS in the presence of a perfect barotropic fluid with a new set of variables. After that a general analysis of the stability of the fixed points has been discussed. In this chapter it is shown that in case of NSS the fixed points can be stable. The stable fixed points can give us late-time attractor which can be helpful in alleviating the cosmic coincidence problem associated with the dark energy.

Chapter-5 contains summary and discussions.

# Chapter 2

# Inflationary universe

## 2.1 Introduction

In this section we briefly review the various aspects of inflationary cosmology. As we have already mentioned that theme of our thesis is spinors in inflation, particularly non-standard spinors, it is necessary that we document the developments in this regard. We first start with the standard model of cosmology based on the FLRW metric (2.2). Then we briefly discuss the drawbacks of the standard model of cosmology and we introduce the idea of inflation which proves to be the potential model for our observed universe (2.3). In section (2.4) we discuss about the present stage of accelerated expansion which can be also be considered as quasi-inflationary stage. After that we present the various scalar field models of inflationary universe such as slow-roll inflation, fast-roll inflation and multi field models of inflation (2.5). In sections (2.6) and (2.7) we discuss the inflationary model based on spinors.

## 2.2 The standard model of cosmology: FLRW universe

The standard model of cosmology is based on the observational fact that the universe is homogeneous and isotropic over length scale greater than 100Mpc.

As we have sufficient evidence that our universe is expanding, geometry of the universe can be described by FLRW metric. To understand the nature of the expansion cosmologists prefer to work using the *comoving formalism.* In this formalism *comoving distance* between the any two points remain same. And the physical distance between the points can be obtained by multiplying *comoving distance* with a scale factor, $a\left(t\right)$. The scale factor contains information about the evolution of our universe: If the scale factor increases with time, the distance between the two points increases and thus describes the expanding universe. If the scale factor decreases with time we have contracting universe. The dependence of scale factor on time varies in different era of evolution (e.g. $\propto t^{1/2}$ in the radiation dominated era, $\propto t^{2/3}$ in the matter dominated era). One of the important quantities that quantify the change in the scale factor is the Hubble parameter defined as $H = \dot{a}\left(t\right)/a\left(t\right)$. Here the 'dot' denotes the derivative with respect to cosmic time $t$. In terms of the Hubble parameter the acceleration can be written as

$$\frac{\ddot{a}}{a} = H^2\left(1 + \frac{\dot{H}}{H^2}\right). \tag{2.2.1}$$

Equation (2.2.1) is an identity and it contains information about geometry. Now the question is what drives the dynamics of the scale factor. The answer lies in the Einstein equations in general relativity, which connects the geometry with the energy-momentum tensor describing the energy density.

The Einstein equation in general space-time is given as

$$R^{\mu}_{\nu} - \frac{1}{2}g^{\mu}_{\nu}R = 8\pi G T^{\mu}_{\nu}. \tag{2.2.2}$$

The left hand side is known as Einstein tensor $G^{\mu}_{\nu} \equiv R^{\mu}_{\nu} - \frac{1}{2}g^{\mu}_{\nu}R$. Here, $G$ is Newton's gravitational constant and $T^{\mu}_{\nu}$ describes the energy-momentum tensor of the matter. The Einstein tensor is a function of the metric and its derivatives and therefore, contains the information of space-time. The term $R^{\mu}_{\nu}$ is the mixed form of the Ricci tensor $R_{\mu\nu}$ which is defined as follows:

$$R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\lambda\eta} - \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\eta}, \tag{2.2.3}$$

where, $\Gamma^{\mu}_{\nu\rho}$ is the Christoffel symbol which can be expressed in terms of the metric $g_{\mu\nu}$ as follows

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left[\partial_{\nu} g_{\sigma\rho} + \partial_{\rho} g_{\sigma\nu} - \partial_{\sigma} g_{\nu\rho}\right]. \tag{2.2.4}$$

The term $R$ is called Ricci scalar, which can be obtained by contracting Ricci tensor into the metric:

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.2.5}$$

$g_{\mu\nu}$ contains the information about the space-time. In four dimension the flat space time the metric $g_{\mu\nu}$ is given by the diagonal Minkowski metric $\eta_{\mu\nu} = diag\left(1, -1, -1, -1\right)$, where the first component corresponds to temporal part and the last three corresponds to the spatial part. As in the case of isotropic and homogeneous expanding universe the physical distance is proportional to the scale factor, one can write the metric by multiplying the space part of $\eta_{\mu\nu}$ with the scale factor:

$$g_{\mu\nu} = diag\left\{1, -a^2\left(t\right), -a^2\left(t\right), -a^2\left(t\right)\right\}. \tag{2.2.6}$$

The metric (2.2.6) is known as FLRW metric. $g^{\mu\nu}$ is 'the inverse' of the metric (2.2.6) defined as $g^{\mu\nu} = diag\left\{1, -\frac{1}{a^2}, -\frac{1}{a^2}, -\frac{1}{a^2}\right\}$. One can see that in case of the Minkowski metric $\eta_{\mu\nu}$ all the components of Christoffel symbol are zero. However, this is not the case when $g_{\mu\nu}$ is considered. Different components of Christoffel symbol for (2.2.6) are listed in the appendix (A.1.1). The expressions for the Ricci scalar and for the different components of Ricci tensor (2.2.6) are listed in the appendix (A.1.2). For isotropic perfect fluid the energy-momentum tensor can be written as a diagonal matrix where energy density is the time-time component of the energy-momentum tensor and pressure is given by the negative of the space-space components:

$$T^{\mu}_{\nu} = diag\left\{\varepsilon, -p, -p, -p\right\}. \tag{2.2.7}$$

Finally, using (A.1.4) and (A.1.5) one can write the Einstein equation for the energy-momentum tensor (2.2.7) in terms of scale factor as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\varepsilon, \tag{2.2.8}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -8\pi G p. \tag{2.2.9}$$

Combining (2.2.8) and (2.2.9) the acceleration $\ddot{a}$ becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\varepsilon + 3p\right). \tag{2.2.10}$$

From (2.2.10) it can be checked that for ordinary matter (pressure $p = 0$) and for radiation ($p = \varepsilon/3$) the universe expands with deceleration ($\ddot{a} < 0$).

The dynamics of the fluid quantity may come from $\nabla_\mu T^{\mu\nu} = 0$. But the equation of state relating $\varepsilon$ and $p$ is needed in order to close the fluid equation. Setting $\nu = 0$ in $\nabla_\mu = 0$, one can obtain the continuity equation in the FLRW background which can be written as follows:

$$\dot{\varepsilon} + 3H\left(\varepsilon + p\right) = 0. \tag{2.2.11}$$

From the above continuity equation one can understand that in the FLRW universe as the universe expands, in the radiation dominated era the energy density falls as $\propto a^{-4}$ and in the matter dominated era it falls as $\propto a^{-3}$. Substituting this in (2.2.8) one can easily see that in the radiation dominated era the scale factor grows as $\propto t^{1/2}$ and in the matter dominated era the scale factor grows as $\propto t^{2/3}$.

## 2.3 Drawbacks of the standard model of cosmology

Although the FLRW model of our universe satisfactorily describes the contents of our universe, still FLRW universe has some problems. There are two major issues with the standard model of cosmology namely: (i) Horizon puzzle and (ii) flatness puzzle in order to understand these puzzles it is useful to write most general form of FLRW metric in polar coordinate can be written as

$$g_{\mu\nu} = diag\left\{1, -a^2\frac{1}{1-\kappa r^2}, -a^2r^2, -a^2r^2sin^2(\theta)\right\} \tag{2.3.1}$$

where, $\kappa$ is the spatial curvature and have values $\kappa = -1, 0, +1$ depending on whether the universe is open, flat and closed respectively. The Friedmann equation now becomes

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\varepsilon. \tag{2.3.2}$$

*i. Horizon puzzle:*

Today the causal horizon size is $l_0 = ct_o$, where $t_0$ is the age of the universe $t_0 \sim 10^{17}$sec. At Planckian time ($t_{pl} \sim 10^{-43}$sec) from which the universe was originated, must have a size greater than $l_{pl} = l_0\frac{a_{pl}}{a_0} = ct_o\frac{a_{pl}}{a_0}$. Now, the causal horizon size at Planck time is $l_c = ct_{pl}$. Comparing $l_{pl}$ and $l_c$ one can get $\frac{l_{pl}}{l_c} = \frac{t_0}{t_{pl}}\frac{a_{pl}}{a_0}$. As $\frac{a_{pl}}{a_0} \sim 10^{-32}$, it can be shown that the size of the universe at nearly Planck time was at least $10^{28}$ times greater than the size of the causal horizon. Now the puzzle is how a smooth distribution of temperature with fluctuation $\sim 10^{-5}$ is observed in the CMBR over a large number of causally disconnected region.

*ii. Flatness puzzle:*

Friedmann's equation (2.3.2) can be written as: $\Omega(t) - 1 = \frac{\kappa}{(aH)^2}$, where $\Omega = \frac{\varepsilon}{\varepsilon_{\rm cr}}$, $\varepsilon_{\rm cr} = \frac{3H^2}{8\pi G}$ is called the critical energy density. From here we get $\Omega_i(t) - 1 = (\Omega_0(t) - 1)(\frac{\dot{a}_0}{\dot{a}_i})^2$. In the discussion of horizon problem we saw that $\frac{l_i}{l_c} \sim \frac{t_0 a_i}{t_i a_o} \sim 10^{28}$. Now as the scale factor is a function of time only we take $\frac{a}{t} \sim \dot{a}$. Therefore we find that $\Omega_i(t) - 1 \leq 10^{-56}$. This means that in the early universe critical energy density was unity(flat universe) with a fluctuation of $10^{-56}$. $\Omega$ can also be written as the ratio of gravitational potential energy and kinetic energy. Now, for $\Omega$ is >1, <1 or =1 respectively signifies closed, open or flat universe. Thus in standard model of cosmology, the early universe the value of $\Omega$ was so fine tuned that any very very small deviation (of the order $10^{-56}$) from 1 would lead to a closed or open. This problem of fine tuning is called flatness puzzle.

These puzzles can be solved if the universe had an accelerated phase of expansion some time early in its expansion history. This phase of acceleration is called inflation. To retain the success of standard model of decelerated expansion of the universe we say that the inflation started and ended very early. During inflation the universe expanded almost exponentially. Actually due to this accelerated expansion, the region of space that were in causal contact before inflation, became causally disconnected during inflation. This paradigm can also explain the observed flatness ($\kappa = 0$) in the CMB spectrum.

## 2.4 Second inflationary stage: Dark energy

We know from observations of CMB that the total energy density of the universe is very close to its critical value, i.e. $\varepsilon_{\rm tot} = \varepsilon_{\rm cr}$ or $\Omega_{\rm tot} = \frac{\varepsilon_{\rm tot}}{\varepsilon_{\rm cr}} = 1$. Discovery of the dark energy is fairly recent. The late-time acceleration which is still continuing is attributed to the dark energy, the name suggestive of our ignorance about its origin. Evidences from type Ia supernova data [11, 21] have rejected the matter dominated expansion after red shift $z \sim 1$. According to our current understanding at present the total energy-budget of the universe consists of 31.4% matter and 68.6% is inform of dark energy [12]. Dark energy can be regarded to be distributed smoothly (in homogeneous and isotropic fashion) over the entire present universe.

One of the simplest models of the universe which can explain the late-time accelerated expansion is based on inclusion of cosmological constant $\Lambda$ in the Einstein equation. Energy density related with $\Lambda$ does not dilute with expansion of the universe and at some point in time it can start dominating over the other energy densities to give the accelerated expansion. Energy density and pressure of the cosmological constant term are respectively given by $\varepsilon_\Lambda = \Lambda$ and $p_\Lambda = -\Lambda$ and thus have the equation of state $p_\Lambda = -\varepsilon_\Lambda$ necessary to give accelerated expansion. But this model of dark energy suffers one theoretical problem. The non-zero vacuum energy density is believed to be acting as the cosmological constant $\Lambda$. From field theoretic calculation it is found that calculation of zero point energy leads to the value of $\Lambda$ which is 60-120 orders of magnitude higher than the observed value [23, 34]. There is an issue with the dark energy models with finite $\Lambda$ which is called 'coincidence' problem. It asks the question, why the dark energy domination started at $z \sim 1$? One possible answer could be that the initial conditions were such that the dark energy domination started at $z \sim 1$. But this will make the initial conditions extremely fine tuned. In cosmology we avoid this kind of fine tuning. To solve 'coincidence', we consider problem dark energy models with a time dependent scalar field $\varphi(t)$ called quintessence and this quintessence field evolve in such a way that irrespective of its initial condition it starts following the track that we know is needed for this transition to take place at $z \sim 1$. In other word dynamical equations should show some attractor behaviour. This kind of attractor solutions are also called tracking solutions.

The quintessence models [35] follow the similar set of equations as of inflation. So, writing the energy-momentum tensor and identifying its energy density and pressure we find that if $\varphi(t)$ is varying very slowly with time we can neglect $\dot{\varphi}^2$ to get $\frac{p(\varphi)}{\varepsilon(\varphi)} = -1$ which is desired equation of state.

In addition there are some other models of dark energy: Composite scalar models of dark energy. Neutrinos may be one of such candidates which condensates and form scalar [36]. In this thesis we investigate the role of NSS in the accelerated expansion of the universe.

## 2.5 Scalar fields in inflation

In this section we discuss some of the important features of the inflationary universe. At first we discuss the standard slow-roll inflationary scenario in which the smallness of slow-roll parameters $\epsilon \ll 1$ and $\eta \ll 1$ help us in getting the negative pressure required for the accelerated expansion. If the second slow-roll condition is violated ($\eta \sim 1$) still one can have accelerated expansion, which is briefly described in the fast-roll section. In a later section we will also discuss the multi field scenario. Some of the general features will be used in the later chapters of the thesis.

### 2.5.1 Slow-roll

From Friedmann equation we find that to get inflationary universe we need $p < -\frac{1}{3}\varepsilon$. We can understand the this odd equation of state from field-theoretic point of view. Let us consider the simple field-theoretic model where inflation is driven by a scalar field. It is called inflaton which is function of time only. That energy momentum tensor for this inflaton can be identified as that of a perfect fluid. As scalar field is independent of space coordinates, we find energy density and pressure as followed:

$$\varepsilon = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \qquad p = \frac{1}{2}\dot{\varphi}^2 - V(\varphi). \tag{2.5.1}$$

Therefore the equation of state is

$$\frac{p}{\varepsilon} = \frac{\frac{1}{2}\dot{\varphi}^2 - V(\varphi)}{\frac{1}{2}\dot{\varphi}^2 + V(\varphi)}. \tag{2.5.2}$$

So from equation (2.5.2) we get, when $\dot{\varphi}^2 \ll V(\varphi)$, $p = -\varepsilon$. That means, when the scalar field is rolling slowly in a flat potential, its kinetic energy is negligible compared to potential energy, then we get our desired equation of state. Therefore to get the two initial problems solved we need to keep $\dot{\varphi}^2$ much smaller than $V(\varphi)$ for a sufficiently long time, for about 75 e-folds. Now if we write the Klein-Gordon equation for the scalar field it looks like:

$$\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi} = 0. \tag{2.5.3}$$

As the scalar field is rolling very slowly down the potential, we can neglect $\ddot{\varphi}$ compared to $\dot{\varphi}$ and $V(\varphi)$. Therefore using (2.5.3) the first order slow roll parameters in terms of the potential can be defined as:

$$\epsilon_V = \frac{1}{16\pi G}\left(\frac{V_{,\varphi}}{V}\right)^2, \; \eta_V = \frac{1}{8\pi G}\left(\frac{V_{,\varphi\varphi}}{V}\right), \; \delta_V = \eta - \epsilon.$$

Now in terms of slow roll parameters we can say that inflation occurs when $\epsilon_V \ll 1$ and $\eta_V \ll 1$. In the slow-roll case the inflation ends when $\epsilon_V \sim 1$.

•**Hamilton-Jacobi approach:** So far we have treated the inflaton field as the fundamental quantity in case of inflation. It is also possible to define the slow-roll quantities entirely in terms of Hubble parameter $H$. This approach is also known as *Hamilton-Jacobi approach* [37]. One of the advantages of using *Hamilton-Jacobi approach* is that it can remove the explicit time dependence as the independent variable in this case is the field itself. Using (2.5.1) one can write the Friedmann equation and the acceleration equations for flat universe respectively as

$$H^2 = \frac{\kappa^2}{3}\varepsilon = \frac{1}{3}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right]. \tag{2.5.4}$$

$$\dot{H} = -\frac{\kappa^2}{2}(\varepsilon + p) = -\frac{1}{2}\dot{\varphi}^2. \tag{2.5.5}$$

Here in the last lines we have set $\kappa^2 = 8\pi G = 1$. As the expression of $\dot{H}$ tells us that the value of Hubble parameter goes down with time, equation (2.2.1) suggests that $|\frac{\dot{H}}{H^2}| < 1$ to get acceleration ($\ddot{a} > 0$). As $H_{,\varphi} = \dot{H}/\dot{\varphi}$, using (2.5.5) the expression of $H_{,\varphi}$ becomes

$$H_{,\varphi} = -\frac{1}{2}\dot{\varphi}. \tag{2.5.6}$$

Finally, the Friedmann equation (2.5.4) can be written as,

$$H^2(\varphi) = \frac{2}{3}H_{,\varphi}^2 + \frac{V}{3}. \tag{2.5.7}$$

Therefore, using (2.5.5) and (2.5.6) the slow-roll parameters can be defined in terms of Hubble parameters as

$$\epsilon_H = -\frac{\dot{H}}{H^2} = 2\left(\frac{H_{,\varphi}}{H}\right)^2, \qquad \eta_H = \frac{\ddot{\varphi}}{H\varphi} = 2\frac{H_{,\varphi\varphi}}{H}. \tag{2.5.8}$$

During inflation both $\epsilon_H \ll 1$ and $\eta_H \ll 1$. The definition of change in the number of e-folding also can be in terms of the Hubble parameter as

$$\Delta N = -\frac{H}{2H_{,\varphi}}\Delta\varphi = -\frac{\Delta\varphi}{\sqrt{2\epsilon_H}}. \tag{2.5.9}$$

Here we have considered that during inflation the change in the inflaton is very small. The value of inflaton changes towards the end of inflation [38].

Inflation has strong evidence in the Cosmic Microwave Background data from WMAP(Wilkinson Microwave Anisotropy Probe) satellite. WMAP measures the temperature and temperature anisotropy of CMBR. At large scale(early time) it has evidence of inflation. Fluctuation of inflaton $\delta\varphi$ implies fluctuation of energy-momentum tensor($\delta T_{\mu\nu}$) which implies fluctuation in metric. Fluctuation in CMB temperature can be related to gravitational potential(in the metric perturbation) at last scattering surface. So temperature fluctuation can be related to the inflaton fluctuation. Finally if we calculate power spectrum of the inflaton fluctuation we can see it fits very well with the WMAP data.

### 2.5.2 Fast-roll

So far we have seen that in case of slow-roll inflation we need two independent conditions, $\epsilon \ll 1$ and $\eta \ll 1$. The usefulness of slow-roll conditions is mainly twofolds: One is, it produces a very large expansion during inflation and the other one is production of scale invariant power spectra of density perturbation which gives us nearly isotropic universe, consistent with observation. In other words, if the perturbations that we observe today are the primordial perturbations produced during inflation, they must have occurred when inflaton was on the top of the potential. Now the question can be asked is: Can inflation occur if the slow-roll conditions are violated? From Friedmann equations one point is clear that to get scale factor growing exponentially $\epsilon \ll 1$ must be satisfied. If this condition is violated, we can't treat Hubble parameter ($H$) as a constant – hence, exponential expansion will not be possible. Thus, in the fast-roll scenario only possible violation of slow-roll can be $\eta \sim 1$.

For inflaton with mass $m$ the slow-roll condition $\eta \ll 1$ can be translated into $|m^2| \ll H^2$. However, in various models of supergravity theories the mass square terms are observed being of the order of the square of Hubble parameter, $|m^2| = \mathcal{O}(H^2)$ [39, 40]. Non minimal coupling scenarios [41, 42] with conformal coupling constant ($\xi = 1/6$) also gives the correction in the mass square term $\sim H^2$. In reference [43, 44] the authors have studied the inflationary theory in the context of fast-roll. It has been shown that in case of potentials which are unbounded from below (for example $V = V_0 - m^2\varphi^2/2$), under the condition $m^2 = \mathcal{O}(H^2)$, the scale factor can grow exponentially. Fast-roll inflation can be an interesting scenario to study at the beginning and the end of slow-roll inflation. It can also be the possible reason behind the current acceleration of our universe [44].

### 2.5.3 Multi field inflation

So far we have described the standard single field inflationary scenario. In this picture the inflationary scenario has following stages: The scalar field very slowly rolls down the potential which gives acceleration of our universe. Towards the end of inflation, kinetic energy of the field becomes comparable to potential energy and the inflation ends with the violation of the slow-roll. The primordial perturbations that we observe today might have originated at the time of inflation. At the end of the inflation, the inflaton field decays into particles by oscillation near the minimum of the inflaton potential and transfers all its energy to the created particles [45]. The created particles finally come to thermal equilibrium by interacting among themselves and the universe reheats.

It is natural to have more than one scalar field can during the inflation. One of the simplest way to consider the multi field inflationary scenario is provided by the hybrid models where two scalar fields are required to have the inflation. In hybrid inflation one field contributes the most to the total energy density and thus gives accelerated expansion. While the second field remains sub-dominant during inflation and this field would not contribute to the expansion. Hybrid models are extension of new-inflation and hybrid inflation is not eternal [46, 47]. One very popular example of hybrid inflation is the two field case where one field $\varphi$ has potential $V(\varphi) = m^2\varphi^2/2$ with mass $m$ and the other scalar field $\sigma$ (sometimes known as 'waterfall field') has the symmetry breaking potential $V(\sigma) = \frac{1}{4\lambda}\left(M^2 - \lambda\sigma^2\right)^2$, where $M$ is mass and $\lambda$ is the self coupling parameter for $\sigma$. The hybrid potential with the interaction term between the two scalar fields can be written as, $V(\varphi, \sigma) = \frac{1}{4\lambda}\left(M^2 - \lambda\sigma^2\right)^2 + \frac{1}{2}m^2\varphi^2 + \frac{1}{2}g^2\varphi^2\sigma^2$, where $g$ is the coupling constant. The main difference between this model of hybrid model and the single field chaotic model is the end of inflation: In the single field model inflation ends when the $\varphi$ potential becomes steep, but in the hybrid model the inflation ends when the potential in the $\sigma$ direction becomes steep [16]. This may give some freedom in the model building of inflation. Production of curvature perturbation in hybrid inflation can be found in references [48, 49]. Hybrid inflation in case of inflaton field ($\varphi$) non-minimally coupled to gravity has been discussed in [50].

Another popular example of multi field inflation is *Curvaton* model [51]. In a curvaton model there exist two scalar fields: One is called inflaton field which drives the inflation and the other field, known as curvaton, seeds the density perturbation observed today. The curvaton field remains sub-dominant during inflation, so it does not participate during inflation. Therefore, the slow-roll conditions – which are must in case of inflaton field – are not necessary in case of curvaton. The curvature perturbation due to curvaton takes place in two stages: During inflation the quantum fluctuation in curvaton becomes classical perturbation at the time of horizon exit and then the classical perturbation is converted into curvature perturbation. The details of the perturbation theory of curvaton can be found in [52]. In contrast to the standard curvaton scenario, in reference [53] the authors have considered the double inflation with the second phase of inflation is due to slowly rolling curvaton field. It has been shown that inflating curvaton also has a significant contribution in curvature perturbation.

## 2.6 Spinors in curved space-time

While treating spinros in a curved space-time it is useful to introduce the concept of *tetrads* [54, 55].

