A Study on the Application of Dissipative Hydrodynamics to Relativistic Heavy Ion Collisions

A THESIS

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by

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UDAIPUR

Year of submission: 2016

To

My Family

DECLARATION

I, **Avdhesh Kumar***, S/O Shri Om Prakash, resident of K-213 Hostel Block 2, PRL Navarangpura Hostel, Navarangpura, Ahmedabad, 380009, hereby declare that the research work incorporated in the present thesis entitled,* **"A Study on the Application of Dissipative Hydrodynamics to Relativistic Heavy Ion Collisions"** *is my own work and is original. This work (in part or in full) has not been submitted to any University for the award of a Degree or a Diploma. I have properly acknowledged the material collected from secondary sources wherever required. I solely own the responsibility for the originality of the entire content.*

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CERTIFICATE

I feel great pleasure in certifying that the thesis entitled, **“A Study on the Application of Dissipative Hydrodynamics to Relativistic Heavy Ion Collisions”** embodies a record of the results of investigations carried out by Mr. Avdhesh Kumar under my guidance. He has completed the following requirements as per Ph.D regulations of the University.

(a) Course work as per the university rules.

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(c) Regularly submitted six monthly progress reports.

(d) Presented his work in the departmental committee.

(e) Published minimum of one research papers in a refereed research journal.

I am satisfied with the analysis, interpretation of results and conclusions drawn. I recommend the submission of thesis.

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ABSTRACT

This thesis is based on the studies related to the application of dissipative hydrodynamics in the context of relativistic heavy ion collisions. Experimental studies indicates that matter created in a high energy collision ($\sqrt{s} = 200$ GeV at RHIC and $\sqrt{s} = 5$ TeV at LHC) of two heavy nuclei evolves in several stages. Relativistic hydrodynamics can be applied from the stage of local thermodynamical equilibrium to the freeze-out stage. There are several kinds of hydrodynamic models have been developed so far. In this thesis we discuss the relativistic generalization of Navier-Stokes (NS) hydrodynamics (also known as the first order hydrodynamics) and a second order hydrodynamics due to Müller, Israel and Stewart (MIS). We also briefly discuss other kinds of second and third order relativistic hydrodynamics. The NS approach may not be adequate for the relativistic heavy ion collisions as it is found to have problems of acausality and unphysical instabilities, while no such issues arise in the models of second order hydrodynamics which are also referred as the causal hydrodynamics. The formalism to calculate the hydrodynamic fluctuations by applying the Onsager theory to the relativistic NS hydrodynamics is already known. We develop a theory of hydrodynamic-fluctuations for the second order MIS hydrodynamics and its generalization to the third order. We also calculate the fluctuations for several other causal hydrodynamical equations. We show that the forms of the Onsager-coefficients and the correlation-functions remain the same as those obtained by the relativistic NS equation, and these do not depend on any specific model of hydrodynamics. As an illustrative example we apply this result to numerically compute the two point correlation function for the one dimensional boost-invariant flow (Bjorken Flow) using the models of causal hydrodynamics. We find that the qualitative properties of the correlation-functions are similar for all the models of the causal hydrodynamics.

In the next work, we focus on the Weibel and chiral-imbalance instabilities. Weibel instabilities arise because of the anisotropic single particle distribution function and may play an important role in the thermalization of the strongly interacting matter produced in heavy ion collisions experiments. Chiral-imbalance

instabilities arise because of the violation of P and CP symmetries. The observation of the CP-violating correlations in the strongly interacting matter produced in the heavy ion collision experiments suggests that Chiral-imbalance instabilities can occur in the strongly interacting matter. We argue that in many realistic situations, e.g. relativistic heavy-ion collisions, both the instabilities can occur simultaneously and study the interplay between them using the Berry-curvature modified kinetic equation. We find that the Weibel instability depends on the momentum anisotropy parameter ξ and the angle θ_n between the propagation vector and the anisotropy direction. It grows maximally at $\theta_n = 0$ and gets damped at $\theta_n = \pi/2$. The chiral-imbalance instability depends on the difference between the chiral chemical potentials of right and left-handed particles, denoted by μ_5. We show that for $\theta_n = 0$, growth rates of both the instabilities are comparable when $\xi \sim \xi_c$. For the cases $\xi_c < \xi \ll 1$ or $\xi \gtrsim 1$ at $\theta_n = 0$, the Weibel modes dominate over the chiral-imbalance instability if $\mu_5/T \leqslant 1$. However, when $\mu_5/T \geqslant 1$, it is possible to have the dominance of the chiral-imbalance modes at certain values of θ_n for an arbitrary ξ. It is important to note that these instabilities may give rise to the turbulent transport which may enhance the collisional rate; consequently, leading to an additional viscosity in the system called as anomalous viscosity.

In the last work, we consider the case of isotropic chiral plasma and discuss the chiral instability. We show that chiral-imbalance instability may drive the turbulent transport. For the case $\mu_5 \ll T$, we estimate the anomalous shear viscosity arising from the enhanced collisional rate due to turbulence. We show that the ratio $\eta/s \propto \mu_5/T$, which can be a large number depending upon the values of μ_5 and T. This result may be suitable for neutron stars. The case $\mu_5 \geqslant T$ could be important for the heavy ion collisions. Study of such a case is a part of our future plan.

Keywords : Navier-Stokes Hydrodynamics; Causal Hydrodynamics; Fluctuations; Onsager coefficients; Correlation functions; Berry curvature; Momentum anisotropy; CP violation; Chiral-imbalance and Weibel Instabilities; Turbulence; Anomalous viscosity.

LIST OF PUBLICATIONS

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4. *Shear Viscosity of Turbulent Chiral Plasma*,
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List of Abbreviations

QED	Quantum Electrodynamics
QCD	Quantum Chromodynamics
CSC	Color Superconductivity
QGP	Quark Gluon Plasma
RHIC	Relativistic Heavy Ion Collider
LHC	Large Hadron Collider
AGS	Alternating Gradient Synchrotron
SPS	Super Proton Synchrotron
STAR	Solenoidal Tracker
CFL	Color Flavor Locked
1SC	One Flavor Color Superconductivity
2SC	Two Flavor Color Superconductivity
DFS	Deformed Fermi Sphere
LOFF	Larkin, Ovchinnikov, Fulde and Ferrell
CGC	Color Glass Condensate
EoS	Equation of state
ADS	Anti-de-sitter Space
KSS	Kovtun, Son and Starinets
CFT	Conformal Field Theory
NS	Navier-Stokes
MIS	Müller, Israel and Stewart
CME	Chiral Magnetic Effect
P	Parity
CP	Charge Parity
DTT	Divergence Type Theory

Contents

List of Tables

List of Figures

Chapter 1

Introduction

"One of the basic rules of the Universe is that nothing is perfect. Perfection simply does not exist..... Without imperfection, neither you nor I would exist."

– Stephen Hawking

1.1 Nuclear matter and QCD phase diagram

According to our current understanding (both theoretical as well as experimental), there are four types of fundamental interactions: gravitational, electromagnetic, weak and strong. Strong interaction is the strongest of all the four fundamental interactions. Particles which take part in the strong interactions are hadrons or more fundamentally quarks and gluons. Hadrons are regarded as the bound states of three quarks (Baryon) or one quark and antiquark (Meson). Hadrons containing more than three valence quarks [tetra (exotic meson) and penta quark states (exotic baryons)] have also been discovered in the recent years [1–3]. Quarks are spin-1/2 fermions and they come in six flavors namely; up (u), down (d), strange (s), charm (c), beauty (b), top(t), and carry fractional electric charge. The u, c and t quarks carry electric charge of $+2e/3$, while s, b and d of $-e/3$, where e is the magnitude of electron charge. Another important physical attribute that quarks have is the "color-charge". Quarks can also interact with each other by their color charges with the exchange of gauge bosons referred as gluons. Gluons also carry color charge and therefore can interact with each other. This feature of quark-gluon interaction is different from the interaction between electrons and photons described as by quantum electrodynamics (QED), where the gauge bosons (photons) do not interact among themselves. The theory describing interactions of quarks and gluons is called quantum chromodynamics (QCD) [4]. Unlike QED, QCD has two very important properties arising due to the interactions among gluons: the asymptotic freedom and confinement. Results from perturbative QCD at high energy show that the strength of the strong coupling constant is a function of the energy scale. At the one-loop perturbation theory, the QCD running coupling constant is

$$\alpha_s(Q^2) = \frac{g^2(Q)}{4\pi} = \frac{1}{4\pi\beta_0 \ln\left(\frac{Q^2}{\Lambda^2_{\rm QCD}}\right)}, \tag{1.1}$$

where $\beta_0 = (11N_c - 2N_f)/48\pi^2$; N_c and N_f are the number of color and flavor degrees of freedom respectively. Q denotes the momentum transfer and $\Lambda_{\rm QCD}$ is the typical scale of QCD [4–7]. Measurements from high energy scattering experiments suggest, $\Lambda_{\rm QCD} = 213^{+38}_{-35}$ MeV [8]. Thus the interaction between quarks and gluons becomes weak at smaller length scales or higher energies, $Q \gg \Lambda_{\rm QCD}$. This property of QCD is called asymptotic-freedom [5, 6]. Because of this property, it is possible to have well defined perturbative techniques for QCD at energy scale $Q \gg \Lambda_{\rm QCD}$. In this regime QCD is very well tested in scattering experiments [9]. At lower energies $Q \ll \Lambda_{\rm QCD}$ or long distances ($\gtrsim 1$ fm), the interaction becomes strong and theory becomes non-perturbative. In this regime, QCD is expected to give confinement of color degree of freedom [10]. Since there is no observational evidence of free quarks, it is assumed that color degree of freedom is confined within the hadrons of size $\sim$ 1 fm (10^{-15} m). An analytical proof of the confinement has been elusive because of the non-perturbative nature of QCD at long distances. However, in various numerical studies of QCD on a lattice, confining behavior has been realized [7, 11, 12].

The property of asymptotic freedom suggests that under the extreme conditions i.e. temperature $T > \Lambda_{\rm QCD} \sim 10^{12}$ K or density 2-3 times the normal nuclear density n_0 (n_0 is 0.153 fm$^{-3} \approx 2.7 \times 10^{14}$ gm cm^{-3}), QCD is expected to undergo a phase transition where quarks and gluons are no longer confined within a hadronic volume; but, move in a much larger volume. This deconfined phase of matter is referred to as quark matter or quark-gluon plasma (QGP) [13, 14].

It is believed that, QGP at high temperature (hot QGP) might have existed in the early Universe when the Universe was only a few tens of microseconds old, while a cold (high density and low temperature) QGP can be expected in the core of a neutron star where density can be as high as ten times n_0. It would be interesting to create such a matter in laboratories, because the cold ultradense quark matter (cold QGP) is predicted to possess the attribute of being a transparent color superconductor [15]. Until now, it has not been possible to create such a state in laboratories. However, hot quark-gluon matter of extremely high density has been successfully created in heavy ion collider experiments [16–22]. Search for QGP in heavy ion collisions has a long history of almost 35 years. It started from the beginning of relativistic heavy ion program at the Bevalac (Lawrence Berkeley National Laboratory, LBNL) with beam energies of 1-2 GeV/nucleon in early 1980s [23] followed by Schwerionensynchrotron (SIS) at Gesellschaft Schwerionenforschung (GSI) in Darmstadt with similar beam energies. Later on (in the late 1980s) Alternating Gradient Synchrotron (AGS) at Brookhaven National Laboratory (BNL) with Gold (Au) ion beams of energies about 10 GeV/nucleon and Super Proton Synchrotron (SPS) at CERN with lead (Pb) ion beams up to energies of 158 GeV/nucleon were able to produce $\sqrt{s} \approx 5$ GeV and $\sqrt{s} \approx 17.3$ GeV respectively. More recently, the Relativistic Heavy Ion Collider (RHIC) at BNL has produced $\sqrt{s} \approx 200$ GeV in head-on Au+Au collisions [16–19, 21, 22] since its inception in 2000. Currently, using the head on collisions of Pb-Pb nuclei, energies up to $\sqrt{s} \approx 5.5$ TeV has been achieved at Large Hadron Collider (LHC) at CERN. [20]. There was no clear evidence for the QGP formation at energies lower than that used in AGS and SPS, however, a number of signals at SPS (CERN) provide an indirect evidence for a 'new state of matter' in 2000' [21]. More evidences were reported with the commissioning of RHIC, leading to the discovery of QGP [16–22]. Studying the properties of nuclear matter under extreme conditions and exploring this entirely new phase, referred as QGP, is of fundamental importance and a topic of extensive research in recent times. Its study will help us to understand the elusive QCD vacuum structure and its modification via temperature and density. Further, such a study can be helpful in the improvement of our understanding of the confinement and hadronic structure etc.

On the basis of thermodynamic considerations and lattice gauge simulations as well as experimental studies a commonly conjectured form of the phase diagram of QCD is shown in following Fig. 1.1.

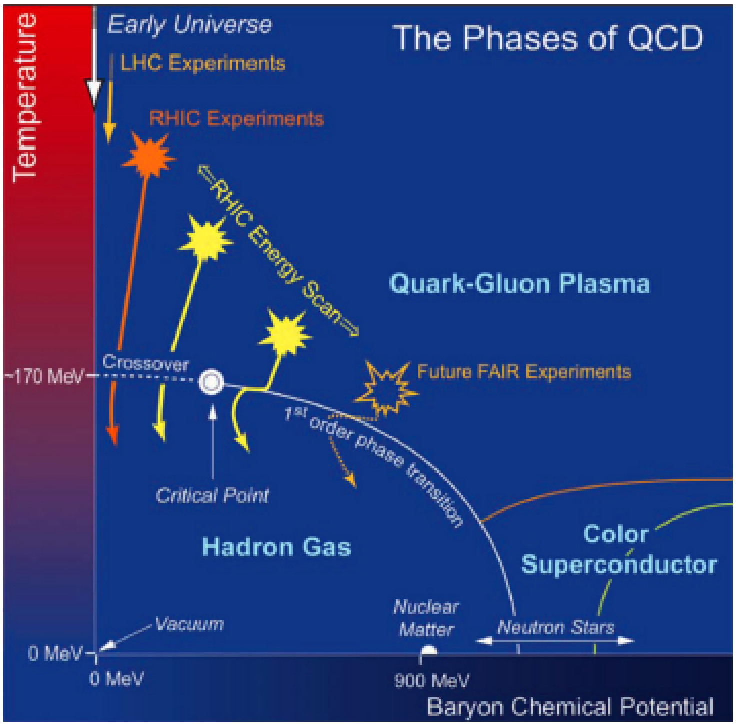

Figure 1.1: The conjectured QCD phase diagram for nuclear matter. Phase boundaries for different phases have been shown by solid lines. Dotted line represents the crossover region. The solid circle points out the critical point. Possible regions of the phase diagram which will be probed in future heavy ion collision experiments have also been shown. Reprinted figure from [STAR collaboration, BES Phase-II Whitepaper , STAR Note SN0598 (2014).]

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In the above figure, the thermodynamic variables are baryon chemical potential μ_B and temperature T. A pair of values of (μ_B, T) defines the various phases of QCD. The point ($T = 0$, $\mu_B = 0$) corresponds to vacuum. As the energy density of the system is increased, either via 'compression' (along μ_B axis) or via 'heating' (T axis), a phase transition from a phase where quarks are confined within hadron (hadron gas) to the deconfined phase of Quark-Gluon plasma may occur. If we move along μ_B axis, $T = 0$ and $\mu_B = 900$MeV defines the boundary which separates the gaseous nuclear phase (hadron gas) at lower μ_B to the nuclear matter at higher μ_B, where first order liquid-gas transitions may take place [24–29]. The first order transition weakens as T increases and it disappears at a critical end point $T = 10$ MeV [30, 31] with a second order phase transition. Above this point, the two phases can not be distinguished. At higher values of μ_B in low temperature regime, there will be more compressed nuclear matter/quark matter such as encountered in the core of neutron star. In this scenario, theoretical predictions suggest for the existence of color superconductivity (CSC) [32–37]. This phenomena is similar to superconductivity in metals described by Bardeen, Cooper, and Schrieffer (BCS) theory [38, 39]. In case of metals, superconductivity is caused by a condensation of Cooper pairs [40] (bound state of electrons by attractive interaction) into a bosonic state. In quark-matter, one gluon potential can be attractive between the two quarks under certain conditions, leading to Cooper pairing and consequently, color superconducting phase. Because of different flavors and colors of quarks along with charge neutrality, there may exist a variety of superconducting phases in a quark matter over a range of chemical potential as well as the difference in chemical potential of different flavors of quarks. At ultra high values of chemical potential ($\mu_B \gg \Lambda_{QCD} \gg m_u, m_d, m_s$) and in low temperature regime there is a transition to deconfined quark matter at unknown value of μ_B, this corresponds to color flavor locked (CFL) color superconductor [15, 32, 41–46] phase due to the pairing among all the three quark flavors. This is because at very high densities along with m_u and m_d, strange quark mass m_s can also be neglected and flavor SU(3) is a good symmetry of the QCD Lagrangian [47]. However, at intermediate densities just above the deconfinement transition strange quark mass can not be ignored and this explicitly breaks the flavor SU(3) symmetry. This may eventually limit CFL phase at an unlocking transition [48–53] and lead to a variety of other color superconducting phases (labeled as non-CFL phase whose nature is still not fully understood) in the QCD phase diagram at intermediate densities between the boundary of deconfinement transition to CFL phase. Studies without the requirement of charge neutrality [48, 49] suggest that due to the high mass of strange quark, its pairing with u and d gets disrupted at the unlocking transition, consequently, condensate involves only two colors and two flavors. This leads to one of the possibility of the existence of 2 flavour CSC (referred as 2SC) phase [15, 44, 46, 54–56]. In a charge neutral system there is a mismatch in the chemical potentials of different flavors due to the introduction of an electron chemical potential. At higher densities such a mismatch may stop inter-species pairing and allow self-paring in different flavors. This may lead to single flavor super conducting (1SC) phase. However, it has also been suggetsed by Larkin, Ovchinnikov, Fulde and Ferrell (LOFF) [57, 58] that there may exist 3 flavor pairing beyond BCS (in which pairing happen only among those quarks whose momenta add to zero) where cooper pairs have non zero total momentum. This kind of pairing has a restricted phase space and spontaneously breaks the rotational and translational symmetry, thereby leading to a crystalline color superconducting state [59–69]. However, in this intermediate density region there could also be other possible phases over the crystalline CSC, as the true ground state is not fully revealed here. For example, there may also exist alternatives to the crystalline color superconducting phase, which may allow pairing even if there is a mismatch in the chemical potentials of different flavors. The first alternative is the deformed Fermi sphere (DFS) superconductor [70, 71]. In DFS phase, two species have unequal Fermi surfaces that are deformed. As a result of the deformation, unequal Fermi surfaces intersect. Pairing occurs in the vicinity of this intersection. In the DFS phase, rotational symmetry is broken; but, on contrary to crystalline (LOFF) phase, translational symmetry remains preserved due to zero total momentum of Cooper pairs. Another alternative is the breached pair color superconductor [72, 73]. In this case both translational and rotational symmetries remain preserved.

At high temperatures and small baryon densities, one obtains the QGP phase;

a phase might have existed in the early universe during the first few microseconds after the Big Bang and to be explored by ongoing experiments like RHIC and LHC. Lattice simulations show that the transition between QGP phase and hadron gas phase is a rapid crossover (as shown by the dashed line in Fig. 1.1) with explicitly broken chiral symmetry in hadronic phase at small μ_B and high T. However, effective models predict that there is a first order phase transition with a critical point appearing at large μ_B and high T. Beyond the critical point the boundary between deconfined QGP and confined hadron phases becomes a sharp line (shown as a solid line in Fig. 1.1). Determining the precise location of critical point and the phase boundaries in heavy ion collision experiments is central to a quantitative understanding of the QCD phase diagram. Recent results from RHIC have given some hints that there is a continuous transition from quark gluon phase to hadron gas. This is shown by the dashed line in Fig. 1.1. To explore the critical point and full phase boundary further experiments are being anticipated to produce the nuclear matter at finite chemical potential and temperature. A broad range of collision energies is required to create such conditions. These conditions will be achieved at RHIC, SPS at CERN and FAIR at GSI.

The aim of heavy ion collision experiments is not simply mapping its phase diagram but also to study several other properties of nuclear matter such as its equation of state, entropy, the nature of its excitations (e.g. quasi-particles and collective modes), transport of energy-momentum, baryons and other conserved quantities, emission of particles, stopping of quark/hadronic projectiles or in other words dissipation of energy. These challenges can only be tackled with correct theoretical modeling of collisions.

1.2 Description of relativistic heavy ion collisions

Relativistic heavy ion collision experiments provide us an unique opportunity to create and study hot/dense nuclear matter. Basically the heavy nuclei e.g.

gold (Au) or lead (Pb) are collided at ultra relativistic speed with center of mass energy per nucleon at RHIC $\approx \sqrt{s}$=200 GeV & at LHC $\approx \sqrt{s}$=5.5 TeV. Due to ultra relativistic speed nuclei are Lorentz contracted (with a Lorentz factor, $\gamma = 100$ for RHIC and even more for LHC) in a direction along collision axis and look like pancake in the laboratory frame. The radius of the pancakes in the direction transverse to the collision axis remains same as of original nuclei ($\sim$ 15 fm) while it becomes around 1 fm in the longitudinal direction (direction of collision axis). Figs. 1.2 and 1.3 respectively show various stages of a relativistic heavy ion collision and its space-time evolution.

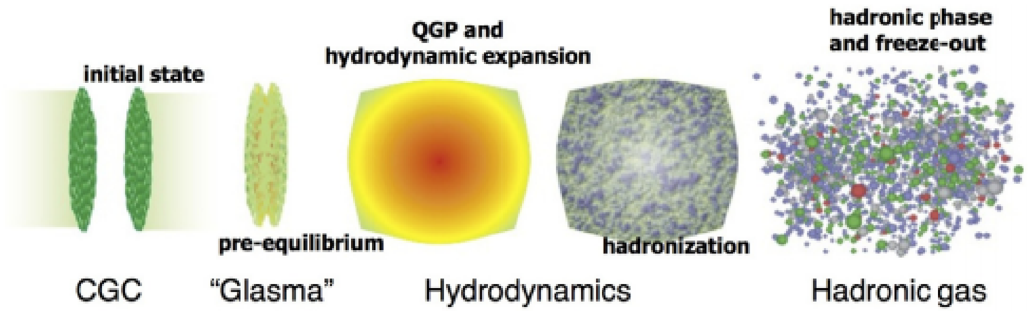


Figure 1.2: Various stages of a heavy ion collision. Reprinted figure from [B. Müller, Phys. Scripta **T 158**, 014004 (2013).] Copyright © 2013 IOP Publishing Ltd.

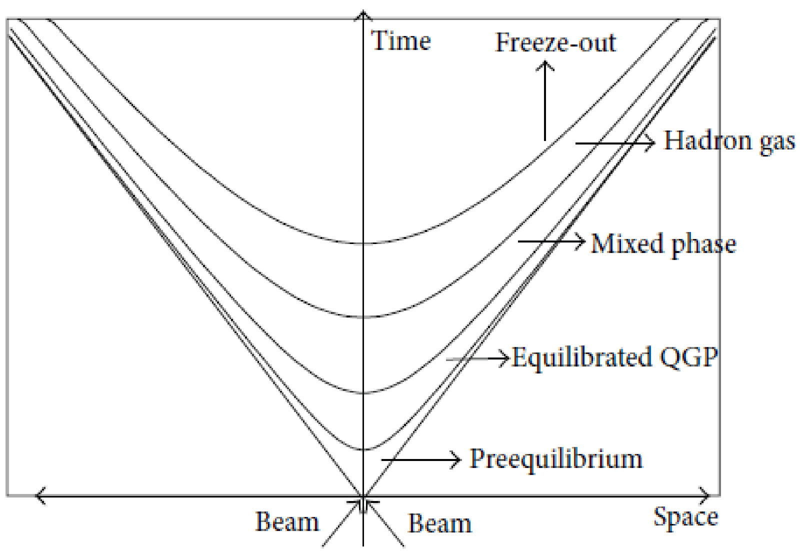


Figure 1.3: Space-time evolution of a heavy ion collision. Reprinted figure from [S. K. Tiwari and C. P. Singh, Adv. High Energy Phys. **2013** 805413 (2013).] Copyright © 2013 by S. K. Tiwari and C. P. Singh.

1.2.1 Initial stage ($\tau < 0$):

Initially, due to ultra relativistic speed the Lorentz contracted nuclei, which are about to smash each other, can be viewed as the sheets of dense gluons so called 'Color Glass Condensates' (CGC) [74–82]. The term 'color' refers to the color charge of quarks and gluons. 'Glass' signifies the analogy with the glassy materials, which behave like a solid on short time scales and liquids on much longer time scales. In the "gluon sheets", the gluons are disordered and do not change their positions rapidly because of time dilation. 'Condensate' means a very high density of massless gluons.

1.2.1.1 Color Glass Condensates (CGC):

The motivation for CGC comes from the deep inelastic lepton-hadron scattering experiments which suggest that a nucleon is made up of hard valence quarks and soft wee partons (gluons and sea-quarks). Soft wee partons carry smaller momentum fraction (x) of the nucleon as compared to hard valence quarks and their distribution $xF(x, Q^2)$ increases as x approaches to zero for large enough momentum transfer Q^2 (probe resolution). In this process, the gluon density outnumbers all the other partons because of the non-ablian nature of QCD. In a variety of theoretical works [83–88], it is shown that gluon density rises as $\log(1/x)$. It is important to note that the gluons at small x are produced by the radiative corrections from gluons at larger x due to the intrinsic nonlinearity of QCD at high energies,. If we consider an infinite momentum frame of reference of hadron or nuclei, the high x gluons will also travel very fast, giving rise to a Lorentz dilation to their natural time scales. This dilation is transferred to the scale of low x gluons, making them evolve on time scales very large compared to their natural time scales. This is the property of a glass. Now, further rise in the gluon density at small x will lead to the saturation at some momentum scale Q_s^2 [74, 82–86, 89–91], which is much larger than Λ_{QCD}. At Q_s, the gluon occupation number becomes of the order of $1/\alpha_s$, which corresponds to a high density and highly coherent Bose condensate of gluons. Due to the high occupation number at saturation scale, system can be considered as weakly coupled ($\alpha_s \ll 1$), which is possible to study from the first principles in QCD [74]. If we go to much higher energies, the saturation momentum increases. This is due to the repulsive gluonic interactions, which forbids further occupancy once the maximum phase density is achieved. Therefore, more gluons should add to the vacant states of higher momentum. Because of these properties the initial stage of high energy colliding nuclei is called Color Glass Condensates.

1.2.2 Pre-equilibrium evolution and thermalization ($0 < \tau < \tau_0$):

In this stage a collision between two nuclei at ultra-relativistic energies excites the QCD vacuum. This results into the production of a very hot and dense partonic matter which is far from its thermodynamic equilibrium. During this process a huge amount of entropy is generated in the central region of the collision. At $\tau = 0$, we still do not have clear understanding about how much entropy is generated. After the collision ($\tau = 0$), it takes some certain time, $\tau_0 < 1$ fm/c, for the matter to reach a local thermodynamic equilibrium. Although the mechanism for primary particle production is well understood in the CGC framework [74], we still do not have clear a understanding about the physical process involved in thermalization. However, it has been suggested that pre-equilibrium evolution in a heavy ion collision may involve a few stages, from the CGC through an anisotropic glasma stage towards initial isotropization and thermalization. Now we shall discuss parton production via breaking of CGC to glasma and isotropization and thermalization.

1.2.2.1 The glasma and its evolution:

CGC description[74, 79–82] suggests that due to high parton density at saturation scale, parton (gluon) can no longer be treated as independent, instead they act coherently and can be described by coherent classical color fields which are created by source of the hard initial partons. Lorentz contracted high energy hadrons look like discs which is orthogonal to the collision axis, the color fields can be described by the Lienard-Wiechart potentials of electrodynamics, which are static in a plane perpendicular to collision axis $\hat{z}$ i.e. $\mathbf{E^a} \perp \mathbf{B^a} \perp \hat{\mathbf{z}}$. As a result of collisions of two color field discs combine and get separated. In this scenario, solution of Classical Yang-Mills equations for the evolution of these fields with CGC initial conditions [79, 81, 92–96] of the color distribution of the valence quarks in the two colliding nuclei, suggest that in a very short time $\Delta t = \exp\left(-\kappa/\alpha_s\right)/Q_s$ [97, 98], where constant κ is of the order 1, the color field changes from transverse to longitudinal. This is due to the generation of color electric and magnetic monopole charge densities of opposite sign on the two discs in a collision. As a result, a flux tube of color electromagnetic fields, stretching between random color sources in the disc, is generated between the two discs, which are passing away each other from the point of collision. Thus, soon after the collision a medium consisting of three dimensional classical color fields is created. This state is called "glasma". In this state the energy momentum tensor takes the form $T^{\mu\nu} = diag(\epsilon, \epsilon, \epsilon, -\epsilon)$ [99, 100]. In this expression negative longitudinal pressure indicates that the initial glasma expansion is highly anisotropic. Also, it is important to note that due to electro-magnetic duality of QCD, both the longitudinal color electric and magnetic fields will be of equal magnitude which may result a non-zero $\mathbf{E} \cdot \mathbf{B}$ [97]. Therefore, there will be a large topological charge density associated with the glasma. In QCD, the field configurations with non-zero topological charges are associated with anomalous mass generation and breaks the chiral symmetry also responsible for violation of P and CP symmetry locally in QCD. It has been suggested in Ref. [101] that it may be possible to observe such an effect in the heavy ion collision experiments. Indeed, preliminary results of STAR collaboration [102–104] at RHIC are found in agreement.

1.2.2.2 Isotropization and thermalization:

Longitudinal electric and magnetic field induced by high energy collisions subsequently decay into nearly on shell partons [105, 106] and the system becomes dilute as a result of anisotropic expansion. One need to describe isotropization and thermalization. For dilute system with incoherent parton configuration, parton cascade model [107–109] was used to describe the thermalization. In PCM model one solves the Boltzmann equation with leading order pQCD collision terms to describe the classical motion of on-shell partons. Initially, on-shell, $2 \rightarrow 2(gg \rightarrow gg)$ [110–112] and later, $2 \rightarrow 3(gg \rightarrow ggg)$ [113–115] scattering processes were used to describe the thermalization. In the 'bottom up' thermalization scenario these scattering processes take a thermalization time of the order of $\tau_{th} = 1/(\alpha_s^{13/5} Q_s)$[115], which is larger than the time scale (< 1 fm/c) required by RHIC data [116, 117]. At very high energies (LHC) parton density will be large and therefore, the partons shall scatter so frequently that they can no longer be treated as on-shell. To handle such a situation quantum transport theory is required. A quark-gluon quantum transport theory was developed in Refs. [118–121]. But due to the extreme computing requirements, it has not yet been utilized. Actual mechanism to describe fast isotropization and thermalization has been a theoretical challenge and it is not clearly understood. It has been suggested that the color fields present in the initial stage of the collision may exhibit instabilities. In fact, in several studies, it has been shown that classical solution of Yang-Mills equation may suffer from instabilities [122–131] analogous to Weibel instabilties [132–141] and may play an important role in fast isotropization and thermalization. Role of these kind of instabilities in the context of heavy ion collisions has been investigated in Refs. [142–155].

In this thesis, we shall be focusing on the instabilities arising because of local CP violation. This instability may occur together with the Weibel instability in the relativistic heavy ion collision experiments.

1.2.3 QGP expansion ($\tau_0 < \tau < \tau_c$):

Once the thermalization is achieved, system (QGP) expansion is driven by the thermal pressure gradient. As a result of expansion, the plasma cools and its energy density decreases. Until temperature becomes $T_c = 170 - 190$ MeV and the energy density decreases to a value approximately 1 GeV/fm^3, the system remains in QGP phase. Since system is in local thermodynamic equilibrium, its evolution from initial time τ_0 can be described by relativistic hydrodynamics. A comparison of hydrodynamical predictions from experimental data can provide a deep insight about collective behavior, transport coefficients and equation of state of QGP etc.

1.2.4 Mixed/Hadronic expansion and freezeout ($\tau_c < \tau < \tau_{fr}$):

At temperature T_c hadronization occurs and system remains in mixed phase for some time. Since the system is going from QGP to hadronic phase, its entropy should decrease. Further, below T_c a complete hadronization occurs and the system acquires a pure hadronic phase. Initially, due to large scattering crossection, produced hadrons will undergo inelastic collisions. As a result, chemical composition of hadron gas may change. However, with further expansion, temperature and the density of gas decreases, which causes the scattering crossection to decrease and the collisions become elastic. This is called chemical freeze-out. Soon after it, the elastic collision and the strong decays of the heavier hadrons take place. A statistical approach is much useful to describe such data. After sometime elastic collisions are also stopped. This is because, mean free path of the particles becomes larger than system size or in other words collision time between the particles is much larger than the expansion time scale. At this juncture local equilibrium is no longer maintained and particles will decouple. This is called thermal freeze-out. At this point applicability of hydrodynamics ceases. It should be noted that the mean free path for the different kind of particles will not be the same therefore their thermal freeze-out time will be different.