• The *tetrad formalism*: In the Minkowski space-time infinitesimal Lorentz transformation is described as

$$x^a \to \tilde{x}^a = \Lambda^a_b x^b, \tag{2.6.1}$$

where, $a$ and $b$ are the Lorentz indices associated with inertial frame and $\Lambda^a_b = (\delta^a_b + \sigma^a_b)$. $\sigma_{ab}$ is an antisymmetric tensor and has the value much smaller than one, $|\sigma_{ab}| \ll 1$. Under the above infinitesimal Lorentz transformation any general field $\mathcal{F}$ transform as

$$\mathcal{F} \to \tilde{\mathcal{F}} = D\,(\Lambda)\,\mathcal{F}. \tag{2.6.2}$$

Here the quantity $D\,(\Lambda)$ can be written as,

$$D\,(\Lambda) = 1 + \frac{1}{2}\sigma_{ab} f^{ab}, \tag{2.6.3}$$

where $f_{ab}$ is the generator of the Lorentz group. The above transformation law is valid for any physical quantity under a Lorentz transformation, it could be scalar, vector, tensor of rank 2 (or above) or it could be a spinor. Depending on our interest the generator takes different forms, for example, in case of scalar $f_{ab} = 0$ and for spinors $f_{ab}$ can be expressed in terms of the $\gamma$ matrices (defined in the next section). To accommodate the general field in the curved space time, respecting the Lorentz transformation, we need to use the *tetrad or vierbien formalism.*

In *tetrad formalism* we erect normal coordinates associated with the local inertial frames ($\xi^a_X$) at each points $X$ of the curved space-time. At each point $X$ in the space-time the coordinate $\xi^a_X$ has the flat Minkowski metric $\eta_{ab}$. But in general coordinate system the metric $g_{\mu\nu}$ can be related with the Minkowski metric as

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}. \tag{2.6.4}$$

Here the quantities $e^a_\mu$ are called *tetrads or vierbiens* and they are defined as

$$e^a_\mu = \frac{\partial \xi^a_X}{\partial x^\mu}. \tag{2.6.5}$$

• Properties of *tetrads*: The *tetrads* transforms as a vector under both the general coordinate and the Lorentz transformations. The transformation rules for the *tetrads* under the general coordinate and Lorentz transformation are respectively,

$$e^a_\mu \to \frac{\partial x^\nu}{\partial x'^\mu} e^a_\nu, \qquad e^a_\mu \to \Lambda^a_b e^b_\mu. \tag{2.6.6}$$

Inverse of the tetrads can be found from the following normalization conditions

$$e^a_\mu e^\nu_a = \delta^\nu_\mu, \quad e^a_\mu e^\mu_b = \delta^a_b. \tag{2.6.7}$$

When any covariant coordinate vector (transforms as a vector under a coordinate transformation) is contracted into a tetrad then, it gives us a quantity which transforms as a vector under a Lorentz transformation, at the same time it transforms as a scalar under a general coordinate transformation. For example, if a four coordinate vector $A_\mu$ is contracted into $e^\mu_a$ we get,

$$A_a = e^\mu_a A_\mu. \tag{2.6.8}$$

$A_a$ is a scalar under a coordinate transformation and a vector under a Lorentz transformation. Similarly, when covariant Lorentz vector $A_a$ is contracted into tetrads, it gives us a Lorentz scalar which at the same time is a coordinate vector. Therefore, using tetrad formalism one can bring the spinors (in general, any other field of arbitrary spin) into the considerations of general relativity.

As there are two kind of transformations, to write a sensible action we have to ensure that the action is invariant under both coordinate as well as the local Lorentz transformations. As the action contains derivatives of the physical quantity of interest and the quantity itself, we have to ensure that the action is coordinate scalar as well as Lorentz scalar in spite of the presence of the derivatives. One way to do this is defining the covariant derivatives which contains tetrads. In case of classical spinors, which follows Dirac equation, the action is linear in space-time derivative – $\partial_\mu$. Since $\partial_\mu$ is a coordinate vector, one can make it a coordinate scalar by contracting it into $e^\mu_a$, i.e., $e^\mu_a\partial_\mu$. When we operate $e^\mu_a\partial_\mu$ on any spinor $\psi$ one can see that the transformation (2.6.2) does not allow $e^\mu_a\partial_\mu\psi$ transform like a Lorentz vector. Thus, one can construct a Lorentz vector by defining a 'coordinate scalar Lorentz vector derivative' $\nabla_a$ which, when operated on a spinor transforms like

$$\nabla_a\psi \to \Lambda^b_a D\left(\Lambda\right)\nabla_b\psi. \tag{2.6.9}$$

This can be achieved when $\nabla_a$ is defined as:

$$\nabla_a = e^\mu_a\left(\partial_\mu + \Gamma_\mu\right), \tag{2.6.10}$$

where, $\Gamma_\mu$ is given by

$$\Gamma_\mu = \frac{1}{2} f^{ab} e^\nu_a\left(\partial_\mu e_{b\nu} - \Gamma^\rho_{\nu\mu} e_{b\rho}\right). \tag{2.6.11}$$

In the above expression $\Gamma^\rho_{\nu\mu}$ is the Christoffel symbol in the curved space-time. Therefore, the invariant action for the Dirac spinor defined in curved space-time is,

$$S_{\text{Dirac}} = \int \sqrt{-g}\left[\frac{i}{2}\left(\bar{\psi}\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma^\mu\psi\right) - V\right] d^4x, \tag{2.6.12}$$

where, $\bar{\psi}$ is the adjoint of $\psi$, $V$ is the potential, $\nabla_\mu = (\partial_\mu + \Gamma_\mu)$ and $\gamma^\mu = e^\mu_a\gamma^a$ is the gamma matrices in the curved space time with anti-commutation relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Thus Dirac equation in a curved space-time becomes

$$i\Gamma^\mu\nabla_\mu\psi - m\psi = 0, \tag{2.6.13}$$

where $\Gamma^\mu$ is the spin connection defined in equation (2.6.11) and $m$ is the mass. In the above equation the form of the potential is chosen as $V = m\bar{\psi}\psi$. In the FLRW background the equation of motion for the homogeneous spinor field becomes

$$\dot{\psi} + \frac{3}{2}H\psi + i\gamma^0 m\psi = 0. \tag{2.6.14}$$

## 2.7 Non-standard Spinors

Recently there is a lot of interests in studying dark or Non-Standard Spinor (NSS). The theory of NSS was first developed in Refs. [24, 56]. Subsequently the NSS models were further developed and investigated by several authors [57–65]. These spinors can be regarded as 'dark' as their dominant interaction is with Higgs and via gravitational field only and they have been extensively applied to study above mentioned problems in cosmology [27–31, 66–68]. Unlike the Dirac, Majorana or Weyl spinors, NSS propagator behaves like $1/p^2$ in the large momentum limit and has mass dimension one. At present the theory of NSS is under development, however, NSS are known to be either violating the Lorentz invariance or locality or both. Basic Lagrangian of NSS can be written as

$$\mathcal{L}_{\text{cosmo}} = \frac{1}{2}\vec{\lambda}\overleftarrow{\nabla}_\mu \nabla^\mu \lambda - V(\vec{\lambda}\lambda), \tag{2.7.1}$$

where, $\vec{\lambda}\overleftarrow{\nabla}_\mu \equiv \partial_\mu\vec{\lambda} + \vec{\lambda}\Gamma_\mu$, $\nabla_\mu\lambda \equiv \partial_\mu\lambda - \Gamma_\mu\lambda$. $\lambda$ and $\vec{\lambda}$ are NSS and its dual respectively. $\Gamma_\mu$ are defined as

$$\Gamma_\mu = \frac{i}{4}\omega_\mu^{ab} f_{ab}, \qquad f^{ab} = \frac{i}{2}\left[\gamma^a, \gamma^b\right] \tag{2.7.2}$$

where index $\mu$ is the space-time index and index $a$ is the spinor index. Here $\omega_\mu^{ab}$ is defined as

$$\omega_\mu^{ab} = e_\nu^a \partial_\mu e^{\nu b} + e_\nu^a e^{\sigma b}\Gamma^\nu_{\mu\sigma}, \tag{2.7.3}$$

where $e_\mu^a$ are tetrads defined as $e_\mu^a e_\nu^b \eta_{ab} = g_{\mu\nu}$. Here

$$g_{\mu\nu} = a^2(\eta) \times diag(1, -1, -1, -1) \tag{2.7.4}$$

is the space-time metric in the conformal time ($\eta$). Unlike (2.2.6), in the conformal time the temporal part is multiplied with the scale factor. Here the relation between conformal time and cosmic time can be written as: $d\eta = \frac{dt}{a}$. $\Gamma^{\nu}_{\mu\sigma}$ are Christoffel symbols of $g_{\mu\nu}$ defined before. $\gamma$-matrices are constructed as

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_{2\times 2} \\ \mathbb{I}_{2\times 2} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \tag{2.7.5}$$

where $\sigma^i$ $(i = 1, 2, 3)$ are Pauli matrices defined as

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.7.6}$$

It should be noted that the form of NSS considered in [24] does not have a positive definite Lorentz invariant norm [30]. This can lead us to negative energy ghost modes. To remove the ghost modes, a term with a special choice of projection operator $P$ is needed in the action (or Lagrangian). But this can make the NSS theory Lorentz violating. However the NSS theory can be made Lorentz invariant with a non-local choice of operator $P$ [30]. But using this kind Lorentz invariant form of the Lagrangian the calculation of energy momentum tensor can be extremely complicated. In spite of that one can treat the action using the Lagrangian (2.7.1) classically and study various areas of NSS cosmology. For simplicity we use the following form of NSS and it dual

$$\lambda = \varphi(\eta)\xi, \qquad \bar{\lambda} = \varphi(\eta)\bar{\xi}, \tag{2.7.7}$$

where, $\varphi(\eta)$ is a scalar quantity. $\xi$ and $\bar{\xi}$ are two constant matrices with $\bar{\xi}\xi = \mathbb{I}$. As, in this case $\bar{\xi}\xi > 0$ the NSS theory can be ghost free as there may not be any negative energy solution. With the above ansatz the components of energy-momentum tensor can be written entirely in terms of the scalar, $\varphi(\eta)$. Therefore, the NSS cosmological theories can be treated as Lorentz invariant.

In [30] it is shown that the energy-momentum tensor $T^{\mu\nu}_{\rm cosmo}$ can be constructed from $\mathcal{L}_{\rm cosmo}$ as

$$T^{\mu\nu}_{\rm cosmo} = \bar{\lambda}\overleftarrow{\nabla}^{(\mu}\nabla^{\nu)}\lambda - g^{\mu\nu}\mathcal{L}_{\rm cosmo} + F^{\mu\nu}, \tag{2.7.8}$$

where $F^{\mu\nu} = \frac{1}{2}\nabla_\rho J^{\mu\nu\rho}$ and $J^{\mu\nu\rho}$ defined as

$$J^{\mu\nu\rho} = -\frac{i}{2}\left[\bar{\lambda}\overleftarrow{\nabla}^{(\mu}f^{\nu)\rho}\lambda + \bar{\lambda}f^{\rho(\mu}\nabla^{\nu)}\lambda\right]. \tag{2.7.9}$$

$F^{\mu\nu}$ is the additional term which did not appear in the earlier models [27, 28, 68] as the authors did not consider the variation of $\Gamma_\mu$ with respect to the metric. Therefore in cosmological perturbation theory this term can bring in some additional features compared to [29]. As it turned out , the cosmological perturbations based on equation (2.7.8) are far more complex than the theory based upon canonical scalar field model. Appearance of $F^{\mu\nu}$ term can give rise to an additional scale $\tilde{F} = \frac{\bar{\lambda}\lambda}{8M_{\rm pl}^2}$ in the problem, where $M_{\rm pl} = \sqrt{\frac{1}{8\pi G}}$ is the reduced Planck mass and $G$ is the gravitational constant.

It is generally assumed that the inflation is driven by a scalar field, which can have the following verifiable predictions: (a) nearly a scale invariant spectrum, (b) existence of gravitational waves and (c) the tensor to scalar ratio of the power spectrum may be of the order of $\epsilon$, where $\epsilon$ is the slow roll parameter [15, 69, 70]. In this thesis we investigate some of the predictions of the inflation theory by assuming that the inflation is driven by a NSS with energy-momentum tensor described by equation (2.7.8).

### 2.7.1 Background equations

Using (2.7.7) in a flat, isotropic and homogeneous space-time unperturbed $\mathcal{L}_{\rm cosmo}$ can be written as

$$\mathcal{L}_{\rm cosmo} = \frac{1}{2a^2}\left[\varphi'^2 + \frac{3}{4}\mathcal{H}^2\varphi^2\right] - V(\varphi), \tag{2.7.10}$$

where prime ($'$) denotes the derivative with respect to conformal time $\eta$ . $V(\varphi)$ is the potential which is a function of $\varphi$. While the Hubble expansion parameter $\mathcal{H}$ is defined as $\mathcal{H} = \frac{a'}{a}$. The relation between $\mathcal{H}$ and $H$ is $\mathcal{H} = aH$. Unperturbed energy momentum tensors and equation of motion for $\varphi$ in FRLW space-time have already been calculated in [30]. We are enlisting them below in conformal time . Let us first define the covariant energy momentum tensor $T^{\mu\nu}_{\rm cosmo}$, which appears into the Einstein's equation, as

$$T^{\mu\nu}_{\rm cosmo} = \bar{T}^{\mu\nu} + \frac{1}{2}\nabla_\rho J^{\mu\nu\rho}, \tag{2.7.11}$$

where,

$$\bar{T}^{\mu\nu} = \vec{\lambda}\overleftarrow{\nabla}^{(\mu}\nabla^{\nu)}\lambda - g^{\mu\nu}\mathcal{L}_{cosmo}. \tag{2.7.12}$$

The non-vanishing components of $J^{\mu\nu\rho}$ are

$$J^{i\eta j} = J^{\eta ij} = \frac{1}{4}\frac{\mathcal{H}}{a^4}\varphi^2\delta_{ij}, \qquad J^{ij\eta} = -\frac{1}{2}\frac{\mathcal{H}}{a^4}\varphi^2\delta_{ij}. \tag{2.7.13}$$

Here $\varphi^2 = \vec{\lambda}\lambda$, is a function of time only. Next, we can write the expressions for the energy density $\varepsilon$ and pressure $p$ as following:

$$\varepsilon = T^\eta_\eta = \bar{T}^\eta_\eta + F^\eta_\eta, \qquad p = -T^i_j\delta_{ij} = -\left(\bar{T}^i_j + F^i_j\right)\delta_{ij}. \tag{2.7.14}$$

Expressions for $\bar{T}^{\mu\nu}$ and $F^{\mu\nu}$ can be written as,

$$\bar{T}^\eta_\eta = \frac{1}{2a^2}\left[\varphi'^2 - \frac{3}{4}\mathcal{H}^2\varphi^2\right] + V \tag{2.7.15}$$

and

$$F^\eta_\eta = \frac{3}{4a^2}\mathcal{H}^2\varphi^2. \tag{2.7.16}$$

From these one can write energy density as

$$\varepsilon = \frac{1}{2a^2}\left[\varphi'^2 + \frac{3}{4}\mathcal{H}^2\varphi^2\right] + V. \tag{2.7.17}$$

It is useful to write the expression for $\varepsilon$ as,

$$\varepsilon = X + V, \tag{2.7.18}$$

where, $X = \left(\nabla^{\eta}\bar{\lambda}\nabla_{\eta}\lambda\right) - g^{\eta}_{\eta}\left(\frac{1}{2}\nabla_{\mu}\bar{\lambda}\nabla^{\mu}\lambda\right) + g_{\eta\eta}\frac{1}{2}\nabla_{\rho}J^{\eta\eta\rho} = \frac{1}{2a^2}\left[\varphi'^2 + \frac{3}{4}\mathcal{H}^2\varphi^2\right]$. Considering the diagonal space-space components of energy-momentum tensor one can write

$$\bar{T}^i_j\delta_{ij} = -\frac{1}{2a^2}\left[\varphi'^2 + \frac{1}{4}\mathcal{H}^2\varphi^2\right] + V, \tag{2.7.19}$$

and

$$F^i_j\delta_{ij} = \frac{1}{4a^2}\mathcal{H}^2\varphi^2 + \frac{1}{4a^2}\left(\mathcal{H}\varphi^2\right)'. \tag{2.7.20}$$

From these one can obtain the expression for pressure as

$$p = \frac{1}{2a^2}\left[\varphi'^2 - \frac{1}{4}\mathcal{H}^2\varphi^2\right] - \frac{1}{4a^2}\left(\mathcal{H}\varphi^2\right)' - V. \tag{2.7.21}$$

It is easy to notice that the pressure is homogeneous and isotropic. All other components of background $T^{\mu}_{\nu}$ are zero. By adding $\varepsilon$ and $p$

$$(\varepsilon + p) = \frac{\varphi'^2}{a^2} + \frac{1}{4a^2}\mathcal{H}^2\varphi^2 - \frac{1}{4a^2}\left(\mathcal{H}\varphi^2\right)'. \tag{2.7.22}$$

For the instance when the last two terms in the above equations are absent, one can recover the expression for $(\varepsilon + p)$ of the canonical scalar-field. Equation of motion for $\varphi$ can be obtained by equating the divergence of $T^{\mu}_{\nu}$ to zero:

$$\varphi'' + 2\mathcal{H}\varphi' - \frac{3}{4}\mathcal{H}^2\varphi + a^2V_{,\varphi} = 0. \tag{2.7.23}$$

It should be emphasized that the above equation for $\varphi$ matches with the equation motion obtained using Euler-Lagrange equation as discussed in [30]. However, in the earlier calculations based on non Lorentz invariant model of NSS there were mismatches between the equation motions calculated using these two methods, e.g. [28]. This is solved because of the additional term $F^{\mu}_{\nu}$ in equation (2.7.8). The modified Friedmann equations can be written as

$$\begin{aligned}
\mathcal{H}^2 &= \frac{1}{1-\tilde{F}}\left[\frac{1}{3M_{\rm pl}^2}\left(\frac{\varphi'^2}{2} + a^2V\right)\right], \\
\mathcal{H}' &= \frac{1}{1-\tilde{F}}\left[\frac{1}{3M_{\rm pl}^2}\left(a^2V - \varphi'^2\right) + \mathcal{H}\tilde{F}'\right],
\end{aligned} \tag{2.7.24}$$

where, $\tilde{F} = \frac{\varphi^2}{8M_{pl}^2}$. One can notice from the above that the condition $\tilde{F} < 1$ is required to be satisfied to ensure the positivity of $\mathcal{H}^2$. Therefore $\varphi$ has to be smaller than $\sqrt{8}M_{\rm pl}$ as mentioned in [30]. It should be emphasized that the introduction of $J^{\mu\nu\rho}$ term in equation (2.7.8) makes the expressions for $\mathcal{H}^2$ and $\mathcal{H}'$ different from those obtained in [29, 66, 67]. In this thesis the expression for $T^{\mu\nu}$ given in equation (2.7.8) is used to study the cosmology of NSS. From what follows the label *cosmo* on the energy-momentum tensor has been dropped.

# Chapter 3

# Perturbation theory

## 3.1 Introduction

After the inflation ends, the inflaton decays and the universe reheats. The universe become full with matter and radiation in thermal equilibrium. As the density of the radiation component, in the expanding universe, falls faster than the energy density of the matter component, the matter-radiation decoupling occurs around 380,000 years after the Big Bang. After this decoupling the 'thermalised' photons travel freely in the space which we see today as a black-body radiation in the microwave range known as CMBR. It should be noted that the CMB photons are not in thermal equilibrium currently. As the CMB photons have not interacted since they left the 'Last Scattering Surface', they carry the information of the universe back to the time of the matter-radiation decoupling. COBE revealed that CMB photons have mean temperature of approximately 2.73K with the fluctuation $\frac{\Delta T}{T} \sim 10^{-5}$. The fluctuation in the temperature of the universe can be connected with the fluctuations during inflation. The cosmological perturbation theory gives us a relation between the temperature fluctuation in the CMB and the metric perturbation during inflation due to the Sachs-Wolf effect [15].

The primordial inhomogeneities produced during inflation are also important from the point of view of structure formation. A rapid expansion during the inflation leaves the universe 'almost' homogeneous and isotropic. The amount of inhomogeneity that is observed in the CMB spectrum is about $\frac{\Delta\varepsilon}{\varepsilon} \sim 10^{-5}$.

The structure formation mechanism lies in the 'tug of war' between the pressure and gravity. The primordial inhomogeneities can be amplified by the gravitational instability and this can seed the formation of structures that we see today [71–73]. The standard Friedmann cosmology works only when the physical length scale is smaller than Hubble radius ($H^{-1}$). However, the relevant astrophysical scale of clusters, galaxies etc. were bigger than the Hubble radius at early epochs. Therefore, to seed the large-scale structure formation at early universe we need a mechanism which can make the wavelength of the density perturbation ($\lambda$) larger than the Hubble radius ($\lambda > H^{-1}$) starting from the time when $\lambda < H^{-1}$. During inflation the proper wavelengths of the perturbations grow exponentially (as $\lambda \propto a$) and at the same time $H$ remains constant allowing the wavelengths to exit the Hubble radius. In this way perturbation during inflation seeds the density perturbation required for the large-scale structure.

Apart from the providing seed for the formation of large-scale structures the inflationary perturbation theory also leaves imprint in the CMB. As mentioned before the CMB radiation can allow us to probe the structure of the universe at the time of the Last Scattering Surface (LSS). There are nearly $10^4$ causally disconnected patches on LSS. The angle sustaining the horizon size on LSS is approximately $1°$, beyond which correlation between the temperature fluctuation produced by any causal process of expansion can not exist. However, there exists nonzero correlation in temperature fluctuation at angular scale $> 1°$. The patches which were once in causal contact, became causally disconnected because of the accelerated expansion during inflation giving us nonzero correlation at large angular scale. The inflationary perturbation theory has been remarkably successful in calculating the power spectra of scalar and tensor perturbation. The theory predicts nearly scale invariant power spectra within the slow-roll paradigm. Data obtained from the missions like WMAP [8], PLANCK [17] confirm some of these general predictions of the inflationary perturbation theory. Power spectrum is the two point correlation of the perturbation. It is observed that over large scales the power spectrum is nearly scale invariant. The scale dependence of the power spectrum is measured by spectral index. The measured spectral index associated with the scalar perturbation is very close to unity and is red tilted, $n_s \lesssim 1$. The exact scale invariance ($n_s = 1$) has been ruled out by PLANCK. The observations also reveal a small amount of gravitational wave. In the inflationary theories the gravitational waves are generated by the divergence less and trace less tensor modes of the metric perturbation. In the single field inflationary models the tensor-to-scalar ratio ($r$) remains small in the slow-roll paradigm. Tensor-to-scalar ratio observed by the PLANCK mission is $< 0.12$ [17]. The minimally coupled canonical scalar field models of inflation produces tensor to scalar ratio $\sim 16\epsilon$, where $\epsilon$ is slow-roll parameter ($\sim 10^{-2}$). As shown in the reference [42] the tensor to scalar ratio in the single scalar field models with non-minimal coupling remains even smaller.

In this chapter we will study the inflationary perturbation theory with NSS. As described in the introduction, because of spinor nature of the inflaton field, the perturbation theory can have some issues in comparison with the canonical single scalar field theory. Therefore, using the ansatz similar to the background case we will reduce the problem of spinor perturbation to scalar perturbation and use the standard tool of scalar perturbation theory. Here we will show that the terms which appears in addition to the standard single canonical scalar field theory, are very small. The calculation of the spectral index will show that NSS can produce nearly scale invariant perturbation consistent with observation. It can also be argued that NSS cosmology can have very small tensor-to-scalar ratio.

## 3.2 Metric perturbation

Our aim is to calculate the cosmological evolution of the linear perturbations for NSS. The first step to do this is perturbing the metric about the background FRW metric at the first order. The full metric in general can be written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \overrightarrow{x}), \tag{3.2.1}$$

where $\bar{g}_{\mu\nu}(\eta)$ is the homogeneous and isotropic background FRW metric and $\delta g_{\mu\nu}$ is the perturbation. Unlike the background FRW metric the perturbed metric in general may have off-diagonal elements. The other property of $\delta g_{\mu\nu}$ is that it is symmetric. In general one can write $\delta g_{\mu\nu}$ at any order. But, in this thesis we are interested in the first order perturbation theory. Therefore, we choose to write $\delta g_{\mu\nu}$ up to linear order.