1.3 Introduction to hydrodynamics

Describing a system with a large number of microscopic constituents interacting with each other is an extremely difficult task. However, if microscopic dynamics drives such systems rapidly to a state of maximum disorder i.e. the microscopic variables fluctuate so rapidly in space and time that they result in a very small changes of the average values, the system's global behavior can then be expressed in terms of a few macroscopic thermodynamic fields. Thermalization happens locally on microscopic time scales which are much smaller than the macroscopic time scales related to the system's reaction to small non-uniformities of the density, temperature and pressure etc. In such a situation, the system can be described by ideal Hydrodynamics. The resulting equations of motion for the macroscopic thermodynamic fields are the continuity and Euler equations or their relativistic generalizations (for a relativistic case). Hydrodynamics is a very efficient and widely discussed theoretical tool in many areas of research including astrophysics, cosmology and relativistic heavy ion collisions. In certain areas of astrophysics (e.g. compact stars and flows around black holes) and cosmology, hydrodynamic description requires consistency with the general theory of relativity. However, in the context of heavy ion collisions, special theory of relativity is sufficient. The beauty of hydrodynamic framework is that a large number of degrees of freedom associated with the microscopic composition of the fluids gets enormously reduced to a few macroscopic hydrodynamic variables, which represent the local property of the fluid. Here, by local property we mean equation of state (EoS) which represents the thermodynamic relations among the hydrodynamic variables and the transport coefficients. It has been a long history since hydrodynamics was applied for the first time to describe the expansion of the strongly interacting matter created in the high energy hadronic collisions [156, 157, 157]. Because of its conceptual simplicity, now a days it has become an important tool to describe the collective behavior of hot and dense strongly interacting matter created in the relativistic heavy ion collisions. Initially, it has been thought that the ideal hydrodynamics should describe the evolution of the matter produced at RHIC quite well. However, a comparision of the elliptic flow coefficient v_2 (a parameter that describes the collective flow) measured in RHIC experiments [158, 159] to the hydrodynamic simulation suggests that one needs to incorporate the viscosity in hydrodynamic framework because no fluid can be ideal due to the uncertainty principle [160].

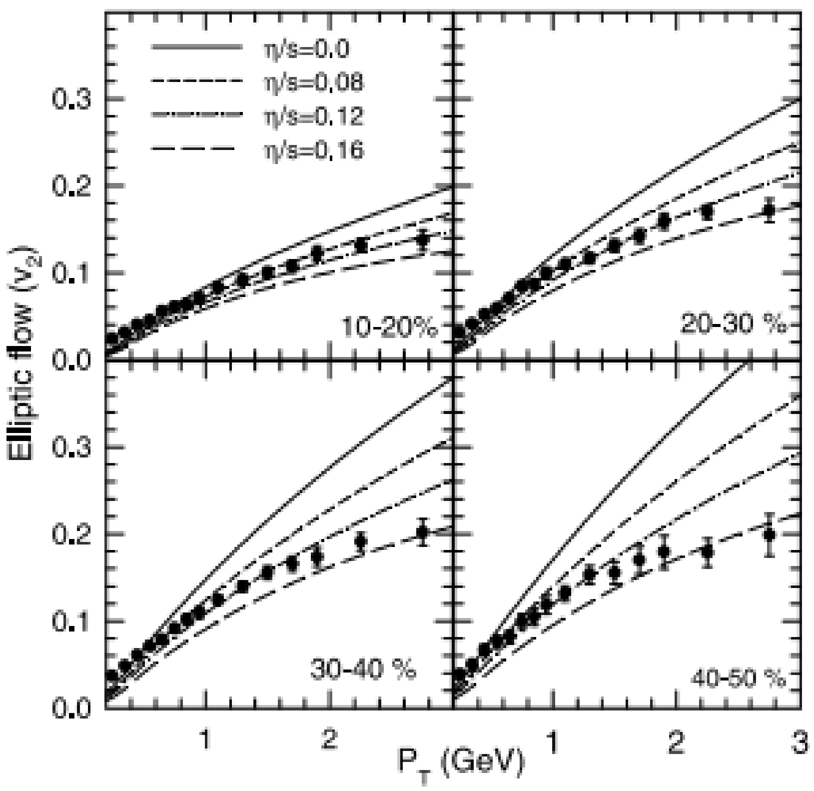


Figure 1.4: Shows a comparison of ALICE measurements for charged particles elliptic flow in 10– 20%, 20–30%, 30–40% and 40–50% Pb + Pb collisions to hydrodynamic model simulation for fluid viscosity $\eta/s = 0, \quad 0.08, \quad 0.12, \quad 0.16$ respectively. Reprinted figure from [Victor Roy and A.K. Chaudhuri, Phys. Lett. **B 703**, 313 (2011).] Copyright © 2011 by Elsevier B.V.

There are several approaches to describe the relativistic viscous hydrodynamics. The first approach is a generalization of the Navier-Stokes(NS) equation to the relativistic regime by Eckart and Landau-Lifshitz [161, 162]. It is also known as the first order theory because it can be derived from the traditional argument of the linear irreversible thermodynamics by assuming that the entropy four–current contains terms up to linear order in dissipative quantities. The generalized NS approach may not be adequate for the relativistic heavy-ion collisions as it is found to have some acausal behavior [163, 164] and unphysical instabilities. The reason for acausality is the linear relation between dissipative fluxes and the thermodynamical forces (to be written in terms of the first order gradient in velocity, temperature and chemical potential) leading to the parabolic evolution of the small perturbation in a physical system, which is initially in equilibrium.

As a consequence, group velocity (v_{gr}) associated with perturbation becomes linearly dependent on the wave number (k) of the corresponding mode. Thus, for a sufficiently high k, group velocity can become superluminal and thus violating the causality. The problem of unphysical instabilities which is even more serious has been pointed out by Hiscock and Lindblom [165–167]. In their analysis it has been shown that the unstable modes diverge exponentially over a time scale of 10^{-34} s which is much less than the time scales that describe any known physical process. Though the causality can be restored in the extended theories due to Grad [168], Müller [169], Israel and Stewart [170–172] (MIS), stability may not be guaranteed [173]. These theories are called the second order causal hydrodynamics, because these are based on the assumption that entropy four-current should have additional contributions from the terms which are of the second order in the dissipative fluxes. In this case the resulting equations of motions are hyperbolic in nature, which lead to the causal propagation of perturbation modes. MIS hydrodynamics has been extensively applied to study the relativistic heavy ion collisions [163, 164, 174, 175] and in cosmology [176]. Later this formulation was extended up to the third order [177]. The standard derivation of causal theories (MIS) using extended irreversible thermodynamics contains some additional unknown transport coefficient (e.g. relaxation time) associated with the dynamical equations of dissipative fluxes. Such a problem can be avoided using kinetic theory [172]. In fact the derivation of causal theories using the underlying kinetic theory, is not unique as there may exist more general set of equations which may give more consistent description to the relativistic heavy ion collisions [178, 179] and allow one to obtain MIS as a special case.

1.4 Hydrodynamics prerequisites for heavy ion collisions

Although there exists various hydrodynamics theories but none of them is complete without the knowledge of the following requirements.

1.4.1 Equation of state (EoS):

An EoS describes the relationship between various macroscopic thermodynamic variables (e.g. energy density, pressure, temperature, number density etc). In the case of relativistic heavy ion collisions, strongly interacting matter with no net baryon chemical potential is expected to be produced which evolves in different phases (as suggested in the section §1.2). If one assumes QGP and hadronic phase (resonance gas) to be connected by the first order phase transition, an ideal equation of state i.e $\epsilon = 3p$ can be used. However, it can be much better to use a more realistic EoS that describes the QGP phase as well as crossover to hadron gas. This can be done by using lattice results. Lattice calculations are much suited at zero baryon chemical potential. In Ref. [180] authors have parameterized the trace anomaly calculated from lattice QCD as follows,

$$\frac{\epsilon - 3p}{T} = \left(1 - \frac{1}{\left[1 + \exp\left(\frac{T - c_1}{c_2}\right)\right]^2}\right)\left(\frac{d_2}{T^2} + \frac{d_4}{T^4}\right). \tag{1.2}$$

The values of coefficients d_2, d_4, c_1 and c_2 were provided with error bars. The central values of these coefficients were obtained by combining the lattice calculation done with the p4 action and hadron resonance gas. The central values obtained are; $d_2 = 0.24$ GeV^2, $d_4 = 0.0054$ GeV^2, $c_1 = 0.2073$ GeV and $c_2 = 0.0172$ GeV. A crossover between hadron resonance gas and QGP phase was found to be in the range 180 MeV $\lesssim T \lesssim$ 200 MeV. An equation of state related to the above trace anomaly is given by

$$\frac{P(T)}{T^4} - \frac{P(T_0)}{T_0^4} = \int_{T_0}^{T} dT' \frac{\epsilon - 3p}{T'^5}. \tag{1.3}$$

T_0 was assumed to be 50 MeV and $p(T_0) = 0$.

1.4.2 Transport coefficients:

Transport coefficients quantify the effect of dissipation in the hydrodynamic evolution. There are several kind of transport coefficients like shear viscosity (η), bulk viscosity (ζ) and thermal conductivity (λ) etc. The η acts against the buildup of flow anisotropy and measures the fluid's resistance to flow. The ζ acts against the buildup of radial flow and measures the fluid's resistance to expansion. The λ measures the fluid's ability to transfer heat.

Computing of the coefficients of shear and bulk viscosities of a strongly interacting matter from various theories and verifying it with the experimental data using hydrodynamic simulation is currently a topic under intensive investigations. A comparison of the elliptic flow parameter using hydrodynamic simulation with the experimental results suggests that the shear viscosity to entropy ratio (η/s) should not be much larger than the KSS bound $\frac{1}{4\pi}$ [181]. This bound was conjectured by using anti-de-sitter space/conformal field theory (ADS/CFT) correspondence. Non-perturbative lattice QCD calculation of shear viscosity for SU(3) gluon dynamics by Mayer [182] suggests an upper bound $\eta/s < 1.0$. He estimated $\eta/s = 0.134$ (33) at ($T = 1.65\ T_c$) and $\eta/s = 0.102$ (56) at ($T = 1.24\ T_c$), which is consistent with VISHNU Hybrid code results [183] obtained at $T_c \leq T \leq 2T_c$. For a weakly coupled QCD i.e. $T >> T_c$, kinetic theory in the relaxation time approximation gives $\eta \sim \frac{T^3}{\alpha_s^2 \ln(1/\alpha_s)}$ [184, 185], where α_s is the strong coupling constant. Calculation of the leading logarithmic contribution from Boltzmann equations was performed in [186]. A full leading order calculation beyond logarithmic contribution using effective kinetic theory, in the hard thermal loop approximation [187] with three massless quark flavors, gives the ratio of shear viscosity to entropy $\frac{\eta}{s} \sim \frac{5.12}{g^4 \ln(2.42/g)}$, where $g(T)$ is the running coupling. For $T << T_c$, the effective degree of freedom are hadrons (Baryons and Mesons). Calculations using theory of massless pions in the low energy chiral limit (massless case of u and d quarks) yield; $\eta/s = \frac{15}{16\pi}\frac{f_\pi^4}{T^4}$, where $f_\pi = 93$ MeV is the pion decay constant. Calculations of η/s for hadronic matter that include both pions and kaons were also performed. It goes beyond the chiral limit and uses intermediate ρ meson. The results qualitatively agree with each other at low temperature, however differ significantly above $T > 100$ MeV due to kaon excitations. All these calculations were performed at zero baryon chemical potential. Studies at finite chemical potential are still need to be done.

The bulk viscosity scales like $(\epsilon - 3p)$ which arises from the response of the trace of the energy momentum tensor (T^{μ}_{μ}) to a uniform expansion. Therefore, at a very high temperature it should go to zero because the system will satisfy ideal EoS on classical level. Thus the system have the conformal symmetry. However, due to quantum effect, conformal symmetry can be broken which may results non-zero bulk viscosity. At high temperature QCD $(T >> T_c)$, an estimate for the bulk viscosity is found in reference [188] which is related with shear viscosity (calculated within weakly coupled QCD) [186, 187] as follows,

$$\zeta \approx 15\eta(1/3 - C_s^2)^2. \tag{1.4}$$

In the case of $T << T_c$, for massless pion gas, it was found that $\zeta/s \propto T^4/f_\pi^4$ [189]. Result for massive pion case was presented in Ref.[190]. Unlike for the massless pion gas, in this case ζ/s was found to be a decreasing function of T. Around T_c, the behavior of ζ/s was investigated in many references [188, 191–195] and found to peak near T_c. In this regime the contribution of bulk viscosity is much larger than that of η/s. The larger values of ζ/s can make the effective pressure of the fluid very small or negative [196–199]. This may cause cavitation in the fluid therefore, may limit the applicability of hydrodynamics.

1.4.3 Initial and final Conditions:

In case of heavy ion collisions, hydrodynamics can be applied to a regime from local thermodynamical equilibrium to freeze-out. Therefore, a typical estimate of time and the energy density/temperature etc that the pre-equilibrium state takes to reach local thermodynamic equilibrium (thermalization) will be helpful in setting the initial conditions for the hydrodynamical evolution of a heavy ion collision. The time (τ_{fr}) and energy density/temperature (ϵ_{fr}/T_{fr}) at which freeze out happens can be used as final conditions. Recently, based on a comparison of the transverse momentum spectrum of direct photons measured in the heavy ion collision experiments (Pb-Pb collisions with $\sqrt{s}$ = 2.76 TeV at LHC and Au-Au collisions with $\sqrt{s}$ = 200 GeV) with the $(3 + 1)$ dimensional ideal hydrodynamic simulations constrained with hadronic data suggests that the thermalization time scale is about 0.6 fm/c [200]. However, till now there is no well established theoretical approach which can give us thermalization time below 1 fm/c. The initial energy density can be calculated using Bjorken model [201]. In Bjorken model, initial conditions were assumed to homogeneous on the constant proper time hyperbolas. However, distribution of nuclear matter in the colliding nuclei is not homogeneous, therefore, model with fluctuating initial conditions were developed. The most commonly used models to describe the fluctuating initial conditions are Glauber-MC Model and the model based on Color Glass Condensates (CGC).

1.5 CP-violation and Chiral Magnetic Effect (CME) in heavy ion collisions

1.5.1 CP-violation

The violation of local P and CP symmetry in strong interactions is currently also a topic of intense discussions. Excitement began from the discovery of topological nature of QCD vacuum. It has been suggested that QCD vacuum solution can be characterized in terms of topological invariants, called topological charges or winding number, defined as: $Q_w = \frac{g^2}{32\pi^2}\int d^4x F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a$, where g is the QCD coupling constant, $F^{\mu\nu}$ is the gluonic field strength tensor and $\tilde{F}^{\mu\nu}_a = \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ is dual of $F^{\mu\nu}$. The existence of vacuum topological solutions in QCD [202] leads to a puzzling question, why QCD does not seem to break the P and CP symmetries (the strong CP problem). However, it has been suggested by Vafa and Witten in 1985 that P and CP symmetries can not be broken in a true ground state of QCD i.e. when $\theta = 0$ [203, 204]. Kharzeev et al. [205–208] have suggested that the configuration with non-zero Q_w violates the P and CP symmetry of QCD and provided a possible mechanism of charge separation which suggests the possibility to have the Chiral magnetic effect (CME) in heavy ion collision. Therefore, CME can be useful to explain P and CP-violation in the heavy ion collisions if it could be observed. The charge separation mechanism and CME can described as follows.

It has been suggested that it is due to quantum axial anomalies [209, 210], the gauge field configurations with non-zero Q_w leads to the non-conservation of flavor singlet axial current $J_{\mu_5} = \sum_f \langle \bar{\psi}_f \gamma_\mu \gamma_5 \psi_f \rangle_A$ even in the chiral limit ($m_f = 0$) as given by the following equation,

$$\partial^\mu J^5_\mu = -\frac{N_f g^2}{16\pi^2} \int d^4x F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a, \tag{1.5}$$

where, N_f denotes the number of quark flavors, ψ_f and m_f respectively denote the quark field and mass for a flavor f. Now if one assumes at $t = -\infty$ there are equal numbers of right and left handed fermions i.e. chiral chemical potential μ_5 is zero then the integration of the above equation will yield finite μ_5 at $t = +\infty$,

$$\mu_5 = (N_L - N_R)_{t=+\infty} = 2N_f Q_w, \tag{1.6}$$

where, $N_{R,L}$ are the number of right-handed and left handed fermions. This equation shows that if we have a non-zero Q_w, it is possible to convert the right handed fermions into the left handed ones or vice-versa.

1.5.2 Chiral Magnetic Effect:

In non-central heavy ion collisions, due to the relativistic motion of the heavy nuclei (Au-Au in case of RHIC and Pb-Pb of LHC) with large positive charges, strong magnetic fields can be generated [207, 211] in a plane perpendicular to the reaction plane. In the recent numerical simulations, it has been shown that the magnitude of the magnetic field at RHIC could be of the order of 10^{18} Gauss, while at LHC it can reach up to 10^{20} Gauss [207, 212–220]. Now, if we assume that initially there are equal number of right and left handed particles (which could be a reasonable assumption for heavy ion collisions) and there is a strong magnetic field ($\mathbf{B}$) present, the spins of quarks will align parallel or anti-parallel to magnetic field (depending upon the sign of electric charge).

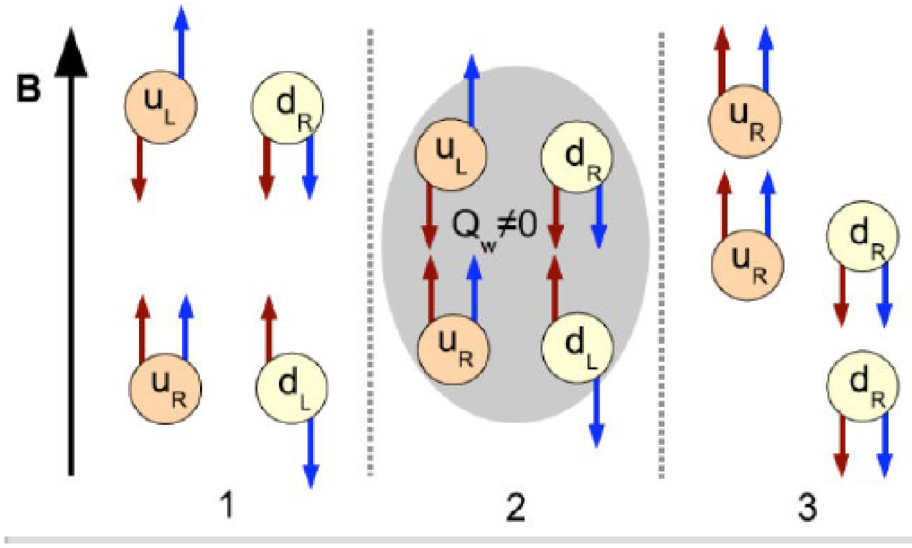


Figure 1.5: Illustrate a mechanism by which configuration with non zero Qw can separate charge in the presence of a background magnetic field leading to CME (Chiral Magnetic effect). The blue and red arrows show the direction of spins and momentum respectively. Reprinted figure from [Kharzeev et al., Nucl. Phys. **A 803**, 227 (2008).] Copyright © 2008 by Elsevier B.V.

Therefore, positively charged right-handed and negatively charged left-handed particles will move in the direction parallel to B and negatively charged right-handed and positively charged left-handed particles shall move anti-parallel to B. Now, due to the topological nature of QCD, quarks will interact with the gauge field configuration with $Q_w \neq 0$, as a result their chirality will change. The only possibility by which particles can change their chirality is to reverse their momenta because spin flip is energetically suppressed as $eB \gg \Lambda^2_{QCD} \gg p^2$ (in case of heavy ion collisions). For simplicity, we consider the case of two right and left-handed up and down quarks (with the blue and red arrows denoting the direction of spins and momentum respectively) as shown in Fig. 1.5. Initially positively charged right-handed up quark and negatively charged left-handed down quark will be moving in a direction parallel to B while positively charged left-handed up quark and negatively charged right-handed down quark will be moving in a direction anti-parallel to B. If $Q_w = -1$ then it will convert the left-handed up/down quark into right-handed up/down quark by reversing direction of momentum. As a result, right handed up quarks will move upward and right-handed down quarks will move downward and charge difference $Q = 2e$ will be created. If there are N_f number of flavors, the electrical charge separation will be of the order of $\mid Q \mid = 2Q_w \sum_f \mid q_f \mid$, where q_f is the electrical charge of quark of flavor f. Therefore, a net electrical current can be generated in the direction of magnetic field. This is called chiral magnetic effect (CME).

In heavy ion collision experiments, a charge separation along the magnetic field vector (in a single event) is described by sine terms in the Fourier decomposition of the charge particle azimuthal distribution given by the following equation,

$$\frac{dN_{\pm}}{d\phi} \sim 1 + 2v_1 \cos(\Delta\phi) + 2v_2 \cos(2\Delta\phi) + ... + 2a_{\pm} \sin\Delta\phi + ... \qquad (1.7)$$

where, $\Delta\phi = \phi - \psi_{RP}$ is the azimuthal angle of particle relative to reaction plane. ϕ and ψ_{RP} are the azimuthal angle of the particle and reaction plane (a plane containing the trajectories of the colliding nuclei and impact parameter). Coefficients v_1 and v_2 account for the directed and elliptic flow and $a_{\pm}$ for chiral magnetic effect; which causes hadrons with opposite charge to be preferentially emitted on the different sides of the reaction plane. The sign of a_+ and a_- fluctuates from event to event. When summed over many events, $\langle a_{\pm} \rangle = 0$. Therefore, the observation of charge separation or local P and CP violation is only possible by measuring the correlator $\langle a_\alpha a_\beta \rangle$, where α, β corresponds to $+$ or $-$ sign. One could measure $\langle a_\alpha a_\beta \rangle$ by calculating the expectation value of $\langle \sin\Delta\phi_\alpha \sin\Delta\phi_\beta \rangle$ over all particles of charge α paired with β. However, this also has a problem of being very sensitive to several parity conserving physics backgrounds. This lead to a proposal of a new observable [221].

$$\langle \cos(\phi_\alpha + \Phi_\beta - 2\psi_{RP}) \rangle = \langle \cos\Delta\phi_\alpha \cos\Delta\phi_\beta \rangle - \langle \sin\Delta\phi_\alpha \sin\Delta\phi_\beta \rangle \qquad (1.8)$$

The observable $\langle \cos(\phi_\alpha + \Phi_\beta - 2\psi_{RP}) \rangle$ is called the three particle azimuthal correlator. This is because the reaction plane is not known. It is estimated by measuring the ‘event plane’ which can be obtained using the three particle azimuthal correlation, where the third particle serves to measure the event plane. In this case the observable,

$$\langle \cos(\phi_\alpha + \Phi_\beta - 2\psi_{RP}) \rangle = \langle \cos(\phi_\alpha + \Phi_\beta - 2\phi_c)/v_{2,c} \rangle \tag{1.9}$$

where, subscript c accounts for the third particle. Very recently STAR Collaboration at RHIC [103, 104] has reported the results for the measurement of three particle azimuthal correlations $\langle \cos(\phi_\alpha + \Phi_\beta - 2\psi_{RP}) \rangle$ with respect to the collision centrality as shown in Fig. 1.6.

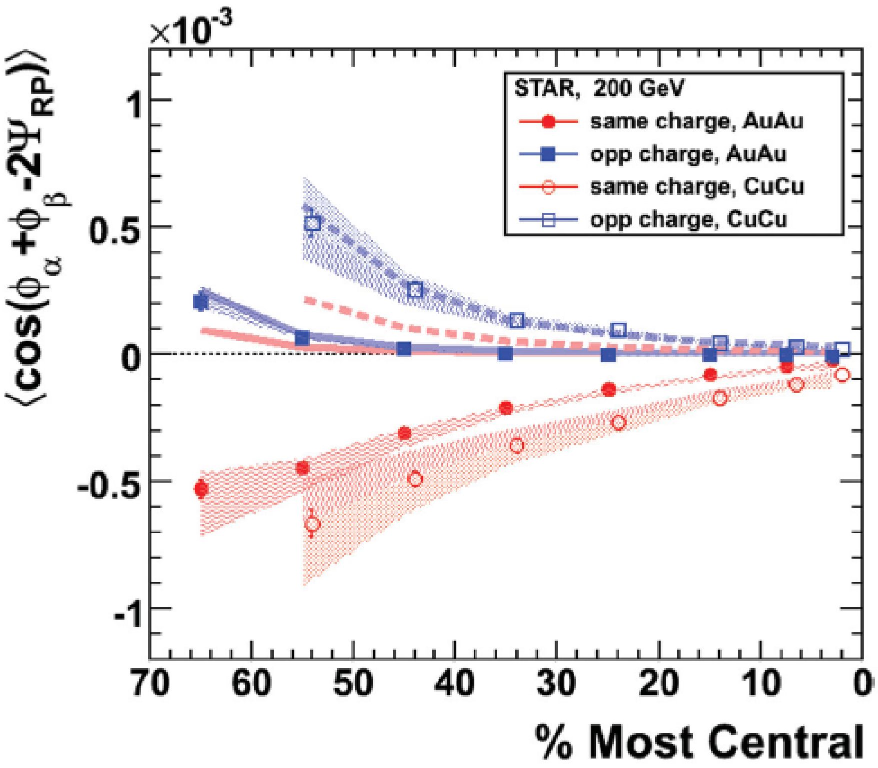


Figure 1.6: STAR Collaboration results for the three particle azimuthal correlations ($\langle \cos(\phi_\alpha + \Phi_\beta - 2\psi_{RP}) \rangle$) in Au-Au and Cu-Cu collisions at $\sqrt{s} = 200$GeV. The solid (Au-Au) and dashed (Cu-Cu) lines represent the HIJING calculations while shaded bands show the uncertainty from the measurement of v_2. Reprinted figure from [B. I. Abelev et al. (STAR Collaboration), Phys. Rev. Lett. **103**, 251601 (2009).] Copyright © 2009 by the American Physical Society.

One can see from the figure that the azimuthal correlations of the particle of opposite charges in both the cases (Au-Au and Cu-Cu) separate out on opposite sides and increases in more peripheral (non central) collisions. The increase of the signal at same collision centrality for the case Cu-Cu is attributed to the fact of increasing multiplicity. This study gives a clear indication of the local violation P and CP symmetry and CME.

1.6 Objectives of this Thesis:

A study of the fluctuations in continuous media is of great interest in physics and it can provide a link between the macroscopic and microscopic points of view. A macroscopic theory such as hydrodynamics provides a simplest possible description of a complicated many body system in terms of space-time evolutions of the mean or average quantities like energy density, pressure and flow velocity etc. However, according to our knowledge of statistical mechanics, a physical quantity which describes a macroscopic body in thermal equilibrium fluctuates about its mean value. Intensity of these fluctuations is determined by equal time correlation functions which represent the correlation between the values of a given quantity from one space time point to another. The fluctuation theory studies small deviations from the mean behavior and helps in calculating the correlation functions of the macroscopic variables [222, 223]. In this thesis our first aim is to apply the theory of quasi-stationary fluctuations due to Onsager [224, 225] to calculate such fluctuations in various relativistic hydrodynamic frameworks (causal or acausal) and to study the behavior of viscous correlation with simple example of boost invariant Bjorken Flow [201] which is very often discussed in the context of relativistic heavy ion collisions. Such a study can be useful in the determination of transport coefficient such as coefficient of shear viscosity (η) and bulk viscosity (ζ).

We would also like to focus on another aspect of heavy ion collisions i.e. P and CP-violation. We have already discussed how and why P and CP-violation or CME can occur and how it can be measured in heavy ion collisions in the previous section. However, it could be interesting if we have a kinetic theoretical formalism which could describe the P and CP-violation or Chiral magnetic effect. Indeed, very recently such a framework has been developed [226–229] by using the Berry curvature [230] corrections. This modified kinetic theory gives the identical results for the parity odd correlation function [229, 231] as computed from the perturbation theory in the next to leading order hard dense loop approximation. At present, there also exist several models of hydrodynamics [232–236] that incorporates the parity odd effects. Recently using the modified kinetic approach it has been shown that the presence of CP-violating effects can lead to an instability (Chiral Imbalance Instability) in the transverse branch of the dispersion relation. Such a study is also possible by hydrodynamics, but a kinetic theory approach is much more general and can be applied in situations whether there is a thermal equilibrium or not. In a realistic situation, such as heavy ion collisions, it is important to consider initial distribution function to be anisotropic in the momentum space. It is well known that momentum anisotropy can lead to so called Weibel instabilities [122, 132–141, 237–239] of transverse waves. Therefore, it is important to study the collective modes of the chiral plasma using the modified kinetic theory in the presence of momentum anisotropy and to study the chiral imbalance and Weibel instabilities together. Weibel instabilities could be important in thermalization of strongly interacting matter created in the heavy ion collisions [142–155]. Recently in the context of heavy ion collisions it was shown that the Weibel instability can drive turbulent transport [240] which can lower the ratio of the shear viscosity to the entropy density for QGP, calculated using binary collisions [240–242] by means of enhanced collisionality due to turbulence. It will also be interesting to calculate the contribution of the CP-violating instability in determining the anomalous shear viscosity. Results that shall be obtained from this study could be useful to explain the total shear viscosity of the strongly interacting matter found in the core of the neutron star as well as the matter produced in the heavy ion collisions.

1.7 Organization of the thesis:

In this thesis we shall focus on the study of hydrodynamic fluctuations, Weibel and chiral instabilities in the context of heavy ion collisions. The thesis is organized as follows :

In Chapter §2, we shall discuss several kind of the relativistic viscous hydrodynamics approaches which are extensively used to describe the evolution of strongly interacting matter known as Quark Gluon Plasma (QGP). In order to incorporate the physics of local CP-violation in the context of heavy ion collisions, we shall also introduce hydrodynamics with triangle anomalies and related kinetic theory.

Chapter §3 will contain the calculation of the hydrodynamic fluctuations in the framework of the causal hydrodynamics of Müller, Israel and Stewart (MIS) also to other related approaches. In particular, we shall try to study the behavior of the correlation functions using one dimensional expanding boost invariant (Bjorken) flow for different kind of viscous hydrodynamics.

In Chapter §4, we shall focus on the study of chiral imbalance instability (arising because of CP-violation) and Weibel instability (arising because of momentum anisotropy in the distribution function) using Berry curvature modified kinetic equation. In particular, we shall try to study how these two instabilities compete with each other.

In Chapter §5, using the fact that a plasma instability can drive the turbulent transport, we shall calculate the anomalous viscosity of chiral plasma due to the chiral-imbalance instability. Such a viscosity may lower the total kinetic viscosity calculated using binary collisions. Result can be important for the case of QGP formed in the heavy ion collision as well as for case of a neutron star.

In Chapter §6, I shall give a brief summary of the thesis by emphasizing the significance of the present work.

Chapter 2

Relativistic Viscous Hydrodynamics, CP-violation and Modified Kinetic Theory

In this chapter we shall introduce the basic theoretical tools required for this thesis. In particular, we shall introduce various hydrodynamic frameworks, and a modified kinetic theory approach to describe the effects of CP-violation. In section §2.1, we shall discuss relativistic ideal hydrodynamics. In section §2.2, we shall consider the and Bjorken Flow and apply it to relativistic ideal hydrodynamics. In section §2.3, we shall go beyond ideal fluid approximation and discuss various formulations of relativistic dissipative hydrodynamics. In section §2.3.2, we shall discuss the formulation of Müller, Israel and Stewart (MIS) and a few other related hydrodynamic frameworks using kinetic theory. We shall also discuss hydrodynamics with triangle anomalies to describe the parity violating physics. In section §2.4, a modified kinetic theory framework using Berry curvature corrections will be introduced.

2.1 Relativistic ideal hydrodynamics

Hydrodynamics is governed by the conservations laws; namely the conservation of energy momentum tensor $T^{\mu\nu}(x)$ and particle four-current $J^{\mu}(x)$,

$$\partial_\mu T^{\mu\nu}(x) = 0, \tag{2.1}$$

$$\partial_\mu J^{\mu}(x) = 0, \tag{2.2}$$

where, x denotes the space-time co-ordinates (t, x). $T^{\mu\nu}(x)$ and $J^{\mu}(x)$ are defined when the system is in local thermodynamical equilibrium. By local thermodynamical equilibrium we mean that the mean free path (λ) of the particles in a system is smaller than its characteristic size R. The ratio λ/R is called the Knudsen Number K_n and the system is considered to be in a local thermodynamics equilibrium if $K_n = \lambda/R \ll 1$. The λ is defined as the distance traveled by a particle between two successive collisions and it can be expressed as,

$$\lambda = \frac{1}{\sigma\rho}. \tag{2.3}$$

For Au-Au collisions ($\sqrt{s}$ = 200 GeV) at RHIC, $\rho \sim 0.153$ fm^{-3} and $\sigma \sim 45$ mb=4.5 fm^2. Thus $\lambda \sim 1.45$ fm. If we take system size $R = 2R_A = 2 \times 1.2\mathrm{A}^{1/3}$ fm (for Au, A=197), we can get $K_n \sim 0.1<1$ which permits the applicability of the hydrodynamics.

For a relativistic fluid, the general form of $T^{\mu\nu}$ and J^{μ} can be constructed from the hydrodynamic degrees of freedom namely energy density $\epsilon(x)$, pressure $P(x)$ and particle number, $n(x)$ and 4-flow velocity u^{μ} which is given by,

$$u^{\mu} = \gamma(1, v(x)), \tag{2.4}$$

where, $\gamma = 1/\sqrt{1 - v^2(x)}$ and $\vec{v}(x)$ is the three-velocity vector. u^{μ} satisfies the normalization condition $u^{\mu}(x)u_{\mu}(x) = 1 = g^{\mu\nu}u^{\mu}u^{\nu}$. Here $g^{\mu\mu} = diag(+1, -1, -1, -1)$. In a local rest frame (LRF) $\vec{v} = 0$ and $u^{\mu} = u^{\mu}_{LRF} = (1, 0, 0, 0)$. In this case $T^{\mu\nu}$ and J^{μ} should have forms similar to the case when a system is in a static equilibrium i.e. in this case there is no flow of energy i.e $T^{0i}_{LRF} = 0$ and pressure exerted by a fluid element is isotropic i.e. $T^{ij}_{LRF} = P\delta^{ij}$ and there is no particle flow $\vec{J} = 0$. Thus we have

$$T^{\mu\nu} = \begin{bmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}, \; j^{\mu} = \begin{bmatrix} n \\ 0 \\ 0 \\ 0 \end{bmatrix}; \tag{2.5}$$

where ϵ, n is the energy density and net baryon density respectively.
Using Lorentz transformations on $u^{\mu}_{LRF} = (1, 0, 0, 0)$ one finds,

$$u^{\mu} = \Lambda^{\mu}_{\nu} u^{\nu}_{LRF}, \tag{2.6}$$

thus, $\Lambda^{\mu}_{0} = u^{\mu}$. Now, using $u^{\mu}(x)u_{\mu}(x) = g^{\mu\nu}u^{\mu}u^{\nu} = 1$, one gets

$$g^{\mu\nu}\Lambda^{\rho}_{\mu}\Lambda^{\sigma}_{\nu} = g^{\rho\sigma}. \tag{2.7}$$

The above equation can be written as,

$$g^{\rho\sigma} = \Lambda^{\rho}_{0}\Lambda^{\sigma}_{0} - \Lambda^{\rho}_{i}\Lambda^{\sigma}_{i}, \tag{2.8}$$

which implies that

$$\Lambda^{\rho}_{i}\Lambda^{\sigma}_{i} = u^{\rho}u^{\sigma} - g^{\rho\sigma}. \tag{2.9}$$

Further, one can also write,

$$T^{\mu\nu} = \Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}T^{\rho\sigma}_{LRF}, \tag{2.10}$$

$$J^{\mu} = \Lambda^{\mu}_{\nu}J^{\nu}_{LRF}. \tag{2.11}$$

Using Eq. (2.5) in above equations one gets,

$$T^{\mu\nu} = \Lambda^{\mu}_{0}\Lambda^{\nu}_{0}\epsilon + \Lambda^{\mu}_{i}\Lambda^{\nu}_{i}P, \tag{2.12}$$

$$J^{\mu} = \Lambda^{\mu}_{0}n. \tag{2.13}$$

Now, using Eq.(2.9) in Eq.(2.12) and $\Lambda_0^\mu = u^\mu$ in Eq.(2.13), following expressions can be obtained,

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - p\Delta^{\mu\nu}, \tag{2.14}$$

$$J^\mu = nu^\mu, \tag{2.15}$$

where, $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$, has the following properties.

$$\Delta^{\mu\nu}u_\nu = 0 = \Delta^{\mu\nu}u_\mu, \quad \Delta^\mu_\alpha\Delta^{\alpha\nu} = \Delta^{\mu\nu}, \quad \Delta^\alpha_\alpha = 3. \tag{2.16}$$

Thus, $\Delta^{\mu\nu}$ can be regarded as a projection operator in a direction perpendicular to u^μ. It is good to write the conservation law given by Eq.(2.1) in a direction parallel and perpendicular to the fluid velocity. This can be done by projecting Eq.(2.1) along u^μ and $\Delta^{\mu\nu}$. By doing so, the equations of motion of ideal hydrodynamics can be written as,

$$u_\nu\partial_\mu T^{\mu\nu} = D\epsilon + (\epsilon + p)\partial_\mu u^\mu = 0, \tag{2.17}$$

$$\Delta^\alpha_\nu\partial_\mu T^{\mu\nu} = (\epsilon + p)Du^\alpha - \nabla^\alpha p = 0, \tag{2.18}$$

$$\partial_\mu J^\mu = Dn + n\partial_\mu u^\mu = 0, \tag{2.19}$$

where $D = u^\mu\partial_\mu$, $\nabla^\alpha = \Delta^{\mu\alpha}\partial_\mu \Rightarrow \partial^\alpha + u^\alpha D$. Here, we have five independent equations and four fields which correspond to six degrees of freedom (one for each n_B, ϵ, P, and 3 for u^μ). Therefore, one additional equation is required to close system of hydrodynamical equations. This additional equation is supplied by an equation of state of the fluid $p = p(n, \epsilon)$. Now, using the following thermodynamic relations,

$$\epsilon + p = Ts + \mu n, \tag{2.20}$$

$$d\epsilon = Tds + \mu dn, \tag{2.21}$$

where, s is the entropy density and μ the chemical potential. It can be shown that four-entropy current $S^\mu = su^\mu$ is conserved i.e.

$$\partial_\mu S^\mu = 0. \tag{2.22}$$

Eqs.(2.17-2.19) are the fundamental equations for an ideal relativistic fluid. In the non-relativistic limit, $|\vec{v}| \ll 1$, $p \ll \epsilon$, and energy density is approximated by mass density i.e. $\epsilon \simeq \rho$. In this case, one can find, $D \simeq \partial_t + \vec{v} \cdot \vec{\partial} + O(|\vec{v}|^2)$ and $\nabla^i \simeq \partial^i + v^i \partial_t$. Thus, with the proper substitution for D, ∇^i and ϵ, one can easily recognize the Eq.(2.17) and Eq.(2.18) as the standard continuity and Euler equations of the non-relativistic physics respectively.

2.2 Bjorken flow:

Bjorken flow was introduced in 1983 by J. D. Bjorken to describe the expansion of the thermalized strongly interacting matter created in the heavy ion collisions. It uses an idea that in heavy ion collision experiments, there should be a plateau in the central rapidity distribution ($\frac{dN}{dy}$) of the produced particles as shown in Fig. 2.1. This means that the particle multiplicity is a boost invariant quantity and alternatively implies that n, ϵ must also be boost invariant. Bjorken suggested that all of this would be true if we assume;

1. Soon after the collision, there is a fast thermalization with no net baryon number.

2. The reaction zone is strongly expanded along the longitudinal direction (direction of collision axis or z-axis).

3. Also, local velocity $u^\mu(x)$ of the fluid has the same form as the free stream of the particles from the origin.

Note that due to the assumption of longitudinal expansion, the transverse spatial dimensions ($x_\perp$=x_1,x_2) can be dropped and the expansion of reaction zone can be described in t and z, (1+1) dimensions. Dimensionality can be reduced to (1+0) (we shall see later, physical quantities do not depend on rapidity) if we

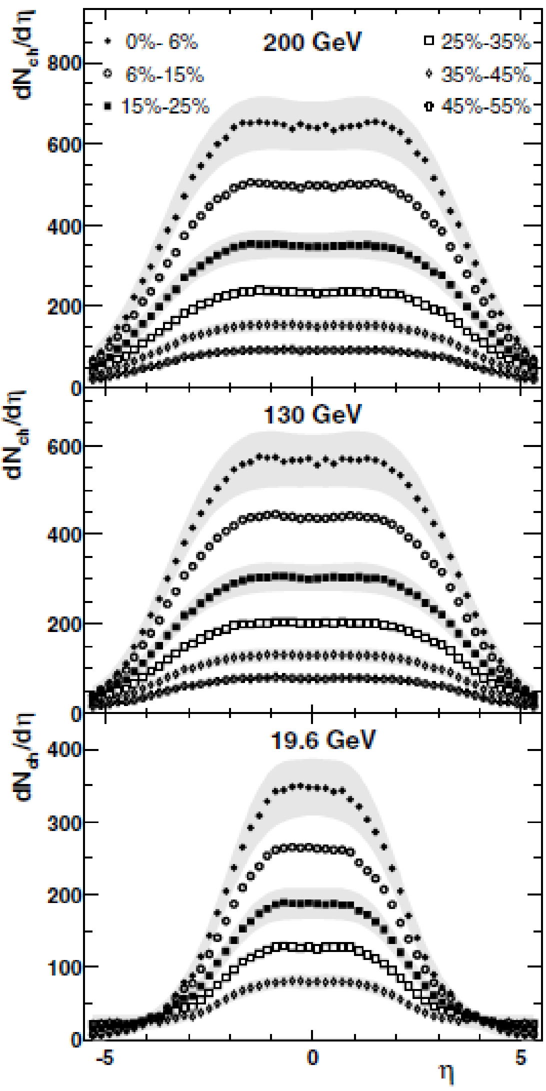


Figure 2.1: Charge particle pseudorapidity distribution. Reprinted figure from [B. B. Back et al., Phys. Rev. Lett. **91** ,052303 (2003).] Copyright © 2003 by the American Physical Society.

use the light cone variables (τ, y) as follows,

$$t = \tau \cosh y, \quad z = \tau \sinh y \tag{2.23}$$

$$\tau = \sqrt{t^2 - z^2}, \qquad y = arc\tanh(z/t) = \frac{1}{2}\ln(\frac{t+z}{t-z}). \tag{2.24}$$

The flow velocity can be written as,

$$u^\mu = \gamma(1,0,0,v_z) = (\frac{t}{\tau},0,0,\frac{z}{\tau}) = (\cosh y, 0, 0, \sinh y). \tag{2.25}$$

Note that here we have taken $v_z = z/t$. This scenario is called scaling or Bjorken flow. In this situation, partial derivatives in time and space can be expressed as,

$$\begin{bmatrix} \partial_t \\ \partial_z \end{bmatrix} = \begin{bmatrix} \cosh y & -\sinh y \\ -\sinh y & \cosh y \end{bmatrix} = \begin{bmatrix} \partial_\tau \\ \frac{1}{\tau}\partial_y \end{bmatrix}. \tag{2.26}$$

Using eq.(2.25) and the transformation of derivatives given by the above equation, we can write the operators D, ∇ such that, $D = u^\mu \partial_\mu = \frac{\partial}{\partial \tau} = \partial_\tau$ and $\partial_\mu u^\mu = \frac{1}{\tau}$. Thus, the fluid Eqs.(2.17-2.18) can be written as,

$$\partial_\tau \epsilon + \frac{(\epsilon + p)}{\tau} = 0, \tag{2.27}$$

$$\partial_y p = 0. \tag{2.28}$$

Note that Eq.(2.19) will not contribute, because in this scenario the net baryon number was considered to be zero (i.e. $n = 0$). Also the equation for entropy (Eq.(2.22)) shall take the form.

$$\partial_\tau s + \frac{s}{\tau} = 0 \tag{2.29}$$

From Eqs.(2.27-2.29), one can clearly see that the quantities ϵ, p, and s do not depend on the rapidity variable. Hence, they all are boost invariant.

Solution of Eq.(2.29) can be written as,

$$s(\tau) = s(\tau_0)\frac{\tau_0}{\tau}, \tag{2.30}$$

where, τ_0 and $s(\tau_0)$ respectively denote the initial proper time and entropy. Note that Eq.(2.27) contains two variables ϵ and p, therefore, an equation of state will be needed to solve it. Assuming *ideal EOS* i.e. $p = c_s^2 \epsilon$ ($c_s = \sqrt{\frac{dp}{d\epsilon}} = 1/\sqrt{3}$ is the speed of sound) one can get the following solution for Eq.(2.27),

$$\epsilon(\tau) = \epsilon(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{1+c_s^2}, \tag{2.31}$$

where, $\epsilon(\tau_0)$ is the initial energy density. If we consider the pressure given by Bag model,

$$p = aT^4; \qquad a = \left(16 + \frac{21}{2}N_f\right)\frac{\pi^2}{90}, \tag{2.32}$$

where, N_f denotes the number of flavors considered. We can easily show,

$$T(\tau) = T(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{c_s^2}, \tag{2.33}$$

where, $T(\tau_0)$ is the initial temperature. Eqs.(2.30-2.31) and Eq.(2.33) show that as a result of expansion s, ϵ and T decrease with time. Note that Bjorken flow is a good approximation during the early stages of the relativistic heavy ion collisions. However, it is not a good approximation in the most realistic situation where the transverse expansion occurs.

2.3 Relativistic dissipative hydrodynamics

For the reasons suggested in section §1.3, it is important to consider the dissipative effects in the hydrodynamic frameworks when applied to the relativistic heavy ion collisions. However, theories of relativistic dissipative hydrodynamics are still under development and there are several models of hydrodynamics available in the literature. In this section we discuss all those one by one.

2.3.1 Navier-Stokes (NS) and Muller, Israel and Stewart (MIS) Hydrodynamics from Covariant Thermodynamics

2.3.1.1 Navier-Stokes (NS) Hydrodynamics:

The basic thermodynamic relations (Euler's relation and the first Law of thermodynamics) in Eqs.(2.20-2.21) can be written as,

$$s = \beta(\epsilon + p) - \alpha n, \tag{2.34}$$

$$ds = \beta d\epsilon - \alpha dn, \tag{2.35}$$

where, $\alpha = \frac{\mu}{T}$, and $\beta = \frac{1}{T}$. Using above two equations one can also obtain,

$$d(p\beta) = nd\alpha - \epsilon d\beta. \tag{2.36}$$

For a relativistic fluid an arbitrary local state is specified phenomenologically by variables $T^{\mu\nu}_{(0)}$, $J^{\mu}_{(0)}$ and $S^{\mu}_{(0)}$ (with some additional variables) which satisfy the conservation laws given by Eqs.(2.1-2.2) and Eq.(2.22). For a non-equilibrium fluid, in general, there can be infinite number of additional variables. Thus, for fluid dynamic description a covariant generalization is needed. The covariant form [170–172] of the above thermodynamic relations for equilibrium dynamics can be written down using $S^{\mu}_{(0)}$, $T^{\mu\nu}_{(0)}$ and $J^{\mu}_{(0)}$ as follows,

$$S^{\mu} = p\beta^{\mu} - \alpha J^{\mu}_{(0)} + \beta_{\nu} T^{\mu\nu}_{(0)}, \tag{2.37}$$

$$dS^{\mu} = -\alpha dJ^{\mu}_{(0)} + \beta_{\nu} dT^{\mu\nu}_{(0)}, \tag{2.38}$$

where, $\beta_{\nu} = \frac{u_{\nu}}{T}$. The above equations represent the covariant form of Euler's relation and the first law of thermodynamics. Using these equations one can easily get the covariant form of generalized Gibbs-Duhem relation,

$$d(p\beta^{\mu}) = J^{\mu}_{(0)} d\alpha - T^{\mu\nu}_{(0)} d\beta_{\nu}. \tag{2.39}$$

The above equations indicate that if we know the equation of state then all the basic variables S^{μ}, $T^{\mu\nu}$ and J^{μ} can be obtained by the fugacity four-vector $p\beta^{\mu}$. Note that in the rest frame of the fluid Eqs.(2.37,2.38,2.39) reduce to Eqs.(2.34,2.35,2.34). Thus, the covariant thermodynamic relations do not have any additional modification to usual thermodynamic relations. Now, using the first law of thermodynamics one can obtain,

$$\partial_{\mu} S^{\mu} = -\alpha \partial_{\mu} J^{\mu}_{(0)} + \beta_{\nu} \partial_{\mu} T^{\mu\nu}_{(0)}. \tag{2.40}$$

Ideal hydrodynamics is described by the equilibrium covariant thermodynamics where the quantities $T^{\mu\nu}$ and J^{μ} are given by Eqs.(2.14-2.15) which satisfied the

Eqs.(2.1-2.2). Thus, from the above equation it is straight forward to see that the four-entropy current remains conserved i.e. $\partial_\mu S^\mu = 0$. However, dissipative effects cause the system to behave irreversibly. Therefore, system should be characterized by the non-equilibrium state. In this situation, the second law of thermodynamics says that its entropy should increase. Non-equilibrium generalization of the covariant thermodynamics provides a beautiful way to achieve the relativistic dissipative hydrodynamics. It is done by incorporating the dissipative fluxes in four-entropy current S^μ as well as in energy momentum tensor $T^{\mu\nu}$, particle four-current J^μ obeying the conservation equations

$$\partial_\mu T^{\mu\nu}(x) = 0, \tag{2.41}$$

$$\partial_\mu J^\mu(x) = 0. \tag{2.42}$$

Once dissipative fluxes have been included, it is necessary to determine their transport equations. This can be done by using the second law of thermodynamics which says that for an irreversible process, $\partial_\mu S^\mu > 0$. The generalized $T^{\mu\nu}$ and J^μ can be written as,

$$T^{\mu\nu} \equiv T^{\mu\nu}_{(0)} + \delta T^{\mu\nu} = \epsilon u^\mu u^\nu - p\Delta^{\mu\nu} + \delta T^{\mu\nu}_{vis} + \delta T^{\mu\nu}_{heat} \tag{2.43}$$

$$J^\mu_B \equiv J^\mu_{B(0)} + \delta J^\mu_B = n u^\mu + \nu^\mu, \tag{2.44}$$

$$S^\mu \equiv S^\mu_{(0)} = s u^\mu + \phi^\mu. \tag{2.45}$$

Note here that due to the introduction of the dissipative quantities, fluid can no longer be treated in the equilibrium fluid, hence, the variables like ϵ and n will be required to define properly. One can define, $\epsilon \equiv u_\mu u_\nu T^{\mu\nu}$ and $n \equiv u_\mu N^\mu$. Note that by doing so we are matching the non-equilibrium energy and particle densities to the corresponding equilibrium quantities. This leads the dissipative quantities to be constrained by the following equations,

$$u_\mu u_\nu \delta T^{\mu\nu} = 0, \tag{2.46}$$

$$u_\mu \nu^\mu = 0. \tag{2.47}$$

These are called matching conditions. Owing to the matching conditions, the $\Delta T^{\mu\nu}$ can be decomposed in terms of its irreducible components i.e. a scalar, a vector and a traceless symmetric second rank tensor as follows,

$$\delta T^{\mu\nu} = \underbrace{\pi^{\mu\nu} - \Pi\Delta^{\mu\nu}}_{\delta T^{\mu\nu}_{vis}} + \underbrace{W^{\mu}u^{\nu} + W^{\nu}u^{\mu}}_{\delta T^{\mu\nu}_{heat}}, \tag{2.48}$$

where, $W^{\mu} = q^{\mu} + \frac{(\epsilon+p)}{n}\nu^{\mu}$ is the net energy flow. Due to matching conditions (Eq.(2.46-2.47)), the dissipative fluxes should satisfy the following conditions,

$$u_{\mu}\pi^{\mu\nu} = 0,\ u_{\mu}W^{\mu} = 0,\ u_{\mu}q^{\mu} = 0,\ u_{\mu}\nu^{\mu} = 0. \tag{2.49}$$

Further, $\pi^{\mu\nu}$ needs to be traceless which imposes additional constraints given by the following equation,

$$\pi^{\alpha}_{\alpha} = 0, \quad \Delta_{\mu\nu}\pi^{\mu\nu} = 0 \tag{2.50}$$

All the irreducible quantities appearing in the tensor decomposition of $T^{\mu\nu}$, N^{μ} and S^{μ} can be defined as,

$$\Pi \equiv -P - \frac{1}{3}\Delta^{\alpha\beta}T^{\alpha\beta} \qquad \textit{Bulk Pressure} \tag{2.51}$$

$$\pi^{\mu\nu} \equiv \Delta^{\mu\nu}_{\alpha\beta}T^{\alpha\beta} \qquad \textit{Stress tensor} \tag{2.52}$$

$$W^{\mu} \equiv u_{\alpha}T^{\alpha\lambda}\Delta^{\nu}_{\lambda} \qquad \textit{Energy Flow} \tag{2.53}$$

$$\nu^{\mu} \equiv \Delta^{\mu}_{\nu}J^{\nu} \qquad \textit{net charge Flow} \tag{2.54}$$

$$\phi^{\mu} \equiv \Delta^{\mu}_{\nu}S^{\nu} \qquad \textit{Entropy Flux} \tag{2.55}$$

where, $\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}(\Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta} + \Delta^{\mu}_{\beta}\Delta^{\nu}_{\alpha}) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$ is traceless and double symmetric (symmetric in indices μ, ν and α, β) projector which is orthogonal to u^{μ}. Now, with the dissipative corrections the conservation equations can be written as,

$$
\begin{aligned}
u_\nu \partial_\mu T^{\mu\nu} &= D\epsilon + (\epsilon + p + \Pi)\nabla_\mu u^\mu - \pi_{\mu\nu}\nabla^{\langle\mu}u^{\nu\rangle} + \nabla_\mu W^\mu \\
&\quad -2W^\mu D u_\mu = 0, \qquad (2.56)\\
\Delta^\alpha_\nu \partial_\mu T^{\mu\nu} &= (\epsilon + p + \Pi)Du^\alpha - \nabla^\alpha(p+\Pi) + \Delta^{\alpha\nu}\nabla^\sigma\pi_{\nu\sigma} - \pi^{\alpha\nu}Du_\nu \\
&\quad +\Delta^{\alpha\nu}DW_\nu + 2W^{(\alpha}\nabla_\nu u^{\nu)} = 0 \qquad (2.57)\\
\partial_\mu J^\mu_B &= Dn_B + n_B\nabla_\mu u^\mu + \partial_\mu \nu^\mu = 0. \qquad (2.58)
\end{aligned}
$$

Note that here we have only 5 equations written above and 14 unknowns n_B, ϵ, Π, W^μ, $\pi^{\mu\nu}$ and u^μ. Therefore, 9 additional equations for dissipative fluxes are required to close the system of equations. These equations can be obtained from extending the equilibrium entropy four-current to the non-equilibrium case [169–172] as follows,

$$
S^\mu = p(\alpha,\beta)\beta^\mu - \alpha J^\mu + \beta_\nu T^{\mu\nu} + Q^\mu(\delta J^\mu, \delta T^{\mu\nu}), \qquad (2.59)
$$

where, α, β's are local equilibrium parameters as defined earlier and $P(\alpha,\beta)$ is the corresponding equilibrium pressure which should satisfy the equilibrium thermodynamic relations given in Eq.(2.39). Q^μ is a function of the non-equilibrium corrections δJ^μ and $\delta T^{\mu\nu}$ to the equilibrium $J^\mu_{(0)}$ and $T^{\mu\nu}_{(0)}$. Using the thermodynamic relation given in Eq.(2.34) , the above equation can be written as,

$$
S^\mu = su^\mu - \frac{\mu}{T}\nu^\mu + \frac{W^\mu}{T} + Q^\mu. \qquad (2.60)
$$

The above equation represents entropy of the dissipative system having a non-equilibrium correction to its equilibrium value $S^\mu_{(0)} = su^\mu$ due to ν^μ, W^μ and Q^μ. Now taking the divergence of out-of-equilibrium current S^μ as given in Eq.(2.59) and using thermodynamic relation Eq.(2.39) and conservation laws Eq.(2.1-2.2) one can obtain:

$$
\partial_\mu S^\mu = -(\delta J^\mu)\partial_\mu\alpha + \delta T^{\mu\nu}\partial_\mu\beta_\nu + \partial_\mu Q^\mu. \qquad (2.61)
$$

Now, substituting for δJ^μ from Eq. (2.44) and $\delta T^{\mu\nu}$ from Eq.(2.43) and using the constraint on dissipative fluxes i.e. $\pi^{\mu\nu}u_\nu = \nu^\mu u_\mu = W^\mu u_\mu = \Delta^{\mu\nu}u_\nu = 0$, we can write the above equation as,

$$\partial_\mu S^\mu = -\nu^\mu \nabla_\mu \alpha + \frac{1}{T}\pi^{\mu\nu}\nabla_{\langle\nu}u_{\mu\rangle} - \frac{1}{T}\Pi\nabla^\alpha u_\alpha + W^\mu\left(\nabla_\mu\left(\frac{1}{T}\right) + \frac{1}{T}Du_\nu\right) + \partial_\mu Q^\mu. \tag{2.62}$$

Now, it is important to note that in the case of an ideal fluid the local rest frame was defined in which there is no net particle or energy flow. In the case of dissipative fluid it is not possible to define such a frame due to the net particle and energy flow,. However, there are two choices for the frames can be made, one is due to Eckart and the other one due to Landau and Lifshitz.

In the Eckart frame velocity u^μ is defined by the particle flow i.e.

$$u^\mu = \frac{N^\mu}{\sqrt{N^\mu N_\mu}} \Rightarrow \nu^\mu = 0 \Rightarrow W^\mu = q^\mu \tag{2.63}$$

In the Landau-Lifshitz frame u^μ is defined by energy flow i.e.

$$u^\mu = \frac{T^\mu_\nu u^\nu}{\sqrt{u^\nu T^\mu_\nu T_{\mu\alpha}u^\alpha}} \tag{2.64}$$

$$\Rightarrow u_\nu T^{\mu\nu} = \epsilon u^\mu \Rightarrow W^\mu = 0 \tag{2.65}$$

Therefore, the divergence of four-entropy current as given in Eq.(2.62) can be written as:

1. In Eckart Frame:

$$\partial_\mu S^\mu = \frac{1}{T}\pi^{\mu\nu}\nabla_{\langle\nu}u_{\mu\rangle} - \frac{1}{T}\Pi\nabla^\alpha u_\alpha - \frac{q^\mu}{T}\left(\frac{\nabla_\mu T}{T} - Du_\nu\right) + \partial_\mu Q^\mu, \tag{2.66}$$

2. In Landau-Lifshitz Frame:

$$\partial_\mu S^\mu = -\nu^\mu\nabla_\mu\alpha + \frac{1}{T}\pi^{\mu\nu}\nabla_{\langle\nu}u_{\mu\rangle} - \frac{1}{T}\Pi\nabla^\alpha u_\alpha + \partial_\mu Q^\mu. \tag{2.67}$$

Specification of Q^μ: Specification of Q^μ in terms of dissipative quantities like $\pi^{\mu\nu}$, Π, W^μ and ν^μ can lead us to NS or MIS hydrodynamics. We can fix the form of Q^μ by finding the entropy flux using Eq.(2.59) as,

$$\phi^\mu = \Delta^\mu_\nu S^\nu = -\alpha\nu^\mu + \frac{W^\mu}{T} + \Delta^\mu_\nu Q^\nu. \tag{2.68}$$

Note that in both Eckart and Landau-Lifshitz theories, Q^μ was assumed to be first order in the dissipative quantities $\pi^{\mu\nu}$, Π, W^μ and ν^μ. But, due to constraint on dissipative quantities (as given by Eq.(2.49)) if we keep Q^μ to be first order in the dissipative quantities then it can be seen from above equation that Q^μ term in the expression of entropy flux vanishes. Therefore, it is not possible to have first order dissipative corrections in the entropy current. Hence, in this case the four divergence of entropy current can be written as,

1. In Eckart Frame:

$$\partial_\mu S^\mu = \frac{1}{T}\pi^{\mu\nu}\nabla_{\langle\nu}u_{\mu\rangle} - \frac{1}{T}\Pi\nabla^\alpha u_\alpha - \frac{q^\mu}{T}\left(\frac{\nabla_\mu T}{T} - Du_\nu\right) \geq 0, \tag{2.69}$$

2. In Landau-Lifshitz Frame:

$$\partial_\mu S^\mu = -\nu^\mu\nabla_\mu\alpha + \frac{1}{T}\pi^{\mu\nu}\nabla_{\langle\nu}u_{\mu\rangle} - \frac{1}{T}\Pi\nabla^\alpha u_\alpha \geq 0. \tag{2.70}$$

The above inequalities can only be satisfied if we have,

1. In Eckart Frame:

$$\pi^{\mu\nu} \equiv 2\eta\nabla^{\langle\nu}u^{\mu\rangle}, \tag{2.71}$$

$$\Pi = -\zeta\nabla^\alpha u_\alpha, \tag{2.72}$$

$$q^\mu = \lambda T\left(\frac{\nabla_\mu T}{T} - Du_\nu\right), \tag{2.73}$$

2. In Landau-Lifshitz Frame:

$$\pi^{\mu\nu} \equiv 2\eta\nabla^{\langle\nu}u^{\mu\rangle}, \tag{2.74}$$

$$\Pi = -\zeta\nabla^\alpha u_\alpha, \tag{2.75}$$

$$-h\nu^\mu = -\lambda T^2 h^{-1}\nabla^\mu\alpha = q^\mu. \tag{2.76}$$

Thus, we have in both the frames,

$$\partial_\mu S^\mu = \frac{\Pi^2}{\zeta T} - \frac{q^\mu q_\mu}{\lambda T^2} + \frac{\pi^{\mu\nu}\pi_{\mu\nu}}{2\eta T} \geq 0. \tag{2.77}$$

Note that in LRF, orthogonality conditions $q^\mu u_\mu = 0$ and $\nu^\mu u_\mu = 0$ imply $q^\mu q_\mu < 0$ and $\nu^\mu \nu_\mu < 0$. Hence the above inequality is guaranteed to be satisfied if the coefficients ζ (bulk viscosity), η (shear viscosity), λ (thermal conductivity) are greater than or equal to zero. The Eqs.(2.56-2.58) along with an appropriate EOS and the equations for dissipative fluxes Eq.(2.71-2.73) or Eq.(2.74-2.76) represent the Navier-Stokes hydrodynamics.

2.3.1.2 Müller, Israel and Stewart(MIS) Hydrodynamics:

NS hydrodynamics was derived from simplest possible assumptions due to Eckart and Landau-Lifshitz that Q^μ is linear in the dissipative quantities. But due to such an assumption the term which are necessary to provide the causality and stability is no longer exists. Therefore, one has to go beyond the first order. Getting motivated by the pioneering work of Müller [169], Israel and Stewart [170–172] gave the most general form of Q^μ in terms of second order in dissipative quantities $\pi^{\mu\nu}$, Π, W^μ and ν^μ as follows,

$$\begin{aligned} Q^\mu &= -\frac{u^\mu}{T}\left(\beta_0\Pi^2 - \left(\beta_1 q^\nu q_\nu + \frac{W^\nu W_\nu}{(\epsilon+p)}\right) + \beta_2\pi^{\mu\nu}\pi_{\mu\nu}\right) \\ &- \frac{1}{T}\left(\left(\alpha_0 q^\mu + \frac{W^\nu}{(\epsilon+p)}\right)\Pi - \left(\alpha_1 q_\nu + \frac{W_\nu}{(\epsilon+p)}\right)\pi^{\mu\nu}\right) \end{aligned} \tag{2.78}$$

where, $\beta's \geq 0$ are the thermodynamic coefficients which accounts for the second order dissipative contribution to entropy four current due to Π, $\pi^{\mu\nu}$ and q^μ. While $\alpha's$ are contribution due to the coupling of q^μ with Π and $\pi^{\mu\nu}$. Using above equation in Eq.(2.60), the expression for four entropy current can be written as,

$$\begin{aligned} S^\mu &= su^\mu - \frac{\mu}{T}\nu^\mu + \frac{W^\mu}{T} - \frac{u^\mu}{T}\left(\beta_0\Pi^2 - \left(\beta_1 q^\nu q_\nu + \frac{W^\nu W_\nu}{(\epsilon+p)}\right) + \beta_2\pi^{\mu\nu}\pi_{\mu\nu}\right) \\ &- \frac{1}{T}\left(\left(\alpha_0 q^\mu + \frac{W^\nu}{(\epsilon+p)}\right)\Pi - \left(\alpha_1 q_\nu + \frac{W_\nu}{(\epsilon+p)}\right)\pi^{\mu\nu}\right). \end{aligned} \tag{2.79}$$

Note that it can be seen from the above Eq.(2.78) that $u_\mu Q^\mu \leq 0$ which guarantees that the entropy density remains maximum in the equilibrium. Note that the final expression for the four-entropy current in Eckart and Landau-Lifshitz frame can be found by choosing $\nu^\nu = 0$ and $W^\mu = 0$ respectively.

1. Choice of Eckart Frame: In Eckart frame $W^\mu = q^\mu$, the expression of Q^μ will look like as,

$$\begin{aligned} Q^\mu &= -\frac{u^\mu}{2T}\left(\beta_0\Pi^2 - \bar{\beta}_1 q^\nu q_\nu + \beta_2\pi^{\alpha\beta}\pi_{\alpha\beta}\right) \\ &- \frac{1}{T}\left(\bar{\alpha}_0 q^\mu\Pi - \bar{\alpha}_1 q_\nu\pi^{\mu\nu}\right) \end{aligned} \tag{2.80}$$

where $\bar{\alpha_0} = \alpha_0 + \frac{1}{(\epsilon+p)}$, $\bar{\beta_1} = \beta_1 + \frac{1}{(\epsilon+p)}$ and $\alpha_1 = \alpha_1 + \frac{1}{(\epsilon+p)}$. Now, taking the divergence of Q^μ and using the Eq.(2.66), one can get the following equation for the divergence of four-entropy current,

$$\begin{aligned} T\partial_\mu S^\mu = &-\Pi\left[\partial_\mu u^\mu + \beta_0 D\Pi + \frac{1}{2}T\partial_\mu\left(\frac{\beta_0}{T}u^\mu\right)\Pi + \bar{\alpha}_0\nabla_\mu q^\mu\right] \\ &- q^\mu\left[\nabla_\mu lnT - Du_\mu - \bar{\beta}_1 Dq_\mu - \frac{1}{2}T\partial_\nu\left(\frac{\bar{\beta}_1}{T}u^\nu\right)q_\mu - \bar{\alpha}_1\partial_\nu\pi^\nu_\mu + \bar{\alpha}_0\partial_\mu\Pi\right] \\ &+ \pi^{\mu\nu}\left[\sigma_{\mu\nu} - \beta_2 D\pi_{\mu\nu} - \frac{1}{2}T\partial_\lambda\left(\frac{\beta_2}{T}u^\lambda\right)\pi_{\mu\nu} + \bar{\alpha}_1\nabla_{\langle\nu}q_{\mu\rangle}\right]. \end{aligned} \tag{2.81}$$

The second law of thermodynamics i.e. $\partial_\mu S^\mu \geq 0$ can be ensured by writing the above equation of the form of Eq.(2.77) which yields,

$$\partial_\mu u^\mu + \beta_0 D\Pi + \frac{1}{2}T\partial_\mu\left(\frac{\beta_0}{T}u^\mu\right)\Pi + \bar{\alpha}_0\nabla_\mu q^\mu = -\frac{\Pi}{\zeta}, \tag{2.82}$$

$$\nabla_\mu lnT - Du_\mu - \bar{\beta}_1 Dq_\mu - \frac{1}{2}T\partial_\nu\left(\frac{\bar{\beta}_1}{T}u^\nu\right)q_\mu - \bar{\alpha}_1\partial_\nu\pi^\nu_\mu + \bar{\alpha}_0\partial_\mu\Pi = \frac{q_\mu}{\lambda T}, \tag{2.83}$$

$$\sigma_{\mu\nu} - \beta_2 D\pi_{\mu\nu} - \frac{1}{2}T\partial_\lambda\left(\frac{\beta_2}{T}u^\lambda\right)\pi_{\mu\nu} + \bar{\alpha}_1\nabla_{\langle\nu}q_{\mu\rangle} = \frac{\pi_{\mu\nu}}{2\eta}. \tag{2.84}$$

Eqs.(2.82-2.84) can be written as,

$$\tau_\Pi D\Pi + \Pi = -\zeta\nabla_\alpha u^\alpha - l_{\Pi q}\nabla_\mu q^\mu - \left(\frac{1}{2}T\zeta\partial_\mu\left(\frac{\tau_\Pi u^\mu}{\zeta T}\right)\right)\Pi, \tag{2.85}$$

$$
\begin{aligned}
\tau_q D q_\mu + q_\mu &= \lambda(\nabla_\mu lnT - Du_\mu) + l_{q\Pi}\nabla_\mu \Pi - l_{q\pi}\nabla_\nu \pi^\nu_\mu \\
&+ \frac{1}{2}\lambda T^2 \partial_\nu \left(\frac{\tau_\pi u^\nu}{\lambda T^2}\right) q_\mu, \qquad (2.86) \\
\tau_\pi D\pi_{\mu\nu} + \pi_{\mu\nu} &= 2\eta\sigma_{\mu\nu} + l_{\pi q}\nabla_{\langle\mu} q_{\nu\rangle} - \eta T \partial_\lambda \left(\frac{\tau_\pi u^\lambda}{2\eta T}\right)\pi_{\mu\nu}, \qquad (2.87)
\end{aligned}
$$

where, $\tau_\Pi = \zeta\beta_0$, $\tau_q = \lambda T\bar{\beta}_1$, $\tau_\pi = 2\eta\beta_2$ are identified as the relaxation times and $l_{\Pi q} = \zeta\bar{\alpha}_0$, $l_{q\Pi} = \lambda T\bar{\alpha}_0$, $l_{q\pi} = \lambda T\bar{\alpha}_1$, $l_{\pi q} = 2\eta\bar{\alpha}_1$ as coupling constants corresponding to Eckart-frame.