In the $(n+1)$ dimensional space-time the linear order metric perturbation can be written as:

$$\delta g_{\mu\nu} = \begin{pmatrix} \delta g_{\eta\eta} & \delta g_{\eta i} \\ \delta g_{i\eta} & \delta g_{ij} \end{pmatrix}, \tag{3.2.2}$$

where $i, j = 1, ..., n$. Here $\delta g_{\eta\eta}$ is one component of the above $((n+1)\times(n+1))$ dimensional matrix, where as $\delta g_{\eta i}$ is the $(1\times n)$ dimension row matrix, $\delta g_{i\eta}$ is the $(n\times 1)$ dimensional column matrix and $\delta g_{ij}$ is the $(n\times n)$ matrix. $\delta g_{\eta\eta}$ is a scalar quantity and we write it as $\delta g_{\eta\eta} = a^2(2\Psi)$. The entries $\delta g_{\eta i}$ and $\delta g_{i\eta}$ are vector in nature as it has one running index $(i)$. One can write this matrix as $\delta g_{\eta i} = a^2(\partial_i B + v_i)$, where $B$ is a scalar quantity and $v_i$ is a real divergence less $(\partial_i v^i = 0)$ vector. Similarly the tensor part $\delta g_{ij}$ can be written as $\delta g_{ij} = a^2\left[2\Phi\delta_{ij} + \frac{1}{2}(\partial_i \Pi_j + \partial_j \Pi_i) + 2\Pi_{,ij} + h_{ij}\right]$, where $\Phi$ and $\Pi$ are scalars, $\Pi_i$ is divergence less vector and $h_{ij}$ is a traceless $(h^i_i = 0)$ and transverse $(\partial_i h^{ij} = 0)$ tensor. The pure tensor modes $h_{ij}$ are also referred as 'gravitational waves'. A detailed discussion on the various degrees of freedom can be seen in references [69, 74].

Let us now consider the scalar degrees of freedom in the perturbed metric in the covariant form,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} = a^2 \begin{pmatrix} (1+2\Psi) & \partial_i B \\ \partial_i B & (-1+2\Phi)\,\delta_{ij} + 2\Pi_{,ij} \end{pmatrix}, \tag{3.2.3}$$

where $i, j = 1, ..., 3$. To calculate the various perturbed quantities it is important to know the contravariant form of the metric perturbation $\delta g^{\mu\nu}$. To calculate

$\delta g^{\mu\nu}$ it is useful to know the orthogonality condition:

$$g^{\mu\sigma} g_{\sigma\nu} = \delta^{\mu}_{\nu}. \tag{3.2.4}$$

Using (3.2.4) $g^{\mu\nu}$ can be written as,

$$g^{\mu\nu} = \bar{g}^{\mu\nu} + \delta g^{\mu\nu} = \frac{1}{a^2} \begin{pmatrix} (1-2\Psi) & \partial^i B \\ \partial^i B & (-1-2\Phi)\,\delta_{ij} - \Pi^{ij}_{,} \end{pmatrix}. \tag{3.2.5}$$

## 3.3 Gauge transformation and Gauge invariant quantities

As described in equation (3.2.1), unlike the background FRW metric the perturbed metric depends also upon the spatial part of the coordinate. All of the perturbed quantities, in general are considered inhomogeneous and anisotropic. Our aim is to study the evolution of these perturbed quantities in the homogeneous and isotropic background.

In analysis of the perturbations in GTR described by the FLRW-cosmology it is important that we work with the quantities which are invariant under general coordinate transformation [75, 76]. In general relativity the unperturbed background quantities are calculated in a preferred coordinate system (which we choose by the symmetry of the background). However for the perturbed quantities we do not have any preferred coordinate. The freedom of gauge or coordinate leads to various 'fictitious' perturbations under coordinate transformation. Therefore to analyze perturbations either we choose a specific gauge or we construct gauge invariant quantities. The construction of gauge invariant quantities are done by determining the transformation laws for the perturbations of scalar, vector and tensor quantities under infinitesimal coordinate transformation

$$x^{\mu} \to \tilde{x}^{\mu} = x^{\mu} + \xi^{\mu}, \tag{3.3.1}$$

and then removing the gauge dependences by choosing the appropriate combinations of the various quantities. Here $\xi^\mu$ is an infinitesimal 4-vector ($\xi^\mu \ll x^\mu$) with the components $\xi^\mu \equiv (\xi^\eta,\, \xi^i)$. $\xi^\eta$ contributes to the scalar perturbation and $\xi^i$ is a 3-vector which can be written in terms of a divergence free vector ($\alpha^i$) and a gradient of a scalar ($\beta$): $\xi^i = \alpha^i + \beta^i_{,}$. The keys to calculate the transformation laws for perturbations are:(i) *Transformation of various quantities under the general coordinate transformation*, and (ii) *the functional form of the background remains same in all coordinates.* Finally, the transformation properties of scalar ($f$), vector ($v_\mu$) and tensor ($S_{\mu\nu}$) perturbations in the linear order can be listed respectively as:

$$\tilde{\delta f} = \delta f - f^{(0)}_{,\gamma}\xi^\gamma, \tag{3.3.2}$$

$$\tilde{\delta v_\mu} \approx \delta v_\mu - v^{(0)}_{\mu,\gamma}\xi^\gamma - v^{(0)}_\sigma \xi^\sigma_{,\mu}, \tag{3.3.3}$$

$$\tilde{\delta S_{\mu\nu}} \approx \delta S_{\mu\nu} - S^{(0)}_{\mu\nu,\gamma}\xi^\gamma - S^{(0)}_{\sigma\nu}\xi^\sigma_{,\mu} - S^{(0)}_{\mu\rho}\xi^\rho_{,\nu}. \tag{3.3.4}$$

The superscript '(0)' denotes the background quantities.

Because of the presence of $\xi^\mu$ in the above expressions the scalar (e.g. inflaton perturbation or density perturbation) and various components of metric perturbation are gauge dependent quantities. Inflaton perturbation and the scalar components of the metric perturbation depend on the temporal part of $\xi^\mu$, i.e. $\xi^\eta$. The gauge dependence of the perturbations $\Psi$ and $\Phi$ can be removed by choosing the following combinations

$$\psi = \Psi - \frac{1}{a}\left[a\left(B - \Pi'\right)\right]', \qquad \phi = \Phi + \mathcal{H}\left(B - \Pi'\right). \tag{3.3.5}$$

It can be straightforward to check that the above quantities are gauge invariant. These expressions of the metric perturbations were first proposed by Bardeen in [75]. The scalar quantities $\psi$ and $\phi$ are also known as *Bardeen potentials.* Using (3.3.2) and transformation properties of metric perturbation the gauge invariant inflaton perturbation can be defined as

$$\delta\varphi = \overline{\delta\varphi} - \varphi'\left(B - \Pi'\right). \tag{3.3.6}$$

It can be easily verified that the combination $-\mathcal{R} = \phi + \frac{\mathcal{H}}{\varphi'}\delta\varphi$ is also a gauge invariant quantity. $\mathcal{R}$ is also known as comoving curvature perturbation. From the perturbed Einstein equations we will see that $\mathcal{R}$ stops evolving once it crosses the horizon.

**Fixing gauge and connection to the gauge invariant quantities:** One can fix the coordinate by choosing a proper gauge condition as we have the freedom to choose the functions $\xi^\eta$ and $\beta$. Fixing gauge sometimes can simplify the scalar equations. So, solving the perturbed equations can be sometimes easier in a particular gauge. But, the only threat is that the perturbed quantities may not be always physical. However, there exist one particular gauge where the perturbations turns out to be gauge invariant. This gauge is called *longitudinal gauge* or *conformal Newtonian gauge.* The longitudinal gauge is given by the gauge condition $B = 0$ and $\Pi = 0$. To achieve this the only choice we have is $\xi^\eta = 0$ and $\beta = 0$. Any other choice of $\xi^\eta$ or $\beta$ can spoil this gauge. From equation (3.3.5) and (3.3.6) it clear that in the longitudinal gauge the gauge invariant perturbations become $\psi = \Psi$, $\phi = \Phi$ and $\delta\varphi = \overline{\delta\varphi}$.

Although the calculation with the gauge invariant quantities can be complicated, one can use a simple trick based on the properties of the Newtonian gauge: *derive the perturbed equations in the Newtonian gauge and solve for gauge invariant quantities after directly substituting the gauge invariant quantities in those equations.* We will follow this trick in the following sections.

We have already explained before that in case of spinors the action can be made invariant under coordinate and Lorentz transformation by properly defining a covariant derivative. In the first-order perturbation theory of spinors we will show that the perturbed energy momentum tensor can be written in terms of various combinations of the product of the perturbed spinors and the background spinors and their derivatives. Because of the presence of spinors Lorentz invariance could be an issue in perturbation while defining proper physical quantities using them. This can be avoided if one considers the following ansatz regarding the structure of the perturbed spinor: $\delta\lambda = \delta\varphi(\eta, \overrightarrow{x})\xi$ and $\delta\bar{\lambda} = \delta\varphi(\eta, \overrightarrow{x})\bar{\xi}$. The advantage of this ansatz is that the perturbation can be written entirely in terms of the scalar field perturbation. This allows one to construct the gauge invariant definition of inflaton perturbation ($\delta\varphi$) and density perturbation ($\delta\varepsilon$) in the standard way described in this section. By doing so we will see that the perturbed equations will pick up some terms, proportional to $\tilde{F}$, along with the standard terms which appear in case of single scalar field theory. Thus, in the next section we briefly describe the essential features of the perturbation theory for single scalar field.

## 3.4 Perturbation theory in single canonical scalar field:Mukhanov-Sasaki equation

In this section we briefly review the cosmological perturbation theory of the single scalar field inflationary scenario based on the gauge invariant formalism discussed in the previous section. In the following sections we generalise the technique learned here for the NSS perturbation theory. To calculate the perturbed equations we have to write the perturbed Einsteins equations, about the homogeneous and isotropic background.

The linearised perturbed Einstein equation, in terms of the gauge dependent quantities can be written as

$$\overline{\delta G^{\mu}_{\nu}} = 8\pi G \overline{\delta T^{\mu}_{\nu}}. \tag{3.4.1}$$

Here $\overline{\delta G^{\mu}_{\nu}}$ are functions of the gauge dependent metric perturbations $\Psi$ and $\Phi$ whereas components of $\overline{\delta T^{\mu}_{\nu}}$ contain inflaton fluctuation, metric perturbations and background quantities. Now we will calculate the perturbation in the conformal Newtonian gauge. Then using the gauge invariant definitions given in the previous section we can write the various components of linearised perturbed Einstein equation in terms of the gauge invariant variables $\psi$ and $\phi$ as (see appendix A.3 for details):

$$\Delta\phi - 3\mathcal{H}\left(\phi' + \mathcal{H}\psi\right) = 4\pi G a^2 \delta T^{\eta}_{\eta}, \tag{3.4.2}$$

$$(\phi' + \mathcal{H}\psi)_{,i} = 4\pi G a^2 \delta T^{\eta}_{i}, \tag{3.4.3}$$

$$-\left[2\phi'' + 2\mathcal{H}(\psi' + 2\phi') + 2\left(2\mathcal{H}' + \mathcal{H}^2\right)\psi + \Delta(\psi - \phi)\right]\delta^i_j + \partial_i\partial_j(\psi - \phi) = 8\pi G a^2 \delta T^i_j, \tag{3.4.4}$$

where $\Delta = \partial_i\partial^i$. For single scalar field theories the perturbed expressions of $\delta T^{\mu}_{\nu}$ will be written in terms of gauge invariant $\delta\varphi$ (as given in appendix B.1). Following the methods given in reference [15] we will only focus on two equations (3.4.2) and (3.4.3). The standard single scalar field theories give us isotropic perturbation, i.e. $\delta T^i_j = 0$ for $i \neq j$. For $i \neq j$, from the equation (3.4.4) we can see that $\psi = \phi$. Using this constraint and substituting the expressions of $\delta T^{\eta}_{\eta}$ and $\delta T^{\eta}_{i}$ from the appendix (B.1) in (3.4.2) and (3.4.3) we can finally write,

$$\Delta\phi = \frac{4\pi G \varphi'^2}{\mathcal{H}}\mathcal{R}', \tag{3.4.5}$$

$$\left(a^2\frac{\phi}{\mathcal{H}}\right)' = \frac{4\pi G a^2}{\mathcal{H}^2}\mathcal{R}, \tag{3.4.6}$$

where $\mathcal{R} = \phi + \mathcal{H}\frac{\delta\varphi}{\varphi'}$ is the comoving curvature perturbation. From (3.4.5) one can see that in the large scale limit ($\Delta \to 0$) the comoving curvature perturbation $\mathcal{R}' = 0$ thus, $\mathcal{R}$ does not evolve in the super-horizon scale.

Equations (3.4.5) and (3.4.6) are coupled equations of two gauge invariant quantities $\phi$ and $\mathcal{R}$. After eliminating $\mathcal{R}$ from these equations we can decouple them. The decoupled equation for $\phi$ in the Fourier space can be written as

$$u'' + \left(k^2 - \frac{\theta''}{\theta}\right)u = 0, \tag{3.4.7}$$

where $u = \frac{\phi}{4\pi G\sqrt{(\varepsilon + p)}}$ and $\theta = \sqrt{\frac{8\pi G}{3}\frac{1}{a}\left(\frac{\varepsilon}{\varepsilon + p}\right)^{1/2}}$. To calculate the power spectrum and the spectral index associated with the perturbation $\phi$ we have to solve the equation (3.4.7). Elimination of $\phi$ from (3.4.5) and (3.4.6) gives us

$$v'' + \left(k^2 - \frac{z''}{z}\right)v = 0, \tag{3.4.8}$$

where $v = a\frac{\varphi'}{H}\mathcal{R}$ and $z = \frac{a^2\sqrt{(\varepsilon + p)}}{\mathcal{H}}$. The power spectrum associated with the metric perturbation can be expressed as:

$$\delta_{\phi}^{2} = \frac{k^{3}}{2\pi^{2}}|\phi_{\mathbf{k}}|^{2}. \tag{3.4.9}$$

Using spectral index associated with the power spectrum, one can measures the scale dependence of the perturbation. The spectral index associated with $\phi$ can be defined as

$$n_{\phi} = 1 + \frac{d\ln\delta_{\chi}^{2}}{d\ln k}. \tag{3.4.10}$$

In canonical single scalar field theories, in the slow-roll paradigm, the last term on the right hand side can be written in terms of the slow-roll parameters giving us nearly scale invariant spectrum. The spectral index that is given by PLANCK collaborators in [17] indicates the 'nearly' scale invariant power spectrum (spectral index $\mathrm{n_s} = 0.9643 \pm 0.0059$) with PLANCK+WP+BAO data.

## 3.5 Perturbation theory in NSS

In this work the gauge invariant approach for treating the cosmological perturbations discussed in section (3.4) is applied to the NSS cosmology. The full metric in terms of gauge invariant quantities can be written as:

$$\bar{g}_{\mu\nu} + \delta g_{\mu\nu} = a^{2}\begin{pmatrix} (1+2\psi) & \mathbb{O} \\ \mathbb{O} & (-1+2\phi)\,\delta_{ij} + 2h_{ij} \end{pmatrix}. \tag{3.5.1}$$

Here $i, j$ denotes space-space components of the metric, $\phi$,$\psi$ are the gauge invariant scalar perturbations and $h_{ij}$ are traceless and divergence-less tensor perturbations. The metric perturbations are functions of space and time. It is necessary to first calculate the perturbations in energy momentum tensor $\delta T^{\mu}_{\nu}$ by including the perturbations in $F^{\mu}_{\nu}$ term. Final equations are obtained by substituting $\delta T^{\mu}_{\nu}$ into the perturbed Einstein's equations. As mentioned in section (3.3), the ansatz regarding the NSS perturbation:

$$\delta\lambda = (\delta\varphi)\,\xi, \qquad \delta\bar{\lambda} = (\delta\varphi)\,\bar{\xi} \tag{3.5.2}$$

can be used to calculate $\delta T^{\mu}_{\nu}$. The constant matrices $\xi$ and $\bar{\xi}$ as given in [29]:

$$\xi = \frac{1}{\sqrt[4]{12}} \begin{pmatrix} -\alpha_1 e^{i\frac{\pi}{4}} \\ \alpha_2 \frac{i}{\sqrt{2}} \\ \alpha_2 \frac{1}{\sqrt{2}} \\ \alpha_1 e^{i\frac{\pi}{4}} \end{pmatrix}, \ \bar{\xi} = \frac{1}{\sqrt[4]{12}} \begin{pmatrix} -\alpha_1 e^{-i\frac{\pi}{4}} & -\alpha_2 \frac{i}{\sqrt{2}} & \alpha_2 \frac{1}{\sqrt{2}} & \alpha_1 e^{-i\frac{\pi}{4}} \end{pmatrix}, \tag{3.5.3}$$

which follows the condition $\bar{\xi}\xi = 1$ provided $\alpha_1 = \alpha_2^{-1} = \sqrt{\frac{1+\sqrt{3}}{2}}$, can be used in the ansatz of perturbed NSS to compute the perturbed energy-momentum tensor. It should be noted here that we have not used the hedgehog ansatz for the unperturbed NSS like the previous study [29]. With this relatively simpler ansatz one can obtain all the equations perturbations given in [29] provided that the $J^{\mu\nu\rho}$ term is ignored from equation (2.7.8). With the above ansatz, one can construct the gauge invariant quantities for NSS using the method similar to the one described in case of standard scalar perturbation theory.

### 3.5.1 Perturbed energy momentum tensors:

Using equations (3.5.1, 3.5.3) we can calculate $\delta T^{\eta}_{\eta}$, $\delta T^{\eta}_{i}$ and $\delta T^{i}_{j} (i \neq j)$ components of the perturbed energy-momentum tensor. Below we have enlisted the different components of the energy-momentum tensor for the scalar perturbations.

i) <u>Perturbation of $\varepsilon = T^{\eta}_{\eta}$</u>: One can write the general expression for energy as

$$\varepsilon = X + V, \tag{3.5.4}$$

where $X$ is the kinetic part dependent on background quantities $\mathcal{H}$, $\varphi$ and $\varphi'$ and $V$ is the potential which is function of $\varphi$ only. $X$ can be written as $X = Y + g_{\eta\eta}F^{\eta\eta}$, where $Y = \left(\nabla^{\eta}\bar{\lambda}\nabla_{\eta}\lambda\right) - g^{\eta}_{\eta}\left(\frac{1}{2}\nabla_{\mu}\bar{\lambda}\nabla^{\mu}\lambda\right)$. From the expression of $\varepsilon$ which is a function of X and V we can write

$$\delta\varepsilon = \varepsilon_{,X}\delta X + \varepsilon_{,\varphi}\delta\varphi. \tag{3.5.5}$$

The continuity equation one can write as:

$$\varepsilon' = \varepsilon_{,X} X' + \varepsilon_{,\varphi} \varphi' = -3\mathcal{H} (\varepsilon + p) . \tag{3.5.6}$$

Eliminating $\varepsilon_{,\varphi}$ from (3.5.5) and (3.5.6)

$$\delta\varepsilon = \varepsilon_{,X} \left( \delta X - X' \frac{\delta\varphi}{\varphi'} \right) - 3\mathcal{H} (\varepsilon + p) \frac{\delta\varphi}{\varphi'} . \tag{3.5.7}$$

The perturbation in Y is

$$\delta Y = \frac{1}{a^2} \left( -\psi\varphi'^2 + \frac{3}{4}\psi\mathcal{H}^2\varphi^2 + \varphi'\delta\varphi' + \frac{3}{4}\psi'\mathcal{H}\varphi^2 - \frac{3}{4}\mathcal{H}^2\varphi\delta\varphi \right) , \tag{3.5.8}$$

while the perturbation in $F^{\eta\eta}$ can be written as

$$\delta F^{\eta\eta} = \frac{1}{2}\delta \left( \nabla_\rho J^{\eta\eta\rho} \right) . \tag{3.5.9}$$

Next, the perturbation in the covariant derivative of $J^{\mu\nu\rho}$ can be written as;

$$\delta \left( \nabla_\rho J^{\eta\eta\rho} \right) = \partial_\rho \delta J^{\eta\eta\rho} + \delta \left( \Gamma^\eta_{\sigma\rho} J^{\sigma\eta\rho} + \Gamma^\eta_{\sigma\rho} J^{\eta\sigma\rho} + \Gamma^\rho_{\sigma\rho} J^{\eta\eta\sigma} \right) . \tag{3.5.10}$$

Therefore we get after substituting for $\delta \left( \nabla_\rho J^{\eta\eta\rho} \right)$

$$\delta F^{\eta\eta} = -\frac{1}{4a^4} (\Delta\psi) \varphi^2 + \frac{3}{2a^4}\mathcal{H}^2\varphi\delta\varphi - \frac{3}{2a^4}\phi'\mathcal{H}\varphi^2 - \frac{3}{a^4}\psi\mathcal{H}^2\varphi^2 . \tag{3.5.11}$$

From this one can calculate $\delta X$

$$\delta X = -\psi (2X) + \frac{1}{a^2}\varphi'\delta\varphi' + \frac{3}{4a^2} (\psi' - 2\phi') \mathcal{H}\varphi^2 + \frac{3}{4a^2}\mathcal{H}^2\varphi\delta\varphi - \frac{1}{4a^2} (\Delta\psi) \varphi^2 , \tag{3.5.12}$$

$$X' = -2\mathcal{H}X + \frac{1}{a^2}\varphi'\varphi'' + \frac{3}{4a^2}\mathcal{H}\mathcal{H}'\varphi^2 + \frac{3}{4a^2}\mathcal{H}^2\varphi\varphi' . \tag{3.5.13}$$

Finally one can write the energy perturbation $\delta\epsilon$ as

$$\begin{aligned}\delta\varepsilon &= \varepsilon_{,X}[2X\left(-\psi+\mathcal{H}\frac{\delta\varphi}{\varphi'}+\left(\frac{\delta\varphi}{\varphi'}\right)'\right)-\frac{3}{4a^2}\mathcal{H}\varphi^2\left(\mathcal{H}\frac{\delta\varphi}{\varphi'}\right)'+\\ &\quad \frac{3}{4a^2}\left(\psi'-2\phi'\right)\mathcal{H}\varphi^2-\frac{1}{4a^2}\left(\Delta\psi\right)\varphi^2]-3\mathcal{H}\left(\varepsilon+p\right)\frac{\delta\varphi}{\varphi'}. \end{aligned} \tag{3.5.14}$$

ii) Perturbation of $T^\eta_i$:

$$\delta T^\eta_i = \delta\bar{T}^\eta_i + \delta F^\eta_i. \tag{3.5.15}$$

Now for scalar perturbation

$$\delta\bar{T}^\eta_i = \left[\frac{1}{a^2}\varphi'\delta\varphi - \frac{1}{4a^2}\left(\mathcal{H}\varphi^2\right)\psi\right]_{,i}, \tag{3.5.16}$$

and

$$\delta F^\eta_i = \left[-\frac{a^2}{8}\left(\frac{\psi\varphi^2}{a^4}\right)' - \frac{1}{8a^2}\mathcal{H}\left(2\varphi\delta\varphi\right) - \frac{1}{4a^2}\left(\psi+\phi\right)\mathcal{H}\varphi^2 + \frac{1}{8a^2}\phi'\varphi^2\right]_{,i}. \tag{3.5.17}$$

And we get

$$\begin{aligned}\delta T^\eta_i &= \left[\frac{1}{a^2}\varphi'\delta\varphi - \frac{1}{4a^2}\left(\mathcal{H}\varphi^2\right)\psi\right]_{,i} + \\ &\quad \left[-\frac{a^2}{8}\left(\frac{\psi\varphi^2}{a^4}\right)' - \frac{1}{8a^2}\mathcal{H}\left(2\varphi\delta\varphi\right) - \frac{1}{4a^2}\left(\psi+\phi\right)\mathcal{H}\varphi^2 + \frac{1}{8a^2}\phi'\varphi^2\right]_{,i}\end{aligned} \tag{3.5.18}$$

iii) Perturbation of $T^i_j\,(i \neq j)$:

$$\delta T^i_j = \delta\bar{T}^i_j + \delta F^i_j.$$

Now for scalar perturbation, $\delta\bar{T}^i_j = 0$ and $\delta F^i_j = -\frac{1}{4a^2}\left(\partial_i\partial_j\phi\right)\varphi^2$ for $i \neq j$. Therefore,

$$\delta T^i_j = -\frac{1}{4a^2}\left(\partial_i\partial_j\phi\right)\varphi^2 \qquad (i \neq j)\,. \tag{3.5.19}$$