2. Choice of Landau-Lifshitz Frame: In the Landau-Lifshitz frame $W^\mu = 0$. Therefore, Q^μ will have the following form,

$$
\begin{aligned}
Q^\mu &= -\frac{u^\mu}{T}\left(\beta_0\Pi^2 - \beta_1 q^\nu q_\nu + \beta_2 \pi^{\mu\nu}\pi_{\mu\nu}\right) \\
&- \frac{1}{T}\left(\alpha_0 q^\mu \Pi - \alpha_1 q_\nu \pi^{\mu\nu}\right). \qquad (2.88)
\end{aligned}
$$

Now taking the divergence of Q^μ and substituting $\nu^\mu = -h^{-1}q^\mu$ in the Eq.(2.67) one can get,

$$
\begin{aligned}
T\partial_\mu S^\mu = &-\Pi\left[\partial_\mu u^\mu + \beta_0\dot{\Pi} + \frac{1}{2}T\partial_\mu\left(\frac{\beta_0}{T}u^\mu\right)\Pi + \alpha_0\nabla_\mu q^\mu\right] \\
&- q^\mu\left[-h^{-1}T\nabla_\mu\left(\frac{\mu}{T}\right) - \beta_1\dot{q}_\mu - \frac{1}{2}T\partial_\nu\left(\frac{\beta_1}{T}u^\nu\right)q_\mu - \alpha_1\partial_\nu\pi^\nu_\mu + \alpha_0\partial_\mu\Pi\right] \\
&+ \pi^{\mu\nu}\left[\sigma_{\mu\nu} - \beta_2\dot{\pi}_{\mu\nu} - \frac{1}{2}T\partial_\lambda\left(\frac{\beta_2}{T}u^\lambda\right)\pi_{\mu\nu} + \alpha_1\nabla_{\langle\nu}q_{\mu\rangle}\right]. \qquad (2.89)
\end{aligned}
$$

Now, imposing the second law of thermodynamics i.e. $\partial_\mu S^\mu \geq 0$, where $\partial_\mu S^\mu$ is given by Eq.(2.77), one can get,

$$
\begin{aligned}
\partial_\mu u^\mu + \beta_0 D\Pi + \frac{1}{2}T\partial_\mu\left(\frac{\beta_0}{T}u^\mu\right)\Pi + \alpha_0\nabla_\mu q^\mu &= -\frac{\Pi}{\zeta}, \qquad (2.90) \\
-h^{-1}T\nabla_\mu\left(\frac{\mu}{T}\right) - \beta_1 Dq_\mu - \frac{1}{2}T\partial_\nu\left(\frac{\beta_1}{T}u^\nu\right)q_\mu - \alpha_1\partial_\nu\pi^\nu_\mu + \alpha_0\partial_\mu\Pi &= \frac{q_\mu}{\lambda T}, \qquad (2.91) \\
\sigma_{\mu\nu} - \beta_2 D\pi_{\mu\nu} - \frac{1}{2}T\partial_\lambda\left(\frac{\beta_2}{T}u^\lambda\right)\pi_{\mu\nu} + \alpha_1\nabla_{\langle\nu}q_{\mu\rangle} &= \frac{\pi_{\mu\nu}}{2\eta}. \qquad (2.92)
\end{aligned}
$$

Above equations can also be written as,

$$\tau_\Pi D\Pi + \Pi = -\zeta\nabla_\alpha u^\alpha - l_{\Pi q}\nabla_\mu q^\mu - \left(\frac{1}{2}T\zeta\partial_\mu\left(\frac{\tau_\Pi u^\mu}{\zeta T}\right)\right)\Pi, \tag{2.93}$$

$$\begin{aligned}\tau_q Dq_\mu + q_\mu &= -\lambda T^2 h^{-1}\nabla_\mu\left(\frac{\mu}{T}\right) + l_{q\Pi}\nabla_\mu\Pi - l_{q\pi}\nabla_\nu\pi^\nu_\mu \\ &+ \frac{1}{2}\lambda T^2\partial_\nu\left(\frac{\tau_\pi u^\nu}{\lambda T^2}\right)q_\mu,\end{aligned} \tag{2.94}$$

$$\tau_\pi D\pi_{\mu\nu} + \pi_{\mu\nu} = 2\eta\sigma_{\mu\nu} + l_{\pi q}\nabla_{\langle\mu}q_{\nu\rangle} - \eta T\partial_\lambda\left(\frac{\tau_\pi u^\lambda}{2\eta T}\right)\pi_{\mu\nu}, \tag{2.95}$$

here, $\tau_\Pi = \zeta\beta_0$, $\tau_q = \lambda T\beta_1$, $\tau_\pi = 2\eta\beta_2$ are identified as the relaxation times and $l_{\Pi q} = \zeta\alpha_0$, $l_{q\Pi} = \lambda T\alpha_0$, $l_{q\pi} = \lambda T\alpha_1$, $l_{\pi q} = 2\eta\alpha_1$ as coupling constants. From the physics point of view, the relaxation time represents the time taken by the corresponding dissipative flux to relax to its steady state value. Relaxation time can not be determined within the irreversible thermodynamics framework. A kinetic theory approach is more general and can be used for determining the relaxation times. Note that the introduction of the relaxation time brings the hyperbolicity and it ensures the causal propagation of the small perturbations in a physical system [164]. Here, we note that sometimes the term with factor 1/2 on the right hand side of Eqs.(2.85-2.87) and (2.93-2.95) are ignored by arguing that the gradient of thermodynamic quantities are small [175, 243].

A combination of Eqs.(2.56-2.58) with an appropriate EoS and the set of equations for dissipative quantities given by Eq.(2.85-2.87) describes the complete MIS hydrodynamics in the Eckart frame. For Landau-Lifshitz frame, the equations for the dissipative quantities are given by Eq.(2.93-2.95). The limit $\tau_\Pi, \tau_q, \tau_\pi, l_{\Pi q}, l_{q\Pi}, l_{q\pi}, l_{\pi q} \to 0$ is the first order limit, which corresponds to the Navier-Stokes case.

2.3.2 Causal viscous hydrodynamics from Kinetic Theory

2.3.2.1 Israel Stewart hydrodynamics:

Kinetic theory is used to describe a system of extremely large number of particles which are in constant, random and rapid motion. It assumes that all the particles have same masses and they are so small that the total volume of the individual particles when added together, is negligible as compared to the system size. The system is so dilute that their interactions can be restricted to 2$\leftrightarrow$2 collisions. Due to a large number of particles, a statistical description was used by introducing a single particle distribution function $n_p(x,p)$ such that the quantity $n_p(x,p)\Delta^3 x\Delta^3 p$ represents the average number of particles at time t, in the phase space volume $\Delta^3 x\Delta^3 p$, having position close to $\vec{x}$ and momenta close to $\vec{p}$.

The evolution of the distribution function is given by Boltzmann equation which for a relativistic particle can be written as,

$$p^\mu \partial_\mu n_p(x,p) = C[n_p], \tag{2.96}$$

where, $p = p^\mu = (p^0 = \sqrt{(\vec{p})^2 + m^2}, \vec{p})$ is the four momentum of the particle which satisfies the on-shell condition i.e. $p^\mu p_\mu = m^2$, m is the rest mass. $C[f]$ is the collisional terms. For a 2$\leftrightarrow$2 collisions $C[f]$ can be written as,

$$C[n_p] = \frac{1}{2}\int d^3\mathbf{p_2} d^3\mathbf{p'_1} d^3\mathbf{p'_2} M_{p_1 p_2 \to p'_1 p'_2}(n_p(x,p'_1,t)n_p(x,p'_2,t) - n_p(x,p_1,t)n_p(x,p_2,t)). \tag{2.97}$$

For a system in equilibrium, $n_p(\vec{x},\vec{p},t) = n_{p_0}(\vec{p})$ is stationary. In this case, the right hand side of Boltzmann equation (Eq.(2.96)) will vanish which implies that $C[n_{p_0}] = 0$. Using this distribution function, one can calculate the particle density and particle current by the following formulae,

$$n(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} n_p(x,p), \quad \vec{j}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \vec{v} n_p(x,p), \tag{2.98}$$

where, $\vec{v} = \frac{\vec{p}}{p^0}$ is the particle velocity. Therefore, particle four-current $j^\mu = (n,\vec{j})$ can be written as,

$$j^\mu = \int \frac{d^3\mathbf{p}}{(2\pi)^3 p^0} p^\mu n_p(x,p). \tag{2.99}$$

Now the average energy density which is represented by T^{00} can be written as,

$$T^{00}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} p^0 n_p(x,p). \tag{2.100}$$

Flow of energy can be written as,

$$T^{0i}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} p^0 v^i n_p(x,p) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} p^i n_p(x,p) \tag{2.101}$$

It is clear from above equation that energy flow is equal to momentum density. Now, knowing the momentum density, one can write the momentum flow as,

$$T^{ij}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} p^i v^j n_p(x,p). \tag{2.102}$$

Combing the Eqs.(2.100-2.102), one can write the complete energy momentum tensor as follows,

$$T^{\mu\nu} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 p^0} p^\mu p^\nu n_p(x,p). \tag{2.103}$$

When the system is out of equilibrium, the distribution function can be written as,

$$n_p = n_{p_0}(1+\delta n_p), \tag{2.104}$$

where, $n_{p_0} \sim [\exp(\beta_0 u \cdot p)]^{-1}$ is the equilibrium, distribution function and $\delta n_p \ll 1$ is the out of equilibrium corrections to the distribution function. $\delta n_p(x,p)$ is a function of space-time and momentum and can be written in a most general form as,

$$\delta n_p(x,p) = \epsilon(x) + \epsilon_\mu p^\mu + \epsilon_{\mu\nu} p^\mu p^\nu + O(p^3) + ... \tag{2.105}$$

Substituting Eq.(2.105) into the Eqs.(2.99,2.103), separating out the non-equilibrium part and imposing matching condition as given in Eqs.(2.46-2.46) as well as using Eqs.(2.51-2.53) one finds the form of $\epsilon(x)$, ϵ_μ and $\epsilon_{\mu\nu}$ in the Eckart frame as,

$$\epsilon(x) = A_0\Pi, \tag{2.106}$$

$$\epsilon_\nu(x) = A_1 u_\nu \Pi - B_0 q_\nu, \tag{2.107}$$

$$\epsilon_{\mu\nu}(x) = A_2(3u_\mu u_\nu - \Delta_{\mu\nu})\Pi - B_1 u_{(\mu} q_{\nu)} + C_0 \pi_{\mu\nu}. \tag{2.108}$$

The values of thermodynamic coefficients A_i, B_i and C_i are given in [244]. Thus the non-equilibrium part of N^μ and $T^{\mu\nu}$ will have 9 (=1+3+5) hydrodynamic variables as $\{\Pi, q^\mu, \pi^{\mu\nu}\}$. While total N^μ and $T^{\mu\nu}$ will have 14 (=1+1+3+1+3+5) variables as $\{n, \epsilon, u^\mu, \Pi, q^\mu, \pi^{\mu\nu}\}$. Thus we need 14 equations to have a complete hydrodynamics, out of these five will be provided by conservation law $\partial_\mu N^\mu = 0$ and $\partial_\mu T^{\mu\nu} = 0$, which can be obtained by taking the moment of the Boltzmann equation as follows,

$$\int d\Gamma p^\mu \partial_\mu n_p(x,p) = \int d\Gamma C[n_p],$$
$$\int d\Gamma p^\mu p^\nu \partial_\mu n_p(x,p) = \int d\Gamma p^\nu C[n_p],$$

while, 9 additional equations for $\{\Pi, q^\mu, \pi^{\mu\nu}\}$ can be derived by the following equation,

$$\int d\Gamma p^\mu p^\nu p^\lambda \partial_\mu n_p(x,p) = \int d\Gamma p^\nu p^\lambda C[n_p] \tag{2.109}$$

where, $\int d\Gamma = \int \frac{d^3\mathbf{p}}{(2\pi)^3 p^0}$. For a simple case if we neglect the bulk pressure Π and heat flow q^μ, the $\delta n_p(x,p)$ will read,

$$\delta n_p(x,p) = \epsilon_{\mu\nu} p^\mu p^\nu \tag{2.110}$$

In this case we need only 5 additional equations, which can be easily obtained by taking a relaxation time approximation for the collision term $C[n_p] = -p^\mu u^\mu \frac{n_p - n_{p0}}{\tau_\pi}$ and carrying out integration over $d\Gamma$ in equation Eq.(2.109). Following MIS equation for $\pi^{\mu\nu}$ has been obtained in Ref. [245],

$$\tau_\pi \Delta^\mu_\alpha \Delta^\nu_\beta D\pi^{\alpha\beta} + \pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle} - \tau_\pi \left(\frac{4}{3}\pi^{\mu\nu}\nabla_\alpha u^\alpha - 2\pi^{\lambda(\mu}\omega^{\nu)}_\lambda + \frac{\pi^{\lambda\langle\mu}\pi^{\nu\rangle}_\lambda}{2\eta} \right), \tag{2.111}$$

where, $\omega_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha)$ is the vorticity term, which can not be obtained by the thermodynamic based arguments provided earlier. Relaxation time τ_π is related to the shear viscosity. For a massless Boltzmann gas it can be shown that $\tau_\pi = \frac{6\eta}{Ts}$. One can clearly see that the above equation will reduce to Navier-Stokes form if $\tau_\pi \to 0$.

2.3.2.2 Extension to Third order:

An extension of MIS equation to the third order has been done by A. El. et. al. [177]. They have also used Grad's 14 moment method to express the off equilibrium distribution function as given by Eqs.(2.104-2.108), but neglected bulk pressure and heat flow part from it. In contrast to MIS case (where evolution equations for dissipative fluxes have been derived from the moments of Boltzmann equation) they have derived the evolution equations using the entropy principle. Basically, they have used expression of four-entropy current S^μ in terms of single particle distribution function. Such an expression was generalized from Boltzmann H-function and given as follows,

$$S^\mu = -\int \frac{d^3\mathbf{p}}{(2\pi)^3 p^0} p^\mu n_p (\ln n_p - 1). \tag{2.112}$$

After expanding $\ln n_p$ to the third order in $\delta n_p = C_0 \pi_{\mu\nu} p^\mu p^\nu$, they have found the following expression for the non-equilibrium entropy four-current,

$$S^\mu = s u^\mu - \frac{\beta_2}{2T} \pi_{\alpha\beta} \Pi^{\alpha\beta} u^\mu + \alpha \frac{\beta_2^2}{T} \pi_{\alpha\beta} \pi^\alpha_\sigma \pi^{\beta\sigma} u^\mu, \tag{2.113}$$

where, $s_0 = -\int \frac{d^3\mathbf{p}}{(2\pi)^3} n_{p_0} (\ln n_{p_0} - 1)$, $\alpha = -\frac{8}{9}$ and $\beta_2 = \frac{9}{4\epsilon}$. The last term on the right hand side of the above equation represents the third order correction to the equation of entropy. In order to fulfill the requirement of the maximal entropy at equilibrium, the third order term must satisfy the condition, $\alpha \frac{\beta_2^2}{T} \pi_{\alpha\beta} \pi^\alpha_\sigma \pi^{\beta\sigma} u^\mu \leq 0$. Divergence of the entropy four-current can be written as,

$$\begin{aligned} \partial_\mu S^\mu =& \frac{1}{T} \pi_{\alpha\beta} \sigma^{\alpha\beta} - \pi_{\alpha\beta} \pi^{\alpha\beta} \partial_\mu \left(\frac{\beta_2}{2T} u^\mu \right) - \frac{\beta_2}{T} \pi_{\alpha\beta} D\pi^{\alpha\beta} \\ &+ \alpha \partial_\mu \left(\frac{\beta_2^2}{T} u^\mu \right) \pi_{\alpha\beta} \pi^\alpha_\sigma \pi^{\beta\sigma} + 3\tau_\pi \theta \alpha \frac{\beta_2^2}{T} \pi_{\alpha\beta} \pi^\alpha_\sigma D\pi^{\beta\sigma} \geq 0. \end{aligned} \tag{2.114}$$

Here, the Knudsen number($=\tau_\pi \theta$) is required to satisfy the condition, $\tau_\pi \theta \ll 1$, for the validity of hydrodynamic approach. Note that here a little trickery has been followed to include the relaxation time term in the above equation (a detailed description for this is given in [177]). Now, for the condition $T\partial_\mu S^\mu \geq 0$ to be satisfied one must have,

$$\partial_\mu s^\mu = \frac{1}{2\eta T}\pi^{\mu\nu}\pi_{\mu\nu}, \tag{2.115}$$

which implies that the form of shear viscous tensor $\pi^{\alpha\beta}$ should be given by,

$$\begin{aligned}\pi^{\alpha\beta} &= 2\eta T\left[\frac{1}{T}\sigma^{\alpha\beta} - \pi^{\alpha\beta}\partial_\mu\left(\frac{\beta_2}{2T}u^\mu\right) - \frac{\beta_2}{T}D\pi^{\alpha\beta} + \alpha\partial_\mu\left(\frac{\beta_2^2}{T}u^\mu\right)\pi^\alpha_\sigma\pi^{\beta\sigma}\right.\\ &\left.\quad +3\tau_\pi\theta\alpha\frac{\beta_2^2}{T}\pi^\alpha_\sigma D\pi^{\beta\sigma}\right].\end{aligned} \tag{2.116}$$

Since $\tau_\pi\theta \sim \frac{\tau_\pi}{\tau}$ is of the same order as $\frac{\pi^{\alpha\beta}}{T^4}$ when τ is large, the last term in the above equation is a fourth order term [177]. Thus neglecting the last term, one can write the above equation as,

$$D\pi^{\alpha\beta} = -\frac{\pi^{\alpha\beta}}{\tau_\pi} + \frac{\sigma^{\alpha\beta}}{\beta_2} - \pi^{\alpha\beta}\frac{T}{\beta_2}\partial_\mu\left(\frac{\beta_2}{2T}u^\mu\right) + \alpha\frac{T}{\beta_2}\partial_\mu\left(\frac{\beta_2^2}{T}u^\mu\right)\pi^\alpha_\sigma\pi^{\beta\sigma}. \tag{2.117}$$

2.3.2.3 Several other second order causal hydrodynamics:

Derivation of the second order hydrodynamics from kinetic theory is not unique, there may also exist several other alternative derivations. For example, in Ref. [179] Denicol et.al. has derived the causal dissipative hydrodynamics from kinetic theory in a completely different way. In their derivation, initially, the same definition of j^ν and T^μ as in Eqs.(2.99) has been used. Later by using decomposition of four-momentum in directions parallel and perpendicular to u^μ as given below,

$$p^\mu = (u\cdot p)u^\mu + \Delta^\mu_\nu p^{\langle\nu\rangle} \tag{2.118}$$

j^ν and T^μ has been decomposed in terms of dissipative quantities ν^μ, Π and $\pi^{\mu\nu}$ defined in terms of a single particle distribution function as follows,

$$\Pi \equiv P_0 - \frac{1}{3}\langle\Delta^{\mu\nu}p_\mu p_\nu\rangle, \quad \nu^\mu = \langle p^{\langle\mu\rangle}\rangle, \quad \pi^{\mu\nu} = \langle p^{\langle\mu}p^{\nu\rangle}\rangle, \tag{2.119}$$

where, $\langle ... \rangle \equiv \int_{d\Gamma} n_p$. $A^\mu \equiv \Delta^{\mu\nu} A_\nu$ and $A^{\mu\nu} \equiv \Delta^{\mu\nu\alpha\beta} A_{\alpha\beta}$ with $\Delta^{\mu\nu\alpha\beta} \equiv (\Delta^{\mu\alpha}\Delta^{\mu\alpha} + \Delta^{\nu\alpha}\Delta^{\mu\beta} - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta})$. Using these definitions, the evolution equation for the dissipative quantities can be found by the following equations,

$$\dot{\Pi} = -\frac{1}{3}\int d\Gamma \dot{\delta n_p}, \tag{2.120}$$

$$\dot{\nu}^{\langle\mu\rangle} = \int d\Gamma p^{\langle\mu\rangle} \dot{\delta n_p}, \tag{2.121}$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = \int d\Gamma p^{\langle\mu} p^{\nu\rangle} \dot{\delta n_p}. \tag{2.122}$$

Now, substituting $\dot{\delta n_p}$ from the Boltzmann equation (2.96) in the above equations and using the 14-moment approximation for single particle distribution function in the following form,

$$n_p = n_{p0} + n_{p0}\tilde{n}_{p0}(\lambda_\Pi \Pi + \lambda_\nu \nu_\alpha p^\alpha + \lambda_\pi \pi_{\alpha\beta} p^\alpha p^\beta), \tag{2.123}$$

where, $n_{p_0} = [\exp(\beta_0 u \cdot p - \alpha_0) + a]^{-1}$ is the equilibrium distribution function and $\tilde{n}_{p0} = 1 - a n_p$, $a = 0, 1, -1$ corresponds to Boltzmann, Fermi and Bose gas respectively. One can obtain the following equations for dissipative fluxes,

$$\begin{aligned} \dot{\Pi} &= -\frac{\Pi}{\tau_\Pi} - \beta_\Pi \theta - l\Pi\nu\partial \cdot \nu - \tau_{\pi\nu}\nu \cdot \dot{u} - \delta_{\Pi\Pi\Pi}\Pi\theta - \lambda_{\Pi\nu}\nu \cdot \nabla\alpha_0 \\ &\quad + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}, \end{aligned} \tag{2.124}$$

$$\begin{aligned} \dot{\pi}^{\langle\mu\nu\rangle} &= -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\alpha^{\langle\mu}\omega^{\nu\rangle\alpha} - \tau_{\pi\nu}\nu^{\langle\mu}\dot{u}^{\nu\rangle} + l_{\pi\nu}\nabla^{\langle\nu}\nu^{\nu\rangle} \\ &\quad - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi_\alpha^{\langle\mu}\sigma^{\nu\rangle\alpha} + \lambda_{\pi\nu}\nu^{\langle\mu}\nabla^{\nu\rangle}\alpha_0 + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}, \end{aligned} \tag{2.125}$$

$$\begin{aligned} \dot{\nu}^{\langle\mu\rangle} &= -\frac{\nu^\mu}{\tau_\nu} + \beta_\nu \nabla^\mu \alpha_0 - \nu_\nu \omega^{\nu\mu} - \delta_{\nu\nu}\nu^\mu\theta - l_{\nu\Pi}\nabla^\mu \Pi, \\ &\quad + l_{\nu\pi}\Delta^{\mu\nu}\partial_\lambda \pi_\nu^\lambda \nu^\nu + \tau_{\nu\pi}\Pi\dot{u}^\mu - \tau_{\nu\pi}\pi_\nu^\mu \dot{u}^\nu - \lambda_{\nu\nu}\nu^\nu\sigma_\nu^\mu \\ &\quad + \lambda_{\nu\Pi}\Pi\nabla^\mu\alpha_0 - \lambda_{\nu\pi}\Pi^{\mu\nu}\nabla_\nu\alpha_0, \end{aligned} \tag{2.126}$$

where, $\theta = \nabla_\alpha u^\alpha$ and $\tau's, \beta's, \delta's, \lambda's$ are the transport coefficients. Coefficient $\beta's$ were calculated and found to be,

$$\beta_\Pi = \left(\frac{1}{3} - c_s^2\right)(\epsilon_0 + P_0) - \frac{2}{9}(\epsilon_0 - 3P_0) - \frac{m^4}{9}\langle (u\cdot p)^{-2}\rangle_0,$$
$$\beta_n = \frac{2}{3\beta_0}\langle 1\rangle_0 + \frac{m^2}{3\beta_0}\langle (u\cdot p)^{-2}\rangle_0 - \frac{n_0}{\beta_0}h_0,$$
$$\beta_\pi = \frac{4}{5}P_0 + \frac{1}{15}(\epsilon_0 - 3P_0) - \frac{m^4}{15}\langle (u\cdot p)^{-2}\rangle_0, \tag{2.127}$$

where, $c_s^2 = \left(\frac{dP_0}{d\epsilon_0}\right)_{\frac{s_0}{n_0}}$ is the sound velocity. Other coefficients are yet to be determined.

In an another work by Jaiswal et. al. [178], equations for the evolution of dissipative quantities were derived by taking divergence of the entropy four-current expressed in terms of a single particle phase space distribution function. Such an expression of entropy four-current is obtained from Boltzmann's H-function and given in Eq.(2.112). The expression for the divergence is as follows,

$$\partial_\mu S^\mu = -\int d\Gamma p^\mu \left[(\partial_\mu n_p)\ln\left(\frac{n_p}{\tilde{n}_p}\right)\right]. \tag{2.128}$$

In the above equation using the Grad's 14-moment approximation as given in Eq.(2.123) one can obtain $\partial_\mu S^\mu$ in terms of dissipative quantities. The evolution equation for dissipative quantities can be found by the requirement $\partial_\mu S^\mu \geq 0$,

$$\Pi = -\zeta\Big[\theta + \beta_0\dot{\Pi} + \beta_{\Pi\Pi}\Pi\theta + \alpha_0\nabla_\mu\nu^\mu + \psi\alpha_{\nu\Pi}\nu_\mu\dot{u}^\mu + \psi\alpha_{\Pi\nu}\nu_\mu\nabla^\mu\alpha\Big], \tag{2.129}$$

$$\pi^{\mu\nu} = 2\eta\Big[\sigma^{\mu\nu} - \beta_2\dot{\pi}^{\langle\mu\nu\rangle} - \beta_{\pi\pi}\theta\pi^{\mu\nu} - \alpha_1\nabla^{\langle\mu}\nu^{\nu\rangle} - \chi\alpha_{\pi\nu}\nu^{\langle\mu}\nabla^{\nu\rangle}\alpha - \chi\alpha_{\nu\pi}\nu^{\langle\mu}\dot{u}^{\nu\rangle}\Big], \tag{2.130}$$

$$n^\mu = \lambda\Big[T\nabla^\mu\alpha - \beta_1\dot{\nu}^{\langle\mu\rangle} - \beta_{\nu\nu}\nu^\mu\theta + \alpha_0\nabla^\mu\Pi + \alpha_1\Delta^\mu_\rho\nabla_\nu\pi^{\rho\nu} + \tilde{\psi}\alpha_{\nu\Pi}\Pi\dot{u}^{\langle\mu\rangle} + \tilde{\psi}\alpha_{\Pi\nu}\Pi\nabla^\mu\alpha + \tilde{\chi}\alpha_{\pi\nu}\pi^\mu_\nu\nabla^\nu\alpha + \tilde{\chi}\alpha_{\nu\pi}\pi^\mu_\nu\dot{u}^\nu\Big], \tag{2.131}$$

where, λ, ζ and $\eta(\geq 0)$ respectively denote the coefficient of charge conductivity, bulk viscosity and shear viscosity. Coefficients $\alpha_i, \beta_i, \alpha_{XY}, \beta_{XX}$ are the additional transport coefficients which depend on β_0, α_0 as well as complicated integrals coefficient which emerge while doing the integration of Eq.(2.128) and the parameters ψ, χ along with $\tilde{\psi} = 1 - \psi$ and $\tilde{\chi} = 1 - \chi$ describe the contributions due to the cross terms of Π and $\pi^{\mu\nu}$ with ν^μ.

2.3.3 Conformal case of the second order dissipative hydrodynamics:

In addition to above, a viscous hydrodynamics framework for the conformal case has also been developed in Ref. [246], where authors have shown that the second order term in $\pi^{\mu\nu}$ can be determined by using the conformal symmetry. For this thesis, a detailed derivation of the same is not required.

2.3.4 Hydrodynamics with triangle anomalies

Triangle anomaly is a widely discussed and important phenomena in quantum field theories. It is believed that the anomalies may affect the macroscopic dynamics of the fluid. The strong support for this, in the context of strongly interacting matter, comes from the observation of anomalous "chiral magnetic currents" at LHC. Therefore, it is important to incorporate the anomalies in the relativistic hydrodynamic framework. A first order hydrodynamic framework which describes the anomaly effect was developed by Son and Surowka [232]. In their derivation they have suggested that in the presence anomalies the conservation laws can be modified as follows,

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda, \tag{2.132}$$

$$\partial_\mu J^\mu = C E^\mu B_\mu, \tag{2.133}$$

where, $E^\mu = F^{\mu\nu} u_\nu$, $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$. Non-zero right hand sides of the above equations indicate that the external field performs work on the system and anomaly. Now in the Landau-Lifshitz frame, energy momentum tensor and the current can be decomposed as follows,

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - p\Delta^{\mu\nu} + \Pi^{\mu\nu}, \tag{2.134}$$
$$J_B^\mu = n u^\mu + \nu^\mu.$$

The form of $\Pi^{\mu\nu}$ and ν^μ can be specified by using $\partial_\mu S^\mu \geq 0$. The expression for $\partial_\mu s^\mu$ can be obtained by evaluating $u_\nu \partial_\mu T^{\mu\nu} - \mu\partial_\mu J_B^\mu$ using eqs.(2.134-2.135) and the thermodynamic relation $\epsilon + p = Ts + \mu n$ as follows,

$$\partial_\mu \left(s u^\mu - \frac{\mu}{T}\nu^\mu \right) = \frac{1}{T}\Pi^{\mu\nu}\partial_\mu u_\nu - \nu^\mu \left(\partial_\mu \frac{\mu}{T} - \frac{E_\mu}{T} \right) - C\frac{\mu}{T}EB \tag{2.135}$$

Taking $C = 0$, one can define $S^\mu = \left(s u^\mu - \frac{\mu}{T}\nu^\mu \right)$ and get the following expression for $\Pi^{\mu\nu}$ and ν^μ,

$$\Pi^{\mu\nu} = 2\eta\nabla^{\langle\nu}u^{\mu\rangle} - \zeta\nabla^\alpha u_\alpha,$$
$$\nu^\mu = -\lambda T\nabla^\mu \left(\frac{\mu}{T} \right) + \lambda E^\mu.$$

However, the presence of the last term in Eq.(2.135) requires the following modification to the S^μ and ν^μ,

$$S^\mu = s u^\mu - \frac{\mu}{T}\nu^\mu - D\omega^\mu - D_B B^\mu,$$
$$\nu^\mu = -\lambda T\nabla^\mu \left(\frac{\mu}{T} \right) + \lambda E^\mu + \xi\omega^\mu + \xi_B B^\mu,$$

where the coefficient D,D_B, ξ and ξ_B are given by,

$$D = \frac{1}{3}C\frac{\mu^3}{T}, \quad D_B = \frac{1}{2}C\frac{\mu^2}{T},$$
$$\xi = C\left(\mu^2 - \frac{2}{3}\frac{n\mu^3}{\epsilon + p} \right), \quad \xi_B = C\left(\mu - \frac{1}{2}\frac{n\mu^2}{\epsilon + p} \right),$$

Later on conformal hydrodynamics with triangle anomaly was also developed [236].

2.4 Modified kinetic theory with Berry curvature and triangle anomalies

The relativistic kinetic theory framework discussed above misses the effects of triangle anomalies [209, 210] which are responsible for P and CP violations. To describe such effects, a modified kinetic theory formalism from the underline quantum field theory [229] has been developed by taking into account the Berry curvature [230]. In this section, we shall give a brief description about the derivation of chiral kinetic equation as discussed in Ref. [229]. To begin with, let us consider a single chiral fermion described by Hamiltonian $H = \sigma \cdot \hat{\mathbf{p}}$. This will satisfy the Weyl equation,

$$(\sigma \cdot \mathbf{p})\mathbf{u_p} = \pm e|p|\mathbf{u_p},$$

where, u_p is the two component spinor. $+/-$ sign corresponds to right/left handed fermions respectively. Now, parameterizing $\sigma \cdot \hat{\mathbf{p}}$ as follows,

$$\sigma \cdot \hat{\mathbf{p}} = \begin{bmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{bmatrix}, \tag{2.136}$$

we can construct the following form for the spinors,

$$u_{p+} = \begin{bmatrix} e^{-i\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}. \end{bmatrix}, \quad u_{p-} = \begin{bmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{bmatrix}. \tag{2.137}$$

With this parameterization, the two component spinors will have a non-zero Berry connection [230] defined by,

$$\mathbf{Q_p} \equiv -i\mathbf{u_p^\dagger}\nabla_\mathbf{p}\mathbf{u_P}. \tag{2.138}$$

Using the above equation, Berry curvature can be calculated as follows,

$$\mathbf{\Omega}(\mathbf{p}) \equiv \nabla_\mathbf{p} \times \mathbf{Q_p} = \pm\hat{\mathbf{p}}/2|p|^2, \tag{2.139}$$

where, $\hat{\mathbf{p}} = \vec{p}/|p|$ is a unit vector. Non-zero Berry connection ($\mathbf{Q_p}$) and curvature ($\mathbf{\Omega}(\mathbf{p})$) can be treated as the fictitious vector potential and magnetic field in the momentum space. Therefore, Berry curvature can affect the motion of chiral fermion in the momentum space and one can write the corresponding action as[247, 248],

$$S(x,p) = \int dt[(p^i + eA^i(x))\dot{x^i} - \mathbf{Q^i}(\mathbf{p})\dot{p^i} - \epsilon_p(x) - A^0(x)], \tag{2.140}$$

where, $\epsilon_p(p)$ is the quasi particle energy and $A_0, A^i(x)$ are the scalar and magnetic vector potential. Above equation can be written in a more compact form as follows,

$$S(\xi) = \int dt[\Sigma_a(\xi)\dot{\xi^a} - H(\xi)], \tag{2.141}$$

where, $\Sigma_a(\xi) = (p^i + eA_i(x), -\mathbf{Q^i}(\mathbf{p}))$, $\xi^a = (x^i, p^i)$ and $H(\xi) = \epsilon_p(p) + A^0(x)$. $\mathbf{Q^i}(\mathbf{p}) = -\mathbf{i}\mathbf{u_p^\dagger}\nabla_\mathbf{p}\mathbf{u_P}$ is the Berry connection for chiral fermion.
Now the equations of motion of the action read as,

$$\Sigma_{ab}\dot{\xi^b} = -\frac{\partial H(\xi)}{\partial \xi^a}, \tag{2.142}$$

where, $\Sigma_{ab} = \frac{\partial \Sigma_a(\xi)}{\partial \xi^b} - \frac{\partial \Sigma_b(\xi)}{\partial \xi^a}$. Further, we rewrite the above equation as,

$$\dot{\xi^a} = -(\Sigma^{-1})^{ab}\frac{\partial H(\xi)}{\partial \xi^b}. \tag{2.143}$$

Hamilton's equation of motion is,

$$\dot{\xi^a} = -\{\xi^a, H(\xi)\} = -\{\xi^a, \xi^b\}\frac{\partial H(\xi)}{\partial \xi^b}. \tag{2.144}$$

Eqs.(2.143-2.144) implies $\{\xi^a, \xi^b\} = (\Sigma^{-1})^{ab}$. Using the above equation, we can write down the explicit form of Poisson brackets for variables x^i, p^i with berry curvature as follows,

$$\{x^i, x^j\} = \frac{\epsilon_{ijk}\Omega_k}{1 + e\mathbf{B}\cdot\mathbf{\Omega_p}}, \quad \{x^i, p^j\} = -\frac{\delta_{ij} + e\Omega_i B_j}{1 + e\mathbf{B}\cdot\mathbf{\Omega_p}}, \quad \{p^i, p^j\} = -\frac{e\epsilon_{ijk}B_k}{1 + e\mathbf{B}\cdot\mathbf{\Omega_p}}, \tag{2.145}$$

where, $B^i = \epsilon^{ijk}\frac{\partial A^k}{\partial x^j}$ and $\mathbf{\Omega_p} = \nabla_\mathbf{p} \times \mathbf{Q_p}$. As a result of the modification of the Poisson Bracket, the invariant phase space gets modified [247, 248],

$$d\Gamma = \sqrt{det\Sigma_{ab}}d\xi = (1 + e\mathbf{B}\cdot\mathbf{\Omega})\frac{dpdx}{(2\pi)^3}. \tag{2.146}$$

Equivalent Liouville's theorem will give,

$$\dot{n}_\mathbf{p} - (\Sigma)^{-1}_{ab}\frac{\partial H(\xi)}{\partial \xi^b}\frac{\partial n_\mathbf{p}}{\partial \xi^a} = 0,$$

where, n_p is the distribution function of chiral fermion. Taking $H = \epsilon_p + A_0$, One can get the following equation,

$$\dot{n}_\mathbf{p} + \dot{\mathbf{x}}\cdot\frac{\partial n_\mathbf{p}}{\partial \mathbf{x}} + \dot{\mathbf{p}}\cdot\frac{\partial n_\mathbf{p}}{\partial \mathbf{p}} = 0, \tag{2.147}$$

where,

$$\dot{\mathbf{x}} = \frac{1}{1+e\mathbf{B}\cdot\mathbf{\Omega_p}}\left(\tilde{\mathbf{v}} + e\tilde{\mathbf{E}}\times\mathbf{\Omega_p} + e(\tilde{\mathbf{v}}\cdot\mathbf{\Omega_p})\mathbf{B}\right),$$
$$\dot{\mathbf{p}} = \frac{1}{1+e\mathbf{B}\cdot\mathbf{\Omega_p}}\left[\left(e\tilde{\mathbf{E}} + e\tilde{\mathbf{v}}\times\mathbf{B} + e^2(\tilde{\mathbf{E}}\cdot\mathbf{B})\mathbf{\Omega_p}\right)\right],$$

Note that here, $\tilde{\mathbf{v}} = \frac{\partial\epsilon_\mathbf{p}}{\partial\mathbf{p}}$, $e\tilde{\mathbf{E}} = e\mathbf{E} - \frac{\partial\epsilon_\mathbf{p}}{\partial\mathbf{x}}$, $\epsilon_\mathbf{p} = p(1 - e\mathbf{B}\cdot\mathbf{\Omega_p})$ and $\mathbf{\Omega_p} = \pm\mathbf{p}/2p^3$. $\pm$ sign respectively corresponds to the right and left handed particles. If $\mathbf{\Omega_p} = 0$, the above equation reduces to Vlasov equation. It is easy to check that Eq.(2.147) gives the anomaly equation as follows,

$$\partial_t n + \nabla\cdot\mathbf{j} = \mathbf{e^2}\int\frac{\mathbf{d^3p}}{(\mathbf{2}\pi)^\mathbf{3}}\left(\mathbf{\Omega_p}\cdot\frac{\partial\mathbf{n_p}}{\partial\mathbf{p}}\right)\mathbf{E}\cdot\mathbf{B}, \tag{2.148}$$

where,

$$n = \int\frac{d^3p}{(2\pi)^3}(1 + e\mathbf{B}\cdot\mathbf{\Omega_p})n_\mathbf{p},$$
$$\mathbf{j} = -e\int\frac{d^3p}{(2\pi)^3}\left[\epsilon_\mathbf{p}\frac{\partial n_\mathbf{p}}{\partial p} + e\left(\mathbf{\Omega_p}\cdot\frac{\partial\mathbf{n_p}}{\partial\mathbf{p}}\right)\epsilon_\mathbf{p}\mathbf{B} + \epsilon_\mathbf{p}\mathbf{\Omega_p}\times\frac{\partial n_\mathbf{p}}{\partial x}\right] + \mathbf{E}\times\boldsymbol{\sigma}.$$

$\boldsymbol{\sigma}$ is given by,

$$\boldsymbol{\sigma} = \int\frac{d^3p}{(2\pi)^3}\mathbf{\Omega_p}n_\mathbf{p}.$$

Performing the integration of the right hand side of Eq.(2.148) is a bit tricky because of the singularity at p=0. At this point motion of particles can be described by quantum mechanics [228]. To carry out such integrals one should exclude the region $|p| < R$ surrounding the point $p = 0$ such that classical description remains applicable outside it. In the classical region $|p| > R$, particles can not be created or destroyed; they can only enter or exit through the surface boundary at R. Therefore, for the region $|p| > R$ we can write Eq.(2.148),

$$\partial_t n + \nabla \cdot \mathbf{j} = \mathbf{e^2} \int_{\mathbf{S^2(R)}} \frac{\mathbf{dS}}{(\mathbf{2}\pi)^\mathbf{3}} \cdot \mathbf{\Omega_p n_p E \cdot B}, \tag{2.149}$$

where, dS is the surface element of the sphere. Now even if we take limit $R \to 0$ which implies that $p = 0$ and carry out surface integral we get,

$$\partial_t n + \nabla \cdot \mathbf{j} = \frac{\mathbf{e^2}}{\mathbf{4}\pi^\mathbf{2}} \mathbf{n_{p=0} E \cdot B}. \tag{2.150}$$

Thus total flux remains finite even at $p = 0$ due to anomaly ($\mathbf{E} \cdot \mathbf{B}$ term) [209, 210]. The presence of $n_{\mathbf{p=0}}$ in the above equation shows that there must be some thermal correction. However, it is important to note that at finite temperature one must also consider anti-fermions. Therefore, if we consider both right-handed and left-handed particles/antiparticles and write the same sort of transport equation as above, we can arrive to the following equation,

$$\partial_\mu J^\mu = \frac{e^2}{4\pi^2} (n^R_{\mathbf{p=0}} + n^{\bar{R}}_{\mathbf{p=0}} + n^L_{\mathbf{p=0}} + n^{\bar{L}}_{\mathbf{p=0}}) \mathbf{E} \cdot \mathbf{B}, \tag{2.151}$$

which simplifies to the following equation,

$$\partial_\mu J^\mu = \frac{e^2}{4\pi^2} \mathbf{E} \cdot \mathbf{B}. \tag{2.152}$$

Thus, chiral anomaly does not receive any thermal corrections, which is well known result in the literatures of quantum field theory [249, 250].

Now, if we define the energy densities and momentum densities of the particles as follows,

$$\epsilon = \int \frac{d^3p}{(2\pi)^3}(1+\mathbf{B}\cdot\mathbf{\Omega_p})\epsilon_\mathbf{p} n_\mathbf{p}, \qquad \pi^i = \int \frac{d^3p}{(2\pi)^3}(1+\mathbf{B}\cdot\mathbf{\Omega_p})p^i n_\mathbf{p}, \tag{2.153}$$

by multiplying Eq.(2.147) by $\epsilon_\mathbf{p}\sqrt{\Sigma_{ab}}$ and $p^i\sqrt{\Sigma_{ab}}$ and performing the integral over momentum $\mathbf{p}$, we can get energy and momentum conservation laws as follows,

$$\partial_\mu T^{0\mu} = E^i j^i, \qquad \partial_\mu T^{i\mu} = nE^i + \epsilon^{ijk}j^j B^k, \tag{2.154}$$

where,

$$\begin{aligned} T^{0i} &= -\int \frac{d^3p}{(2\pi)^3}\left[(\delta^{ij}+B^i\Omega^j)\frac{\epsilon_\mathbf{p}^2}{2}\frac{\partial n_\mathbf{p}}{\partial p^j} + \epsilon^{ijk}\frac{\epsilon_\mathbf{p}^2}{2}\Omega^j\frac{\partial n_\mathbf{p}}{\partial x^k}\right], \\ T^{ij} &= -\int \frac{d^3p}{(2\pi)^3}p^i\left[\epsilon_\mathbf{p}(\delta^{jk}+B^j\Omega^k)\frac{\partial n_\mathbf{p}}{\partial p^k} + \epsilon^{jkl}\Omega^k\left(E^l n_\mathbf{p} + \epsilon_\mathbf{p}\frac{\partial n_\mathbf{p}}{\partial x^l}\right)\right] - \delta^{ij}\epsilon. \end{aligned} \tag{2.155}$$

In the above equations expression of $\epsilon_\mathbf{p}$ is still not known. It can be determined using the constraint due to Lorentz invariance, which demands that the energy flux is equal to the momentum density i.e.

$$T^{0i} = \pi^i. \tag{2.156}$$

According to Lorentz invariance above equation is valid at any order of perturbation. Using the expression of for T^{0i} and π^i from Eqs.(2.155) and (2.153), writing down the final equation to the first order in perturbations in the quantities $n_\mathbf{p}$ and $\epsilon_\mathbf{p}$, from Eq.(2.156) one can obtain,

$$\epsilon_\mathbf{p}^0 = p - \frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{2p}. \tag{2.157}$$

This is the dispersion relation of chiral fermions near Fermi surface in the presence of magnetic field [229].

In the upcoming chapters we use the theoretical tools as we have discussed above to study the fluctuations in causal hydrodynamics, chiral-imbalance and

Weibel instabilities and to estimate η/s arising because of turbulence due to chiral-imbalance instability.

Chapter 3

Theory of Fluctuations in Relativistic Causal Hydrodynamics

In this chapter we aim to apply the fluctuation-dissipation theorem, for the first time, to calculate Onsager coefficients [222–225] and hydrodynamic fluctuations (basically the two point correlation functions) for the relativistic causal hydrodynamics.

In the context of the relativistic hydrodynamics, results of the fluctuation-dissipation theorem have been studied in Refs. [251, 252]. In Ref. [251] authors have studied the fluctuation in the context of general relativistic Navier-Stokes theory. A more general framework of hydrodynamics described as the divergence type theory (DTT) [253] was considered in Ref. [252]. It ought to be noted that recently in an interesting work in Ref. [254], the authors have applied results of the fluctuation-dissipation theorem to the relativistic *Navier-Stokes* theory of hydrodynamics and calculated the two particles correlators for the one-dimensional hydrodynamics (Bjorken) flow relevant for the relativistic heavy-ion collision experiments at RHIC and LHC. The authors obtained several analytical results for the two particle correlation functions. Further, in Ref. [255], the authors have studied the effect of thermal conductivity on the correlation function using the Bjorken-flow. It is well-known that the relativistic Navier-Stokes theory exhibits acausal behavior and it can give rise to unphysical instabilities [165–167]. However, the causality can be restored if the terms with higher orders in the spatial derivative of the fluid velocities are included in the hydrodynamics as indicated by the Maxwell-Cattaneo law [164]. Indeed, these issues do not arise in the second-order causal hydrodynamics theory developed by Müller, Israel and Stewart (MIS) [169–172]. Form of the NS as well as MIS equations can be determined from the second law of thermodynamics $\partial_\mu S^\mu \geq 0$. For NS case, S^μ denotes the equilibrium entropy current. However, in general it is not possible for an out-of-equilibrium fluid to have an equilibrium entropy current [164]. In MIS hydrodynamics, out-of-equilibrium current can have contributions from dissipative processes like the effect of viscosity and the heat conduction. This has an interesting analogy with the irreversible thermodynamics [256–258]. Further, MIS formalism was extended to include the effect of the third order terms in the gradient expansion [177]. Recently, it has been shown that the derivation of the MIS equations from the underlying kinetic equation may not be unique, there may exist a more general set of hydrodynamic equations which may allow one to obtain MIS equations as a special case [178, 179]. It should also be mentioned here that although the divergent type theory (DTT) of relativistic fluid of Geroch-Lindblom [253] allows for a consistent proof of causality and stability of its solutions, it is far from direct thermodynamic intuition. Moreover, the connection between the DTTs and MIS or other causal hydrodynamics theories is not yet clearly established.

We shall consider the second order causal hydrodynamics of Muller, Israel and Stewart (MIS) and its generalization to the third order. We shall also consider several other related causal hydrodynamic frameworks to compute the Onsager coefficients and the two point correlation functions using the fluctuation dissipation theorem. In order to understand the qualitative behavior of the the two point correlation functions, we shall numerically study evolution of the correlation function using the one dimensional boost-invariant (Bjorken) flow and try to compare the correlation functions obtained using the causal hydrodynamics with the correlation-function for the relativistic Navier-Stokes equation. This chapter is organized as follows.

In section §3.1, we shall discuss the fluctuation-dissipation theorem. In section §3.2, we shall discuss fluctuation in hydrodynamic framework, in particular we calculate the viscous correlation functions for the case of MIS in Landau-Lifshitz frame and Eckart frame. In this section §3.2, we shall also calculate the viscous correlations for other related models [177–179, 246]. In section §3.3, we apply these results to the case of $1+0$ dimensional Bjorken flow. In section §3.4, we shall discuss our results. Finally, in section §3.5 we shall conclude.

3.1 Fluctuation-dissipation theorem

In thermodynamic equilibrium, entropy of the system S which is a function of the additive quantities, x_k becomes maximum and it satisfies the condition $X_k = -\frac{\partial S}{\partial x_k} = 0$. However, when the system is slightly away from the equilibrium, the generalized forces $X_k \neq 0$ and $\frac{dx_i}{dt} = -\gamma_{ik} X_k + \xi_i$, the summation convention is implied, describes the flux associated with the quantity x_i. Here ξ_i are the random forces or the noise term and γ_{ik} are the Onsager coefficients. The Onsager reciprocity relations imply that $\gamma_{ik} = \gamma_{ki}$. In this phenomenological theory, time rate of change of the total entropy $\frac{dS}{dt}$ is given by [252],

$$\frac{dS}{dt} = -\frac{dx_i}{dt} X_i, \tag{3.1}$$

which can also be written as,

$$\frac{dS}{dt} = \gamma_{ik} X_k X_i - \xi_i X_i. \tag{3.2}$$

Correlation between ξ_i's is given by the formula,

$$\langle \xi_i(t_1) \xi_k(t_2) \rangle = (\gamma_{ik} + \gamma_{ki}) \delta(t_1 - t_2). \tag{3.3}$$

Thus, if we know γ_{ij}, it is easy to calculate correlations [223, 254]. In the next section we shall apply these results for case of various causal hydrodynamic frameworks.

3.2 Fluctuations and correlations in hydrodynamics

In the hydrodynamic framework, fluctuations can be characterized by incorporating stochastic terms in $T^{\mu\nu}$ and J^μ as follows,

$$T^{\mu\nu} = T^{\mu\nu}_{id} + \Delta T^{\mu\nu} + S^{\mu\nu}, \tag{3.4}$$

$$J^\mu = nu^\mu + \nu^\mu + I^\mu, \tag{3.5}$$

where, $T^{\mu\nu}_{id} = \epsilon u^\mu u^\nu - p\Delta^{\mu\nu}$ is the ideal part of the energy momentum tensor. $\Delta T^{\mu\nu} = \Delta T^{\mu\nu}_{vis} + \Delta T^{\mu\nu}_{heat}$ with $\Delta T^{\mu\nu}_{vis} = \pi^{\mu\nu} - \Delta^{\mu\nu}\Pi$ and $\Delta T^{\mu\nu}_{heat} = W^\mu u^\nu + W^\nu u^\mu$, is the dissipative part of the energy momentum tensor and $S^{\mu\nu}$ is the stochastic term arising due to the local thermal fluctuations [254]. Similarly, ν^μ and I^μ denote the dissipative (non-equilibrium) and the stochastic terms in baryon current density respectively. The relevant conservation equations for the hydrodynamics can be written as,

$$Dn + n\nabla_\mu u^\mu + \partial_\mu \nu^\mu + \partial_\mu I^\mu = 0, \tag{3.6}$$

$$D\epsilon + (\epsilon + p + \Pi)\nabla_\mu u^\mu - \pi_{\mu\nu}\nabla^{\langle\mu}u^{\nu\rangle} + \nabla_\mu W^\mu - 2W^\mu D u_\mu - S_{\mu\nu}\nabla^{\langle\mu}u^{\nu\rangle} = 0, \tag{3.7}$$

$$(\epsilon + p + \Pi)Du^\alpha - \nabla^\alpha(p + \Pi) + \Delta^{\alpha\nu}\nabla^\sigma\pi_{\nu\sigma} - \pi^{\alpha\nu}Du_\nu + \Delta^{\alpha\nu}DW_\nu + 2W^{(\alpha}\nabla_\nu u^{\nu)} + \Delta^{\alpha\nu}\nabla^\sigma S_{\nu\sigma} - S^{\alpha\nu}Du_\nu = 0. \tag{3.8}$$

3.2.1 Viscous correlations for MIS and TO hydrodynamics in Landau-Lifshitz frame

Viscous correlations are linearly related to the Onsager coefficients γ_{ij}. To find out γ_{ij} one needs to know the $\frac{dS}{dt}$ for the underlying hydrodynamical theory together with identification of the generalized forces and fluxes. Note that in section §2.3 we have discussed that equations for dissipative fluxes, irrespective of the models of hydrodynamics (e.g. NS, MIS, TO), were derived from the expression of $T\partial_\mu S^\mu$ (see Eqs.2.81, 2.114). In order to ensure that the second law is satisfied, it was assumed that $T\partial_\mu S^\mu$ must have the following general tensorial structure,

$$T\partial_\mu S^\mu = \frac{\Pi^2}{\zeta} - \frac{q^\mu q_\mu}{\lambda T} + \frac{\pi^{\mu\nu}\pi_{\mu\nu}}{2\eta} \geq 0, \tag{3.9}$$

where, $q^\mu q_\mu < 0$ [163]. We use the above expression to identify the generalized forces and fluxes. Using the identity $\Delta^{\mu\nu}\Delta_{\mu\nu} = 3$ and the condition $\Delta_{\mu\nu}\pi^{\mu\nu} = 0$, one can write Eq.(3.9) in the Landau-Lifshitz frame as follows,

$$\partial_\mu S^\mu = \frac{\Delta T^{\mu\nu}_{vis}}{T}\left(\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta}\right) - \frac{q^\mu q_\mu}{\lambda T^2}. \tag{3.10}$$

Upon integrating over the whole volume Eq.(3.10) can be written as,

$$\frac{dS}{dt} = \int d^3x \left[\frac{\Delta T^{\mu\nu}_{vis}}{T}\left(\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta}\right) - \frac{q^\mu q_\mu}{\lambda T^2}\right]. \tag{3.11}$$

Following identification between the phenomenological variables ($\dot{x}_1$, $\dot{x}_2$) and the hydrodynamical variables can be made [254],

$$\dot{x_1} \to \Delta T^{\mu\nu}_{vis}\ ,\ \dot{x_2} \to q^\mu. \tag{3.12}$$

A comparison of Eq.(3.11) with the phenomenological equation Eq.(3.1) will give,

$$\begin{aligned} X_1 &= -\frac{1}{T}\left(\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta}\right)\Delta V, \\ X_2 &= \frac{q_\mu}{\lambda T^2}\Delta V. \end{aligned} \tag{3.13}$$

Now neglecting the stochastic term in Eq.(3.2) and comparing it with Eq.(3.11), one can get,

$$\gamma_{11}X_1 = -\Delta T^{\mu\nu}_{vis}, \tag{3.14}$$

$$\gamma_{22}X_2 = -q^\mu, \tag{3.15}$$

$$\gamma_{12} = \gamma_{21} = 0. \tag{3.16}$$

The coefficients γ_{12} and γ_{21} are zero, because the dissipative fluxes due to heat and viscosity are considered to be independent. Coefficients γ_{11} and γ_{22} are rank-four tensors and they can be parameterized as follows,

$$\gamma_{11} = \left[A\Delta^{\mu\nu\alpha\beta} + B\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\frac{1}{\Delta V}, \quad \gamma_{22} = \frac{C\Delta^{\mu\nu}}{\Delta V}, \tag{3.17}$$

where, $\Delta^{\mu\nu\alpha\beta} = \Delta^{\mu\alpha}\Delta^{\nu\beta} - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}$. Now using Eqs.(3.14, 3.15) one can determine the coefficients $A = 2\eta T$, $B = \zeta T$ and $C = -\lambda T^2$. Thus, one can write,

$$\begin{aligned}\gamma_{11} = 2T\left[\left(\eta\Delta^{\mu\alpha}\Delta^{\nu\beta} - \frac{1}{3}\eta\Delta^{\mu\nu}\Delta^{\alpha\beta}\right) + \frac{1}{2}\zeta\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\frac{1}{\Delta V},\\ \gamma_{22} = -\frac{\lambda T^2\Delta^{\mu\nu}}{\Delta V}.\end{aligned} \tag{3.18}$$

From above expression of γ_{11} one can see that there is an additive contribution of shear and bulk viscosity i.e one can write it as $\gamma_{11} = (\gamma_{11})_\eta + (\gamma_{11})_\zeta$. Now following Eq.(3.2), the correlation functions can be written as,

$$\langle S^{\mu\nu}_{vis}(x_1)S^{\alpha\beta}_{vis}(x_2)\rangle = 2T\left[\eta(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}) + (\zeta - \frac{2}{3}\eta)\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\delta(x_1 - x_2), \tag{3.19}$$

$$\langle I^\mu(x_1)I^\nu(x_2)\rangle = -2\lambda T^2\Delta^{\mu\nu}\delta(x_1 - x_2), \tag{3.20}$$

$$\langle S^{\mu\nu}_{vis}(x_1)I^\alpha(x_2)\rangle = 0. \tag{3.21}$$

These are the stochastic or noise auto-correlation functions for the MIS and third order (TO) hydrodynamics in the Landau-Lifshitz frame.

3.2.2 Viscous correlations for MIS and TO hydrodynamics in Eckart frame

In the Eckart frame, Eq.(3.9) can be written in the following form,

$$T\partial_\mu S^\mu = \Delta T^{\mu\nu}\left[\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta} - \frac{1}{2\lambda T}(u_\nu q_\mu + u_\mu q_\nu)\right]. \tag{3.22}$$

Upon integrating over the whole volume, Eq.(3.22) can be written as,

$$\frac{dS}{dt} = \int d^3x \frac{\Delta T^{\mu\nu}}{T}\left[\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta} - \frac{1}{2\lambda T}(u_\nu q_\mu + u_\mu q_\nu)\right], \tag{3.23}$$

which can be rearranged in the following form,

$$\frac{dS}{dt} = \int d^3x \left[\frac{\Delta T^{\mu\nu}_{vis}}{T}\left(\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta}\right) + \frac{\Delta T^{\mu\nu}_{heat}}{T}\left[\left(\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta}\right) - \frac{1}{2\lambda T}\left(u_\nu q_\mu + u_\mu q_\nu\right)\right]\right]. \tag{3.24}$$

In this case also one can make the identifications as before,

$$\dot{x}_1 \rightarrow \Delta T^{\mu\nu}_{vis}\ ,\ \ \dot{x}_2 \rightarrow \Delta T^{\mu\nu}_{heat}. \tag{3.25}$$

The comparison between Eqs. (3.24) and (3.1) will give,

$$X_1 = -\frac{1}{T}\left(\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta}\right)\Delta V, \tag{3.26}$$

$$X_2 = -\frac{1}{T}\left[\left(\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta}\right) - \frac{1}{2\lambda T}(u_\nu q_\mu + u_\mu q_\nu)\right]\Delta V.$$

Again neglecting the stochastic term in Eq.(3.2) and comparing it with Eq.(3.24) one can get,

$$\gamma_{11}X_1 = -\Delta T^{\mu\nu}_{vis}, \tag{3.27}$$

$$\gamma_{22}X_2 = -\Delta T^{\mu\nu}_{heat}, \tag{3.28}$$

$$\gamma_{12} = \gamma_{21} = 0. \tag{3.29}$$

One can use the following parameterization for γ_{11} and γ_{22},

$$\gamma_{11} = \left[A\Delta^{\mu\nu\alpha\beta} + B\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\frac{1}{\Delta V},\quad \gamma_{22} = \left[\bar{A}\Delta^{\mu\alpha}u^\nu u^\beta + \bar{B}\Delta^{\nu\beta}u^\mu u^\alpha\right]\frac{1}{\Delta V}. \tag{3.30}$$

Since we know the forms of (X_1, X_2) and $(\Delta T^{\mu\nu}_{vis}, \Delta T^{\mu\nu}_{heat})$, therefore using Eqs.(3.30) and Eqs.(3.27-3.28), one can determine the coefficients $A = 2\eta T$, $B = \zeta T$ and

$\bar{A} = \bar{B} = -2\lambda T^2$. Thus γ_{11} and γ_{22} can be written as,

$$\gamma_{11} = 2T\left[\left(\eta\Delta^{\mu\alpha}\Delta^{\nu\beta} - \frac{1}{3}\eta\Delta^{\mu\nu}\Delta^{\alpha\beta}\right) + \frac{1}{2}\zeta\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\frac{1}{\Delta V}, \tag{3.31}$$

$$\gamma_{22} = -2\lambda T^2\left[\Delta^{\mu\alpha}u^\nu u^\beta + \Delta^{\nu\beta}u^\mu u^\alpha\right]\frac{1}{\Delta V}. \tag{3.32}$$

Thus one can write the correlation functions using Eq.(3.3) as,

$$\langle S^{\mu\nu}_{vis}(x_1)S^{\alpha\beta}_{vis}(x_2)\rangle = 2T\left[\eta(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}) + (\zeta - \frac{2}{3}\eta)\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\delta(x_1 - x_2), \tag{3.33}$$

$$\begin{aligned}\langle S^{\mu\nu}_{heat}(x_1)S^{\alpha\beta}_{heat}(x_2)\rangle &= -2\lambda T^2\Big[\Delta^{\mu\alpha}u^\nu u^\beta + \Delta^{\nu\beta}u^\mu u^\alpha + \Delta^{\mu\beta}u^\nu u^\alpha \\ &+ \Delta^{\nu\alpha}u^\mu u^\beta\Big]\delta(x_1 - x_2),\end{aligned} \tag{3.34}$$

$$\langle S^{\mu\nu}_{vis}(x_1)S^{\alpha\beta}_{heat}(x_2)\rangle = 0. \tag{3.35}$$

Form of these correlations is very similar to the correlations obtained for the relativistic Navier-Stokes case [254]. The relaxation time for the dissipative fluxes do not appear explicitly in the expressions for the correlation. However, the evolution of the correlations can be very different as demonstrated later.

3.2.3 Viscous correlations for other hydrodynamic models in Landau-Lifshitz frame

3.2.3.1 DKR (Denicol, Koide and Rischke) hydrodynamics

In chapter §2, we have demonstrated that the derivation of relativistic viscous hydrodynamic equations is not unique, there can be several kinds of other second order causal hydrodynamics models which can be derived from the kinetic theory. Here, we consider the case of DKR as discussed in section §2.3.2.3. For simplicity, we consider the case of fluid with no net Baryon number ($n = 0$). In this case, the equations (Eqs.(2.124)-(2.125)) for dissipative quantities reduce to the following,

$$\dot{\Pi} = -\frac{\Pi}{\tau_\Pi} - \beta_\Pi\theta - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}, \tag{3.36}$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\alpha^{\langle\mu}\omega^{\nu\rangle\alpha} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi_\alpha^{\langle\mu}\sigma^{\nu\rangle\alpha} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}, \quad (3.37)$$

where $\theta = \nabla_\alpha u^\alpha$, and $\tau's, \beta's, \delta's, \lambda's$ are the transport coefficients.

It should be noted that Eq.(3.37) contains vorticity term $\omega^{\alpha\beta} = \frac{1}{2}(\nabla^\alpha u^\beta - \nabla^\beta u^\alpha)$. Note that while writing the above equations we have considered the fluid with no net baryon number. Thus the Eq.(3.7) with no net baryon number can be written as,

$$\partial_\mu(su^\mu) = \frac{\pi^{\mu\nu}\sigma_{\mu\nu}}{T} - \frac{\Pi\nabla_\alpha u^\alpha}{T}. \quad (3.38)$$

From Eq.(3.36) and (3.37) it is easy to write,

$$\nabla_\alpha u^\alpha = -\frac{\dot{\Pi}}{\beta_\Pi} - \frac{\Pi}{\beta_\Pi\tau_\Pi} - \frac{\delta_{\Pi\Pi}\Pi\nabla_\alpha u^\alpha}{\beta_\Pi} + \frac{\lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}}{\beta_\Pi}, \quad (3.39)$$

$$\sigma_{\mu\nu} = \frac{\dot{\pi}^{\langle\mu\nu\rangle}}{2\beta_\pi} + \frac{\pi^{\mu\nu}}{2\beta_\pi\tau_\pi} - \frac{\pi_\alpha^{\langle\mu}\omega^{\nu\rangle\alpha}}{\beta_\pi} + \frac{\delta_{\pi\pi}\pi^{\mu\nu}\nabla_\alpha u^\alpha}{2\beta_\pi} + \frac{\tau_{\pi\pi}\pi_\alpha^{\langle\mu}\sigma^{\nu\rangle\alpha}}{2\beta_\pi} - \frac{\lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}}{2\beta_\pi}. \quad (3.40)$$

Now substituting Eq.(3.39) and (3.40) in Eq.(3.38) one can write,

$$\begin{aligned}\partial_\mu(su^\mu) =& \frac{\pi_{\mu\nu}}{T}\left[\frac{\dot{\pi}^{\langle\mu\nu\rangle}}{2\beta_\pi} + \frac{\pi^{\mu\nu}}{2\beta_\pi\tau_\pi} - \frac{\pi_\alpha^{\langle\mu}\omega^{\nu\rangle\alpha}}{\beta_\pi} + \frac{\delta_{\pi\pi}\pi^{\mu\nu}\nabla_\alpha u^\alpha}{2\beta_\pi} + \frac{\tau_{\pi\pi}\pi_\alpha^{\langle\mu}\sigma^{\nu\rangle\alpha}}{2\beta_\pi} - \frac{\lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}}{2\beta_\pi}\right] \\ &- \frac{\Pi}{T}\left[-\frac{\dot{\Pi}}{\beta_\Pi} - \frac{\Pi}{\beta_\Pi\tau_\Pi} - \frac{\delta_{\Pi\Pi}\Pi\nabla_\alpha u^\alpha}{\beta_\Pi} + \frac{\lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}}{\beta_\Pi}\right], \quad (3.41)\end{aligned}$$

After substituting back for $\nabla_\alpha u^\alpha$ and $\sigma_{\mu\nu}$ again from Eq.(3.39) and (3.40) into Eq.(3.41) one can see the terms with the coefficients $\delta's, \tau's$, and $\lambda's$ are of $O(\pi^3)$ or of the higher order, therefore, one can neglect these terms.