### 3.5.2 Perturbed Einstein's Equation:

Perturbed Einstein's equation can be written as:

$$\delta G^{\mu}_{\nu} = 8\pi G \delta T^{\mu}_{\nu}, \tag{3.5.20}$$

where $\delta G^{\mu}_{\nu}$ is the perturbed Einstein's tensor. The scalar part of perturbed Einstein's equations are given below,

$$\Delta\phi - 3\mathcal{H}\left(\phi' + \mathcal{H}\psi\right) = 4\pi G a^2 \delta T^{\eta}_{\eta}$$

$$\begin{aligned} - & \left[2\phi'' + 2\frac{a'}{a}\left(\psi' + 2\phi'\right) - 2\left\{\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right\}\psi + \Delta\left(\psi - \phi\right)\right]\delta_{ij} + \partial_i\partial_j\left(\psi - \phi\right) \\ = & \ 8\pi G a^2 \delta T^i_j \end{aligned}$$

$$\left(\phi' + \mathcal{H}\psi\right)_{,i} = 4\pi G a^2 \delta T^{\eta}_{i}, \tag{3.5.21}$$

In the previous sub-section we have already calculated the scalar perturbations for the various components of the energy-momentum tensor. The tensor part of the perturbed Einstein's equations can be written as,

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \Delta h_{ij} = -16\pi G a^2 \delta T^i_{j(T)}, \tag{3.5.22}$$

where subscript $T$ on the energy-momentum tensor denotes the tensor part. Next, consider the space-space components of the Einstein equation with $(i \neq j)$.

i) $\underline{\delta G^i_j = 8\pi G \delta T^i_j}$: Using the expression of $\delta T^i_j$ when $i \neq j$ from equation (3.5.19) one can write,

$$\partial_i\partial_j\left(\psi - \phi\right) = \partial_i\partial_j\left(-2\tilde{F}\phi\right), \tag{3.5.23}$$

where $\tilde{F} = \pi G \varphi^2 = \frac{\varphi^2}{8M_{\rm PL}^2}$. In the case of the standard inflation driven by a canonical scalar-field, $\delta T^i_j = 0$ for $i \neq j$ and $\phi = \psi$. However, this is no longer true for a NSS driven inflation. The above equation implies that the condition $\psi = (1 - 2\tilde{F})\phi$ needs to be satisfied by the metric and the NSS perturbations. In the previous study using a NSS field [29], $\delta \bar{T}^i_j = 0$ for $i \neq j$. That's why in [29] the authors got $\phi = \psi$. Here inequality between $\psi$ and $\phi$ arises because of the extra $F^{\mu\nu}$ term in the energy-momentum tensor. As explained in the introduction, we consider $\tilde{F}$ to be a very small quantity and from here onwards we will write the equations up to the linear order in $\tilde{F}$.

ii) $\underline{\delta G^\eta_i = 8\pi G \delta T^\eta_i}$: Using the last equation of (3.5.21) we get

$$\frac{2}{a^2}\left(\phi' + \mathcal{H}\psi\right)_{,i} = 8\pi G[\frac{1}{a^2}\varphi'\delta\varphi - \frac{1}{4a^2}\left(\mathcal{H}\varphi^2\right)\psi - \frac{a^2}{8}\left(\frac{\psi\varphi^2}{a^4}\right)' - \frac{1}{8a^2}\mathcal{H}\left(\delta\bar{\lambda}\lambda + \bar{\lambda}\delta\lambda\right) - \frac{1}{4a^2}\left(\psi + \phi\right)\mathcal{H}\varphi^2 + \frac{1}{8a^2}\phi'\varphi^2]_{,i}, \tag{3.5.24}$$

or,

$$\left(\phi' + \mathcal{H}\psi\right) = 4\pi G a^2\left(\frac{\varphi'^2}{a^2}\right)\frac{\delta\varphi}{\varphi'} - \mathcal{H}\tilde{F}\phi - \left(\frac{\psi' - \phi'}{2}\right)\tilde{F} - \frac{\tilde{F}'}{2}\left(\mathcal{H}\frac{\delta\varphi}{\varphi'} + \psi\right). \tag{3.5.25}$$

Substituting $\psi = (1 - 2\tilde{F})\phi$ in the right hand side of the above equation,

$$\left(\phi' + \mathcal{H}\psi\right) \simeq 4\pi G a^2\left(\varepsilon + p\right)\frac{\delta\varphi}{\varphi'} + \left[\left(\mathcal{H}\tilde{F}\right)' - \mathcal{H}^2\tilde{F} - \frac{\mathcal{H}\tilde{F}'}{2}\right]\frac{\delta\varphi}{\varphi'} - \left(\mathcal{H}\tilde{F} + \frac{\tilde{F}'}{2}\right)\phi. \tag{3.5.26}$$

Again setting $\psi = (1 - 2\tilde{F})\phi$ in the left hand side and multiplying both sides by $\frac{a^2}{\mathcal{H}}$ one may obtain,

$$\left(\frac{a^2}{\mathcal{H}}\phi\right)' \simeq \left[\frac{4\pi G a^4}{\mathcal{H}^2}\left(\varepsilon + p\right) + \frac{a^2}{\mathcal{H}}\left\{\frac{\left(\mathcal{H}\tilde{F}\right)'}{\mathcal{H}} - \mathcal{H}\tilde{F} - \frac{\tilde{F}'}{2}\right\}\right]\left(\mathcal{H}\frac{\delta\varphi}{\varphi'} + \phi\right) + \left(2\mathcal{H}\tilde{F} - \frac{\left(\mathcal{H}\tilde{F}\right)'}{\mathcal{H}}\right)\frac{a^2\phi}{\mathcal{H}}. \tag{3.5.27}$$

iii) $\underline{\delta G^\eta_\eta = 8\pi G \delta T^\eta_\eta = 8\pi G \delta\delta\varepsilon}$: Now the first equation in (3.5.21) implies

$$\Delta\phi - 3\mathcal{H}\left(\phi' + \mathcal{H}\psi\right) = 4\pi G a^2 \delta\varepsilon. \tag{3.5.28}$$

Using the expression of $(\phi' + \mathcal{H}\psi)$ from equation (3.5.26) we get

$$\begin{aligned}\Delta\phi \;&-\; 3\mathcal{H}\left[4\pi G a^2\left(\varepsilon+p\right)\frac{\delta\varphi}{\varphi'}+\left\{\left(\mathcal{H}\tilde{F}\right)'-\mathcal{H}^2\tilde{F}-\frac{\mathcal{H}\tilde{F}'}{2}\right\}\frac{\delta\varphi}{\varphi'}-\left(\mathcal{H}\tilde{F}+\frac{\tilde{F}'}{2}\right)\phi\right]\\ &\simeq\; 4\pi G a^2\delta\varepsilon. \qquad (3.5.29)\end{aligned}$$

Similarly, using the expression of $\psi$ from equation (3.5.26) in the expression of $\delta\varepsilon$ we get,

$$\begin{aligned}\delta\varepsilon \;\simeq\;& \varepsilon_{,X}\frac{2X}{\mathcal{H}}\left[\left(\mathcal{H}\frac{\delta\varphi}{\varphi'}+\phi\right)'-\left\{(\mathcal{H}\tilde{F})'-\mathcal{H}^2\tilde{F}-\frac{\mathcal{H}\tilde{F}'}{2}\right\}\frac{\delta\varphi}{\varphi'}+\left(\mathcal{H}\tilde{F}+\frac{\tilde{F}'}{2}\right)\phi\right]-\\ & \varepsilon_{,X}\frac{3}{4a^2}\mathcal{H}\varphi^2\left(\mathcal{H}\frac{\delta\varphi}{\varphi'}+\phi\right)'+\varepsilon_{,X}\frac{3}{4a^2}\left(\psi'-\phi'\right)\mathcal{H}\varphi^2-\varepsilon_{,X}\frac{1}{4a^2}\left(\Delta\psi\right)\varphi^2-\\ & 3\mathcal{H}\left(\varepsilon+p\right)\frac{\delta\varphi}{\varphi'}. \qquad (3.5.30)\end{aligned}$$

Finally using $\psi = (1-2\tilde{F})\phi$ in the above expression of $\delta\varepsilon$, up to linear order in $\tilde{F}$ the Einstein's equation becomes,

$$\begin{aligned}\left(1+\varepsilon_{,X}\tilde{F}\right)\Delta\phi \;\simeq\;& \left(4\pi G a^2\varepsilon_{,X}\frac{2X}{\mathcal{H}}-\varepsilon_{,X}3\mathcal{H}\tilde{F}\right)\left(\mathcal{H}\frac{\delta\varphi}{\varphi'}+\phi\right)'+\\ & \left(3\mathcal{H}-4\pi G a^2\varepsilon_{,X}\frac{2X}{\mathcal{H}}\right)\left\{\frac{\left(\mathcal{H}\tilde{F}\right)'}{\mathcal{H}}-\mathcal{H}\tilde{F}-\frac{\tilde{F}'}{2}\right\}\left(\mathcal{H}\frac{\delta\varphi}{\varphi'}+\phi\right)-\\ & \left(3\mathcal{H}-4\pi G a^2\varepsilon_{,X}\frac{2X}{\mathcal{H}}\right)\frac{\left(\mathcal{H}\tilde{F}\right)'}{\mathcal{H}}\phi. \qquad (3.5.31)\end{aligned}$$

In order to calculate the power spectrum for $\phi$ and $\delta\varphi$, we have to solve equations (3.5.27, 3.5.31). These equations are highly coupled and one may need to decouple them. For simplicity we first write equations (3.5.27, 3.5.31) in a different notations as given below:

$$x' = A_1 y + B_1 x, \qquad (3.5.32)$$

$$A_2\Delta x = B_2 y' + C_2 y - D_2 x. \qquad (3.5.33)$$

where the quantities $x$ and $y$ are

$$
\begin{aligned}
x &= \left(\frac{a^2\phi}{\mathcal{H}}\right), \\
y &= \left(\mathcal{H}\frac{\delta\varphi}{\varphi'} + \phi\right).
\end{aligned}
$$

One can quickly identify that the variable $y$ is the gauge invariant comoving curvature perturbation ($\mathcal{R}$) defined before. The coefficients associated with (3.5.32) and (3.5.33) are

$$
\begin{aligned}
A_1 &= \frac{4\pi G a^4}{\mathcal{H}^2}(\varepsilon + p) + \frac{a^2}{\mathcal{H}}\left[\frac{(\mathcal{H}\tilde{F})'}{\mathcal{H}} - \mathcal{H}\tilde{F} - \frac{\tilde{F}'}{2}\right], \\
B_1 &= \left(2\mathcal{H}\tilde{F} - \frac{\left(\mathcal{H}\tilde{F}\right)'}{\mathcal{H}}\right), \\
A_2 &= \left(1 + \varepsilon_{,X}\tilde{F}\right), \\
B_2 &= \frac{a^2}{\mathcal{H}}\left(4\pi G a^2 \varepsilon_{,X}\frac{2X}{\mathcal{H}} - \varepsilon_{,X}3\mathcal{H}\tilde{F}\right), \\
C_2 &= \frac{a^2}{\mathcal{H}}\left(3\mathcal{H} - 4\pi G a^2 \varepsilon_{,X}\frac{2X}{\mathcal{H}}\right)\left[\frac{\left(\mathcal{H}\tilde{F}\right)'}{\mathcal{H}} - \mathcal{H}\tilde{F} - \frac{\tilde{F}'}{2}\right], \\
D_2 &= \left(3\mathcal{H} - 4\pi G a^2 \varepsilon_{,X}\frac{2X}{\mathcal{H}}\right)\frac{\left(\mathcal{H}\tilde{F}\right)'}{\mathcal{H}}.
\end{aligned} \tag{3.5.34}
$$

It can be easily seen that when we set the terms containing $\tilde{F}$ to be zero, the coefficients $B_1$, $C_2$ and $D_2$ are all zero; and equations (3.5.32, 3.5.33) reduces to the standard canonical single scalar field equations (3.4.5, 3.4.6). Now, following the procedure similar to single scalar field case, $y$ can be eliminated from equation (3.5.33) by using equation (3.5.32) and the decoupled equation for $x$ can be written as:

$$
x'' - \frac{A_1A_2}{B_2}\Delta x + \left[A_1\left\{\left(\frac{1}{A_1}\right)' - \frac{B_1}{A_1}\right\} + \frac{C_2}{B_2}\right]x' - \left[A_1\left(\frac{B_1}{A_1}\right)' + C_2\frac{B_1}{B_2} + D_2\frac{A_1}{B_2}\right]x = 0. \tag{3.5.35}
$$

Next, it is useful to substitute $x = u(\eta, \overrightarrow{x})f(\eta)$ in the above equation

$$u'' - \frac{A_1A_2}{B_2}\Delta u + \left[2\frac{f'}{f} + \left\{A_1\left(\frac{1}{A_1}\right)' - B_1 + \frac{C_2}{B_2}\right\}\right]u' +$$
$$\left[\frac{f''}{f} + \left\{A_1\left(\frac{1}{A_1}\right)' - B_1 + \frac{C_2}{B_2}\right\}\frac{f'}{f} - \left\{A_1\left(\frac{B_1}{A_1}\right)' + C_2\frac{B_1}{B_2} + D_2\frac{A_1}{B_2}\right\}\right]u = 0. \tag{3.5.36}$$

By equating the coefficient of $u'$ to zero one gets

$$\begin{aligned} f &= \exp\left[-\frac{1}{2}\int\left\{\mathrm{A}_1\left(\frac{1}{\mathrm{A}_1}\right)' - \mathrm{B}_1 + \frac{\mathrm{C}_2}{\mathrm{B}_2}\right\}\mathrm{d}\eta\right] \\ &= \sqrt{A_1}\exp\left[\frac{1}{2}\int\left(\mathrm{B}_1 - \frac{\mathrm{C}_2}{\mathrm{B}_2}\right)\mathrm{d}\eta\right] \end{aligned} \tag{3.5.37}$$

Here $f'$ and $f''$ can be written as

$$\frac{f'}{f} = -\frac{1}{2}\left[A_1\left(\frac{1}{A_1}\right)' - B_1 + \frac{C_2}{B_2}\right] \tag{3.5.38}$$
$$\frac{f''}{f} = \left[-\frac{1}{2}\left\{A_1\left(\frac{1}{A_1}\right)' - B_1 + \frac{C_2}{B_2}\right\}\right]^2 - \frac{1}{2}\left[A_1\left(\frac{1}{A_1}\right)' - B_1 + \frac{C_2}{B_2}\right]' \tag{3.5.39}$$

Finally the generalized Mukhanov-Sasaki equation can be written as

$$u'' - \frac{A_1A_2}{B_2}\Delta u + \left[-\frac{1}{4}\left\{A_1\left(\frac{1}{A_1}\right)' - B_1 + \frac{C_2}{B_2}\right\}^2 - \frac{1}{2}\left\{A_1\left(\frac{1}{A_1}\right)' - B_1 + \frac{C_2}{B_2}\right\}' -\right.$$
$$\left.\left\{A_1\left(\frac{B_1}{A_1}\right)' + C_2\frac{B_1}{B_2} + D_2\frac{A_1}{B_2}\right\}\right]u = 0, \tag{3.5.40}$$

which one may write in a more simplified form as

$$u'' + (1 + A)\,k^2u - \left(\frac{\theta''}{\theta} + B\right)u = 0, \tag{3.5.41}$$

Here both $A$ and $B$ are functions of $\tilde{F}$ and its derivatives. In the limit $\tilde{F} \to 0$ both $A, B \to 0$ and one recovers the standard Mukhanov-Sasaki equation for a canonical scalar-field [15].

The coefficient of the $k^2$ term in equation (3.5.41) can be regarded as the square of sound speed ($c_s^2$), which implies $c_s^2 = (1 + A) = \frac{A_1A_2}{B_2}$. Using the expressions of $A_1$,$A_2$ and $B_2$ in terms of background quantities, we can write $c_s^2$ after some algebra :

$$c_s^2 \approx 1 + \tilde{F} - \frac{1}{3}\frac{\tilde{F}'}{\mathcal{H}}\frac{1}{\left(1+\frac{p}{\varepsilon}\right)_{\text{can}}} \tag{3.5.42}$$

where, $\left(1+\frac{p}{\varepsilon}\right)_{\text{can}}$, which represents the quantity for canonical scalar field, can be obtained by setting $\tilde{F}$ terms to zero in equations (2.7.17, 2.7.21), i.e. $\left(1+\frac{p}{\varepsilon}\right)_{\text{can}} = \frac{\varphi'^2}{3M_{pl}^2\mathcal{H}^2}$. In the cosmic time $c_s^2$ can be expressed as: $c_s^2 \approx 1 + \tilde{F} - \frac{1}{4}\frac{\varphi}{\dot{\varphi}/H}$. The terms $\varphi$ and $\dot{\varphi}/H$ can be expressed in terms of $\tilde{F}$ and slow-roll parameter $\epsilon_V$ as: $\varphi = 2\sqrt{2}M_{pl}\sqrt{\tilde{F}}$ and $\dot{\varphi}/H = \sqrt{2}M_{pl}\sqrt{\epsilon_V}$. Finally the expression of $c_s^2$ can be written as $c_s^2 \approx 1 + \tilde{F} - \frac{1}{2}\sqrt{\frac{\tilde{F}}{\epsilon_V}}$. Later we will show that the spectral index ($n_s$) will be in the observed range if $\tilde{F} < 10^{-4}$. Therefore, for a typical value of $\epsilon_V \sim 10^{-2}$ one can see that just like canonical scalar field, in case of NSS we get $c_s^2 \simeq 1$.

#### 3.5.2.1 Solutions and the power spectrum

From the solutions of equation (3.5.41) the power spectrum for the scalar-perturbations can be calculated. In what follows we closely follow the method of the power-spectrum calculations given in [15] for a canonical scalar-field.

i) <u>Short wavelength(large $k$) region</u> : For a short wavelength regime (or large $k$), we can neglect $\left(\frac{\theta''}{\theta} + B\right)$ term with respect to $(1+A)\,k^2$ term in equation (3.5.41) and write

$$u'' + (1+A)\,k^2 u = 0. \tag{3.5.43}$$

Here we would like to comment on the choice of vacuum in the perturbation theory with NSS. From equation (3.5.43) it is clear that in the small scale the equation behaves like a harmonic oscillator if $A$ is constant in time. However, compared to the canonical single scalar field case (B.2) frequency of the oscillator is differed by terms proportional to $\tilde{F}$. Therefore, in case of NSS one can get Bunch-Davies vacuum with a modified frequency $\omega(\eta) = \sqrt{(1+A)}k$. Later we will show that there is an upper bound on $\tilde{F}$ in order to give a nearly scale invariant spectrum. In that case the Bunch-Davies vacuum for NSS and single scalar field will match with each other as the term $A$ practically becomes very small. One may look for the solution of equation (3.5.43) in the form $u = c\,(\eta)\, exp\left[ik \int \sqrt{1+A}d\eta\right]$. Substituting this back into equation(3.5.43) we get a 2nd order equation for $c(\eta)$,

$$c'' + ikc'\left(1 + \frac{A}{2}\right) + ikc\frac{A'}{2} = 0, \tag{3.5.44}$$

where we have considered $A$ to be a small quantity and write $\sqrt{1+A} \simeq 1 + \frac{A}{2}$. Next, We look for an approximate solution of equation(3.5.44) by regarding $A$ and $A'$ to be small. Thus we consider $c \approx c_0 + \bar{c}$ with $|c_0| > |\bar{c}|$ and $\bar{c}$ is of the same order of $A$ and $A'$. Equations for $c_0$ and $\bar{c}$ can be written as follows,

$$\begin{aligned} & c_0'' + ikc_0' = 0 \\ & \bar{c}'' + ikc_0'\frac{A}{2} + ik\bar{c}' + ikc_0\frac{A'}{2} = 0. \end{aligned} \tag{3.5.45}$$

The solution for $c_0$ can be written as

$$c_0 = b_2 - \frac{b_1 e^{-ik\eta}}{ik}, \tag{3.5.46}$$

where $b_1$ and $b_2$ are the constants of integration. Solution for $\bar{c}$ can be obtained as

$$\bar{c} = e^{-ik\eta} \int \left(b_1 - ikb_2 e^{ik\eta}\right) \frac{A}{2} d\eta. \tag{3.5.47}$$

Finally we get,

$$c(\eta) = b_2 - \frac{b_1 e^{-ik\eta}}{ik} + e^{-ik\eta} \int \left(b_1 - ikb_2 e^{ik\eta}\right) \frac{A}{2} d\eta. \tag{3.5.48}$$

Since in the limit when $A \to 0$ one should get the solution of the canonical scalar-field [15], we set $b_1 = 0$ and $b_2 = -\frac{i}{k^{\frac{3}{2}}}$. Thus one can write solution of equation (3.5.43) as

$$u = -\frac{i}{k^{\frac{3}{2}}} \left[1 - ike^{-ik\eta} \int e^{ik\eta} \frac{A}{2} d\eta\right] exp\left[ik \int \sqrt{1+A}d\eta\right]. \tag{3.5.49}$$

Finally one can obtain

$$\begin{aligned}\phi &= -\frac{i}{k^{\frac{3}{2}}}\left[1 - ike^{-ik\eta}\int e^{ik\eta}\frac{A}{2}d\eta\right]\left\{\frac{\mathcal{H}}{a^2}\sqrt{A_1}exp\left[\frac{1}{2}\int\left(B_1 - \frac{C_2}{B_2}\right)d\eta\right]\right\}\times \\ &\quad exp\left[ik\int\sqrt{1+A}d\eta\right]. \end{aligned} \tag{3.5.50}$$

From this expression of $\phi$ the power spectrum for $\phi$ in case of large $k$(small wavelength) can be found to be

$$\begin{aligned}\delta_\phi^2 &\propto |\phi|^2 k^3 \\ &= \left\{\frac{\mathcal{H}^2}{a^4}A_1 exp\left[\int\left(B_1 - \frac{C_2}{B_2}\right)d\eta\right]\right\}\left[1 - ike^{-ik\eta}\int e^{ik\eta}\frac{A}{2}d\eta\right]^2 \end{aligned} \tag{3.5.51}$$

The at large $k$ (or small wavelength) scales the power spectrum of $\phi$ is not a scale-invariant. However, there is a possibility that it becomes a scale invariant when $A$ can be regarded as a constant.

ii) Large wavelength(Small $k$) region : In case of small $k$ regime one can neglect $(1+A)k^2$ term with respect to $(\frac{\theta''}{\theta} + B)$ term. In this case we write the equation (3.5.41) can be written as

$$u'' - \left(\frac{\theta''}{\theta} + B\right)u = 0. \tag{3.5.52}$$

It is useful to look for the solution of $u$ in the form $u = u_{can}g$ where, $u_{can}$ is the solution when $B = 0$ i.e. no effect of non standard spinor is considered. This implies that in the $B \to 0$ limit $g = 1$. Now substituting for $u$ into equation(3.5.52) we get the equation for $g$ as

$$g'' + 2\left(\frac{u'_{can}}{u_{can}}\right)g' - Bg = 0. \tag{3.5.53}$$

For the case when $B = B(\tilde{F})$ is a small quantity, an approximate solution of $g \approx (g_0 + \bar{g})$ with $|g_0| > |\bar{g}|$ can be obtained in a manner similar to that discussed in the previous section. From equation (3.5.53) we get

$$g_0'' + 2\left(\frac{u'_{can}}{u_{can}}\right) g_0' = 0$$
$$\bar{g}'' + 2\left(\frac{u'_{can}}{u_{can}}\right) \bar{g}' = B g_0. \tag{3.5.54}$$

From the equation for $g_0$ we get

$$g_0 = c_1 + \int \left(\frac{c_2}{u_{can}^2}\right) d\eta, \tag{3.5.55}$$

where $c_1$ and $c_2$ are constants of integration. Plugging this solution of $g_0$ into the equation for $\bar{g}$ and solving the inhomogeneous equation, we can write get $\bar{g}$ as

$$\bar{g} = \int \frac{1}{u_{can}^2}\left[\int B u_{can}^2 d\eta\right] d\eta. \tag{3.5.56}$$

Thus we get,

$$g = c_1 + \int \left(\frac{c_2}{u_{can}^2}\right) d\eta + \int \frac{1}{u_{can}^2}\left[\int B u_{can}^2 d\eta\right] d\eta. \tag{3.5.57}$$

Since $g = 1$ when $B = 0$, one can set $c_1 = 1$ and $c_2 = 0$. The approximate solution for $u$ can be written as

$$u \simeq u_{can}\left(1 + \int \frac{1}{u_{can}^2}\left[\int B u_{can}^2 d\eta\right] d\eta\right). \tag{3.5.58}$$