One can easily show that $\dot{\pi}^{\langle\mu\nu\rangle} = \dot{\pi}^{\mu\nu} + \pi_\beta^\mu u^\nu Du^\beta + \pi_\alpha^\nu u^\mu Du^\alpha$. This would imply that,

$$\pi_{\mu\nu}\dot{\pi}^{\langle\mu\nu\rangle} = \pi_{\mu\nu}\dot{\pi}^{\mu\nu}. \quad (3.42)$$

Now neglecting the terms with the coefficients $\delta's, \tau's$, and $\lambda's$ from Eq.(3.41) for the reason mentioned above, using Eq.(3.42) and the identity, $\pi_{\mu\nu}\pi_\alpha^{\langle\mu}\omega^{\nu\rangle\alpha} = 0$, one can get,

$$\partial_\mu S^\mu = \left[-\partial_\mu\left(\frac{u^\mu}{4\beta_\pi T}\right) + \frac{1}{2\beta_\pi\tau_\pi T}\right]\pi^{\alpha\beta}\pi_{\alpha\beta} + \left[-\partial_\mu\left(\frac{u^\mu}{2\beta_\Pi T}\right) + \frac{1}{\beta_\Pi\tau_\Pi T}\right]\Pi^2, \tag{3.43}$$

where, S^μ is the non-equilibrium entropy current for DKR hydrodynamics and has the form,

$$S^\mu = \left(su^\mu - \frac{\pi^{\alpha\beta}\pi_{\alpha\beta}u^\mu}{4\beta_\pi T} - \frac{\Pi^2 u^\mu}{2\beta_\Pi T}\cdots\cdots\right). \tag{3.44}$$

Note that $\beta_{\pi,\Pi} = \frac{\eta}{\tau_{\pi,\Pi}}$. In Eq.(3.43) the terms with gradients of velocity field can be neglected as $\partial_\mu\left(\frac{u^\mu}{4\beta_\pi T}\right) = \frac{\tau_{\pi,\Pi}\theta}{\eta T} << \frac{1}{\eta T}$, where $\theta = \partial_\mu u^\mu$ is the inverse of the expansion scale and $\tau_{\pi,\Pi}$ is relaxation time scale. For the system to be in the relaxation regime, one must have $\tau_{\pi,\Pi}\theta << 1$ (see Ref. [175, 177]). Therefore from Eq.(3.43) one obtains,

$$\frac{dS}{dt} = \int d^3x\left[\left(\frac{1}{2\beta_\pi\tau_\pi T}\right)\pi^{\alpha\beta}\pi_{\alpha\beta} + \left(\frac{1}{\beta_\Pi\tau_\Pi T}\right)\Pi^2\right]. \tag{3.45}$$

Further, Eq.(3.45) can be written in the following form,

$$\frac{dS}{dt} = \int d^3x\left[\frac{\Delta T^{\alpha\beta}_{vis}}{T}\left(\frac{\pi_{\alpha\beta}}{2\beta_\pi\tau_\pi} - \frac{\Delta_{\alpha\beta}\Pi}{3\beta_\Pi\tau_\Pi}\right)\right]. \tag{3.46}$$

A comparison of the above expression with the phenomenological equation (Eq.(3.1)) yields,

$$\dot{x} \to \Delta T^{\alpha\beta}_{vis}\ ,\ X \to -\frac{1}{T}\left[\left(\frac{\pi_{\alpha\beta}}{2\beta_\pi\tau_\pi} - \frac{\Delta_{\alpha\beta}\Pi}{3\beta_\Pi\tau_\Pi}\right)\right]\Delta V. \tag{3.47}$$

Again by comparing Eq.(3.46) with Eq.(3.2)(when $\xi = 0$) one can get,

$$\gamma X = -\Delta T^{\mu\nu}_{vis}, \tag{3.48}$$

where, γ is a rank four tensor and can be written as,

$$\gamma = 2T\left[\beta_\pi\tau_\pi\Delta^{\mu\nu\alpha\beta} + \frac{1}{2}\beta_\Pi\tau_\Pi\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\frac{1}{\Delta V}. \tag{3.49}$$

Thus, the viscous correlations are,

$$\langle S^{\mu\nu}_{vis}(x_1)S^{\alpha\beta}_{vis}(x_2)\rangle = 2T\left[\beta_\pi\tau_\pi(\Delta^{\mu\alpha}\Delta^{\nu\beta}+\Delta^{\mu\beta}\Delta^{\nu\alpha})+(\beta_\Pi\tau_\Pi-\frac{2}{3}\beta_\pi\tau_\pi)\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\delta(x_1-x_2). \tag{3.50}$$

3.2.3.2 JBP (Jaiswal, Bhalerao and Pal) hydrodynamics

In this case, the expression for the entropy four-current S^μ was generalized from the Boltzmann's H-function and then the expression for its divergence has been found as follows [178],

$$\begin{aligned}\partial_\mu S^\mu = &-\frac{\Pi}{T}\left[\theta+\beta_0\dot{\Pi}+\beta_{\Pi\Pi}\Pi\theta+\alpha_0\nabla_\mu n^\mu+\psi\alpha_{n\Pi}n_\mu\dot{u}^\mu+\psi\alpha_{\Pi n}n_\mu\nabla^\mu\alpha\right]\\ &-\frac{n^\mu}{T}\Big[T\nabla_\mu\alpha-\beta_1\dot{n}^\mu-\beta_{nn}n_\mu\theta+\alpha_0\nabla_\mu\Pi+\alpha_1\nabla_\nu\pi^\nu_\mu+\tilde{\psi}\alpha_{n\Pi}\Pi\dot{u}_\mu\\ &+\tilde{\psi}\alpha_{\Pi n}\Pi\nabla_\mu\alpha+\tilde{\chi}\alpha_{\pi n}\pi^\nu_\mu\nabla_\nu\alpha+\tilde{\chi}\alpha_{n\pi}\pi^\nu_\mu\dot{u}_\nu\Big]\\ &+\frac{\pi^{\mu\nu}}{T}\left[\sigma_{\mu\nu}-\beta_2\dot{\pi}_{\mu\nu}-\beta_{\pi\pi}\theta\pi_{\mu\nu}-\alpha_1\nabla_{\langle\mu}n_{\nu\rangle}-\chi\alpha_{\pi n}n_{\langle\mu}\nabla_{\nu\rangle}\alpha-\chi\alpha_{n\pi}n_{\langle\mu}\dot{u}_{\nu\rangle}\right],\end{aligned} \tag{3.51}$$

where, $\theta=\partial_\mu u^\mu$. The second law of thermodynamics $T\partial_\mu S^\mu\geq 0$ is guaranteed to be satisfied if we have,

$$T\partial_\mu S^\mu = \frac{\Pi^2}{\zeta}-\frac{n^\mu n_\mu}{\lambda}+\frac{\pi^{\mu\nu}\pi_{\mu\nu}}{2\eta}, \tag{3.52}$$

This give the equations for π, n^μ and $\pi^{\mu\nu}$. Onsager coefficients in this case too, can be obtained using the parameterization[see Eq.(3.17)],

$$\gamma_{11} = 2T\left[\eta\Delta^{\mu\alpha}\Delta^{\nu\beta}+\frac{1}{2}(\zeta-\frac{2}{3}\eta)\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\frac{1}{\Delta V}, \tag{3.53}$$

$$\gamma_{22} = -\frac{\lambda T\Delta^{\mu\nu}}{\Delta V}. \tag{3.54}$$

It should be noted that in Ref. [178] the authors have used $\frac{n^\mu}{\lambda}$ in Eq.(3.52) instead of $\frac{n^\mu}{\lambda T}$ and therefore Onsager coefficients differ by a factor of T (see for example, Eqs.(3.18 and 3.54)). The correlation functions can be written as,

$$\langle S^{\mu\nu}_{vis}(x_1) S^{\alpha\beta}_{vis}(x_2)\rangle = 2T\left[\eta(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}) + (\zeta - \frac{2}{3}\eta)\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\delta(x_1 - x_2), \tag{3.55}$$

$$\langle I^{\mu}(x_1) I^{\nu}(x_2)\rangle = -2\lambda T \Delta^{\mu\nu}\delta(x_1 - x_2), \tag{3.56}$$

$$\langle S^{\mu\nu}_{vis}(x_1) I^{\alpha}(x_2)\rangle = 0. \tag{3.57}$$

3.2.3.3 Conformal viscous hydrodynamics

The entropy current for the conformal hydrodynamics [246] can be written as,

$$S^{\mu} = \left(su^{\mu} - \frac{\tau_{\Pi}}{4\eta T}\Pi_{\alpha\beta}\Pi^{\alpha\beta}u^{\mu}\right). \tag{3.58}$$

One can easily find the following expressions for the Onsager coefficient and the correlation function,

$$\gamma = 2\eta T\left[\Delta^{\mu\alpha}\Delta^{\nu\beta} - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\frac{1}{\Delta V}, \tag{3.59}$$

$$\langle S^{\mu\nu}_{vis}(x_1) S^{\alpha\beta}_{vis}(x_2)\rangle = 2\eta T\left[(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}) - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\delta(x_1 - x_2). \tag{3.60}$$

3.3 Calculation of correlation functions in boost-invariant hydrodynamics

As an example, we apply the results obtained in the previous sections to the relativistic heavy ion collisions for the Bjorken flow. According to Bjorken scenario in heavy ion collisions, the reaction volume is strongly expanded in the longitudinal direction, i.e along the collision axis (z-axis). So, one can assume that there is no transverse flow. Thus, one can describe flow in $1+0$ dimension [201] using the light cone variable y and proper time τ as defined in Eqn.(2.24). The flow velocity under the scaling assumption can be given by Eq.(2.25). We consider only longitudinal flow fluctuations and parameterize the flow velocity [259] as,

$$u^{\mu} = (\cosh\bar{\theta}, \sinh\bar{\theta}), \tag{3.61}$$

where $\bar{\theta} = y + \delta\bar{\theta}(y, \tau)$ and $\delta\bar{\theta}(y, \tau)$ are the fluctuations in the longitudinal flow. In scaling limit, $\bar{\theta} = y$. With this parameterization and using the transformation of derivatives one can introduce the operators D, ∇ such that,

$$\begin{bmatrix} D \\ \nabla \end{bmatrix} = \begin{bmatrix} \cosh(\bar{\theta} - y) & \sinh(\bar{\theta} - y) \\ \sinh(\bar{\theta} - y) & \cosh(\bar{\theta} - y) \end{bmatrix} = \begin{bmatrix} \partial_\tau \\ \frac{1}{\tau}\partial_y \end{bmatrix}. \tag{3.62}$$

In the scaling limit, $D = u^\mu \partial_\mu = \frac{\partial}{\partial \tau} = \partial_\tau$ and $\partial_\mu u^\mu = \nabla\bar{\theta} = \frac{1}{\tau}$.
Since $S^{\mu\nu}$ satisfies the condition,

$$u_\mu S^{\mu\nu} = 0, \tag{3.63}$$

one can write $S^{\mu\nu}$ as [254],

$$S^{\mu\nu} = w(\tau) f(y, \tau) \Delta^{\mu\nu}, \tag{3.64}$$

where, $w = \epsilon + p = Ts$ and f is a dimensionless quantity which satisfies, $\langle f \rangle = 0$, where $\langle\rangle$ denotes the 'average value'. In heavy-ion collision experiments at LHC or RHIC a baryon free quark gluon plasma is expected to be produced, therefore, $q^\mu = 0$. Thus, only viscous correlations are of interest, which for MIS, JBP and TO can be written as,

$$\langle f(y_1, \tau_1) f(y_2, \tau_2) \rangle = \frac{2T(\tau_1)}{A\tau_1 w^2(\tau_1)} \left[\frac{4}{3}\eta(\tau_1) + \zeta(\tau_1)\right] \delta(\tau_1 - \tau_2)\delta(y_1 - y_2), \tag{3.65}$$

where $\delta(x_1 - x_2)_{Transverse}$ is replaced by the effective transverse area A of the colliding nuclei. Note that these correlations are the same as that obtained by authors in Ref. [254] for Navier-Stokes case. Similarly, for DKR case one can write the viscous correlations as,

$$\langle f(y_1, \tau_1) f(y_2, \tau_2) \rangle = \frac{2T(\tau_1)\left(\frac{4}{3}\beta_\pi \tau_\pi + \beta_\Pi \tau_\Pi\right)}{A\tau_1 w^2(\tau_1)} \delta(\tau_1 - \tau_2)\delta(y_1 - y_2). \tag{3.66}$$

By defining $\eta = \beta_\pi \tau_\pi$ as in Ref. [179] and neglecting the bulk viscosity for the correlation functions, one can rewrite the correlations for all the models of hydrodynamics that considered here as,

$$\langle f(y_1,\tau_1)f(y_2,\tau_2)\rangle = \frac{X(\tau_1)_{[E]}}{A}\delta(\tau_1-\tau_2)\delta(y_1-y_2), \tag{3.67}$$

where,

$$X(\tau_1)_{[E]} = \frac{8}{3\tau_1 w(\tau_1)}\left(\frac{\eta(\tau_1)}{s(\tau_1)}\right)_{[E]}. \tag{3.68}$$

Here, subscript $[E]$ denotes the particular type of hydrodynamics model considered from the set of hydrodynamics models, for example $[E]$ =[MIS,JBP,DKR,TO, NS].

It is useful to study the correlation function normalized by the initial value of the correlation obtained using the Navier-Stokes theory i.e. $C(\tau)_{[E]} = \frac{w^2(\tau)X(\tau)_{[E]}}{w^2(\tau_0)_{NS}X(\tau_0)_{NS}}$, where, τ_0 is the initial time for the hydrodynamics. $C(\tau)_{[E]}$ can also be written as,

$$C(\tau)_{[E]} = \frac{\left(\frac{\tau_0}{\tau}\right)\left(\frac{\eta(\tau)}{s(\tau)}\right)_{[E]}\frac{w(\tau)}{w(\tau_0)}}{\left(\frac{\eta(\tau)}{s(\tau)}\right)_{NS}}. \tag{3.69}$$

Further, we neglect the effect of bulk-viscosity by considering the initial temperature T_i to be much larger than the critical temperature, $T_c = 0.190$ GeV. Now, in the Landau-Lifshitz frame, the energy and the momentum conservation laws are given by,

$$u_\nu\partial_\mu T^{\mu\nu} = D\epsilon + (\epsilon+p)\nabla\bar{\theta} - \pi_{\mu\nu}\nabla^{\langle\mu}u^{\nu\rangle} - S_{\mu\nu}\nabla^{(\mu}u^{\nu)} = 0, \tag{3.70}$$

$$\Delta^\alpha_\nu\partial_\mu T^{\mu\nu} = (\epsilon+p)Du^\alpha - \nabla^\alpha p + \Delta^{\alpha\nu}\nabla^\sigma\pi_{\nu\sigma} - \pi^{\alpha\nu}Du_\nu + \Delta^{\alpha\nu}\partial^\sigma S_{\sigma\nu} = 0, \tag{3.71}$$

where, $\pi^{\alpha\beta}$ is the shear stress tensor and the dynamical equation for $\pi^{\alpha\beta}$ can be different for different models of hydrodynamics.

In the scaling limit, one can write the above equations as [179],

$$\partial_\tau\epsilon = -\frac{(\epsilon+p)}{\tau} + \frac{\pi}{\tau}. \tag{3.72}$$

Here, $\pi = \pi^{00} - \pi^{zz}$, and the noise term is considered to be much smaller than the background quantities and therefore neglected.

Now equation for π in the scaling limit, for DKR and JBP hydrodynamics can be written as,

$$\partial_\tau \pi + \frac{\pi}{\tau_\pi} = \beta_\pi \frac{4}{3\tau} - \lambda \frac{\pi}{\tau}. \tag{3.73}$$

For JBP case, coefficients β_π, τ_π and λ are as follows,

$$\beta_\pi = \frac{2p}{3}, \ \tau_\pi^{-1} = \frac{5}{9}\frac{\sigma p}{T}, \ \lambda = 4/3, \tag{3.74}$$

where, σ is the total cross-section [179] and it is assumed to be independent of energy [177, 260]. For DKR hydrodynamics, the parameters β_π, τ_π [179] and λ [261] are,

$$\beta_\pi = \frac{4p}{5}, \ \tau_\pi^{-1} = \frac{3}{5}\frac{\sigma p}{T}, \ \lambda \equiv \frac{1}{3}\tau_{\pi\pi} + \delta_{\pi\pi} = \frac{38}{21}. \tag{3.75}$$

Similarly, the equations for π, in the scaling limit MIS and third order hydrodynamics respectively can be written as,

$$\partial_\tau \pi + \frac{\pi}{\tau_\pi} = \frac{\eta}{\tau_\pi}\frac{4}{3\tau} - \frac{1}{2}\pi\left(\frac{1}{\tau} + \frac{\eta T}{\tau_\pi}\frac{\partial}{\partial\tau}\left(\frac{\tau_\pi}{\eta T}\right)\right), \tag{3.76}$$

$$\partial_\tau \pi + \frac{\pi}{\tau_\pi} = \frac{\eta}{\tau_\pi}\frac{4}{3\tau} - \frac{4}{3}\frac{\pi}{\tau} - \frac{\pi^2}{p\tau}, \tag{3.77}$$

where, $\frac{\eta}{\tau_\pi} = \frac{2p}{3}$ and $\tau_\pi^{-1} = \frac{5}{9}\frac{\sigma p}{T}$.

In what follows we consider the ideal equation of state, $\epsilon = 3p$ with the pressure given by the bag model, $p = \frac{\pi^2}{30}T^4$. Further, we consider the initial temperature $T_i = 0.310$ GeV and initial viscous stress π to be either zero or has the Navier-Stoke value i.e. $\pi = \frac{4}{3}\frac{\eta}{\tau}$ for all the causal hydrodynamics and numerically solve Eqs. (3.72,3.73), Eqs. (3.72,3.76) and Eqs. (3.72,3.77) for evaluating the correlations (3.69) in the case of MIS, JBP, DKR and TO hydrodynamics. However, for the Navier-Stokes hydrodynamics one needs to solve only Eq.(3.72) with same value of initial temperature and the viscous stress given by,

$$\pi = \eta\frac{4}{3\tau}. \tag{3.78}$$

The results of the numerical work are presented in the following section.

3.4 Results and discussions

We have studied fluctuations in various models of relativistic causal viscous hydrodynamics. Eqs.(3.19-3.21), (3.33-3.35), (3.55-3.57), (3.50) and (3.60) represent our main results describing the correlation functions for various models of the relativistic causal hydrodynamics. First, we should like to note here that the form of the correlation functions given by Eqs.(3.19-3.21), (3.33-3.35), (3.55-3.57), (3.50) and (3.60) are strikingly similar to the correlation functions obtained using relativistic Navier-Stokes theory [251, 254]. The correlations do not explicitly depend upon the relaxation times that appear in the causal theories of hydrodynamics. This indicates a kind of universality of the correlations given by Eqs. (3.19-3.21), (3.33-3.35), (3.55-3.57), (3.50) and (3.60). One can notice from Eqs.(3.19) that the viscous correlation depends on $\epsilon + p - \mu n$ and the ratio of viscous coefficients to the entropy density. The universality can be understood by the positivity argument of four entropy current i.e. $T\partial_\mu S^\mu = \frac{\Pi^2}{\zeta} - \frac{q^\mu q_\mu}{\lambda T} + \frac{\pi^{\mu\nu}\pi_{\mu\nu}}{2\eta} \geq 0$, which is used to write the expression for $\frac{ds}{dt}$ by using the following properties of dissipative fluxes: $\Delta_{\mu\nu}\pi^{\mu\nu} = 0$, $q_\mu u^\mu = 0$ and $u_\mu \pi^{\mu\nu} = 0$. These constraints are universal and satisfied in the case of Navier-Stokes as well as all causal hydrodynamics no matter what form of $\pi_{\mu\nu}$, q^μ and Π is. The determination of Onsager coefficients [using Eq.(3.2)] also depend on these constraints leading to the same form for all hydrodynamic theories and consequently the correlation function remains the same for all the theories. But in the case of DTT kind of hydrodynamics, it is not clear if divergence of the entropy four-current can be expressed directly in terms of scalar product of the viscosity and heat-flux tensors.

In order to understand the evolution of the correlation functions in some detail, we have calculated the normalized correlation functions given by Eq.(3.69) for an expanding one-dimensional boost-invariant (Bjorken) flow. In this case, all the correlations are proportional to $(\epsilon + p)/\tau$. However, the details of temporal evolution of $\epsilon + p$ varies with the choice of different hydrodynamical models. In Figs. 3.1-3.2, we plot the normalized correlation function $C(\tau)_{[E]}$ (Eq.3.69) as a function of time τ, where [E] stands for MIS, JBP, DKR, TO and NS hydrodynamics. Each figure has five kind of curves: the solid (red) curve describes the Navier-Stokes case while the dotted-dashed (blue), the dashed (purple), the dotted (green) and large-dashed (black) curves respectively describe MIS, JBP, DKR and TO cases. The left panel shows the case when the initial value for the viscous stress $\pi = 0$, while the right panel represents the case when the initial value of π is same as the Navier-Stoke case. There are two possible comparisons between the correlation functions $C(\tau)_{[E]}$. In one such comparison the energy-independent cross-section σ [see Eqs. (3.74, 3.75)] is kept same for all the different versions of the hydrodynamics [179]. Following Ref. [179, 260], one can write the viscosity coefficient for the different models of hydrodynamics as $\eta_{DKR} = \frac{4T}{3\sigma}$, $\eta_{MIS} = \frac{6T}{5\sigma} = \eta_{JBP} = \eta_{TO}$ and $\eta_{NS} = 0.8436\frac{3T}{2\sigma}$. Thus the relation between different η are given by the scaling: $\eta_{MIS} = \eta_{JBP} = \eta_{TO} = 9/10\eta_{DKR}$ and $\eta_{NS} = \frac{7.59}{8}\eta_{DKR}$. In another way of comparing $C(\tau)_{[E]}$, the ratio η/s is kept the same for the different models of the hydrodynamics, while the σ is varied for the different models.

Fig. 3.1 shows the case when the transport cross-section is kept the same for all the models of hydrodynamics. The inset figure in all the diagrams shows the plots of correlation functions with better resolution in τ range between $3fm/c$ to $6fm/c$. Cases 3.1(a)-3.1(b), 3.1(c)-3.1(d) and 3.1(e)-3.1(f) correspond to $\frac{\eta_{DKR}}{s} = 0.08, 0.56$ and 1.60. Values of η/s for the other models can be found using the scaling relation discussed above. The initial temperature T_i and initial time τ_i are respectively chosen to be 310 MeV and 0.5fm/c. One can notice for figures 3.1(a)-3.1(b) that when η_{DKR}/s is close to the minimum possible value $(1/4\pi)$, all the correlations overlap with each other. This is expected as all the viscous hydrodynamics models should approach the ideal hydrodynamics limit when $\eta/s \approx 1/4\pi$. Figures 3.1(c)-3.1(d) corresponds to the case when $\frac{\eta_{DKR}}{s} = 0.56$, i.e. almost seven times larger than the most minimum value, the correlations only marginally differ from each other. Overall difference between the correlation functions obtained using initial condition $\pi = 0$ and $\pi \neq 0$ is not significant. However, when $\pi = 0$, Navier-stoke correlation slightly dominates

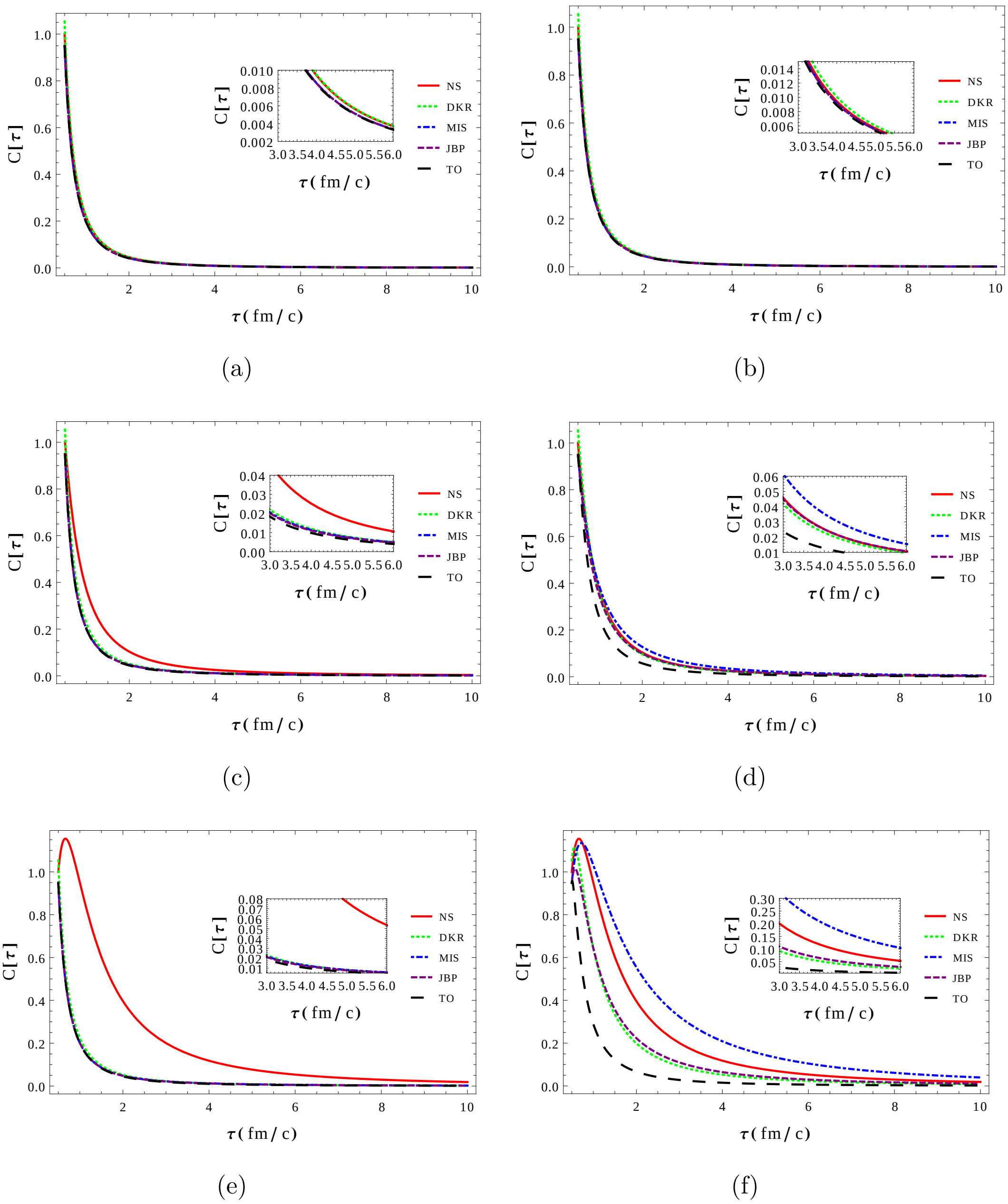


Figure 3.1: 3.1(a), 3.1(b), 3.1(c), 3.1(d), 3.1(e) and 3.1(f) show time evolution of the function $C(\tau)_{[E]}$ [see Eq.(3.69)] with same initial temperature $T_i = 310\ MeV$ where, [E] corresponds to NS, MIS, DKR, JBP and TO hydrodynamics. The coefficient of viscosity is calculated using $\eta_{DKR} = 4T/3\sigma$. The scaling $\eta_{MIS} = \eta_{JBP} = \eta_{TO} = 9/10\eta_{DKR}$ and $\eta_{NS} = 7.59/8\eta_{DKR}$ ensure that the cross-section remains same in the comparison between the models of hydrodynamics. Cases 3.1(a), 3.1(c) and 3.1(e) corresponds to $\eta_{DKR}/s = 0.08,\ 0.56,\ 1.60$ respectively with initial time $\tau_0 = 0.5 fm/c$ and $\pi_0 = 0.0$ for all causal approaches. While the cases 3.1(b), 3.1(d), and 3.1(f) corresponds to same η_{DKR}/s and τ_0 as in the former cases but with π_0 equal to Navier-Stokes initial value for all the hydrodynamic approaches.

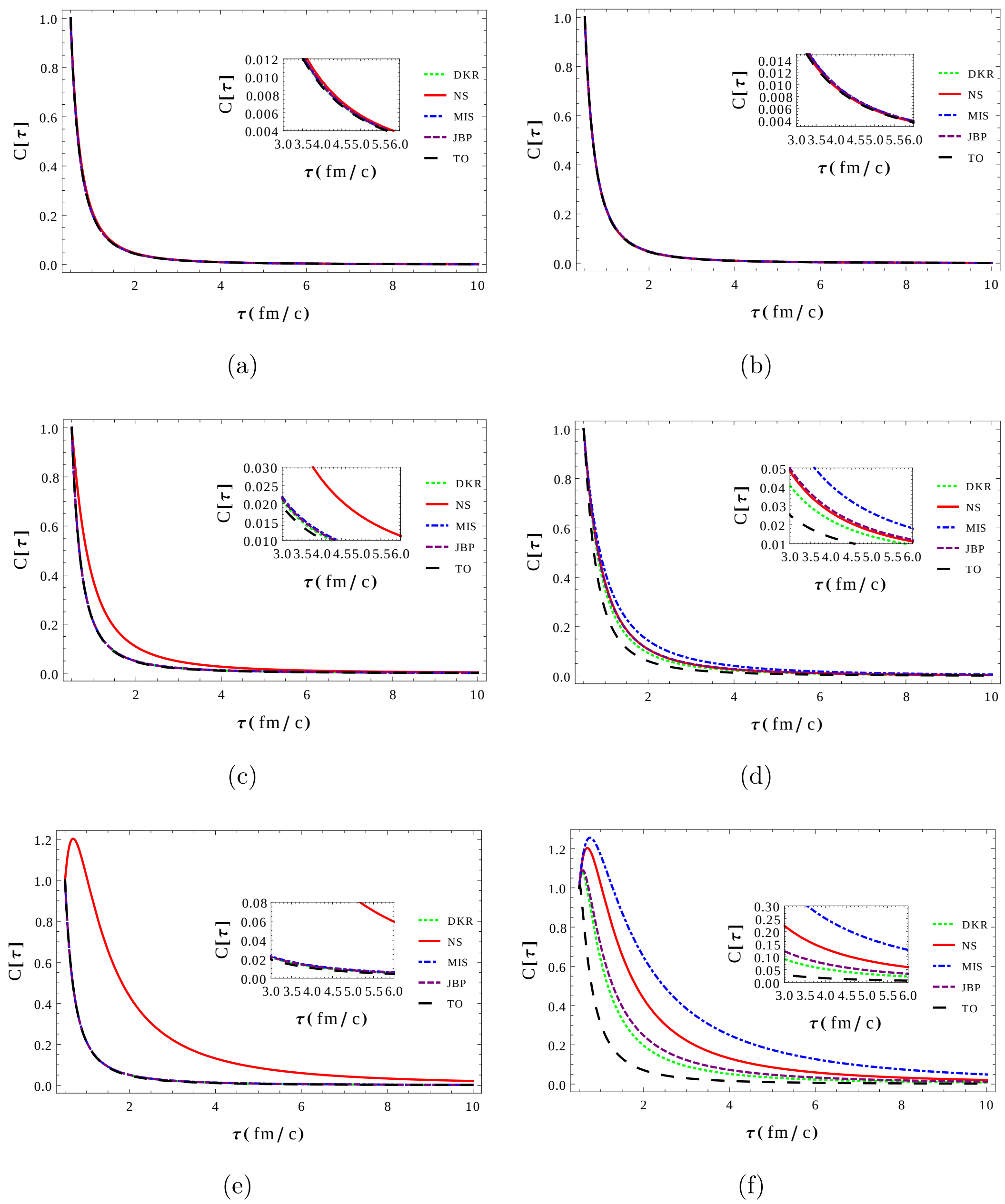


Figure 3.2: 3.2(a), 3.2(c), 3.2(b), 3.2(d), 3.2(e) and 3.2(f) show the time evolution of the function $C(\tau)_{[E]}$[see Eq.(3.69)] with same initial temperature $T_i = 310\ MeV$, where, [E] is corresponds to NS, MIS, DKR, JBP and TO hydrodynamics. Note that in all the figures the ratio of the viscosity to entropy density is kept same for all the Hydrodynamic approaches. Fig. 3.2(a), 3.2(c) and 3.2(e) corresponds to $\eta/s = 0.08,\ \ 0.56,\ \ 1.60$ respectively with initial time $\tau_0 = 0.5$ fm/c and $\pi_0 = 0.0$ for all causal approaches. While Fig. 3.2(b), 3.2(d) and 3.2(f) corresponds to the same η/s and τ_0 as in the former cases but with π_0 equal to Navier-Stokes initial value for all the hydrodynamic approches.

over the correlation functions obtained using the causal models. While for the case when the initial value of π is the same as Navier-Stokes value, it is the MIS correlation function which dominates over the other correlation functions. Figs. 3.1(e)-3.1(f), correspond to the case when η/s almost twenty times larger than the minimum value. In Fig. 3.1(e), the Navier-Stokes correlation first increases with time and then decreases. However, for all the causal models correlations decrease with time. Rise in the Navier-Stokes correlation can be attributed to the unphysical behavior noted in Ref. [245]. In this case it may be possible to distinguish between the correlation function from the Navier-Stoke theory from the causal hydrodynamics models. However, the correlation functions of the causal models overlap with each other. But when the Navier-Stokes value for the initial stress Π_0 is chosen for the causal models, all the correlation functions first increase with time and later the plummet with time. This case can be considered to be unphysical as for all the hydrodynamics models initially $\epsilon + p < \Pi$. The condition $\epsilon + p < \Pi$ violates the validity of the second order hydrodynamics.