One can notice from the expression of $u$ in the long wavelength(small $k$) regime that the resultant power spectrum is a scale invariant quantity. Therefore in a long wavelength regime we can write

$$\phi \simeq \frac{\mathcal{H}}{a^2}\sqrt{A_1} exp\left[\frac{1}{2}\int\left(B_1 - \frac{C_2}{B_2}\right) d\eta\right] u_{can}\left(1 + \int \frac{1}{u_{can}^2}\left[\int B u_{can}^2 d\eta\right] d\eta\right). \tag{3.5.59}$$

Finally we get power spectrum of $\phi$ as

$$\begin{aligned}\delta_\phi^2 &\propto |\phi|^2 k^3 \\ &= \delta^2_{\phi(can)}\left[1-\frac{\mathcal{H}\tilde{F}'}{8\pi G a^2\left(\varepsilon+p\right)_{can}}\right] exp\left[\int\left(B_1-\frac{C_2}{B_2}\right)d\eta\right]\times \\ &\left(1+\int\frac{1}{u^2_{can}}\left[\int B u^2_{can} d\eta\right]d\eta\right)^2 \end{aligned} \tag{3.5.60}$$

### 3.5.2.2 Spectral indices and the bound

Now as $\int \frac{1}{u^2_{can}}\left[\int B u^2_{can} d\eta\right]d\eta$ are $k$ independent, we get that power spectrum of $\phi$ for large wavelength(small $k$) is scale independent. Taking logarithm on both side we get

$$\begin{aligned}\ln\delta_\phi^2 &= \ln\delta^2_{\phi(can)}+\ln\left[1-\frac{\mathcal{H}\tilde{F}'}{8\pi G a^2\left(\varepsilon+p\right)_{can}}\right]+\left[\int\left(B_1-\frac{C_2}{B_2}\right)d\eta\right]+ \\ &2\ln\left(1+\int\frac{1}{u^2_{can}}\left[\int B u^2_{can} d\eta\right]d\eta\right). \end{aligned} \tag{3.5.61}$$

Spectral index for scalar perturbation can be written as

$$n_s-1=\frac{d\ln\left(\delta_\phi^2\right)}{d\ln k}. \tag{3.5.62}$$

At the time of horizon crossing ($c_s k \simeq aH$). Therefore, derivative with respect to ($\ln k$) can be approximated as $d\ln k \simeq \mathcal{H}d\eta$ (here we have neglected that variation of sound velocity and Hubble parameter with respect to cosmic time $t$ as they are very small). Therefore in the expression for the spectral index all the logarithmic derivatives can be replaced with time derivatives and finally we get

$$\begin{aligned}n_s-1 &\approx \frac{1}{\mathcal{H}}\left(\ln\delta^2_{\phi(can)}\right)'+\frac{1}{\mathcal{H}}\left(\ln\left[1-\frac{\mathcal{H}\tilde{F}'}{8\pi G a^2\left(\varepsilon+p\right)_{can}}\right]\right)'+\frac{1}{\mathcal{H}}\left(B_1-\frac{C_2}{B_2}\right)+ \\ &\frac{2}{\mathcal{H}}\left[\ln\left(1+\int\frac{1}{u^2_{can}}\left[\int B u^2_{can} d\eta\right]d\eta\right)\right]'. \end{aligned} \tag{3.5.63}$$

In the case of a slow-roll approximation, for some quantity $M$ its time derivative can be very small compared to $HM$, i.e. $\frac{\dot{M}}{HM} \ll 1$. So we argue that in the above expression we can neglect the second and last term. In the case of single canonical scalar-field we can write the first term in equation (3.5.63) following reference [15] as

$$\frac{1}{\mathcal{H}}\left(\ln \delta^2_{\phi(can)}\right)' \simeq -3\left(1+\frac{p}{\varepsilon}\right)_{\rm can}. \tag{3.5.64}$$

But in the case of NSS the correction term $\frac{1}{\mathcal{H}}(B_1 - \frac{C_2}{B_2})$ can be approximated as

$$\frac{1}{\mathcal{H}}\left(B_1 - \frac{C_2}{B_2}\right) \simeq \frac{3\mathcal{H}^2}{4\pi G\varphi'^2}\tilde{F}. \tag{3.5.65}$$

Using the Friedmann's equation and keeping the terms up to linear order in $\tilde{F}$ we write

$$\frac{1}{\mathcal{H}}\left(B_1 - \frac{C_2}{B_2}\right) \simeq 2\frac{1}{\left(1+\frac{p}{\varepsilon}\right)_{\rm can}}\tilde{F}. \tag{3.5.66}$$

Finally using (3.5.64) and (3.5.66) we get spectral index for scalar perturbation as

$$n_s = 1 - 3\left(1+\frac{p}{\varepsilon}\right)_{\rm can} + 2\frac{1}{\left(1+\frac{p}{\varepsilon}\right)_{\rm can}}\tilde{F}. \tag{3.5.67}$$

On galactic scale the canonical terms $\left(1+\frac{p}{\varepsilon}\right)_{\rm can}$ can be estimated as $10^{-2}$ and $\varepsilon_{\rm can}$ can be estimated as $10^{-12}$ * of the Planckian density [15]. Then equation (3.5.67) can be written as

$$n_s - 1 = -0.03 + 200\tilde{F}. \tag{3.5.68}$$

WMAP 7 years data suggests $n_s = 0.968 \pm 0.012$ with 68 % CL [8]. Therefore from equation (3.5.68) we can understand that, to get $n_s$ closer to the observed value, $\tilde{F}$ has to be smaller than $10^{-4}$. $\tilde{F}$ is the only new feature which NSS driven inflation brings over the inflationary scenario driven by canonical scalar field. Although $\tilde{F}$ is not a part of potential in the theory, its value may be estimated from $V$. As the potential $V(\varphi)$ is the dominant term in $\varepsilon_{\rm can}$, we can write $\varepsilon_{\rm can}/\varepsilon_{\rm PL} \sim \frac{V}{\varepsilon_{\rm PL}} \sim \frac{V}{M_{\rm PL}^4} \sim 10^{-12}$. Now from different models of potentials we can estimate $\tilde{F}$. For example, if we consider $\varphi^4$ kind of potential then $\varepsilon_{\rm can}/\varepsilon_{\rm PL}$ becomes $\tilde{F}^2$ and from the value of $\varepsilon_{\rm can}$ we can estimate $\tilde{F} \sim 10^{-6}$ which is

*One can note that $\left(1+\frac{p}{\varepsilon}\right)_{\rm can}$ is nothing but slow-roll parameter $\epsilon_V$

consistent with the NSS model.

In the case of a canonical scalar-field inflation it is well known that at a large scale the power-spectrum of tensor perturbation is [15] $\delta^2_{\mathrm{h(can)}} \simeq \frac{8}{\pi}H^2$. But for the present case the power-spectrum for the tensor perturbation can be modified to

$$\delta^2_{\mathrm{h}} \simeq \frac{8}{\pi}H^2 \times f\left(\tilde{F}\right). \tag{3.5.69}$$

Thus when $\tilde{F} \to 0$, $f\left(\tilde{F}\right) \to 1$ and we get the power-spectrum of the tensor perturbations for a canonical scalar-field. Since $\tilde{F}$ is a small quantity, the tensor to scalar ratio of the power spectrum for a NSS still be very small. With the upper bound on $\tilde{F}$ the tensor-to-scalar ratio for NSS can be very close to the canonical single scalar field case $r \sim 16\epsilon_{can}$. In the scalar field theories the detectability of the gravitational wave ($r > 0.07$) requires field variation $\Delta\varphi \geq 0.4M_{pl}$ during the last 4-5 e-foldings [38]. This lower bound on the variation of the scalar field is also known as the *Lyth bound*. Because of the upper bound on $\tilde{F}$ the NSS driven inflationary scenario becomes very similar to the standard single scalar field theories of inflation. Thus, we speculate that for the detectability of the gravitational waves, the *Lyth bound* can be valid in case of NSS driven inflationary models also provided $\tilde{F} < 10^{-4}$ is satisfied.

In conclusion,we have studied the cosmological perturbations generated by the inflation driven by a Lorentz invariant NSS model. We find that the the usual condition for the gravitational potentials $\phi$ and $\psi$ for scalar-perturbations i.e. $\delta T^i_j = 0$ giving $\psi = \phi$ is modified to $\psi = \left(1 - 2\tilde{F}\right)\phi$. We have also shown that the perturbations are nearly scale invariant and the hedgehog ansatz is not required. More importantly we have calculated the power-spectrum and spectral index for the metric perturbation. The model predicts the running spectral index which allows for a wide range of $\tilde{F}$. For the case $\tilde{F} = 0$ one gets back the expressions for the power spectrum and spectral index for a canonical scalar-field. Further our analysis shows that the calculated value of the spectral index $n_s$ can match to the value obtained from the WMAP data if there is an upper bound on the parameter $\tilde{F} < 10^{-4}$. Our analysis shows that the sound speed of the perturbation is not a constant but dependent on time. However, the expression of $c_s^2 \simeq 1 + \tilde{F} - \frac{1}{2}\sqrt{\frac{\tilde{F}}{\epsilon_V}}$ and the upper bound on $\tilde{F}$ imply that $c_s^2 \sim 1$. Finally the tensor to scalar ratio of the power spectrum remains much smaller as in the case of a scalar-field inflation due the upper bound on $\tilde{F}$. Here we would like to comment on the choice of vacuum in the perturbation theory with NSS. From equation (3.5.43) it is clear that in the small scale the equation behaves like a harmonic oscillator if $A$ is constant in time. However, compared to the canonical single scalar field case (B.2) frequency of the oscillator is differed by terms proportional to $\tilde{F}$. Therefore, in case of NSS one can get Bunch-Davies vacuum with a modified frequency. When $\tilde{F}$ satisfies the upper bound, the Bunch-Davies vacuum for NSS and single scalar field will match with each other.

# Chapter 4

# Early universe attractors

## 4.1 Introduction

Although the observations provide us the useful informations about dynamics of the inflaton field (like slow-roll etc.), in the inflationary theory we do not know the initial conditions [77–79]. The nature of the solutions of the dynamical equation may change with initial conditions. It is useful to have an important property associated with the dynamical equations, that allows wide class of solutions with different initial conditions to have similar asymptotic behaviour, i.e. attractor. There exist models of cosmology that allow to have an attractor solutions during inflation [15, 37].

The knowledge of initial conditions can provide crucial information about the nature of the fields and their interactions with the known matter fields [80, 81]. For instance, it is usually assumed that the inflaton is a fundamental scalar field. However, we do not know the nature of the scalar fields or its interaction with other fields. Similarly, it is not clear what are the properties of NSS and how they interact with the other fields. If the observations do provide evidence that the inflation occurred due to the one of these fields, the initial conditions of these fields will provide information about the nature of interactions with standard model particles. This in turn can be useful for model building which can be verified in high-energy experiments.

Though we have discussed so far the inflation but above issues are relevant in the context of dark energy also. It is unclear what dynamical fields drive the current accelerating universe. Even if the observations reveal the nature of the field, it is still not possible to know what were the initial condition that has lead to the current acceleration. This is referred to as a cosmic coincidence problem. The constraints on the interaction of these fields (inflaton and dark energy) with standard model particles will provide information about the initial condition that lead to acceleration. The attractor nature is also important in the context of dark energy. In case of dark energy, as explained in section (2.4), to alleviate the fine tuning problem (also known as the cosmic coincidence problem) associated with the initial condition it is important to have attractor behaviour in various dark energy models. In the reference [82] the authors have extensively studied the attractor nature in the various dark energy models.

In this chapter we investigate the following questions: If dynamical field during inflation is a condensate of the non-standard spinor whether a large set of initial conditions lead to inflation. If it does, then, can it constraint the interactions between the spinor fields and the matter particles. Recently NSS was proposed as a candidate of dark matter [83]. In Ref. [31,84], the authors could not find any stable fixed point with various kind of potentials. One of the draw-backs of their analyses is the choice of variables. Specifically, they have assumed $\varphi$ and $V(\varphi)$ to be independent variables. In reference [33] a combination of $\dot{\varphi}$, $H$ and $V(\varphi)$ as variables has been chosen and fixed points of the dynamical equations have been found. It has been shown that in these newly defined variables the dynamical equations have stable fixed points for a wide class of potentials and interactions between ELKO and matter. The general analysis of fixed points as an attractor is given in the appendix (D). In this chapter the scalar field is considered evolving in the presence of a barotropic fluid with equation of state $p_m = (\gamma - 1)\,\varepsilon_m$ ( where $0 \leq \gamma \leq 2$, $\gamma = \frac{4}{3}$ for radiation and $\gamma = 1$ for dust). Analysing the dynamical equations we show that in general there exist two sets of possibilities where one can find stable fixed points. One can also find fixed points which are stable during the time of inflation when there were no other matters and the only constituent of the universe was the inflaton field. We will argue that these fixed points are stable when the slow-roll inflationary conditions are satisfied. Thus we can have *early time attractor* during inflation. In the presence of the barotropic fluid with some specific form of interactions between the NSS and matter the stable fixed points can be achieved provided there are conditions on the coupling parameters of interactions.

## 4.2 Attractors in the scalar field theories

Attractor solutions in the scalar field models of inflation were studied by various authors. In the standard inflationary theory with $m^2\varphi^2$ potential the dynamical equations lead to an attractor where solutions for various initial conditions eventually converges [15]. In [37] the author has shown that in case of minimally coupled canonical scalar fields, under slow-roll conditions during inflation, the perturbations in the Hubble parameters decays exponentially concluding the existence of attractor. From equation (2.5.7) a deviation from any solution $\tilde{H}$ can be written as

$$\frac{\delta H_{,\varphi}}{\delta H} \approx \frac{3}{2}\frac{\tilde{H}}{\tilde{H}_{,\varphi}}. \tag{4.2.1}$$

Therefore, using the expression of $\Delta N$ given in (2.5.9) the solution of this equation can be written as

$$\delta H_f \approx \delta H_i \exp\left[-\frac{3}{2}\Delta N\right], \tag{4.2.2}$$

where the subscripts $i$ and $f$ denotes the initial value and some final value respectively. From equation (2.5.9) it can be noticed that as during inflation $\epsilon_H \ll 1$, the solution of (4.2.2) will vanish rapidly irrespective of the initial value. Thus, the inflationary period will behave like an attractor. Using the similar arguments in reference [85] the authors has shown that there exists attractor solutions in case of brane-inflation scenario in the FRW background. In reference [86] the author has studied the attractor behaviour in the new inflationary scenario. It is shown that in case of new inflationary scenario a small part of the attractor corresponds to the inflation as the rest does not give us e-folding greater than 60. In contrast, the chaotic inflationary scenario gives the inflationary attractor which attracts most of the solutions. In cosmology the role of exponential potentials of the form $V \propto \exp(-\sigma\varphi)$, where $\sigma$ is a constant, were investigated by various authors [87]. The steep nature of the exponential potential prevents it from driving inflation [80, 88, 89]. Depending on the choice of the constant $\sigma$ the attractor solutions can be obtained in different time of evolution. Although exponential potentials do not give good inflationary model, in contrast to standard inflationary models, the steepness in the potential may allow a significant amount of energy density at the time of nucleosynthesis [80]. In reference [90] inflationary scenario with general scalar tensor theory was considered. It was shown that in the scalar tensor theory, in general, attractors exist. It was further shown that as long as the potential has the form $V \propto f^M(\varphi)$, where $M$ is some non-negative number and $f(\varphi)$ is the general coupling term, in the attractor one can get the scale independent spectrum of density perturbation irrespective of the functional choice of $f(\varphi)$. The super-inflationary scenario was considered in [81], where under fast-roll conditions the scale invariant power spectrum was obtained.

## 4.3 Attractors in NSS cosmology

In this section we study attractor solutions in NSS equations.

### 4.3.1 Background equations

In section (2.7.1) we have already given the background equations in the conformal time. In this chapter we will work in cosmic time. Using the transformation between cosmic time and conformal time ($dt = ad\eta$) the expression of the energy density and pressure can be written as

$$\begin{aligned}
\varepsilon_\varphi &= \frac{1}{2}\dot{\varphi}^2 + \frac{3}{8}H^2\varphi^2 + V(\varphi), \\
p_\varphi &= \frac{1}{2}\dot{\varphi}^2 - \frac{1}{4}\left(H\varphi^2\right)^{\cdot} - \frac{3}{8}H^2\varphi^2 - V(\varphi), \qquad (4.3.1)
\end{aligned}$$

where $H$ is the Hubble parameter defined in the cosmic time. In the cosmic time the Friedmann equation and the acceleration equation can be written as follows:

$$\dot{H} = -\frac{\kappa^2}{2\left(1-\tilde{F}\right)}\left[\dot{\varphi}^2 - \frac{1}{2}H\varphi\dot{\varphi}\right], \tag{4.3.2}$$

$$H^2 = \frac{\kappa^2}{3\left(1-\tilde{F}\right)}\left[\frac{1}{2}\dot{\varphi}^2 + V\right] \tag{4.3.3}$$

where, $\kappa^2 = 8\pi G = \frac{1}{M_{pl}^2}$. The equation of motion can be written as:

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{3}{4}H^2\varphi + V_{,\varphi} = 0, \tag{4.3.4}$$

It is interesting to see that the above equations are similar to that of 'teleparallel' dark-energy [91, 92] with a particular value of coupling $\xi = -\frac{1}{8}$. However the physical reason behind this similarity is not known yet and can be a good problem for future.

### 4.3.2 Slow-roll parameters for NSS

Due to the presence of the $H^2\,\varphi^2$ term in the density and pressure, one has be careful in defining the slow-roll parameters for the ELKO condensate. In this section, we give the expressions for the slow-roll parameters. From the expression of time-time component of Einstein's equation one can write the Friedman's equation for NSS as:

$$H^2 = \frac{1}{3M_{pl}^2}\left(\frac{\dot{\varphi}^2}{2D} + \hat{V}\right) \tag{4.3.5}$$

where $D = 1 - \tilde{F} = 1 - \frac{\varphi^2}{8M_{pl}^2}$ and $\hat{V} = \frac{V}{D}$. Taking the time derivative on the both the sides of equation (4.3.5) one can write the slow-roll parameter $\epsilon$ as

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2}\frac{\dot{\varphi}^2/D}{\dot{\varphi}^2/2D + \hat{V}} + \frac{\dot{D}}{HD} = \epsilon_V + \alpha. \tag{4.3.6}$$

where, $\epsilon_V = \frac{3}{2}\frac{\dot{\varphi}^2}{\dot{\varphi}^2/2+V} \simeq \frac{3}{2}\frac{\dot{\varphi}^2}{V}$ (when $\dot{\varphi}^2 \ll V$). The slow-roll parameter $\epsilon_V$ also appears in the canonical single scalar field models of inflation. $\alpha = \frac{\dot{D}}{HD}$ is the additional parameter arising entirely due to the NSS. Thus the slow-roll conditions in NSS cosmology are modified. In what follows, the other slow-roll parameters are written.

Substituting for $\left(\dot{\varphi}^2/2D + \tilde{V}\right)$ from (4.3.6) into (4.3.5) one can write

$$\dot{\varphi}^2 = 2M_{pl}^2 H^2 D \left(\epsilon - \alpha\right). \tag{4.3.7}$$

Now in this case we define $\delta = \frac{\ddot{\varphi}}{H\dot{\varphi}}$. Taking the time derivative on both the sides of (4.3.7) we get,

$$\frac{\ddot{\varphi}}{H\dot{\varphi}} = \delta = -\epsilon + \frac{\alpha}{2} + \frac{\left(\epsilon - \alpha\right)^{\cdot}}{2H\left(\epsilon - \alpha\right)} \tag{4.3.8}$$

The last term can be dropped as it is the time derivative of the slow-roll parameters. Therefore finally one can write the definition of $\delta$ as:

$$\delta = -\epsilon + \frac{\alpha}{2} = -\epsilon_{\text{can}} - \frac{\alpha}{2}. \tag{4.3.9}$$

A closer inspection of above expression immediately suggests that $\delta$ is negative definite. For canonical scalar field, it is positive definite.

### 4.3.3 Early-time inflationary attractor in NSS

Following the procedure in canonical single scalar field case, using equation (4.3.2) and $H_{,\varphi} = \frac{\dot{H}}{\dot{\varphi}}$, for NSS one can write the following expression:

$$H_{,\varphi} = -\frac{\kappa^2}{2\left(1 - \tilde{F}\right)}\left[\dot{\varphi} - \frac{1}{2}H\varphi\right]. \tag{4.3.10}$$

By taking the square of the above equation and rearranging the terms, $\dot{\varphi}^2$ can be expressed as following:

$$\frac{\kappa^2}{6\left(1 - \tilde{F}\right)}\dot{\varphi}^2 = \frac{2\left(1 - \tilde{F}\right)}{3\kappa^2}H_{,\varphi}^2 + \frac{1}{6}\frac{\kappa^2}{\left(1 - \tilde{F}\right)}\left[H\varphi\dot{\varphi} - H^2\varphi^2\right]. \tag{4.3.11}$$

In the previous chapter (also in reference [32]) we have shown that, although perturbation theory tells us that there could be anisotropy associated with NSS, the amount of anisotropy is small and can be neglected. We have argue that the anisotropy terms are proportional to $\tilde{F}$, where $\tilde{F} = \frac{\kappa^2}{8}\varphi^2$. To get scale invariant spectrum one must have an upper bound $\tilde{F} < 10^{-4}$. Thus, during inflation $\tilde{F} = \frac{\kappa^2}{8}\varphi^2 \ll 1$. Under the consideration $\tilde{F} \ll 1$ and the slow-roll condition $\dot{\varphi}^2 \ll V$ the Friedmann equation can be approximated as

$$H^2 \approx \frac{\kappa^2}{3}V. \tag{4.3.12}$$

Using the approximate expression of $H^2$ and the definition of $\epsilon_V$ one can write $\kappa\frac{\dot{\varphi}}{H} \approx \sqrt{2}\sqrt{\epsilon_V}$. Therefore, replacing $\dot{\varphi}^2$ term in the Friedmann equation (4.3.3) with the right-hand-side of (4.3.11), for NSS one can write the relation between $H^2$ and $H_{,\varphi}^2$ as:

$$H^2\left[1+\frac{4\tilde{F}}{3\left(1-\tilde{F}\right)}-\frac{2}{3\left(1-\tilde{F}\right)}\sqrt{\epsilon_V}\sqrt{\tilde{F}}\right] \approx \frac{2}{3}\frac{\left(1-\tilde{F}\right)}{\kappa^2}H_{,\varphi}^2+\frac{\kappa^2}{3\left(1-\tilde{F}\right)}V. \tag{4.3.13}$$

Assuming that $\tilde{H}$ is a solution of (4.3.13), let us consider a small perturbation $\delta H$ around $\tilde{H}$, i.e.

$$H = \tilde{H} + \delta H. \tag{4.3.14}$$

Linearising the equation (4.3.13) one can find that for NSS, the relation equivalent to the canonical scalar field case (4.2.1) becomes

$$\frac{\delta H_{,\varphi}}{\delta H} \approx \frac{3}{2}\frac{1+\frac{4\tilde{F}}{3\left(1-\tilde{F}\right)}-\frac{2}{3\left(1-\tilde{F}\right)}\sqrt{\epsilon_V}\sqrt{\tilde{F}}}{\left(1-\tilde{F}\right)}\kappa^2\frac{\tilde{H}}{\tilde{H}_{,\varphi}}. \tag{4.3.15}$$

When $\tilde{F} \ll \epsilon_V \Rightarrow \varphi < \frac{\dot{\varphi}}{H}$, one can safely ignore the last term on the right hand side of (4.3.10). Therefore, using (4.3.10) one can write the following relation

$$\kappa^2\frac{H}{H_{,\varphi}}\Delta\varphi \approx -2\left(1-\tilde{F}\right)\frac{\Delta\varphi}{\dot{\varphi}/H} = -\left(1-\tilde{F}\right)\Delta N. \tag{4.3.16}$$

Here $\Delta N = N_f - N_i$ is the change in the e-folds during inflation where, the subscripts $i$ and $j$ denote the start and end of the inflation. Finally, using (4.3.16) the solution of (4.3.15) can be expressed as

$$(\delta H)_f \approx (\delta H)_i \exp\left[-\frac{3}{2}\left\{1+\frac{4\tilde{F}}{3\left(1-\tilde{F}\right)}-\frac{2}{3\left(1-\tilde{F}\right)}\sqrt{\epsilon_V}\sqrt{\tilde{F}}\right\}\Delta N\right] \tag{4.3.17}$$

During inflation, the number of e-folds rapidly expands ($\Delta N \sim 60$). Therefore, it can be seen that under the condition $\tilde{F} < 10^{-4}$ and for a typical value of $\epsilon_V \sim 10^{-2}$ the perturbation in the Hubble parameter $H$ decreases exponentially. Hence the dynamical equations of NSS behave like an attractor which is very similar to the canonical scalar field case provided $\tilde{F} < 10^{-4}$.