Fig. 3.2 corresponds to the case when the ratio of the viscosity coefficient to the entropy density is kept the same for all the five models of hydrodynamics. The cases 3.2(a)-3.2(b), 3.2(c)-3.2(d) and 3.2(e)-3.2(f) respectively correspond to the situation when $\frac{\eta}{s}$ equal to $0.08, 0.56$ and 1.60. The initial temperature and the initial times are kept the same as in the case for Fig. 3.1. One can notice that as $C(\tau)$ in Eq.(3.69) remains same for all the hydrodynamical models, all the correlation functions, start at the same initial value. This was not the case in Fig. 3.1. Otherwise the general features about the correlation function remain the same as in Fig. 3.1. Moreover, we have changed the values of initial temperature and initial time. In these cases also the general features of the correlation functions remain similar to those discussed in Fig. 3.1.

Finally we would like to discuss the importance of our results. We first like the readers to note that in the present work we have extended the formalism to calculate hydrodynamic fluctuations given in Ref. [223] to the relativistic causal theories. We have demonstrated that the form of the correlation functions in causal hydrodynamic theories remains same as in the relativistic Navier-Stokes case [254]. This result is not expected apriori, as the underlying hydrodynamic equations for the causal theories [164, 169–172, 177–179, 246] are very different than the Navier Stokes equation. Eqs. (3.19-3.21), (3.33-3.35), (3.55-3.57), (3.50) and (3.60) can be employed to calculate the two particle correlators (see Ref. [254]), which can be compared with the experimental data. However, this would require the solution of inhomogeneous (with noise term) hydrodynamical equations (of different types) in 3-dimension. Further, in the present example we have dealt with boost invariant one dimensional flow. However, for a non-central heavy ion collision, the vorticity can play a significant role [262]. The presence of finite vorticity can cause the difference in the evolution in the correlation function for the different models of hydrodynamics remains to be seen. One can notice from Eq.(3.37) that vorticity can drive dynamics of the viscous stress in DKR hydrodynamics. However this will require to solve hydrodynamical equation in $2+1$ or $3+1$ dimensions. This is at present, beyond the scope of this work. Finally in the numerical example that we have considered here, we plot the correlation function vs time. However, this numerical result can not be compared with the experimental data. But, this can give us some idea about how the correlation-function of different hydrodynamics compare with each other. We find that the correlation functions obtained using various causal theories do not significantly differ from each other for a variety of values of initial conditions and η/s. However, the correlation function obtained using NS-theory can have unphysical behavior for higher values of η/s and the NS-correlation function differ from the correlation functions obtained using the causal hydrodynamics.

3.5 Conclusions

We have studied the fluctuations in various models of relativistic causal hydrodynamics. We have found that the general properties of the dissipative part of the energy-momentum tensor due to the viscosity and heat-flux play an important role in determining the Onsager coefficients and the correlation functions. We find that the analytic form of the correlation functions remains the same for all the causal hydrodynamics that is considered here and do not depend explicitly on the relaxation time. Further our numerical investigations also suggest that the qualitative behavior of the correlation functions for the various models of the causal hydrodynamics remains similar to those of the Navier-Stokes theory at least for a one dimensional boost-invariant flow.

Chapter 4

Instabilities in Anisotropic Chiral Plasmas

In the context of heavy ion collisions thermalization process has been described using various theoretical models[107–115]. Surprisingly none of them actually was able to describe the fast thermalization as suggested by the comparison of hydrodynamic simulations with RHIC data[116, 117]. Weibel type of instabilities [122, 132–141, 144, 237, 238, 263] that can occur during the initial stage of the heavy ion collisions are considered to be promising in explaining the fast thermalization. Such instabilities arise because of the anisotropic single particle distribution function satisfying the kinetic (Boltzmann or Vlasov) equation. The conventional Boltzmann or Vlasov equations imply that the vector current associated with the gauge charges is conserved. But, till recently a very important class of physical phenomena associated with the CP-violation or the triangle-anomaly was left out of the purview of a kinetic theory. In such a phenomenon the axial current is not conserved. Therefore, it is highly desirable to have a proper framework of kinetic theory to tackle the CP-violating effect. Recently, there has been a lot of progress in development of such a kinetic theory. In Refs. [226–228, 247, 264, 265] it was shown that if the Berry curvature [230] has a non-zero flux across the Fermi-surface, the particles on the surface can exhibit a chiral anomaly in presence of an external electromagnetic field. In this formalism, chiral-current j^μ is not conserved and it can be attributed to

Adler-Bell-Jackiw anomaly [209, 210, 266]. The idea that a Berry-phase can influence the electronic properties [e.g. [267] and the references cited therein] is well-known in condensed matter literature and it can have applications in Weyl semimetal [268] and graphene [269] etc. There exists a deep connection between a CP-violating quantum field theory and the kinetic theory with the Berry curvature corrections. In Ref. [229], it was shown that the parity-odd and parity-even correlations calculated using the modified kinetic theory are identical with the perturbative results obtained in next-to-leading order hard dense loop approximation. Recently, using Berry curvature modified kinetic theory, it has been suggested that a net difference between the chiral chemical potentials of right and left handed fermions (quarks) in the QGP can arise (due to parity violation caused by quantum anomalies), which may lead to the chiral-imbalance instability[270]. The existence of the chiral-imbalance instability in the strongly interacting matter seems to be well supported by the possibility of observing the parity violating correlations (as discussed in the section §1.5) in the heavy ion collision experiments[103, 104].

In this chapter, we argue that the chiral-imbalance instability can occur simultaneously with the Weibel instability during the initial stages of the heavy ion collisions and can be studied using Berry-curvature modified kinetic theory discussed in the section §2.4. In particular, we aim to analyze the collective modes in an anisotropic chiral plasma and study how the chiral-imbalance and Weibel instabilities can influence each other. We believe that the results presented in this chapter will be useful in the study of the quark-gluon plasma created in relativistic heavy-ion collisions. It is important to note that the physics of such instabilities depend on the chiral chemical potential μ_5, anisotropy parameter ξ and the angle θ_n between the propagation vector $\mathbf{k}$ and anisotropy direction $\mathbf{n}$. This chapter is organized as: In section §4.1, we shall discuss the basic equations required for such a study. In section §4.2, we shall derive the expression for the propagator using the Maxwell's equation. In section §4.3, we obtain the expression of self energy using the linear response analysis of anisotropic chiral plasma. In section §4.4, we shall obtain the dispersion relation by finding the poles of the propagator and study the collective modes. Finally, in section §4.5, we shall conclude our results.

4.1 Basic equations

We consider weak gauge Field limit and assume the following power counting scheme: $\partial_x = O(\delta)$ and $A^\mu = O(\epsilon)$. Here, ϵ and δ are small independent parameters. In this senario, we use the modified collisonless kinetic (Vlasov) equation [Eq.(2.147)], which at the leading order in A^μ becomes,

$$(\partial_t + \mathbf{v}\cdot\partial_{\mathbf{x}})n_{\mathbf{p}} + (e\mathbf{E} + e\mathbf{v}\times\mathbf{B} - \partial_{\mathbf{x}}\epsilon_{\mathbf{p}})\cdot\partial_{\mathbf{p}}n_{\mathbf{p}} = 0 \tag{4.1}$$

where, $\mathbf{v} = \mathbf{p}/p$, $\epsilon_{\mathbf{p}} = p(1 - e\mathbf{B}\cdot\mathbf{\Omega}_{\mathbf{p}})$ and $\mathbf{\Omega}_{\mathbf{p}} = \pm\mathbf{p}/2p^3$. Here $\pm$ sign corresponds to the right and left handed fermions respectively. In the absence of the Berry curvature term (i.e. $\Omega_{\mathbf{p}}$=0) $\epsilon_{\mathbf{p}}$ is independent of x, and the Eq.(4.1) reduces to the standard Vlasov equation.

In this case, current density $\mathbf{j}$ is defined as;

$$\mathbf{j} = -e\int\frac{d^3p}{(2\pi)^3}\Big[\epsilon_{\mathbf{p}}\partial_{\mathbf{p}}n_{\mathbf{p}} + e\left(\mathbf{\Omega}_{\mathbf{p}}\cdot\partial_{\mathbf{p}}n_{\mathbf{p}}\right)\epsilon_{\mathbf{p}}\mathbf{B} + \epsilon_{\mathbf{p}}\mathbf{\Omega}_{\mathbf{p}}\times\partial_{\mathbf{x}}n_{\mathbf{p}}\Big] + e\mathbf{E}\times\sigma, \tag{4.2}$$

where, $\partial_{\mathbf{P}} = \frac{\partial}{\partial\mathbf{p}}$ and $\partial_{\mathbf{x}} = \frac{\partial}{\partial\mathbf{x}}$. The last term on the right hand side of the above equation represents the anomalous Hall current with σ given as follows:

$$\sigma = e\int\frac{d^3p}{(2\pi)^3}\mathbf{\Omega}_{\mathbf{p}}n_{\mathbf{p}}. \tag{4.3}$$

4.2 Maxwell equation and the propagator

Maxwell's equation can be written as,

$$\partial_\nu F^{\nu\mu} = j^\mu_{ind} + j^\mu_{ext}. \tag{4.4}$$

Here j^μ_{ext} is an external current. The induced current j^μ_{ind} can be expressed in terms of gauge field $A_\nu(k)$ via linear response theory in Fourier space as,

$$j^\mu_{ind} = \Pi^{\mu\nu}(K)A_\nu(K), \tag{4.5}$$

where $\Pi^{\mu\nu}(K)$ is the retarded self energy in Fourier space. Here, we have denoted a Fourier transform of any quantity $F(x,t)$ by $F(K) = \int d^4x e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}F(x,t)$. Now one can write Eq.(4.4) in the Fourier space as

$$[K^2 g^{\mu\nu} - K^\mu K^\nu + \Pi^{\mu\nu}(K)] = -j^i_{ext}(K). \tag{4.6}$$

By choosing temporal gauge $A_0 = 0$ we can write the above equation as,

$$[\Delta^{-1}(K)]^{ij}E^j = [(k^2 - \omega^2)\delta^{ij} - k^i k^j + \Pi^{ij}(K)]E^j = i\omega j^i_{ext}(k). \tag{4.7}$$

From this one can define

$$[\Delta^{-1}(K)]^{ij} = (k^2 - \omega^2)\delta^{ij} - k^i k^j + \Pi^{ij}(K). \tag{4.8}$$

Here, $[\Delta^{-1}(K)]^{ij}$ is the inverse of the propagator. The expression for $\Pi^{ij}(K)$ can be obtained by the linear response analysis of Eqs.(4.1) and (4.2). Dispersion relation can be obtained by finding the poles of the propagator $[\Delta(K)]^{ij}$. First of all we obtain the expressions for the

4.3 Linear response analysis and the expression for $\Pi^{ij}(K)$

Let us first concentrate on the right handed fermions with the chemical potential μ_R. We consider the background distribution of the form $n^0_{\mathbf{p}} = 1/[e^{(\epsilon_{\mathbf{p}}-\mu_R)/T}+1]$. In a linear response theory, we are interested in the induced current by a linear-order deviation in the gauge field. We follow the power counting scheme for gauge field A_μ and derivatives ∂_x as discussed earlier, and consider deviations in the current and the distribution function up to $O(\epsilon\delta)$. In this case, we can write the distribution in Eq.(4.1) as follows,

$$n_{\mathbf{p}} = n^0_{\mathbf{p}} + e(n^{(\epsilon)}_{\mathbf{p}} + n^{(\epsilon\delta)}_{\mathbf{p}}) \tag{4.9}$$

where, $n^0_{\mathbf{p}}$ is the background distribution function in the presence of Berry curvature while $n^{(\epsilon)}_{\mathbf{p}}$ and $n^{(\epsilon\delta)}_{\mathbf{p}}$ are the pertubations of the order $O(\epsilon)$ and $O(\epsilon\delta)$ around $n^0_{\mathbf{p}}$. Since $n^0_{\mathbf{p}}$ contains the Berry curvature contribution (due to $\epsilon_{\mathbf{p}}$) therefore it can also be split into order $O(0)$ and $O(\epsilon\delta)$ i.e., $n^0_{\mathbf{p}} = n^{0(0)}_{\mathbf{p}} + en^{0(\epsilon\delta)}_{\mathbf{p}}$, where $n^{0(0)}_{\mathbf{p}} = 1/[e^{(p-\mu_R)/T}+1]$ is the part of background distribution function without Berry curvature correction, while $n^{0(\epsilon\delta)}_{\mathbf{p}} = (\mathbf{B}\cdot\mathbf{v}/2pT)\,(e^{(p-\mu_R)/T}/[e^{(p-\mu_R)/T}+1]^2)$ is the part of background distribution with Berry curvature correction. In order to bring in effect of anisotropy we follow the arguments of Ref.[138]. It is assumed that the anisotropic equilibrium distribution function can be obtained from a spherically symmetric distribution function by rescaling of one direction in the momentum space. We consider that there is a momentum anisotropy in the direction of a unit vector $\hat{\mathbf{n}}$. Noting that $p = |\mathbf{p}|$, we replace $p \to \sqrt{\mathbf{p}^2 + \xi(\mathbf{p}\cdot\hat{\mathbf{n}})^2}$ in the expression of $n^0_{\mathbf{p}}$ to get anisotropic distribution function. Here ξ is an adjustable anisotropy parameter satisfying a condition $\xi > -1$. It is convenient to define a new variable $\tilde{p}$ such that $\tilde{p} = p\sqrt{1+\xi(\mathbf{v}\cdot\hat{\mathbf{n}})^2}$. Using this new variable one can write $n^{0(0)}_{\mathbf{p}} = 1/[e^{(\tilde{p}-\mu_R)/T}+1]$ and $n^{0(\epsilon\delta)}_{\mathbf{p}} = (\mathbf{B}\cdot\mathbf{v}/2\tilde{p}T)\left(e^{(\tilde{p}-\mu_R)/T}/[e^{(\tilde{p}-\mu_R)/T}+1]^2\right)$.

The anomalous Hall current term in the Eq.(4.2) can be vanished if the distribution function is spherically symmetric in the momentum space. However, for an anisotropic distribution function this may not be true in general. Since the Hall-current term depends on electric field, it can be of the order $O(\epsilon\delta)$ or higher. As we are interested in finding the deviations in current and distribution function up to order $O(\epsilon\delta)$, only $n^{0(0)}_{\mathbf{p}}$ would contribute to the Hall current term. Next, we consider σ from Eq.(4.3) which can be written as,

$$\sigma = \frac{e}{2}\int d\Omega d\tilde{p}\frac{\mathbf{v}}{[1+\xi(\mathbf{v}\cdot\hat{\mathbf{n}})]^{1/2}}\frac{1}{(1+e^{(\tilde{p}-\mu_R)/T})}. \tag{4.10}$$

Since $\mathbf{v}$ is a unit vector, one can express, $\mathbf{v} = (sin\theta cos\phi, sin\theta sin\phi, cos\theta)$ in spherical coordinates. By choosing $\hat{\mathbf{n}}$ in $z-$direction, without any loss of generality, one can have $\mathbf{v}\cdot\hat{\mathbf{n}} = cos\theta$. Thus the angular integral in the above equation becomes $\int d(cos\theta)d\phi\left(\mathbf{v}/(1+\xi cos^2\theta)^{1/2}\right)$. Therefore, σ_x and σ_y components of Eq.(4.10) will be vanished as $\int_0^{2\pi} sin\phi d\phi = 0$ and $\int_0^{2\pi} cos\phi d\phi = 0$. While σ_z will be vanished because of the integration with respect to $\cos\theta$ variable. Consequently, the anomalous Hall current term will not contribute for the problem at the hand.

Now the kinetic equation (4.1) can be split into two equations valid at $O(\epsilon)$ and $O(\epsilon\delta)$ scales of distribution function as written below,

$$(\partial_t + \mathbf{v}\cdot\partial_{\mathbf{x}})n^{(\epsilon)}_{\mathbf{p}} = -(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\partial_{\mathbf{p}}n^{0(0)}_{\mathbf{p}} \tag{4.11}$$

$$(\partial_t + \mathbf{v}\cdot\partial_{\mathbf{x}})(n^{0(\epsilon\delta)}_{\mathbf{p}} + n^{(\epsilon\delta)}_{\mathbf{p}}) = -\frac{1}{e}\partial_{\mathbf{x}}\epsilon_{\mathbf{p}}\cdot\partial_{\mathbf{p}}n^{0(0)}_{\mathbf{p}}. \tag{4.12}$$

Similarly, the current defined in Eq.(4.2) can also be split into $O(\epsilon)$ and $O(\epsilon\delta)$ scales as given below,

$$\mathbf{j}^{\mu(\epsilon)} = e^2\int\frac{d^3p}{(2\pi)^3}v^\mu n^{(\epsilon)}_{\mathbf{p}} \tag{4.13}$$

$$\mathbf{j}^{\mathbf{i}(\epsilon\delta)} = e^2\int\frac{d^3p}{(2\pi)^3}\left[v^i n^{(\epsilon\delta)}_{\mathbf{p}} - \left(\frac{v^j}{2p}\frac{\partial n^{0(0)}_{\mathbf{p}}}{\partial p^j}\right)B^i - \epsilon^{ijk}\frac{v^j}{2p}\frac{\partial n^{(\epsilon)}_{\mathbf{p}}}{\partial x^k}\right] \tag{4.14}$$

After adding the contribution from all type of species i.e. right/left fermions with charge e and chemical potential μ_R/μ_L as well as right/left handed antifermions with charge $-e$ and chemical potential $-\mu_R/\mu_L$, using the expression $j^\mu_{ind} = \Pi^{\mu\nu}(K)A_\nu(K)$ and Eqs. (4.11 -4.14) one can obtain the expression for self energy, $\Pi^{ij} = \Pi^{ij}_+ + \Pi^{ij}_-$. Here we would like to mention that Π^{ij}_+ and Π^{ij}_- respectively denote the parity-even and parity-odd parts of the polarization tensor or self energy given by following equations,

$$\Pi^{ij}_+(K) = m_D^2\int\frac{d\Omega}{4\pi}\frac{v^i(v^l + \xi(\mathbf{v}\cdot\hat{\mathbf{n}})\hat{n}^l)}{(1+\xi(\mathbf{v}\cdot\hat{\mathbf{n}})^2)^2}\left(\delta^{jl} + \frac{v^j k^l}{\mathbf{v}\cdot\mathbf{k} + i\epsilon}\right), \tag{4.15}$$

$$\Pi^{im}_{-}(K) = C_E \int \frac{d\Omega}{4\pi}\left[\frac{i\epsilon^{jlm}k^l v^j v^i(\omega+\xi(\mathbf{v}\cdot\hat{\mathbf{n}})(\mathbf{k}\cdot\hat{\mathbf{n}}))}{(\mathbf{v}\cdot\mathbf{k}+i\epsilon)(1+\xi(\mathbf{v}\cdot\hat{\mathbf{n}})^2)^{3/2}} + \left(\frac{v^j+\xi(\mathbf{v}\cdot\hat{\mathbf{n}})\hat{n}^j}{(1+\xi(\mathbf{v}\cdot\hat{\mathbf{n}})^2)^{3/2}}\right) i\epsilon^{iml}k^l v^j \right.$$
$$\left. - i\epsilon^{ijl}k^l v^j \left(\delta^{mn} + \frac{v^m k^n}{\mathbf{v}\cdot\mathbf{k}+i\epsilon}\right)\left(\frac{v^n+\xi(\mathbf{v}\cdot\hat{\mathbf{n}})\hat{n}^n}{(1+\xi(\mathbf{v}\cdot\hat{\mathbf{n}})^2)^{3/2}}\right)\right] \tag{4.16}$$

where,

$$m_D^2 = -\frac{e^2}{2\pi^2}\int_0^\infty d\tilde{p}\tilde{p}^2\left[\frac{\partial n^{0(0)}_{\tilde{\mathbf{p}}}(\tilde{p}-\mu_R)}{\partial\tilde{p}} + \frac{\partial n^{0(0)}_{\tilde{\mathbf{p}}}(\tilde{p}+\mu_R)}{\partial\tilde{p}}\right.$$
$$\left. + \frac{\partial n^{0(0)}_{\tilde{\mathbf{p}}}(\tilde{p}-\mu_L)}{\partial\tilde{p}} + \frac{\partial n^{0(0)}_{\tilde{\mathbf{p}}}(\tilde{p}-\mu_L)}{\partial\tilde{p}}\right]$$
$$C_E = -\frac{e^2}{4\pi^2}\int_0^\infty d\tilde{p}\tilde{p}\left[\frac{\partial n^{0(0)}_{\tilde{\mathbf{p}}}(\tilde{p}-\mu_R)}{\partial\tilde{p}} - \frac{\partial n^{0(0)}_{\tilde{\mathbf{p}}}(\tilde{p}+\mu_R)}{\partial\tilde{p}}\right.$$
$$\left. - \frac{\partial n^{0(0)}_{\tilde{\mathbf{p}}}(\tilde{p}-\mu_L)}{\partial\tilde{p}} + \frac{\partial n^{0(0)}_{\tilde{\mathbf{p}}}(\tilde{p}-\mu_L)}{\partial\tilde{p}}\right]. \tag{4.17}$$

We would like to mention that the total induced current is, $\mathbf{j} = \mathbf{j}^\epsilon + \mathbf{j}^{\epsilon\delta}$ where, $\mathbf{j}^\epsilon$ gives the contribution of the order of the square of plasma frequency or m_D^2. The plasma frequency contains additive contribution from the densities of all species i.e. right-handed particle/antiparticles and left-handed particles/antiparticles. The current $\mathbf{j}^{\epsilon\delta}$ arises due to the chiral imbalance and its contribution from each plasma specie depends upon $e\vec{\Omega}_p$. Since $e\vec{\Omega}_p$ can change sign depending on the plasma specie, the definition of C_E contains both positive and negative signs. Consequently the relative signs of fermion and anti-fermion are different in m_D^2 and C_E. After performing above integrations one can get $m_D^2 = e^2\left((\mu_R^2+\mu_L^2)/2\pi^2 + T^2/3\right)$ and $C_E = (e^2\mu_5)/4\pi^2$, where $\mu_5 = \mu_R - \mu_L$. It should be emphasized here that $C_E = 0$ when there is no chiral imbalance, whereas $m_D^2 \neq 0$. It should also be noted that the terms with anisotropy parameter ξ are contributing in the parity-odd part of the self-energy given by Eq.(4.16). An introduction of chemical potential μ_5 for chiral fermions requires some qualification. Physically a chiral chemical potential implies an imbalance between the right handed and left handed fermion. This in turn is related to the topological charge [271, 272]. It should be noted here that chiral chemical potential is not associated with any conserved charge due to the axial anomaly. It can still be regarded as 'chemical potential' if its variation is sufficiently slow [270].

4.4 Finding the poles of $[\Delta(K)]^{ij}$ or the dispersion relation

In order to get the expression for the propagator Δ^{ij}, it is necessary to write Π^{ij} in a tensor decomposition. For the present problem, we need the six independent projectors. For an isotropic parity-even plasmas one may need the transverse $P_T^{ij} = \delta^{ij} - k^i k^j/k^2$ and the longitudinal $P_L^{ij} = k^i k^j/k^2$ tensor projectors. Due to the presence of *anisotropy* vector $\hat{\mathbf{n}}$, one needs two more projectors $P_n^{ij} = \tilde{n}^i\tilde{n}^j/\tilde{n}^2$ and $P_{kn}^{ij} = k^i\tilde{n}^j + k^j\tilde{n}^i$ [273]. To account for the parity odd effect we have included two anti-symmetric operators $P_A^{ij} = i\epsilon^{ijk}\hat{k}^k$ and $P_{An}^{ij} = i\epsilon^{ijk}\tilde{n}^k$ where, $\tilde{n}^i = (\delta^{ij} - k^i k^j/k^2)\hat{n}^j$. Thus we write Π^{ij} into the basis spanned by the above six operators as:

$$\Pi^{ij} = \alpha P_T^{ij} + \beta P_L^{ij} + \gamma P_n^{ij} + \delta P_{kn}^{ij} + \lambda P_A^{ij} + \chi P_{An}^{ij} \tag{4.18}$$

where, α,β, γ, δ, λ and χ are some scalar functions of k and ω and are yet to be determined. Similarly we can write $[\Delta^{-1}(k)]^{ij}$ appearing in Eq.(4.7) as,

$$[\Delta^{-1}(K)]^{ij} = C_T P_T^{ij} + C_L P_L^{ij} + C_n P_n^{ij} + C_{kn} P_{kn}^{ij} + C_A P_A^{ij} + C_{An} P_{An}^{ij}. \tag{4.19}$$

Using Eqs.(4.7, 4.18, 4.19), one can find relation between C's and the scalar functions appearing in Eq.(4.18) as,

$$\begin{aligned} C_T = k^2 - \omega^2 + \alpha, \qquad & C_L = -\omega^2 + \beta, \\ C_n = \gamma, \qquad & C_{kn} = \delta, \\ C_A = \lambda, \qquad & C_{An} = \chi. \end{aligned} \tag{4.20}$$

For $\xi \to 0$, using Eqs.(4.15-4.16), one finds $\alpha_{|\xi=0} = \Pi_T$, $\beta_{|\xi=0} = \frac{\omega^2}{k^2}\Pi_L$, $\gamma_{|\xi=0} =$

0, $\delta_{|\xi=0} = 0$, $\lambda_{|\xi=0} = -\frac{\Pi_A}{2}$ and $\chi_{|\xi=0} = 0$, where

$$\Pi_T = m_D^2 \frac{\omega^2}{2k^2}\left[1 + \frac{k^2-\omega^2}{2\omega k}\ln\frac{\omega+k}{\omega-k}\right], \qquad \Pi_L = m_D^2\left[\frac{\omega}{2k}\ln\frac{\omega+k}{\omega-k} - 1\right],$$
$$\Pi_A = -2kC_E\left(1-\frac{\omega^2}{k^2}\right)\left[1-\frac{\omega}{2k}\ln\frac{\omega+k}{\omega-k}\right]. \tag{4.21}$$

Scalar functions Π_T, Π_L and Π_A respectively describe the transverse, longitudinal and the axial parts of the self-energy decomposition when $\xi = 0$[270].

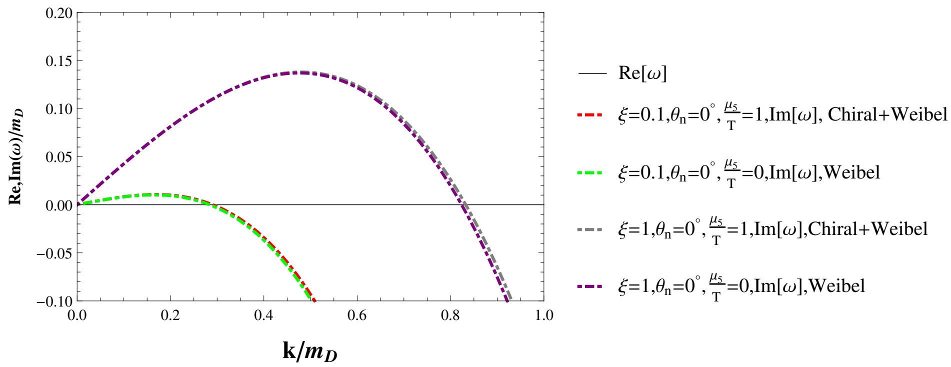
Using the orthogonality condition, $[\Delta^{-1}(K)]^{ij}[\Delta(K)]^{jl} = \delta^{il}$, $[\Delta(K)]^{jl}$ can be determined. Poles of $[\Delta(K)]^{jl}$ are given by the following equation,

$$\begin{aligned} &2k\tilde{n}^2 C_A C_{An} C_{kn} + C_A^2 C_L + \tilde{n}^2 C_{An}^2 (C_n + C_T) \\ &+ C_T(k^2\tilde{n}^2 C_{kn}^2 - C_L(C_n + C_T)) = 0. \end{aligned} \tag{4.22}$$

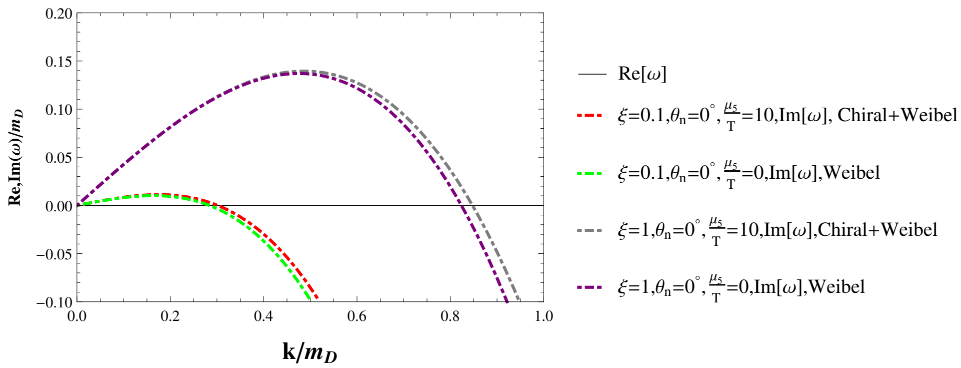
Eq.(4.22) is the general dispersion relation and it is quite complicated to solve analytically or numerically. Here we would like to ascertain that α, β, γ and δ appearing in C's are the same as those given in Ref. [138]. The new contributions come in terms of λ and χ which contain the effect of parity violation. The standard criterion for the Weibel instability [144] is not applicable here due to the parity violating effect. First we note that the chiral instability occurs in the quasi-stationary regime i.e $|\omega| \ll k$, if the initial distribution function of the plasma is isotropic. While, the Weibel instability occurs due to an anisotropy in the initial momentum distribution in the plasma and the instability can also be present in the quasi-stationary regime.

4.4.1 Analysis of collective modes

We study numerical solutions of Eq.(4.22) in quasi-stationary limit. Further, we note that when $C_A, C_{An} = 0$, there is no chiral-imbalance and one can get the pure Weibel modes from Eq.(4.22). The pure chiral-imbalance modes can be obtained by setting $C_n, C_{kn}, C_{An} = 0$ in Eq.(4.22). In order to obtain the growth-rates for the instabilities, one needs to solve Eq.(4.22) for ω. By setting $\frac{\partial\omega}{\partial k} = 0$ one can find k_{max} for which the instability can grow maximally. Upon substituting k_{max} in the expression for ω and using $\omega = i\Gamma$, one can find the growth rate Γ for the instability. Figs. 4.1 and 4.2 depict a comparison between the pure Weibel modes (i.e. $\mu_5 = 0$) with the mixed modes i.e. when both chiral-imbalance and momentum-anisotropy are present.



(a)



(b)

Figure 4.1: Shows plots of real and imaginary part of the transverse dispersion relation for the case when the angle θ_n between the propagation vector $\mathbf{k}$ of the perturbation and the anisotropy direction is zero. The modes are purely imaginary and the real part of frequency $\omega = 0$. Fig. 4.1(a) shows comparison between pure Weibel modes (μ_5=0) with the cases when both the Weibel and chiral-imbalance instabilities are present when μ_5/T =1 and ξ =0.1,1 . Fig. 4.1(b) depicts the similar comparison when μ_5/T =10. It shows that by increasing μ_5/T the chiral-imbalance instability become stronger.

Before we discuss the results, it should be noted that the direction between the propagation vector $\mathbf{k}$ and the anisotropy vector $\hat{\mathbf{n}}$ is quantified by angle θ_n i.e. $\mathbf{k} \cdot \hat{\mathbf{n}} = k \cos\theta_n$ where, k is magnitude of vector $\mathbf{k}$.

In Figs. 4.1(a)-4.1(b) we have considered the case $\theta_n = 0$, where the values

$\mu_5/T = 1$ and $\mu_5/T = 10$ correspond to the mixed modes while $\mu_5/T = 0$ is for pure Weibel modes. These figures show that the Weibel modes become strong with increasing values of anisotropy parameter ξ. It can also be seen that by increasing μ_5/T the chiral-modes become stronger, leading to the enhancement of the mixed modes. In the discussion below, we have obtained analytic results for $\xi \ll 1$ and found a critical value ξ_c at $\theta_n = 0$ such that for $\xi < \xi_c$, the chiral modes will dominate while for $\xi > \xi_c$, the Weibel instability can dominate. Fig. 4.2 depicts the case when $\theta_n = \pi/2$. Here the pure Weibel modes are damped which is a well known result. The damping is increasing with increasing ξ, but it can become weaker by increasing μ_5/T. It is important to note that there also exists a situation for $\xi \gg 1$ when the chiral-imbalance instability can play a dominant role in anisotropic plasma. This is because the Weibel instability growth rate is dependent on θ_n and it is possible to find a particular value of $\theta_n = \theta_{nc}$ when the growth rate of the pure Weibel mode is close to zero.