### 4.3.4 Dynamical equations of NSS in the presence of a barotropic perfect fluid

The expressions of energy density and pressure (4.3.1) can be written in terms of newly defined quantities $X$ and $\tilde{V}$ as following

$$\varepsilon_\varphi = X + \tilde{V}, \qquad p_\varphi = X - \tilde{V}, \tag{4.3.18}$$

where

$$X = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{8}\left(H\varphi^2\right)^{\cdot} \tag{4.3.19}$$
$$\tilde{V} = \frac{1}{8}\left(H\varphi^2\right)^{\cdot} + \frac{3}{8}H^2\varphi^2 + V(\varphi)\,. \tag{4.3.20}$$

$X$ can be considered as the kinetic energy of the NSS and $\tilde{V}$ can be considered as its potential.

Friedmann equation can be written as

$$H^2 = \frac{\kappa^2}{3}\varepsilon_{tot} = \frac{\kappa^2}{3}\left(\varepsilon_\varphi + \varepsilon_m\right), \tag{4.3.21}$$

where $\varepsilon_m$ is the matter density and $\kappa^2 = 8\pi G$. Using equation (4.3.18) we can write the Friedmann equation(4.3.21) as

$$x^2 + y^2 + v^2 = 1, \tag{4.3.22}$$

where $x$,$y$ and $v$ can be defined as $x = \frac{\kappa\sqrt{X}}{\sqrt{3}H}$, $y = \frac{\kappa\sqrt{\tilde{V}}}{\sqrt{3}H}$ and $v = \frac{\kappa\sqrt{\varepsilon_m}}{\sqrt{3}H}$. Now, if we consider that the matter and dark energy are interacting only with themselves then the continuity equation

$$\dot{\varepsilon}_{\rm tot} + 3H(\varepsilon_{\rm tot} + p_{\rm tot}) = 0 \tag{4.3.23}$$

can be written as two separate equations

$$\dot{\varepsilon}_\varphi + 3H(\varepsilon_\varphi + p_\varphi) = -Q, \tag{4.3.24}$$

$$\dot{\varepsilon}_{\rm m} + 3H(\varepsilon_{\rm m} + p_{\rm m}) = Q, \tag{4.3.25}$$

where $Q$ is the interaction term. In terms of the variables $x$, $y$, $v$ equations (4.3.24,4.3.25) can be written respectively as

$$x' = (\epsilon - 3)\, x - \frac{\lambda}{H}\frac{y^2}{x} - \frac{Q_1}{x}, \tag{4.3.26}$$

$$v' = \left(\epsilon - \frac{3}{2}\gamma\right) v + \frac{Q_1}{v}. \tag{4.3.27}$$

Here $'$ is the derivative with respect to time divided by $H$, i.e. $\prime = \frac{d}{Hdt}$, $\epsilon = -\frac{\dot{H}}{H^2}$ and $\lambda = \frac{\dot{\tilde{V}}}{\tilde{V}}$, $Q_1 = \frac{\kappa^2 Q}{6H^3}$. To derive the above equations we have used the relation $p_m = (\gamma - 1)\, \varepsilon_m$, where $\gamma$ can take values 1 or $\frac{4}{3}$ depending on whether the universe is filled with cold matter or radiation respectively. Derivative of the variable $y$ with respect to time give us

$$y' = \left(\epsilon + \frac{\lambda}{2H}\right) y. \tag{4.3.28}$$

$\dot{H}$ can be written as

$$\dot{H} = -\frac{\kappa^2}{2}\left[\varepsilon_\varphi + p_\varphi + \varepsilon_m + p_m\right] \tag{4.3.29}$$

Therefore we have three dynamical equations (4.3.26), (4.3.27) and (4.3.28) with one constraint (4.3.22). It is important to contrast the above set of variables with those used earlier [31]. The two variables $X$ and $\tilde{V}$ are independent of each other. However, in Wei's analysis [31], the two variables $y$ and $u$ are not independent.

In the rest of this work, we study the stability of fixed points with equations (4.3.26,4.3.27,4.3.28). Using these equations it is demonstrated that the NSS cosmology has a new sets of fixed points that can not be identified with the fixed points of a canonical scalar field.

### 4.3.5 Fixed points and stability analysis: General Analysis

Fixed points are the points where the dynamical variables stop evolving, i.e., at fixed point $(\bar{x}, \bar{y}, \bar{v})$ the time derivative of $x$, $y$ and $v$ are zero. At fixed points, dynamical equations (4.3.26, 4.3.27, 4.3.28) can be written as:

$$(\bar{\epsilon} - 3)\,\bar{x} - \frac{\lambda}{H}\frac{\bar{y}^2}{\bar{x}} - \frac{Q_1}{\bar{x}} = 0, \tag{4.3.30}$$

$$\left(\bar{\epsilon} - \frac{3}{2}\gamma\right)\bar{v} + \frac{Q_1}{\bar{v}} = 0, \tag{4.3.31}$$

$$\left(\bar{\epsilon} + \frac{\lambda}{2H}\right)\bar{y} = 0. \tag{4.3.32}$$

Eq. (4.3.32) leads to the following two set of fixed points:

1. Case I: $\bar{y} = 0$ and $\bar{\epsilon} \neq -\frac{\lambda}{2H}$
2. Case II: $\bar{y} \neq 0$ and $\bar{\epsilon} = -\frac{\lambda}{2H}$

In the rest of this section, the above two cases are considered with general interaction term $Q_1$. In the following section, we consider special cases for the interaction term and discuss the nature of fixed points.

#### 4.3.5.1 Case I

Substituting $\bar{y} = 0$ in equation (4.3.30) we get

$$\bar{\epsilon} = 3 + \frac{Q_1}{\bar{x}^2}. \tag{4.3.33}$$

The above form of $\epsilon$ gives crucial information about the class of interaction terms between the NSS and matter fields that can lead to attractor behavior. In particular, it immediately shows that $Q_1 \propto x^2$ may not lead to stable attractor points. It also provides an upper bound on the coupling constant. We discuss these in the next section.

General expression for $\epsilon$ can be written as

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2}\gamma + \left(3 - \frac{3}{2}\gamma\right) x^2 - \frac{3}{2}\gamma y^2. \tag{4.3.34}$$

Therefore, setting $\bar{y} = 0$, at fixed points $\bar{\epsilon}$ can be written as $\bar{\epsilon} = \frac{3}{2}\gamma + \left(3 - \frac{3}{2}\gamma\right) \bar{x}^2$. $\bar{\epsilon}$ has to be a positive to ensure to have an accelerated expansion of the universe. Finally one can write an important relation for $\bar{\epsilon}$ which will be used later

$$\bar{\epsilon} - 3 = \left(\frac{3}{2}\gamma - 3\right)\left(1 - \bar{x}^2\right). \tag{4.3.35}$$

Once we get the fixed points, we need to study the stability of the fixed point to ensure that the fixed points are actually giving us an attractor. If the fixed points are stable then we can have an attractor. The existence of an attractor will help in alleviating the 'cosmic coincidence' problem. To analyse the stability of these fixed points we perturb the system about the fixed point, $x \to \bar{x} + \delta x$ and $y \to \bar{y} + \delta y$ and study the evolution of the perturbations. If we have a growing solution of the perturbations then the fixed points are not stable. However if one finds a decaying solution one can say that the fixed points are stable. Substituting these values of $x$ and $y$ in equation (4.3.26) and (4.3.28) we get the perturbed equations of $x$ and $y$ as follows:

$$\delta x' = \left[(\bar{\epsilon}-3)+(6-3\gamma)\,\bar{x}^2+\frac{Q_1}{\bar{x}^2}-\frac{1}{\bar{x}}\frac{\partial Q_1}{\partial x}\right]\delta x - \left(\frac{1}{\bar{x}}\frac{\partial Q_1}{\partial y}\right)\delta y, \tag{4.3.36}$$

$$\delta y' = \left(\bar{\epsilon}+\frac{\lambda}{2H}\right)\delta y. \tag{4.3.37}$$

Here we have used $\delta\epsilon = [(6-3\gamma)\,\bar{x}]\,\delta x$ and $\bar{y}=0$. Equations (4.3.36) and (4.3.37) can be written as

$$\begin{pmatrix}\delta x'\\ \delta y'\end{pmatrix} = (M)\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}, \tag{4.3.38}$$

where

$$M = \begin{pmatrix}(\bar{\epsilon}-3)+(6-3\gamma)\,\bar{x}^2+\frac{Q_1}{\bar{x}^2}-\frac{1}{\bar{x}}\frac{\partial Q_1}{\partial x} & \frac{1}{\bar{x}}\frac{\partial Q_1}{\partial y}\\ 0 & \left(\bar{\epsilon}+\frac{\lambda}{2H}\right)\end{pmatrix}.$$

Two eigenvalues of the matrix $M$ are

$$\mu_1 = \left(\bar{\epsilon}+\frac{\lambda}{2H}\right), \tag{4.3.39}$$

$$\mu_2 = (\bar{\epsilon}-3)+(6-3\gamma)\,\bar{x}^2+\frac{Q_1}{\bar{x}^2}-\frac{1}{\bar{x}}\frac{\partial Q_1}{\partial x}. \tag{4.3.40}$$

Stability around the fixed points depend upon the nature of the eigen values $\mu_1$ and $\mu_2$. When $\mu_1 < 0$, $\mu_2 < 0$ the fixed points are stable and we can get an attractor solution. If $\mu_1 > 0$, $\mu_2 > 0$, the fixed points are unstable and we can not have any attractor. If one of them is positive and other one is negative, we get a saddle point which says that at one direction the fixed points are stable and at the other direction the fixed points are unstable.

It is already shown that accelerated expansion of the universe can occur only when the pressure is negative, i.e. the equation of state $w_\varphi = \frac{p_\varphi}{\varepsilon_\varphi} < 0$. In case of standard canonical scalar fields, in the region $-1 \le w_\varphi < 0$ the kinetic energy term $\dot{\varphi} \ge 0$. But the phantom modes (or ghosts) can appear with negative kinetic terms ($\dot{\varphi} < 0$) when the equation of state $w_\varphi < -1$. In Ref. [31] it was noted that in case of NSS, equation-of-state parameter $w_\varphi \ge -1$ when $\dot{\varphi}^2 \ge \frac{1}{4}\left(H\varphi^2\right)^{\cdot}$

and the phantom modes appear ($w_\varphi < -1$) if $\dot{\varphi}^2 \leq \frac{1}{4}\left(H\varphi^2\right)^{\cdot}$. Therefore in the region $w_\varphi \geq -1$ we always get $X > 0$ and in the phantom region $X < 0$. Now from Friedmann equation (4.3.21) one gets

$$H^2 = \frac{8\pi G}{3}\left(X + \tilde{V} + \varepsilon_m\right), \tag{4.3.41}$$

which implies that $H^2 > \frac{8\pi G}{3}\tilde{V}$. Finally taking logarithmic time derivative on both the sides of this inequality we get

$$\epsilon + \frac{\lambda}{2H} < 0. \tag{4.3.42}$$

This means that the eigenvalue $\mu_1$ is always negative as far as the condition $w_\varphi \geq -1$ region is satisfied. Therefore it is possible to have a stable fixed point, if $\mu_2$ becomes negative for some interaction $Q$. In the next section, we analyse the stability for three types of interactions with the positive kinetic term $X$ for NSS.

In Ref. [93], the authors have shown that for the $\varphi^4$ potential in case of NSS produces small primordial non-Gaussianity $f_{NL}$ (of the order of slow-roll parameters). The authors also investigated the anisotropy caused due to non-standard spinors by introducing two different expansion parameters. However, it is important to note that the Friedmann's equations for the condensate given in [93] and in the present work are different. In the present work we are using the expressions obtained in [30], whereas in [93] the authors used the old results. Although, the final conclusions about the anisotropy and $f_{NL}$ may still remain valid in our case. With the above mentioned potential, one can show that the dynamical equations can give stable fixed points. Using the definition of $\epsilon$ equation (4.3.19) can be written as

$$X = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{8}\left(\epsilon H^2\varphi^2 - 2H\varphi\dot{\varphi}\right). \tag{4.3.43}$$

As, the above potential suggests that during inflation $\dot{\varphi} < 0$ when $\varphi > 0$, we get $X > 0$. Thus, potential $-V = a^2\varphi^2 + b\varphi^4$, where $a^2$ is the mass term and $b$ is the self coupling– can give us the equation-of-state $w_\varphi \geq -1$ and the inequality

(4.3.42) remains valid in this case.

#### 4.3.5.2 Case II

Substituting the value of $\bar{\epsilon} = -\frac{\lambda}{2H}$ and constraint (4.3.22) in Eqs. (4.3.26) and (4.3.27), we get for ($\gamma = 1$):

$$
\begin{aligned}
x^{'} &= (\epsilon - 3)x - \frac{\lambda}{H}\frac{y^2}{x} - \frac{Q_1}{x} && (4.3.44)\\
v^{'} &= \left(\epsilon - \frac{3}{2}\right)v + \frac{Q_1}{v} && (4.3.45)
\end{aligned}
$$

Substituting for $\lambda$, we get,

$$
\begin{aligned}
\bar{\epsilon} &= -\frac{\lambda}{2H} = 3\bar{x}^2 + \frac{3}{2}\bar{v}^2 && (4.3.46)\\
\delta\epsilon &= 6\bar{x}\delta x + 3\bar{v}\delta v && (4.3.47)
\end{aligned}
$$

The perturbed equations about the fixed point are:

$$
\begin{aligned}
\delta x^{'} &= \left(3 - 9\bar{x}^2 - \frac{15}{2}\bar{v}^2 - 3\frac{\bar{v}^2}{\bar{x}^2} + 3\frac{\bar{v}^4}{\bar{x}^2}\right)\delta x + \left(6\frac{\bar{v}}{\bar{x}} - 12\frac{\bar{v}^3}{\bar{x}} - 15\bar{x}\bar{v}\right)\delta v -\\
&\quad \delta(\frac{Q_1}{\bar{x}}) && (4.3.48)\\
\delta v^{'} &= 6\,\bar{x}\,\bar{v}\,\delta x + \left(3\bar{x}^2 + \frac{9}{2}\bar{v}^2 - \frac{3}{2}\right)\delta v + \delta\left(\frac{Q_1}{v}\right) && (4.3.49)
\end{aligned}
$$

These attractor points are unique to ELKO cosmology regarding which we would like to stress the following points:

1. The perturbed equations do not explicitly depend on the potential. Hence, these equations can be realised for any potential provided $\bar{\epsilon} = -\lambda/(2H)$ is satisfied.

2. In case of $\gamma \geq \frac{2}{3}$ the fixed points $x \ll 1$ and $v \to 1$ (or vice-versa), Eq. (4.3.46) implies that $\epsilon > 1$ *. The parameter $\epsilon > 1$ means the decelerated

*In [33] it was claimed that this corresponds to fast-roll inflation. But $\epsilon > 1$ does not imply fast-roll inflation. If $\gamma < \frac{2}{3}$ then we get $\epsilon < 1$ and this case can give us fast-roll inflation provided the slow-roll condition ,$\frac{\ddot{\varphi}}{H\dot{\varphi}} \ll 1$, is violated for some special potentials [43,44]. It will be interesting to study fast-roll inflationary scenario for the NSS field.

period of evolution of universe. This period will be important in case of matter or radiation dominated era which are non-inflationary in nature ($\ddot{a} < 0$).

### 4.3.6 Special cases of the interaction term

In the previous section, we have obtained the condition for the existence of fixed point for general interaction. However, the analysis for a general interaction term is complicated. Here, for two cases, we take simple form of the interaction term and show explicitly the nature of the fixed points. In particular, the following interactions are considered: $Q_1 = \beta v^2 x$, $Q_1 = \beta v^2 x^2$ and $Q_1 = \beta v x^2$ where $\beta$ is the coupling constant.

#### 4.3.6.1 Case I:

- $Q_1 = \beta v^2 x$

In this case the fixed point $\bar{y}$ is zero. The fixed point $\bar{x}$ and $\bar{v}$ can be found using the equation (4.3.30). Substituting $\bar{y} = 0$ in equation (4.3.30) and using $\bar{v}^2 = 1 - \bar{x}^2$ from equation (4.3.22) we can write

$$(\bar{\epsilon} - 3)\bar{x} - \frac{Q_1}{\bar{x}} = 0. \tag{4.3.50}$$

Which gives us two solutions for $\bar{x}$

$$\bar{x} = \pm 1, \qquad \bar{x} = -\frac{\beta}{\left(3 - \frac{3}{2}\gamma\right)}. \tag{4.3.51}$$

Now $\bar{x} = \pm 1$ can not be a scaling solution because that will make the universe completely kinetic energy dominated. These solutions may be important in the time earlier than inflation. Therefore, the only possible scaling solution is $\bar{x} = -\frac{\beta}{\left(3-\frac{3}{2}\gamma\right)}$ which is negative as $\beta$ and $\left(3 - \frac{3}{2}\gamma\right)$ are both positive.

In the present case the eigenvalue $\mu_2$ of the matrix $M$ can be written as

$$\mu_2 = \left[-\left(3 - \frac{3}{2}\gamma\right)\left(1 - 3\bar{x}^2\right) + 2\beta\bar{x}\right]. \tag{4.3.52}$$

Now substituting the solution of $\bar{x}$ from (4.3.51) in the above expression of $\mu_2$ one can get the following expression of $\mu_2$:

$$\mu_2 = -\left(3 - \frac{3}{2}\gamma\right) + \frac{\beta^2}{\left(3 - \frac{3}{2}\gamma\right)}, \tag{4.3.53}$$

From the above expression of $\mu_2$ it can be understood that when the first term dominates over the last term one can get $\mu_2 < 0$. Therefore the condition for having a stable fixed point for this kind of interaction is:

$$\beta < \left(3 - \frac{3}{2}\gamma\right). \tag{4.3.54}$$

Therefore, in the presence of a barotropic perfect fluid, dynamical equations of NSS can show attractor behaviour for this kind of interaction provided the fixed coupling constant satisfy an upper bound.

The expression of $\bar{\epsilon}$ in terms of the fixed points is

$$\bar{\epsilon} = 3\bar{x}^2 + \frac{3}{2}\gamma\left(1 - \bar{x}^2\right). \tag{4.3.55}$$

The above analysis tells us that when $\gamma = 0$ we get stable attractor when $\beta < 3$. When the coupling $\beta$ is very weak ($\beta \ll 3$), the expression of $\bar{x}$ is also very small ($\bar{x} \ll 1$). In this case equation (4.3.55) suggests $\bar{\epsilon} \ll 1$. On this attractor the dominant contribution in total energy will come from the barotropic fluid which behave as a cosmological constant with negative pressure ($p_m = -\varepsilon_m$). Thus, this attractor can be the late-time attractor which is driven by cosmological constant to give accelerated expansion of the universe today.

- $Q_1 = \beta v^2 x^2$

In this case equation (4.3.35) and (4.3.50) tell us that

$$\left(\frac{3}{2}\gamma - 3\right)\left(1 - \bar{x}^2\right)\bar{x} - \beta\left(1 - \bar{x}^2\right)\bar{x} = 0. \tag{4.3.56}$$

So, the only solution of $\bar{x} = (0, \pm 1)$. Here we get barotropic fluid dominated and kinetic energy dominated universe respectively. When $\bar{x} = 0$, in contrast to the previous case of interaction, $\mu_2$ will become negative and the fixed point will be stable attractor for any value of coupling $\beta$.

- $Q_1 = \beta v x^2$

Following the similar method as described above using (4.3.50) for this kind of interaction one can find that at fixed point the only solution for $x$ is:

$$\bar{x} = \pm\sqrt{1 - \frac{\beta^2}{\left(3 - \frac{3}{2}\gamma\right)^2}}, \tag{4.3.57}$$

Here we have considered $\bar{x} \neq (0, \pm 1)$. Substituting the above expression of $Q_1$ in the expression of $\mu_2$ one can get

$$\mu_2 = (6 - 3\gamma)\,\bar{x}^2 + \beta\frac{\bar{x}^2}{\bar{v}}. \tag{4.3.58}$$

Using the definition of $\bar{v}^2 = 1 - \bar{x}^2$ and the expression of $\bar{x}$ from (4.3.57) one can write the expression of $\mu_2$ in terms of the coupling $\beta$ as:

$$\mu_2 = 3\left[\left(3 - \frac{3}{2}\gamma\right) - \frac{\beta^2}{\left(3 - \frac{3}{2}\gamma\right)}\right], \tag{4.3.59}$$

Therefore in this case the $\mu_2$ will be negative only when $\beta > \left(3 - \frac{3}{2}\gamma\right)$. However from (4.3.57) one can see that this condition will make $\bar{x}$ imaginary. Therefore we can not find a physical stable fixed point in this case.

#### 4.3.6.2 Case II:

- $Q_1 = \beta v^2 x$

For this interaction, the perturbed equations of $x$ and $v$ are:

$$\begin{aligned}
\delta x' &= [3 - 9\bar{x}^2 - \frac{15}{2}\bar{v}^2 - 3\frac{\bar{v}^2}{\bar{x}^2} + 3\frac{\bar{v}^4}{\bar{x}^2}]\delta x + \\
&\quad [6\frac{\bar{v}}{\bar{x}} - 12\frac{\bar{v}^3}{\bar{x}} - 15\bar{x}\bar{v} - 2\beta\bar{v}]\delta v. &(4.3.60)\\
\delta v' &= [6\bar{x}\bar{v} + \beta\bar{v}]\delta x + [3\bar{x}^2 + \frac{9}{2}\bar{v}^2 - \frac{3}{2} + \beta\bar{x}]\delta v &(4.3.61)
\end{aligned}$$

The two eigen-values corresponding to the above set of equations are negative.

Fig. (1a) shows that for different initial conditions $v \to 1$ and $x \ll 1$ is an attractor point.

- $Q_1 = \beta v^2 x^2$

For this interaction, the perturbed equations of $x$ and $v$ are:

$$
\begin{aligned}
\delta x^{'} &= [3 - 9\bar{x}^2 - \frac{15}{2}\bar{v}^2 - 3\frac{\bar{v}^2}{\bar{x}^2} + 3\frac{\bar{v}^4}{\bar{x}^2} - \beta\bar{v}^2]\delta x + \\
&\quad [6\frac{\bar{v}}{\bar{x}} - 12\frac{\bar{v}^3}{\bar{x}} - 15\bar{x}\bar{v} - 2\beta\bar{v}\bar{x}]\delta v. \qquad (4.3.62) \\
\delta v^{'} &= [6\bar{x}\bar{v} + 2\beta\bar{v}\bar{x}]\delta x + [3\bar{x}^2 + \frac{9}{2}\bar{v}^2 - \frac{3}{2} + \beta\bar{x}^2]\delta v \qquad (4.3.63)
\end{aligned}
$$

Here again, both the eigenvalues corresponding to the above set of equations are negative. The eigenvalues are negative for all ranges of $\beta$ for which $x$ and $v$ are real. Fig. (1b) shows that for different initial conditions $v \to 1$ and $x \ll 1$ is an attractor point.

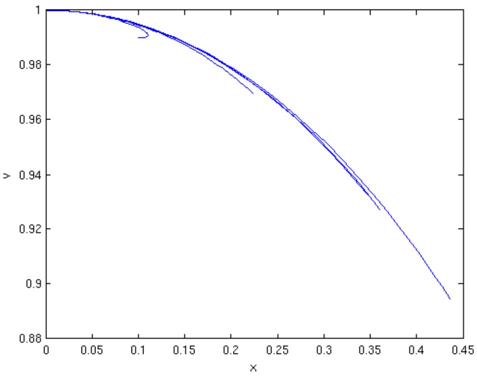
- $Q_1 = \beta\, v\, x^2$

For this interaction, the perturbed equations of $x$ and $v$ are:

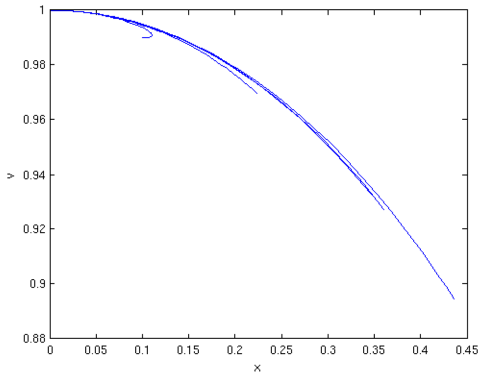
$$
\begin{aligned}
\delta x^{'} &= [3 - 9\bar{x}^2 - \frac{15}{2}\bar{v}^2 - 3\frac{\bar{v}^2}{\bar{x}^2} + 3\frac{\bar{v}^4}{\bar{x}^2} - \beta\bar{v}]\delta x + \\
&\quad [6\frac{\bar{v}}{\bar{x}} - 12\frac{\bar{v}^3}{\bar{x}} - 15\bar{x}\bar{v} - \beta\bar{x}]\delta v. \qquad (4.3.64) \\
\delta v^{'} &= [6\bar{x}\bar{v} + 2\beta\bar{x}]\delta x + [3\bar{x}^2 + \frac{9}{2}\bar{v}^2 - \frac{3}{2}]\delta v \qquad (4.3.65)
\end{aligned}
$$

Here again, in contrast to case-I, both the eigenvalues corresponding to the above set of equations are negative for all values of $\beta$ where $x$ and $v$ are real. Fig. (1c) shows that for different initial conditions $v \to 1$ and $x \ll 1$ is an attractor point.

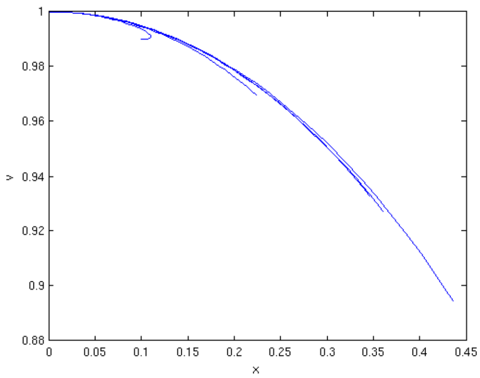
As it is known that it is extremely difficult to know the initial conditions for the field that drives the inflation. In addition it is also desirable to have an inflationary model which does not require any finely tuned initial conditions to have an inflationary regime. Hence it is important for a model of inflation to have attractor points in the space of matter field variables. In earlier works, it was not possible to show explicitly that the NSS based inflationary models can




(a)




(b)




(c)

Figure 4.1: The late attractor for three interactions (a) $Q_1 = \beta v^2 x$, (b) $Q_1 = \beta v^2 x^2$ and (c) $Q_1 = \beta v x^2$. The figures show that $v \to 1$ and $x \ll 1$ is a stable fixed point.

support a late-time attractor. In this section we have shown that by rewriting the background field equations in terms of new variables $x$ and $y$, the NSS based model can have such attractor solution. We have also shown that under the condition $\tilde{F} < 10^{-4}$ the attractor solution can be found for the early-time inflationary era. The search for stable fixed points is made by considering the evolution of NSS in the presence of barotropic perfect fluid. It is shown that the attractor points can be found without any specific choice of potential. Here we have shown that the dynamical equations can give us fixed points for the following two cases: In Case-I we have $y = 0$ and $\epsilon + \frac{\lambda}{2H} \neq 0$, and in Case-II we have $y \neq 0$ and $\epsilon + \frac{\lambda}{2H} = 0$. Stability of the fixed points or the negativity of the eigenvalues are shown to depend on the form of interaction between NSS and matter. In this thesis, for both Case-I and Case-II we have considered three types of interactions between the NSS and the barotropic fluid: $Q_1 = \beta v^2 x$, $Q_1 = \beta v^2 x^2$ and $Q_1 = \beta v x^2$. In Case-I, for the interaction term $Q_1 = \beta v^2 x$, one must put an upper bound on the coupling parameter $\beta < \left(3 - \frac{3}{2}\gamma\right)$ to get stable fixed points. For other types of interactions no such upper bound is required for Case-I. In Case-II it is shown that $x \ll 1$ and $v \to 1$ are stable fixed points. In this case the stable fixed points can be obtained without any condition on the coupling parameter $\beta$.

# Chapter 5

# Summary and Conclusions

We have noted earlier that in previous literature on NSS-cosmology the equation of motion obtained from the energy-momentum tensor was not matching with the equation of motion obtained from varying the action. In this work we use the consistent NSS theory developed in reference [30] to study NSS-cosmology. In particular we have addressed two important aspects of NSS-cosmology in this thesis: The first order inflationary perturbation theory and attractor behaviour. We have used a simple ansatz for the NSS-field $\lambda$ and its dual $\vec{\lambda}$ (3.5.2) to study the NSS-cosmology. This ansatz helps in defining gauge-invariant quantities. It should be noted that the term $F^{\mu}_{\nu}$ in equation (2.7.8), arises due to variation of spin-connection $\Gamma_{\mu}$ with respect to the metric. Perturbation of $F^{\mu\nu}$ would imply that for the NSS $\delta T^{i}_{j} \neq 0$ for $i \neq j$. For a canonical scalar field case $\delta T^{i}_{j} = 0$ for $i \neq j$. Thus the metric perturbations $\phi$ and $\psi$ for the NSS one can write $\psi = \left(1 - 2\tilde{F}\right)\phi$ in contrast with a scalar field theory where $\psi = \phi$. We have shown that the perturbations are nearly scale invariant. More importantly we have calculated the power-spectrum and spectral-index for the metric perturbation. We show that the running spectral-index allows for a wide range of values for $\tilde{F}$. When all of the terms containing $\tilde{F}$ are dropped one gets back the expressions for the power spectrum and spectral index for a canonical scalar-field inflation. It should be noted that our analysis shows that the calculated value of the spectral-index $n_s$ for NSS-field can match with the WMAP data provided $\tilde{F}$ satisfies an upper bound $\tilde{F} < 10^{-4}$. Our analysis also shows that the sound speed of the perturbation is not a constant but dependent on time. However, the expression of $c_s^2 \simeq 1+\frac{2}{3}\tilde{F}$ together with the upper bound on $\tilde{F}$, implies that $c_s^2 \sim 1$. Ratio of power-spectrum of the tensor perturbation and scalar perturbation may remain as small as the ratio calculated from a scalar field inflationary model due to the upper bound on $\tilde{F}$. Thus, from the first order perturbation theory point of view, it may not be possible to distinguish between predictions of NSS field inflationary model and the predictions of single scalar field inflationary model.

The attractor behaviour of NSS cosmology has been studied in this thesis. As we don't know the initial conditions exactly, the attractor behaviour is very important from the point of view of the robustness of the models associated with the inflation and the dark energy. The search for having stable fixed points as attractors in the context of NSS dark energy were pursued in reference [31] with various models of NSS potentials. It was argued that there exist no fixed points and the dark energy models of NSS field suffer from the cosmic coincidence problem. In this thesis we have shown that the dynamical equations can give us early-time inflationary attractor which corresponds to 60 e-folding when the condition $\tilde{F} < 10^{-4}$ is satisfied. We have also demonstrated that for NSS field, stable fixed points can be obtained if we choose the variables in a different way than used in reference [31]. We have shown that in case of NSS dark energy model if we work with the new variables $x$ and $y$ we can get stable fixed point. In this analysis the evolution equations of NSS has been studied in the presence of a barotropic perfect fluid with equation of state $p = (\gamma - 1)\,\varepsilon$. The analysis shows that we can have two cases: Case-I is when $y = 0$ and $\epsilon + \frac{\lambda}{2H} \neq 0$, Case-II is when $y \neq 0$ and $\epsilon + \frac{\lambda}{2H} = 0$. We have also considered three types of interaction terms between NSS and the fluid: $Q_1 = \beta v^2 x$, $Q_1 = \beta v^2 x^2$ and $Q_1 = \beta v x^2$. In Case-I, when the equation of state satisfies the condition $w_\phi \geq -1$, $X > 0$. Therefore, the negativity of the eigenvalue $\mu_1 = s + \frac{\lambda}{2H} < 0$ comes naturally from Friedmann equation. In this case for interaction $Q_1 = \beta v^2 x$ we get the coupling constant must have an upper bound $\beta < \left(3 - \frac{3}{2}\gamma\right)$ to have stability, i.e. $\mu_2 < 0$. There is no such upper bound for the other types of interactions in Case-I. Case-II shows that $x \ll 1$ and $v \to 1$ are the stable fixed points. In Case-II the stability can be achieved for any range of $\beta$. In this work the variables we have worked with are not a simple transformation of the variables chosen in reference [31]. The stability can be achieved by redefining the potential and the kinetic part.

Cosmology with NSS is a very new field. There are many directions to which the future research can be done. In this thesis we have done the first order perturbation theory. As we are in the era of precision cosmology, it is important to study the inflationary theory of NSS numerically, without any slow-roll approximations, to match various cosmological parameters with the *Planck* data. In the first order perturbation theory one important assumption is that the perturbations are Gaussian random fields in nature. Deviation from Gaussianity can be due to various reasons, for example, presence of multi field, violation of slow-roll, exited initial state (non-Bunch-Davies vacua) etc. Recent observational data from *Planck* mission has constrained the local bispectrum amplitude associated with non-Gaussianity as $f_{NL} = 2.7 \pm 5.8$ [94]. It can be interesting to study non-Gaussianity with the NSS field to check whether the theoretical predictions matches with the observation. In order to calculate bispectrum for NSS, calculation of the second order perturbation theory is required. As we have seen that NSS satisfies the Klein-Gordon equation which is second order differential equation in time, study of parametric oscillations in the reheating theories can be exciting problems in future. As we do not know how NSS interacts with the observable matter, at present it is not clear how NSS can decay to matter and reheats. One possibility could be that NSS can decay to ordinary matter through Higgs.

# Appendix A

# Basics of FLRW metric

## A.1 Background

### A.1.1 Christoffel symbol in case of FLRW metric

From equation (2.2.4) it can be seen that the Christoffel symbol contains the derivative of the metric. Therefore, for those metrics whose components are constants of space and time, the christoffel symbol vanishes. Because of the presence of the time dependent scale factor, there will be some non-zero components of Christoffel symbol for the FLRW metric (2.2.6). It can be shown that in case of FLRW metric the cmponents which contain two same spatial indices are non-zero and all other components will vanish. The non-zero components are given as:

$$\Gamma^t_{ij} = \delta_{ij} a\dot{a}, \qquad \Gamma^i_{tj} = \Gamma^i_{jt} = \delta_{ij}\frac{\dot{a}}{a}, \tag{A.1.1}$$

where the index $t$ is the temporal index and $i, j$ are three spatial indices. In case of the metric in conformal time (2.7.4) the expressions of the Christoffel symbols changes a little, the Christoffel symbol with all three temporal indices becomes nonzero and all of the non-zero components of Christoffel symbol are same:

$$\Gamma^\eta_{\eta\eta} = \frac{a'}{a}, \qquad \Gamma^\eta_{ij} = \delta_{ij}\frac{a'}{a}, \qquad \Gamma^i_{\eta j} = \Gamma^i_{j\eta} = \delta_{ij}\frac{a'}{a}, \tag{A.1.2}$$

### A.1.2 Ricci tensor and Ricci scalar for FLRW metric

From the expression of Ricci tensor in (2.2.3) one can write the time-time component of Ricci tensor for the metric (2.2.6) as

$$R_{tt} = -\partial_t \Gamma^{\lambda}_{t\lambda} - \Gamma^{\eta}_{t\lambda}\Gamma^{\lambda}_{t\eta}, \tag{A.1.3}$$

where the first term and the third term becomes zero, as the Christoffel symbol with two temporal indices are zero. Substituting the expressions of Christoffel symbol (A.1.1) in the above expression one can write

$$R_{tt} = -3\frac{\ddot{a}}{a}. \tag{A.1.4}$$

In the similar manner the space-space component of the Ricci tensor can be written as

$$R_{ij} = \delta_{ij}\left(2\dot{a}^2 + a\ddot{a}\right) \tag{A.1.5}$$

Contracting Ricci tensor into the metric the expression of Ricci scalar in terms of Ricci tensor can be written as

$$R = R_{tt} - \frac{1}{a^2}\delta^{ii}R_{ii}, \tag{A.1.6}$$

where $i$ index will be summed over. Therefore, the expression of Ricci scalar becomes

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). \tag{A.1.7}$$

In conformal time (2.7.4) the non-vanishing components of background Ricci tensor and Ricci scalar can be written as

$$R_{\eta\eta} = -3\left[\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right], \qquad R_{ij} = \left[\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right]\delta_{ij}, \qquad R = -\frac{1}{a^2}\left(6\frac{a''}{a}\right). \tag{A.1.8}$$

All other components of Ricci tensor are zero.

It is worth mentioning that, as the Christoffel symbol is quadratic in metric, the Christoffel symbol and Ricci tensor both remains unchanged under the change of signature of the metric (2.2.6). However, the expression of Ricci scalar will pick up opposite sign if we change the signature of the metric. For example, here the signature of the metric is $\{+,-,-,-\}$, but if we use $\{-,+,+,+\}$ the Ricci scalar (A.1.7) becomes $R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)$.

## A.2 Perturbation

To calculate the perturbations in the Ricci scalar and tensor the first step is calculation of the perturbation of Christoffel symbol.

### A.2.1 Perturbation in the Christoffel symbol

The general expression of the perturbed Christoffel symbol for inhomogeneous metric perturbation (3.2.1) can be written as

$$\delta\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\bar{g}^{\alpha\rho}\left[\partial_\beta\left(\delta g_{\rho\gamma}\right) + \partial_\gamma\left(\delta g_{\rho\beta}\right) - \partial_\rho\left(\delta g_{\gamma\beta}\right)\right] + \frac{1}{2}\delta g^{\alpha\rho}\left[\partial_\beta\bar{g}_{\rho\gamma} + \partial_\gamma\bar{g}_{\rho\beta} - \partial_\rho\bar{g}_{\beta\gamma}\right], \tag{A.2.1}$$

where the *bar* over the metric denotes the unperturbed background metric.

As we have seen that gauge invariant perturbation can be calculated by directly writing the perturbation in *longitudinal Newtonian gauge* and replacing the gauge dependent quantities by gauge invariant quantities. The general metric, in conformal time, in terms of gauge invariant quantities can be given as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} = a^2\begin{pmatrix} 1+2\psi & \mathbb{O} \\ \mathbb{O} & \left(-1+2\phi\right)\delta_{ij} + 2h_{ij} \end{pmatrix}, \tag{A.2.2}$$

and

$$g^{\mu\nu} = \bar{g}^{\mu\nu} + \delta g^{\mu\nu} = \frac{1}{a^2}\begin{pmatrix} 1-2\psi & \mathbb{O} \\ \mathbb{O} & \left(-1-2\phi\right)\delta_{ij} - 2h_{ij} \end{pmatrix} \tag{A.2.3}$$

where $\psi$ and $\phi$ are the gauge invariant scalar perturbations. The above metric is obtained by replacing gauge dependent variables $\Psi$ and $\Phi$ with gauge invariant variables $\psi$ and $\phi$ respectively in (3.2.3) and (3.2.5). Here we have added $h_{ij}$ as the symmetric tensor perturbations with properties $h_{yy} = -h_{zz} = h_+$, $h_{yz} =$

$h_{zy} = h_\times$, and all other tensor parts are zero. It should be mentioned here that the tensor perturbations are gauge invariant quantities, therefore they don't need any treatment like scalar perturbations.

Using (A.2.2) and (A.2.3) one can write the different components of the perturbed Christoffel symbol (A.2.1) as,

$$\begin{aligned}
&\delta\Gamma^\eta_{\eta\eta} = \psi', \qquad \delta\Gamma^\eta_{\eta i} = \partial_i\psi, \qquad \delta\Gamma^i_{\eta\eta} = \partial_i\psi, \qquad \delta\Gamma^i_{\eta j} = -\phi'\delta_{ij} - h'_{ij},\\
&\delta\Gamma^\eta_{ij} = -\left(2\frac{a'}{a}\psi + 2\frac{a'}{a}\phi + \phi'\right)\delta_{ij} - \left(h'_{ij} + 2\frac{a'}{a}h_{ij}\right),\\
&\delta\Gamma^i_{jk} = \left[(\partial_i\phi)\,\delta_{jk} - (\partial_j\phi)\,\delta_{ik} - (\partial_k\phi)\,\delta_{ij}\right] + \left[\partial_i h_{jk} - \partial_j h_{ik} - \partial_k h_{ij}\right]. \quad \text{(A.2.4)}
\end{aligned}$$

### A.2.2 Perturbed Ricci tensor and Ricci scalar

The expression of perturbed Ricci scalar and Ricci tensor are respectively

$$\delta R = (\delta g^{\mu\nu})\,R_{\mu\nu} + \bar{g}^{\mu\nu}\,(\delta R_{\mu\nu})\,, \tag{A.2.5}$$

$$\delta R_{\mu\nu} = \left(\delta\Gamma^\alpha_{\mu\nu}\right)_{,\alpha} - \left(\delta\Gamma^\alpha_{\mu\alpha}\right)_{,\nu} + \delta\left(\Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu}\right) - \delta\left(\Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu}\right) \tag{A.2.6}$$

Using the expressions of perturbed Christoffel symbol (A.2.4) the different components of perturbed Ricci tensor can be listed as

$$\begin{aligned}
\delta R_{\eta\eta} &= 3\phi'' + 3\frac{a'}{a}\left(\psi' + \phi'\right) + \Delta\psi, \qquad \delta R_{\eta i} = 2\left(\phi' + \frac{a'}{a}\psi\right)_{,i},\\
\delta R_{ij} &= -\left[\phi'' + \frac{a'}{a}\left(\psi' + 5\phi'\right) + 2\frac{a''}{a}\left(\psi + \phi\right) + 2\left(\frac{a'}{a}\right)^2\left(\psi + \phi\right) - \Delta\phi\right]\delta_{ij} -\\
&\quad \partial_i\partial_j\left(\psi - \phi\right), \qquad \text{(A.2.7)}
\end{aligned}$$

where $\Delta = \partial_i\partial^i$. The perturbed Ricci scalar in terms of the metric perturbations becomes

$$\delta R = \frac{1}{a^2}\left[6\phi'' + 12\frac{a''}{a}\psi + 6\frac{a'}{a}\left(\psi' + 3\phi'\right) + 2\Delta\left(\psi - 2\phi\right)\right]. \tag{A.2.8}$$

## A.3 Perturbed Einstein tensor

The Einstein tensor in the mixed form is written as,

$$G^{\mu}_{\nu} = g^{\mu\rho}\left(R_{\rho\nu} - \frac{1}{2}g_{\rho\nu}R\right). \tag{A.3.1}$$

Using the background expressions of Ricci tensor and Ricci scalar the non-zero components of Einstein tensor can be written as

$$G^{\eta}_{\eta} = \frac{3}{a^2}\left(\frac{a'}{a}\right)^2, \qquad G^{i}_{j} = -\frac{1}{a^2}\left[-2\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right]\delta^i_j. \tag{A.3.2}$$

Perturbation in the Einstein tensor can be written as

$$\delta G^{\mu}_{\nu} = \delta g^{\mu\rho}\left(R_{\rho\nu} - \frac{1}{2}\bar{g}_{\rho\nu}R\right) + \bar{g}^{\mu\rho}\left(\delta R_{\rho\nu} - \frac{1}{2}\delta g_{\rho\nu}R - \frac{1}{2}\bar{g}_{\rho\nu}\delta R\right) \tag{A.3.3}$$

which after substituting the perturbed metric, Ricci tensor and Ricci scalar gives us the following components

$$\begin{aligned}
\delta G^{\eta}_{\eta} &= -\frac{2}{a^2}\left[3\frac{a'}{a}\left(\phi' + \frac{a'}{a}\psi\right) - \Delta\phi\right], \\
\delta G^{\eta}_{i} &= \frac{2}{a^2}\left(\phi' + \frac{a'}{a}\psi\right)_{,i}, \\
\delta G^{i}_{j} &= -\frac{1}{a^2}\left[2\phi'' + 2\frac{a'}{a}\left(\psi' + 2\phi'\right) - 2\left\{\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right\}\psi + \Delta\left(\psi - \phi\right)\right]\delta^i_j + \\
&\quad \frac{1}{a^2}\partial^i\partial_j\left(\psi - \phi\right).
\end{aligned} \tag{A.3.4}$$

Therefore, using perturbed Einstein tensor (A.3.4) one can quickly write the perturbed Einstein equation as (3.4.2), (3.4.3) and (3.4.4).

# Appendix B

# Perturbations in the scalar field theories

## B.1 Perturbed energy-momentum tensor in the scalar field theories

To describe inflationary scenario with single scalar field one need to write the action of the scalar field along with the gravitational Einstein-Hilbert action. In general the gravitation part in the action can have a coupling with the scalar field (non-minimal coupling). For simplified models we consider the scalar field minimally coupled with gravity. The minimally coupled action with the canonical kinetic term in the scalar field can be written as

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}M_{PL}^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V\left(\varphi\right)\right]. \tag{B.1.1}$$

Taking the variation with the metric $g^{\mu\nu}$ the energy-momentum tensor can be written as

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\varphi\partial_\sigma\varphi - V\left(\varphi\right)\right). \tag{B.1.2}$$

One can identify the second term under the parenthesis is the Lagrangian associated with the scalar field, $\mathcal{L} = \left(\frac{1}{2}\partial^\sigma\varphi\partial_\sigma\varphi - V\left(\varphi\right)\right)$. Using the expression of the energy-momentum tensor for the perfect fluid (2.2.7) one can get the following expressions of energy density and pressure for homogeneous and isotropic scalar field

$$
\begin{aligned}
\varepsilon &= \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \qquad & p &= \frac{1}{2}\dot{\varphi}^2 - V(\varphi), \qquad &\text{(cosmic time)}\\
\varepsilon &= \frac{1}{2a^2}\varphi'^2 + V(\varphi), \qquad & p &= \frac{1}{2a^2}\varphi'^2 - V(\varphi). \qquad &\text{(conformal time)}
\end{aligned} \tag{B.1.3}
$$

The perturbed energy momentum tensor in the mixed form can be expressed as

$$
\delta T^{\mu}_{\nu} = \bar{g}^{\mu\sigma}\delta T_{\sigma\nu} + \delta g^{\mu\sigma} T_{\sigma\nu}, \tag{B.1.4}
$$

where $\delta T_{\mu\nu}$ is the perturbation in (B.1.2). Using (A.2.2), (A.2.3) and (B.1.4) the different components of perturbed energy momentum tensor in conformal time, which has been used in section (3.4), can be written as

$$
\begin{aligned}
\delta T^{\eta}_{\eta} &= \frac{1}{a^2}\left[\delta\varphi'\varphi' - \psi\varphi'^2 + a^2 V_{,\varphi}\delta\varphi\right],\\
\delta T^{\eta}_{i} &= \frac{1}{a^2}\left(\delta\varphi\varphi'\right)_{,i},\\
\delta T^{i}_{j} &= -\frac{1}{a^2}\left[\delta\varphi'\varphi' - \psi\varphi'^2 - a^2 V_{,\varphi}\delta\varphi\right]\delta^{i}_{j}.
\end{aligned} \tag{B.1.5}
$$

## B.2 Quantisation of the perturbation

To quantize the perturbations one can follow the standard methods of quantisation, i.e., find the canonically conjugate momentum corresponding to the perturbed quantities and then satisfy the following commutation relations at any particular time:

$$
\begin{aligned}
&\left[v\left(\eta,\mathbf{x}\right), v\left(\eta,\mathbf{y}\right)\right] = \left[\pi\left(\eta,\mathbf{x}\right), \pi\left(\eta,\mathbf{y}\right)\right] = 0,\\
&\left[v\left(\eta,\mathbf{x}\right), \pi\left(\eta,\mathbf{y}\right)\right] = i\delta\left(\mathbf{x}-\mathbf{y}\right).
\end{aligned} \tag{B.2.1}
$$

Here $\pi = v'$ is the canonically conjugate momentum of $v$ associated with the Lagrangian

$$
\mathcal{L} = \frac{1}{2}\left[v'^2 - \left(k^2 - \frac{z''}{z}\right)v^2\right], \tag{B.2.2}
$$

Expanding the solution of (3.4.8) in terms of creation and annihilation operators $(a^\dagger, a)$ we get

$$v = \int \frac{d^3k}{2\pi^{3/2}} \left[ v\left(\eta\right)_{\mathbf{k}} e^{-i\mathbf{k.x}} a^\dagger_{\mathbf{k}} + v^*\left(\eta\right)_{\mathbf{k}} e^{i\mathbf{k.x}} a_{\mathbf{k}} \right], \tag{B.2.3}$$

where $v_{\mathbf{k}}$ and $v^*_{\mathbf{k}}$ are the two independent solutions of equation (3.4.8) which are dependent only up on time. One can write the equation satisfied by the modes $v_{\mathbf{k}}$ as:

$$v''_{\mathbf{k}} + \omega^2\left(\eta\right) v_{\mathbf{k}} = 0, \tag{B.2.4}$$

where $\omega\left(\eta\right) = \sqrt{k^2 - \frac{z''}{z}}$ is the time dependent frequency of the simple harmonic oscillator when $k^2 > \frac{z''}{z}$.

The creation and annihilation operators $a^\dagger$ and $a$ satisfy the bosonic commutation relation

$$\left[a_{\mathbf{p}}, a_{\mathbf{p}'}\right] = \left[a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}\right] = 0, \quad \text{and} \quad \left[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}\right] = \delta\left(\mathbf{p} - \mathbf{p}'\right). \tag{B.2.5}$$

Substituting (B.2.3) in the commutation relation (B.2.1) one can get the following relation among the modes $v_{\mathbf{k}}$ and $v^*_{\mathbf{k}}$:

$$v'_{\mathbf{k}} v^*_{\mathbf{k}} - v_{\mathbf{k}} v^{*\prime}_{\mathbf{k}} = i. \tag{B.2.6}$$

The normalisation condition (B.2.6) is very useful to obtain the nature of the solution $v_{\mathbf{k}}$ at an early time $(\eta_i)$.

Parametrising the solution $v_{\mathbf{k}}$ as

$$v_{\mathbf{k}} = r\, e^{i\alpha}, \tag{B.2.7}$$

where $r$ and $\alpha$ are the time dependent real parameters, the normalisation condition (B.2.6) gives us the condition between $r$ and $\alpha$ as

$$r^2 \alpha' = \frac{1}{2}. \tag{B.2.8}$$

For the modes with $k^2 \gg \frac{z''}{z}$, the expression of the energy corresponding to the harmonic oscillator (B.2.4) can be expressed as

$$E_{\mathbf{k}} = \frac{1}{2}\left(r'^2_{\mathbf{k}} + \frac{1}{r^2_{\mathbf{k}}} + k^2 r^2_{\mathbf{k}}\right). \tag{B.2.9}$$

At some early time $(\eta_i)$ the initial condition of $r_{\mathbf{k}}(\eta_i)$ can be found out by minimizing the Energy $E_{\mathbf{k}}$, i.e., by choosing a vacuum. This choice of vacuum is also known as Bunch-Davis vacuum. Bunch-Davis vacuum corresponds to the choice of initial condition where $r'_{\mathbf{k}}(\eta_i) = 0$ and $r_{\mathbf{k}}(\eta_i) = \frac{1}{\sqrt{2k}}$. Therefore, equation (B.2.8) gives us the value of the parameter $\alpha = k\eta$, where we have set the constant of integration to be zero. Finally, for Bunch-Davis vacuum one gets,

$$v_{\mathbf{k}(BD)} = \frac{1}{\sqrt{2k}}\, e^{ik\eta}, \qquad v'_{\mathbf{k}(BD)} = i\sqrt{\frac{k}{2}}\, e^{ik\eta} \tag{B.2.10}$$

## B.3 Power spectrum and spectral indices

The power spectrum is a statistical property associated with gaussian random fields. It measures the two point correlation function of the random fields. In case of small perturbations about the FRW background one can consider the perturbations to be gaussian. Power spectrum measures scale dependence of the perturbations. In other words it measures *how large the field fluctuations are on different scales.* The remarkable feature of inflation is that inflation predicts nearly a scale invariant power spectrum which is in good agreement with the observation. For any generic field $\xi(\eta, \mathbf{x})$ can be written in the Fourier space as

$$\xi(\eta, \mathbf{x}) = \int \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}.\mathbf{x}} \xi_{\mathbf{k}}(\eta) d^3k \tag{B.3.1}$$

The two point correlation function associated with the field $\xi$ can be written as

$$\langle 0|\xi^2|0\rangle = \int \frac{dk}{k} \delta^2_{\chi}(k), \tag{B.3.2}$$

where, $\delta_\chi^2 = \frac{k^3}{2\pi^2}|\xi_{\mathbf{k}}|^2$ is called the power spectrum associated with $\xi$. Using (B.3.2) the power spectrum associated with the metric perturbation and the spectral index respectively becomes: $\delta_\phi^2 = \frac{k^3}{2\pi^2}|\phi_{\mathbf{k}}|^2$ and $n_\phi = 1 + \frac{d\ln\delta_\chi^2}{d\ln\phi}$

Now let us calculate the power spectrum for the metric perturbation $\phi$. We start with equation (3.4.7). Let us first consider the small wavelength region (also known as *subhorizon mode* characterised by $k\eta > 1$) where $k^2 \gg \frac{\theta''}{\theta}$,

$$u''_{\mathbf{k}} + k^2 u_{\mathbf{k}} = 0. \tag{B.3.3}$$

Under WKB approximation the solution of (B.3.3) can be written as

$$u_{\mathbf{k}} \approx c_1 e^{ik\eta} + c_2 e^{-ik\eta}, \tag{B.3.4}$$

where $c_1$ and $c_2$ are constants which are determined by the initial conditions. Following the section (B.2) one can choose the initial condition which is consistent with the Bunch-Davis vacuum (B.2.10). Therefore, from the initial conditions, using (3.4.5) and (B.2.10), the coefficients $c_1$ and $c_2$ can be fixed as

$$c_1 = -\frac{i}{k^{3/2}}, \qquad \text{and} \qquad c_2 = 0. \tag{B.3.5}$$

Thus, in the region $k^2 \gg \frac{\theta''}{\theta}$ the solution of $u_{\mathbf{k}}$ becomes

$$u_{\mathbf{k}} \approx -\frac{i}{k^{3/2}} e^{ik\eta}. \tag{B.3.6}$$

Although (B.3.6) is the solution in the small scale region, the solution of the perturbation after crossing the horizon will be applicable for a small time period $\frac{\epsilon}{k} < \eta < \frac{1}{k}$. Where $\epsilon$ is the slow-roll parameter. The lower bound is obtained using the fact that $\frac{\theta''}{\theta} \approx \frac{\epsilon}{\eta^2}$ and $k\eta < 1$ implies the entry into the *super-horizon mode (or large scale mode)* after horizon crossing. Thus, in the small time interval $\frac{\epsilon}{k} < \eta < \frac{1}{k}$ the exponent of the solution (B.3.6) does not evolve hence it freezes. From (3.4.9) the power spectrum in the small wavelength region becomes

$$\delta_\phi^2 = 8G^2\left(\varepsilon + p\right). \tag{B.3.7}$$

In the super horizon mode $k^2 \ll \frac{\theta''}{\theta}$ the equation of motion of $u_{\mathbf{k}}$ becomes

$$u''_{\mathbf{k}} - \frac{\theta''}{\theta} u_{\mathbf{k}} = 0, \tag{B.3.8}$$

The solution of (B.3.8) can be written as (for details see appendix (B.4))

$$u_{\mathbf{k}} = \frac{A_0}{4\pi G\sqrt{(\varepsilon + p)}} \left(1 - \frac{H}{a} \int a dt\right). \tag{B.3.9}$$

Where, the quantity $A_0$ can be found by comparing (B.3.9) with (B.3.6), ignoring the exponential term, during the small time period at the time of horizon crossing ($k\eta \simeq 1$ or $k \simeq aH$). After some simplification (B.4) the expression of $u_{\mathbf{k}}$ becomes

$$u_{\mathbf{k}} \approx A_0 \frac{\sqrt{\varepsilon + p}}{H^2}. \tag{B.3.10}$$

Therefore, the expression of $A$ can be written as

$$A_0 \simeq -\frac{i}{k^{3/2}} \left(\frac{H^2}{\sqrt{\varepsilon + p}}\right)_{k \simeq aH}. \tag{B.3.11}$$

Using the Friedmann equation the power spectrum in the super horizon scale can be written as

$$\delta_\phi^2 \approx \frac{32}{9} G^2 \left[\frac{\varepsilon}{1 + p/\varepsilon}\right]_{k \simeq aH} \left(1 - \frac{H}{a} \int a dt\right)^2. \tag{B.3.12}$$

Taking the logarithm on both the sides of (B.3.12) we get

$$\ln \delta_\phi^2 \approx \ln \left(\frac{32}{9} G\right) + \ln \varepsilon - \ln (1 + p/\varepsilon) + 2 \ln M, \tag{B.3.13}$$

Where $M = \left(1 - \frac{H}{a} \int a dt\right)$. Now using the fact that during horizon crossing $k \simeq aH$ one can have

$$\frac{d \ln k}{dt} \approx H, \tag{B.3.14}$$

where we have ignored the slow-roll parameter $\epsilon = -\frac{\dot{H}}{H^2}$ compared to Hubble parameter $H$. Finally, taking the logarithmic time derivative of equation (B.3.13) the final expression of spectral index can be obtained as

$$n_\phi - 1 \approx -2\epsilon - \frac{\dot{\epsilon}}{H\epsilon} + \frac{\dot{M}}{HM}. \tag{B.3.15}$$

This expression is obtained using the Friedmann equation. The last two terms are also slow-roll parameters which have small values. Therefore, one can see that the single scalar field inflationary models give us the spectral index $n_\phi \sim 1$.

In the next section we calculate the perturbation theory for NSS based on this section. We will show that the effect of the NSS is small compared to the single scalar field theories of inflation. Hence, one can calculate the power spectrum and spectral index as a small deviation around the results obtained in this section. Finally we will confront the expression of spectral index with the observed value and put a bound on the additional term $\tilde{F}$ that appears in case of NSS.

## B.4 Solutions in the super horizon mode

Let us consider the equation (B.3.8) in the super horizon scale. One obvious solution of this equation is $u_{\mathbf{k}} = c_1\theta$. This can be checked that $u_{\mathbf{k}} = c_2\theta \int \frac{d\eta}{\theta^2}$ is also a solution of (B.3.8). Here $c_1$ and $c_2$ are the constants. Therefore the complete solution can be written as,

$$u_{\mathbf{k}} = c_2\theta \left[\frac{c_1}{c_2} + \int \frac{d\eta}{\theta^2}\right], \tag{B.4.1}$$

As, the ratio $\frac{c_1}{c_2}$ is a number, one can always absorb this ratio into the integral by changing the limit of it appropriately. Therefore, we would like to consider the second term of (B.4.1) as the solution of (B.3.8),

$$u_{\mathbf{k}} = c_2\theta \int \frac{d\eta}{\theta^2}, \tag{B.4.2}$$

Using the Freidmann and accelration equations it can be shown that $\theta = \frac{1}{a}\sqrt{\frac{4\pi G}{1-\frac{\mathcal{H}'}{\mathcal{H}^2}}}$. Using this expression of $\theta$ one can write

$$\int \frac{d\eta}{\theta^2} = \frac{1}{4\pi G}\int a^2\left[1+\left(\frac{1}{\mathcal{H}}\right)'\right]d\eta, \tag{B.4.3}$$

After integrating by parts one can write the expression of $u_{\mathbf{k}}$ as

$$u_{\mathbf{k}} = A\left[1-\frac{\mathcal{H}}{a^2}\int a^2 d\eta\right] = A\left[1-\frac{H}{a}\int a dt\right], \tag{B.4.4}$$

where all other terms are absorbed in $A = c_2\frac{a^2\theta}{4\pi G\mathcal{H}}$. After integrating by parts the last term of (B.4.4) and ignoring the higher time derivatives, finally one can write

$$u_{\mathbf{k}} \simeq -\frac{A_0}{4\pi G\sqrt{\varepsilon+p}}\frac{\dot{H}}{H^2}, \tag{B.4.5}$$

after replacing $c_2 = A_0$, which after using Friedmann equation gives (B.3.10).

# Appendix C

# Perturbations in tetrad and $\Gamma_\mu$

## C.1 Tetrad or vierbien Field

Vierbien field is related to the metric as

$$e^\mu_a e^\nu_b \eta^{ab} = g^{\mu\nu}, \tag{C.1.1}$$

where $\{a, b\} = \{0, 1, 2, 3\}$ are tetrad indices and $\{\mu, \nu\} = \{\eta, x, y, z\}$ are space-time indices. $\eta^{ab} = \begin{pmatrix} 1 & \mathbb{O} \\ \mathbb{O} & -1\delta^{ij} \end{pmatrix}$ is Minkowski metric and $g^{\mu\nu} = \frac{1}{a^2} \begin{pmatrix} 1 & \mathbb{O} \\ \mathbb{O} & -1\delta^{ij} \end{pmatrix}$ is FRW metric.

### C.1.1 Unperturbed vierbien

Unperturbed vierbien in conformal time are written as

$$e^\mu_a = diag\left\{\frac{1}{a}, \frac{1}{a}, \frac{1}{a}, \frac{1}{a}\right\}, \qquad e^a_\mu = diag\left\{a, a, a, a\right\}. \tag{C.1.2}$$

### C.1.2 Perturbed Veirbiens

Perturbed vierbiens are defined as follows,

$$\delta e^\mu_a = -\sigma^b_a e^\mu_b, \qquad \delta e^a_\mu = \sigma^a_b e^b_\mu, \tag{C.1.3}$$

where $\sigma^a_b$ is the perturbation. In case of vierbiens perturbation, we can raise or lower the tetrad and space-time indices with the unperturbed Minkowski metric and FRW metric respectively where as we can transform between tetrad and coordinate frames with unperturbed vierbiens. Therefore we can write,

$$\begin{aligned}\delta e^\mu_a &= -\sigma^b_a e^\mu_b = -\eta^{bc}\sigma_{ca}e^\mu_b,\\ \delta e^a_\mu &= \sigma^a_b e^b_\mu = \eta^{ac}\sigma_{cb}e^b_\mu.\end{aligned} \tag{C.1.4}$$

Using equation (C.1.1) one can write that the perturbed FRW metric in the following form,

$$\begin{aligned}\delta g_{\mu\nu} &= \eta_{ab}(\delta e^a_\mu)e^{b(0)}_\nu + \eta_{ab}e^{a(0)}_\mu(\delta e^b_\nu)\\ &= \sigma_{\nu\mu} + \sigma_{\mu\nu}.\end{aligned} \tag{C.1.5}$$

Once we know the expression of $\sigma_{\mu\nu}$ we can contract it with the unperturbed vierbien and find the expression of $\sigma_{ab}$ which we can use to find the perturbed vierbien using (C.1.4). Using the expression of $\delta g_{\mu\nu}$ in (A.2.2) and (A.2.3) one can write the perturbation in the vierbien $\sigma_{\mu\nu}$ and $\sigma_{ab}$ as

$$\sigma_{\mu\nu} = a^2\begin{pmatrix}\psi & 0 & 0 & 0\\ 0 & \phi & 0 & 0\\ 0 & 0 & \phi+h_+ & h_\times\\ 0 & 0 & h_\times & \phi-h_+\end{pmatrix}, \qquad \sigma_{ab} = \begin{pmatrix}\psi & 0 & 0 & 0\\ 0 & \phi & 0 & 0\\ 0 & 0 & \phi+h_+ & h_\times\\ 0 & 0 & h_\times & \phi-h_+\end{pmatrix}. \tag{C.1.6}$$

Finally, using the above results we can write the complete expression of the perturbed vierbien as

$$\delta e^a_\mu = a\begin{pmatrix}\psi & 0 & 0 & 0\\ 0 & -\phi & 0 & 0\\ 0 & 0 & -(\phi+h_+) & -h_\times\\ 0 & 0 & -h_\times & -(\phi-h_+)\end{pmatrix}, \tag{C.1.7}$$

and

$$\delta e_a^\mu = \frac{1}{a}\begin{pmatrix} -\psi & 0 & 0 & 0 \\ 0 & \phi & 0 & 0 \\ 0 & 0 & (\phi+h_+) & h_\times \\ 0 & 0 & h_\times & (\phi-h_+) \end{pmatrix}. \tag{C.1.8}$$

## C.2 Unperturbed and perturbed $\Gamma_\mu$'s

As mentioned previously the the general expression of $\Gamma_\mu$ is given by

$$\Gamma_\mu = \frac{1}{4}e_\nu^b(\nabla_\mu e_c^\nu)\gamma_b\gamma^c, \tag{C.2.1}$$

where the covariant derivative of vierbien $e_c^\mu$ is given by

$$\nabla_\mu e_c^\nu = \partial_\mu e_c^\nu + \Gamma^\nu_{\mu\rho}c_c^\rho. \tag{C.2.2}$$

### C.2.1 Unperturbed components of $\Gamma_\mu$

Using the background vierbiens (C.1.2) the components components of $\Gamma_\mu$ in FRW background are written as

$$\begin{aligned} \Gamma_\eta = 0, \qquad \Gamma_x &= \frac{1}{4}\frac{a'}{a}(\gamma^0\gamma^1-\gamma^1\gamma^0) = \frac{a'}{a}f^{01}, \\ \Gamma_y = \frac{1}{4}\frac{a'}{a}(\gamma^0\gamma^2-\gamma^2\gamma^0) = \frac{a'}{a}f^{02}, \qquad \Gamma_z &= \frac{1}{4}\frac{a'}{a}(\gamma^0\gamma^3-\gamma^3\gamma^0) = \frac{a'}{a}f^{03}, \end{aligned} \tag{C.2.3}$$

where $f^{ab} = \frac{1}{4}\left[\gamma^a,\gamma^b\right]$ are the generators. $\gamma^a$ are the Dirac gamma matrices which follows the anticommutation relation $\left\{\gamma^a,\gamma^b\right\} = 2\eta^{ab}$. Here we have used a convenient definition of $\Gamma_\mu$, which apparently looks different compared to the expression given in (2.7). However it can be easily checked that the $\Gamma_\mu$ calculated in both the ways will match exactly for both background and perturbation.

### C.2.2 Perturbed components of $\Gamma_\mu$

Perturbing the expression of $\Gamma_\mu$ in (C.2.1) one can write the general expression of perturbed $\Gamma_\mu$ as,

$$\delta\Gamma_\mu = \frac{1}{4}\left[\left(\delta e^b_\nu\right)\left(\nabla_\mu e^\nu_c\right) + e^b_\nu \delta\left(\nabla_\mu e^\nu_c\right)\right]\gamma_b\gamma^c, \tag{C.2.4}$$

where the perturbed covariant derivative of the vierbien can be expressed as

$$\delta\left(\nabla_\mu e^\nu_c\right) = \partial_\mu\left(\delta e^\nu_c\right) + \Gamma^\nu_{\mu\rho}\left(\delta e^\rho_c\right) + \left(\delta\Gamma^\nu_{\mu\rho}\right)e^\rho_c. \tag{C.2.5}$$

Finally, the components of $\delta\Gamma_\mu$ can be calculated as

$$\begin{aligned}
\delta\Gamma_\eta &= \frac{1}{2}\left[\left(\partial_x\psi\right)\gamma^0\gamma^1 + \left(\partial_y\psi\right)\gamma^0\gamma^2 + \left(\partial_z\psi\right)\gamma^0\gamma^3\right], \\
\delta\Gamma_x &= \frac{1}{2}\left[-\left(\frac{a'}{a}\psi + \frac{a'}{a}\phi + \phi'\right)\gamma^0\gamma^1 + \left(\partial_y\phi\right)\gamma^1\gamma^2 + \left(\partial_z\phi\right)\gamma^1\gamma^3\right], \\
\delta\Gamma_y &= \frac{1}{2}\left[-\left(\frac{a'}{a}\psi + \frac{a'}{a}\phi + \phi' + \frac{a'}{a}h_+ + h'_+\right)\gamma^0\gamma^2 - \left(h'_\times + \frac{a'}{a}h_\times\right)\gamma^0\gamma^3\right] - \\
&\quad \frac{1}{2}\left[\left(\partial_x\phi + \partial_x h_+\right)\gamma^1\gamma^2 - \left(\partial_x h_\times\right)\gamma^1\gamma^3 + \left(\partial_z\phi + \partial_z h_+ - \partial_y h_\times\right)\gamma^2\gamma^3\right] \\
\delta\Gamma_z &= \frac{1}{2}\left[-\left(h'_\times + \frac{a'}{a}h_\times\right)\gamma^0\gamma^2 - \left(\frac{a'}{a}\psi + \frac{a'}{a}\phi + \phi' - \frac{a'}{a}h_+ - h'_+\right)\gamma^0\gamma^3\right] - \\
&\quad \frac{1}{2}\left[\left(\partial_x h_\times\right)\gamma^1\gamma^2 - \left(\partial_x\phi - \partial_x h_+\right)\gamma^1\gamma^3 - \left(\partial_y\phi - \partial_y h_+ - \partial_z h_\times\right)\gamma^2\gamma^3\right].
\end{aligned} \tag{C.2.6}$$

# Appendix D

# Fixed points and stability

## D.1 Fixed points

Let us consider two linearised dynamical equations of $x$ and $y$ as

$$x' = a_1 x + a_2 y, \quad \text{and,} \quad y' = b_1 x + b_2 y, \tag{D.1.1}$$

where $'$ denotes the derivative with respect to e-folding $N$. As $dN = Hdt$ one can calculate the prime by simply taking time derivative and dividing by Hubble parameter. Fixed points are those solutions of the dynamical equations when the dynamical quantities stop evolving, i.e., $x' = 0$ and $y' = 0$.

## D.2 Stability of fixed points

Let us say that the fixed points associated with the equations (D.1.1) are $\bar{x}$ and $\bar{y}$. Now let us perturb the solutions about the fixed points:

$$x = \bar{x} + \delta x, \quad \text{and,} \quad y = \bar{y} + \delta y. \tag{D.2.1}$$

Substituting (D.2.1) in (D.1.1) and linearising the perturbed equations we get

$$\delta x' = a_1 \delta x + a_2 \delta y, \quad \text{and,} \quad \delta y' = b_1 \delta x + b_2 \delta y. \tag{D.2.2}$$

Finally, the perturbed equations can be written in a matrix form as

$$\begin{pmatrix} \delta x' \\ \delta y' \end{pmatrix} = \mathcal{M} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}, \tag{D.2.3}$$

where $\mathcal{M} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$. The general solution of the equation (D.2.3) can be written as

$$\delta x = \delta x_1 e^{\mu_1 N} + \delta x_2 e^{\mu_2 N}, \quad \text{and,} \quad \delta y = \delta y_1 e^{\mu_1 N} + \delta y_2 e^{\mu_2 N} \tag{D.2.4}$$

where $\mu_1$ and $\mu_2$ are the eigenvalues of the matrix $\mathcal{M}$. From the expressions (D.2.4) it can be understood that the stability criteria depends entirely of the sign of the eigenvalues $\mu_1$ and $\mu_2$. The different possibilities regarding the solutions based on the sign of $\mu_1$ and $\mu_2$ are listed below:

1. $\mu_1 > 0$ and $\mu_2 > 0$, no stable solutions, i.e., perturbations will grow and will be away from the fixed points.

2. $\mu_1 > 0\,(< 0)$ and $\mu_2 < 0\,(> 0)$, saddle point, i.e., perturbations will be stable in one direction and will be unstable in the other direction.

3. $\mu_1 < 0$ and $\mu_2 < 0$, stable solutions, i.e., perturbations will decay exponentially and all the solutions irrespective of the initial conditions will fall on the fixed points after some times.

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[94] **Planck Collaboration** Collaboration, P. Ade *et. al.*, *Planck 2013 Results. XXIV. Constraints on primordial non-Gaussianity*, arXiv:1303.5084.

# LIST OF PUBLICATIONS

## Publications related to the thesis work

1. **"Lorentz invariant dark-spinor and inflation"**
   **Abhishek Basak**, and Jitesh R. Bhatt
   *JCAP* 06 (2011) 011
   arXiv:1104.4574.

2. **"Attractor behaviour in ELKO cosmology"**
   **Abhishek Basak**, Jitesh R. Bhatt, S. Shankaranarayanan, and K. V. P. Verma
   *JCAP* 04 (2013) 025
   arXiv:1212.3445.