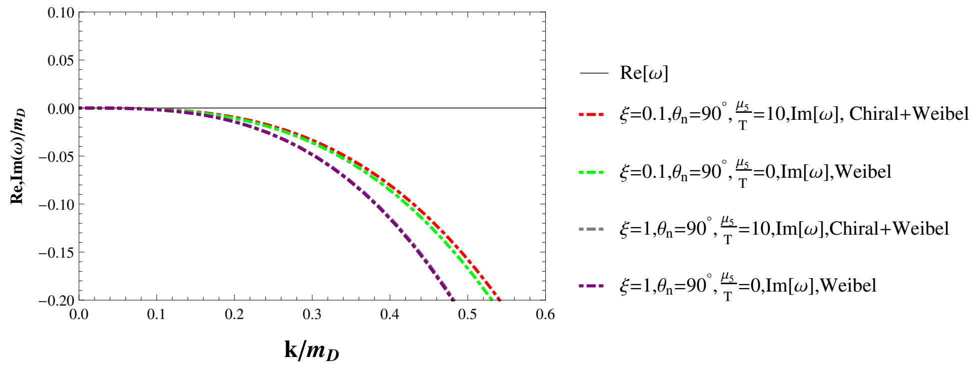


Figure 4.2: Shows plots of the dispersion relation when $\theta_n = \pi/2$. The pure Weibel modes are known to give damping when $\theta_n = \pi/2$. For the instances when both the chiral-imbalance and Weibel instabilities are present (μ_5/T =10 and $\xi = 0.1$,1) the damping can become weaker.

By setting $\omega = 0$ in the pure Weibel dispersion relation, one can find for $\xi \gg 1$, $\theta_{nc} \sim \left(\pi m_D^2/2k^2\xi^{1/2}\right)^{1/2}$. In the regime $\xi < 1$, but closer to unity at $\theta_n = 0$, a comparison between the growth rates of the chiral-imbalance (Γ_{ch}) and Weibel (Γ_w) instabilities is given in the following table:

ξ	0.6	0.7	0.8	0.9
$\frac{\Gamma_{ch}}{\Gamma_w}$	$\frac{0.0088\alpha_e^{3/2}\mu_5^3}{T^3}$	$\frac{0.0076\alpha_e^{3/2}\mu_5^3}{T^3}$	$\frac{0.0067\alpha_e^{3/2}\mu_5^3}{T^3}$	$\frac{0.0060\alpha_e^{3/2}\mu_5^3}{T^3}$

Table 4.1: Ratio of Γ_{ch} to Γ_w for various values of ξ closer to unity

Thus the ratio Γ_{ch}/Γ_w decreases by increasing the values of ξ while keeping μ_5/T fixed. This is because Γ_w increases by increasing ξ. For $\alpha_e = 1/137$ and $\mu_5/T \leq 1$ one can clearly see from the table that the ratio $\Gamma_{ch}/\Gamma_w \ll 1$. Thus Weibel modes dominate in this case. However when $\mu_5/T \gg 1$ chiral modes can also dominate.

4.4.2 Analysis of the collective modes in small ξ limit

Now we consider the case $\xi \ll 1$. This approximation is valid when the initial momentum anisotropy is weak or the Weibel instability has already nearly thermalized (or isotropized) the plasma. This may not be an unlikely scenario in the heavy ion collisions as the growth rates for the Weibel instabilities can be much larger than the chiral instability. In this case it is possible to evaluate all the integrals in the dispersion relation analytically and one can express α, β, γ, δ, λ and χ up to linear order in ξ as follows,

$$\begin{aligned}
\alpha = \Pi_T + \xi\Big[&\frac{z^2}{12}(3+5\cos 2\theta_n)m_D^2 - \frac{1}{6}(1+\cos 2\theta_n)m_D^2 \\
&+ \frac{1}{4}\Pi_T\Big((1+3\cos 2\theta_n) - z^2(3+5\cos 2\theta_n)\Big)\Big]; \\
z^{-2}\beta = \Pi_L + \xi\Big[&\frac{1}{6}(1+3\cos 2\theta_n)m_D^2 + \Pi_L\Big(\cos 2\theta_n - \frac{z^2}{2}(1+3\cos 2\theta_n)\Big)\Big]; \\
\gamma &= \frac{\xi}{3}(3\Pi_T - m_D^2)(z^2-1)\sin^2\theta_n; \\
\delta &= \frac{\xi}{3k}(4z^2 m_D^2 + 3\Pi_T(1-4z^2))\cos\theta_n; \\
\lambda = -\frac{\mu_5 k e^2}{4\pi^2}&\Big[(1-z^2)\frac{\Pi_L}{m_D^2}\Big] - \xi\frac{\mu_5 k e^2}{32\pi^2}\Bigg[(1-z^2)\frac{\Pi_L}{m_D^2}\times \\
&\Big((1+7\cos 2\theta_n) - 3z^2(1+3\cos 2\theta_n)\Big) \\
&+\frac{1}{3}(1+11\cos 2\theta_n) - z^2(3+5\cos 2\theta_n)\Bigg]; \\
\chi &= \xi\left[f(\omega,k)\right],
\end{aligned} \tag{4.23}$$

where $z = \omega/k$ and $f(\omega, k)$ is some function of k and ω. However, in the present analysis, exact form of $f(\omega, k)$ is not required. Using the above equations with Eqs. (4.20, 4.21) one can finally express Eq.(4.22) in terms of k and ω. One can notice from Eq.(4.23) that the most significant contribution for γ, δ, λ and χ is $O(\xi)$. Thus in the present scheme of approximation one can write Eq.(4.22) up to $O(\xi)$ as:

$$C_A^2 C_L - C_T C_L (C_n + C_T) = 0, \tag{4.24}$$

which in turn can give the following two branches of the dispersion relation,

$$C_A^2 - C_T^2 - C_n C_T = 0, \tag{4.25}$$

$$C_L = 0. \tag{4.26}$$

Note that when $C_A = 0$, Eqs.(4.25-4.26) reduce to exactly the same dispersion relation discussed in Ref.[138] for the Weibel instability in an anisotropic plasma when there is no parity violating effect. Let us consider Eq.(4.25), it can be written as:

$$(k^2 - \omega^2)^2 + (k^2 - \omega^2)(2\alpha + \gamma) + \alpha^2 + \alpha\gamma - \lambda^2 = 0. \tag{4.27}$$

This equation is a quadratic equation in $(k^2 - \omega^2)$ and it's solutions can be written as:

$$(k^2 - \omega^2) = \frac{-(2\alpha + \gamma) \pm 2\lambda}{2}. \tag{4.28}$$

Now, it is of particular interest to consider the quasi-static limit $|\omega| \ll k$. In this limit, expressions for $\alpha \sim \Pi_T$, $\beta \sim \frac{\omega^2}{k^2}\Pi_L$ and $\lambda \sim -\frac{\Pi_A}{2}$. Π_L, Π_T and Π_A can be obtained by expanding Eq.(4.21) in the quasi static limit as:

$$\begin{aligned}
\Pi_{T|_{|\omega| \ll k}} &= \left(\mp i \frac{\pi}{4} \frac{\omega}{k}\right) m_D^2; \\
\Pi_{L|_{|\omega| \ll k}} &= m_D^2 \left[\mp i \frac{\pi}{2} \frac{\omega}{k} - 1\right] \\
\Pi_{A|_{|\omega| \ll k}} &= \frac{\mu_5 k e^2}{2\pi^2} \left(\frac{\Pi_{L|_{|\omega| \ll k}}}{m_D^2}\right)
\end{aligned} \tag{4.29}$$

Thus in the quasi-stationary limit one can write positive branch of the transverse modes given by Eq.(4.28) as:

$$\rho(k) = \frac{\left(\frac{4\alpha_e\mu_5}{\pi^2 m_D^2}\right) k^2 \left[1 - \frac{\pi k}{\mu_5 \alpha_e} + \frac{\xi(1+5\cos 2\theta_n)}{12} + \frac{\xi(1+3\cos 2\theta_n)}{12}\frac{\pi m_D^2}{\mu_5\alpha_e k}\right]}{\left[1 + \frac{2\mu_5\alpha_e k}{\pi m_D^2}(1-\frac{\xi}{4}) + \xi\cos 2\theta_n\left(1 - \frac{7\mu_5\alpha_e k}{2\pi m_D^2}\right)\right]} \tag{4.30}$$

Here we have used $\omega = i\rho(k)$ and defined $\alpha_e = e^2/4\pi$ as the electromagnetic coupling. It is clear from Eq.(4.30) that ω is purely an imaginary number and its real-part is zero i.e. $Re(\omega) = 0$. Positive $\rho(k) > 0$ implies an instability as $e^{-i(i\rho(k))t} \sim e^{+\rho(k)t}$. From Eq.(4.30), in the limit $\xi \to 0$ and $\mu_5 \to 0$ one gets $\rho(k) = -4k^3/\pi m_D^2$. Thus for an isotropic plasma (of massless particles), without any chiral-imbalance there is no unstable propagating mode when $\omega \ll k$. This is consistent with the fact that without any source of free energy there should not be any unstable mode.

Now let us first consider that the quasi-static limit, $|\omega| \ll k$, is indeed satisfied for Eq.(4.30). Since we have already assumed that $\xi \ll 1$ and $\alpha_e < 1$, also for $\mu_5 \ll T$ one has $\frac{\mu_5}{m_D} \approx \frac{1}{2\alpha_e^{1/2}}\left(\frac{\mu_5}{T}\right)$. It is then rather easy to show that $\rho/k \ll 1$, if the condition $\frac{k^2}{m_D^2} \ll 1$ is satisfied. In this case denominator of Eq.(4.30) can be approximated to unity. Now we write the above equation as:

$$\rho(k) = \frac{4}{\pi}\frac{k^2}{m_D^2}\left[\frac{\alpha_e\mu_5}{\pi} - k + \frac{\alpha_e\xi\mu_5}{12\pi}(1+5\cos 2\theta_n) + \frac{\xi}{12}\frac{m_D^2}{k}(1+3\cos 2\theta_n)\right]. \tag{4.31}$$

Here we emphasize that when $\xi = 0$, the first two terms in the square bracket survive and the Eq.(4.31) matches with the dispersion relation of the chiral instability given in Ref.[270]. When $\mu_5 = 0$, the second and the last term survive to give the Weibel modes considered in Ref.[138]. Term with $\alpha_e\xi\mu_5$ factor arises due to the interaction between the Weibel and chiral-imbalance modes.

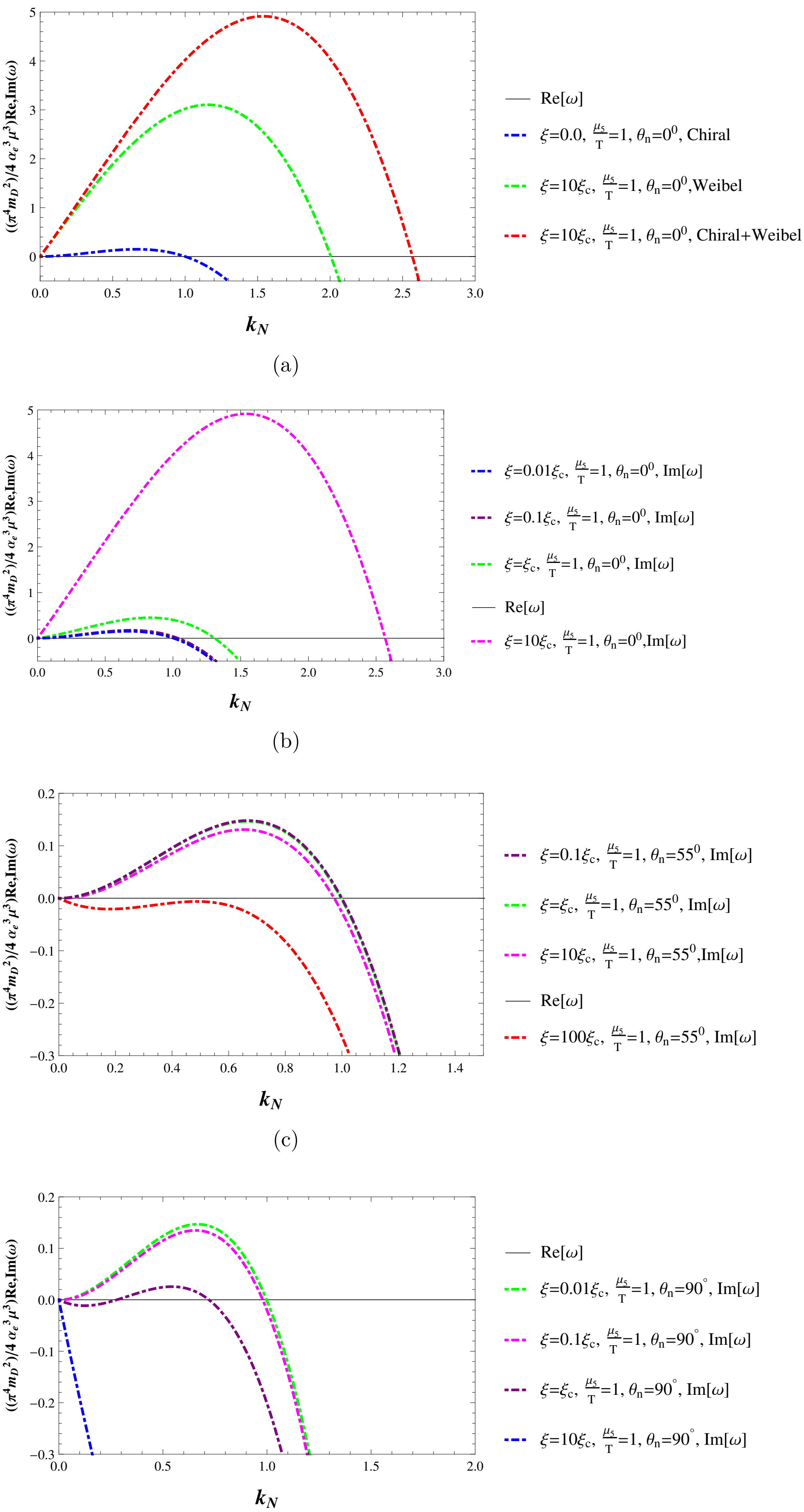
Before we analyse the interplay between the chiral-imbalance and the Weibel instabilities, it is instructive to qualitatively understand their origin. Let us first consider the chiral-imbalance instability. For such a plasma 'chiral-charge' density n is given by $\partial_t n + \nabla \cdot \mathbf{j} = \frac{2\alpha_e}{\pi}\mathbf{E}\cdot\mathbf{B}$. From this one can estimate the axial charge density $n \sim \alpha_e k A^2$, where A is the gauge-field. Assuming that there are only the right handed particles i.e. ($\mu_5 \sim \mu_R$), then the number and energy densities of the plasma are respectively given by $\mu_5 T^2$ and ${\mu_5}^2 T^2$. The fermionic number density associated with the gauge field can be estimated from the Chern-Simon term to be $\alpha_e k A^2$. The number densities associated with the fields and particles have the same value for $k_1 \sim \mu_5 T^2/\alpha_e A^2$. The typical energy for the gauge field is $\epsilon_A \sim k^2 A^2$. For this particular value of k_1 it can be seen that $\epsilon_A = {\mu_5}^2 T^2 \frac{T^2}{\alpha_e^2 A^2}$. Thus there exists a state satisfying the condition $T^2/\alpha_e^2 < A^2$ for which energy in the gauge field is lower than particle energy. This leads to the chiral-imbalance instability[270, 274]. The Weibel instability arises when the equilibrium distribution function of the plasma has anisotropy in the momentum space[132, 239]. The anisotropy in the momentum space can be regarded as anisotropy in temperature. Suppose there is plasma which is hotter in the y-direction than the x or z direction, one may write $n_p^0 = 1/(1+e^{-(\sqrt{p_x^2+(1+\xi)p_y^2+p_z^2})/T})$. If in this situation, a disturbance with a magnetic field $B = B_0 cos(kx)$ arises, say from noise, one can write the Lorentz force term in the kinetic equation as $e(v \times B)\cdot \partial_p n_p^0 = e[\xi(v_z B_x - v_x B_z)\frac{p_y}{T}]\left(-e^{-(\sqrt{p_x^2+(1+\xi)p_y^2+p_z^2})/T}/(1+e^{-(\sqrt{p_x^2+(1+\xi)p_y^2+p_z^2})/T})\right)$. This Lorentz force can produce current sheets where the magnetic field changes its sign. The current-sheet in turn enhances the original magnetic field [132, 239].

The Weibel instability is known to grow maximally for $\theta_n = 0$. In the quasi-static limit, the instability has the maximum growth rate $\Gamma_w \sim \frac{8\xi^{3/2}}{27\pi} m_D$ for $k = \frac{\sqrt{\xi}}{3} m_D$. For the chiral imbalance instability, the maximum growth rate $\Gamma_{ch} \sim \frac{16\alpha_e^3}{27\pi^4}\left(\frac{\mu_5}{m_D}\right)^2 \mu_5$, occurs at $k \sim \frac{2\alpha_e}{3\pi}\mu_5$[270]. Thus the ratio $\frac{\Gamma_{ch}}{\Gamma_w} \sim \frac{2}{\pi^3}\left(\frac{\alpha_e}{\xi^{1/2}}\right)^3 \left(\frac{\mu_5}{m_D}\right)^3 \sim \frac{1}{4\pi^3}\left(\frac{\alpha_e}{\xi}\right)^{3/2}\left(\frac{\mu_5}{T}\right)^3$, where we have used $\frac{\mu_5}{m_D} \approx \frac{1}{2\alpha_e^{1/2}}\left(\frac{\mu_5}{T}\right)$. The ratio $\frac{\Gamma_{ch}}{\Gamma_w}$ becomes unity when $\xi_c \approx 2^{2/3}\left(\frac{\alpha_e}{4\pi^2}\right)\left(\frac{\mu_5}{T}\right)^2$. When $\mu_5 \sim T$ and $\alpha_e = 1/137$ (QED), one can estimate $\xi_c < 10^{-3}$. ξ_c will change if coupling varies (QCD case). Thus for $\xi_c < \xi \ll 1$ the Weibel instability can dominate over the chiral-imbalance modes. However, it may still be possible to see the chiral-imbalance modes if we consider θ_n dependence of the instability as described by Eq.(4.31). In Eq.(4.31) the Weibel instability term vanishes if $\theta_n \sim \frac{1}{2}\cos^{-1}(1/3) \sim 55°$. For this value of θ_n the interaction term between the Weibel and the chiral modes becomes negative and tries to suppress the unstable mode. However this term is very small in comparison to the pure chiral term.

In Fig. 4.3 we plot the dispersion relation given by Eq.(4.30) as a function of $k_N = \frac{\pi}{\alpha_e \mu_5} k$ for various values of ξ which is given in units of ξ_c and the propagation angle θ_n. y-axis shows the $Re[\omega]$ and $Im[\omega]/\left(\frac{4\alpha_e^3 \mu_5{}^3}{\pi^4 m_D^2}\right)$. Note that the real part of the frequency $Re[\omega]$ is zero. For the case when $\xi = 0$ there is no Weibel mode and the only the chiral-imbalance can give the instability, whereas when $\mu_5 = 0$, only Weibel instability will contribute. From the condition $\rho(k) > 0$, one can obtain the range of the instability which can be stated as,

$$k_N = 1 + \frac{\xi\,(1 + cos2\theta_n)}{12} + \left[\left(1 + \frac{\xi\,(1 + cos2\theta_n)}{12}\right)^2 + \frac{\pi^2 \xi\,(1 + 3cos2\theta_n)}{3\alpha_e}\right]^{1/2} \tag{4.32}$$

In Fig. 4.3(a) we have shown for $\theta_n = 0$, the pure Weibel case ($\xi = 10\xi_c$ and $\mu_5 = 0$) and the pure chiral-case ($\xi = 0$ and $\mu_5 \neq 0$) along with the case when both the instabilities are present (i.e. $\xi = 10\xi_c$ and $\mu_5 \neq 0$). The plot shows that the pure Weibel modes dominate over the pure chiral case. However, the combined effect of both the instabilities is much more pronounced. The maximum growth rate and the range of the instability are altered significantly for the combined case. In Figs. 4.3(b)-4.3(d) we study the cases where both the instabilities are present and ξ and θ_n vary when $\mu_5/T = 1$. It is important to note that in this analysis we are showing the plots of the dispersion relation following the same normazation as used in Ref.[270] so that we can compare our results. Due to the normalization of dispersion relation, Weibel term picks up a factor μ_5/T. Therefore, Weibel instability appears to be also dependent on μ_5/T apart from the variables ξ and θ_n. However, in order to take limit $\mu_5 \to 0$ one need to undo the normalization in terms of $Im[\omega]$ and k. Fig. 4.3(b) clearly shows for $\theta_n = 0$ when condition $\xi \ll \xi_c$ is satisfied, the chiral instability dominates over the Weibel modes. However, such values of ξ are extremely small.



$((\pi^4 m_D^2)/4\,\alpha_e^3\mu^3)Re,Im(\omega)$
k_N
Re[ω]
ξ=0.0, $\frac{\mu_5}{T}$=1, θ_n=0^0, Chiral
ξ=10ξ_c, $\frac{\mu_5}{T}$=1, θ_n=0^0,Weibel
ξ=10ξ_c, $\frac{\mu_5}{T}$=1, θ_n=0^0, Chiral+Weibel

(a)

$((\pi^4 m_D^2)/4\,\alpha_e^3\mu^3)Re,Im(\omega)$
k_N
ξ=0.01ξ_c, $\frac{\mu_5}{T}$=1, θ_n=0^0, Im[ω]
ξ=0.1ξ_c, $\frac{\mu_5}{T}$=1, θ_n=0^0, Im[ω]
ξ=ξ_c, $\frac{\mu_5}{T}$=1, θ_n=0^0, Im[ω]
Re[ω]
ξ=10ξ_c, $\frac{\mu_5}{T}$=1, θ_n=0^0,Im[ω]

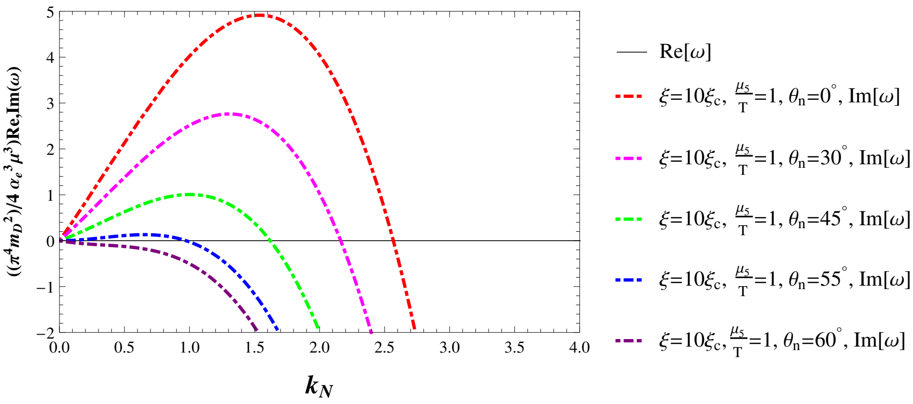
(b)

$((\pi^4 m_D^2)/4\,\alpha_e^3\mu^3)Re,Im(\omega)$
k_N
ξ=0.1ξ_c, $\frac{\mu_5}{T}$=1, θ_n=55^0, Im[ω]
ξ=ξ_c, $\frac{\mu_5}{T}$=1, θ_n=55^0, Im[ω]
ξ=10ξ_c, $\frac{\mu_5}{T}$=1, θ_n=55^0,Im[ω]
Re[ω]
ξ=100ξ_c, $\frac{\mu_5}{T}$=1, θ_n=55^0, Im[ω]

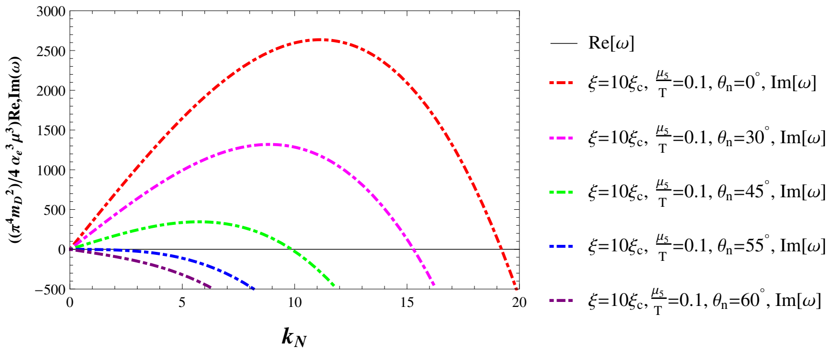
(c)

$((\pi^4 m_D^2)/4\,\alpha_e^3\mu^3)Re,Im(\omega)$
k_N
Re[ω]
ξ=0.01ξ_c, $\frac{\mu_5}{T}$=1, θ_n=90°, Im[ω]
ξ=0.1ξ_c, $\frac{\mu_5}{T}$=1, θ_n=90°, Im[ω]
ξ=ξ_c, $\frac{\mu_5}{T}$=1, θ_n=90°, Im[ω]
ξ=10ξ_c, $\frac{\mu_5}{T}$=1, θ_n=90°, Im[ω]

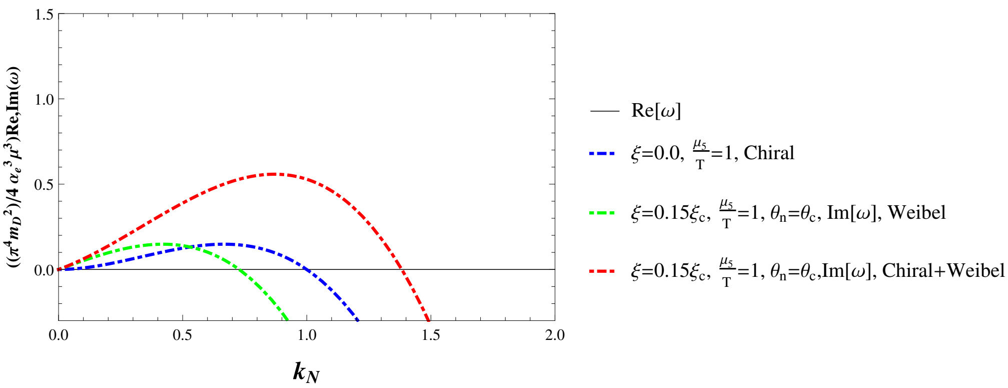
(d)



(e)



(f)



(g)

Figure 4.3: Shows plots of real and imaginary part of the dispersion relation. Here θ_n is the angle between the wave vector k and the anisotropy vector. Real part of dispersion relation is zero. Fig. 4.3(a) show plots for three cases: (i) pure chiral (no anisotropy), (ii) pure Weibel (chiral chemical potential=0) and (iii) when both chiral and Weibel instabilities are present. Fig. 4.3(b)-4.3(d) represent the case when both the instabilities are present but the anisotropy parameter varies at different values of θ_n for fixed $\mu_5/T = 1$. Fig. 4.3(e)-4.3(f) represents the case when both the instabilities are present for a fixed anisotropy parameter at different values of θ_n when $\mu_5/T = 1$ and $\mu_5/T = 0.1$ respectively. Fig. 4.3(g) represents the case when for a particular value of $\theta_n \sim \theta_c$ both the instabilities have equal growth rates. Here frequency is normalized in the unit of $\omega/\left(\frac{4\alpha_e^3 {\mu_5}^3}{\pi^4 m_D^2}\right)$ and wave-number k by $k_N = (\pi k)/(\mu_5 \alpha_e)$.

For the cases when $\xi \geq \xi_c$, the Weibel modes are dominating. Contribution from the Weibel modes is maximum for $\theta_n = 0$ and the modes are strongly damped at $\theta_n = \pi/2$. Angular part in the dispersion relation for the pure Weibel modes becomes zero when $\theta_n \approx 55^0$. In this case, one can see that the chiral modes can remain dominant. This case is shown in Fig. 4.3(c). It should be noted that for the case when $\xi \gg \xi_c$ the contribution from the coupling term between the Weibel and chiral modes become sufficiently strong and it can again suppress the instability. In Fig. 4.3(d) we have shown the case when $\theta_n = \pi/2$. The modes with $\xi \geq \xi_c$ are strongly damped and there is no instability. Here, the coupling term between the two modes also contribute in the damping of the instability. In Figs. 4.3(e)-4.3(f) we have plotted the unstable modes for $\xi = 10\xi_c$ for different values of θ_n when $\mu_5/T = 1$ and 0.1 respectively. When $\mu_5/T = 0.1$ (i.e. $\mu_5 \ll T$) the instability increases enormously. Now, by comparing the growth rates of the pure Weibel and pure chiral modes, when $\mu_5/T = 1$, one can find that they become equal at $\theta_c = \frac{1}{2}\cos^{-1}\left[\left(\frac{2}{27}\right)^{2/3}\frac{3\alpha_e}{\xi\pi^2} - \frac{1}{3}\right]$. Fig. 4.3(g) represents this case where we have shown that the growth rate of the pure Weibel case at $\xi = 0.15\xi_c$ becomes comparable to the pure chiral mode with $\xi = 0$. The topmost (red) curve in this figure shows the case when both the modes operate together. This case shows that the combined effect of the instability can significantly alter the range and the growth rate of the instability.

4.5 Conclusions

In conclusion, we have studied collective modes in an anisotropic chiral plasma where the both Weibel and chiral-imbalance instabilities are present. Out of these two instabilities which one will dominate in a given physical situation depends upon three parameters, θ_n, ξ and μ_5/T. We have demonstrated that for $\theta_n = 0$ and $\mu_5/T \sim 1$, when $\xi \geqslant 1$, $\xi < 1$ but closer to unity or $\xi \sim \xi_c \ll 1$, the Weibel modes dominate over the chiral-imbalance instability. It was shown analytically that for $\theta_n = 0$ and $\mu_5/T \sim 1$, only for a very small values of the anisotropy parameter $\xi \sim \xi_c \ll 1$, growth rates of both the instabilities are comparable.

It was also demonstrated numerically that for $\xi < \xi_c$, $\mu_5/T \sim 1$, there exists a critical angle $\theta = \theta_c$ at which growth rates of the two instabilities can also be comparable. We have also shown for the case when $\xi \gg 1$, the chiral-imbalance can dominate over Weibel modes when $\theta = \theta_{nc}$.

Chapter 5

CP Violation, Turbulence and Anomalous Viscosity

The suggestion that the strongly interacting matter created in the relativistic heavy ion collision experiments can have local P and CP violations has created a lot of excitement. According to Refs. [205–208] the proposed P and CP violations in QCD can be due to finite non-zero topological charges present at high temperature and density. In the presence of a very strong magnetic field (that can be created during the heavy ion collision) the non-zero topological charge can induce a net chiral imbalance leading to a phenomenon known as 'chiral magnetic effect' (CME) [207, 211, 271]. In a different context, this phenomenon has also been considered in the field of cosmology [275–277]. Theoretical models that study these aspects of strongly interacting matter consider a plasma of massless fermions which interacts with each other in chiral invariant way. There exist both the hydrodynamical and kinetic theory based models describing such a plasma in which the quantum mechanical nature of the chiral anomaly can have a macroscopic consequences.

It should be noted here that the effect of parity violation due to weak-interaction is considered to be important in the context of the core collapsing supernova and the formation of neutron stars[278, 279] e.g. the peculiar velocity of pulsar [280] or in the generation of magnetic field during the core collapsing neutron star [281–283]. However, the role of parity violating effects due to the strong sector in a quark star is not fully explored. In the present work, we consider the chiral-imbalance instability, which may arise either in core collapsing supernova due to weak process [282] or in a quark matter in the interior of a neutron star due to a strong process. Such instabilities have been studied in the context of electromagnetic and quark-gluon plasma at finite temperature using the Berry-curvature modified kinetic equation [270, 284]. A similar kind of instability can exist in the case of a electroweak plasma and early universe [282]. In Ref. [270] it was argued that the chiral-imbalance instability can lead to the growth of Chern-Simons number (or magnetic-helicity plasma physics parlance) at expense of the chiral imbalance. Subsequently in Refs.[282, 285], it was shown that the generation of magnetic helicity in the presence of chiral instability may lead to a huge magnetic field of the order of 10^{16} G in the core of a compact star. Such kind of instabilities were mentioned in Refs. [231, 272, 274, 286, 287] in a different context and may be seen in the heavy ion collisions.

In this chapter, we calculate the coefficient of shear viscosity arising due to the Chiral-imbalance instability generated turbulent transport in an isotropic chiral plasma (which can found in the core of a neutron star or QGP created in heavy ion collisions). By definition, η measures the ratio of stress to velocity gradient. Stress in a medium arises because of momentum transfer/diffusion generated by a velocity gradient [288]. The momentum transfer in a medium is usually governed by collision. However, in the case of a turbulence interaction between particles and field can enhance the decorrelation frequency and the effective viscosity can be written as,

$$\eta \sim \frac{Stress}{\nu_{collision} + \nu_{decorrelation}}, \tag{5.1}$$

where, $\nu_{collision}$ and $\nu_{decorrelation}$ respectively denote the collision and decorrelation frequencies. In the case of a neutron star, collision frequency can become very small as temperature T becomes small [289] and thus the decorrelation frequency can have dominant contribution in the determination of η.

This chapter is organized is as follows. In section §5.1 we discuss the chiral instability. In section §5.2 we discuss how chiral instability leads to a turbulent transport. In section §5.3 we give a typical estimate to anomalous shear viscosity.

Finally in section §5.4 we conclude our results.

5.1 Chiral-imbalance instability

In chapter §4 we have considered the case of anisotropic chiral plasma and obtained a general dispersion relation that shows both types of instabilities (Weibel and the chiral-imbalance). Here we consider the case of isotrpic chiral plasma. In this case, the dispersion relation can be obtained by taking the isotropic limit i.e. $\xi \to 0$ of Eq.(4.30) and it is as follows,

$$\omega = i\left(\frac{4\alpha_e\mu_5}{\pi^2 m_D^2}\right)k^2\left[1-\frac{\pi k}{\mu_5\alpha_e}\right], \tag{5.2}$$

It is clear from above equation that ω is purely an imaginary number and its real-part is zero i.e. $Re(\omega) = 0$. One can also notice from above equation that ω can be positive if $1 > \pi k/\mu_5\alpha_e$. Positive ω ($\omega > 0$) implies an instability as $e^{-i(i\rho(k))t} \sim e^{+\rho(k)t}$ due to net chiral chemical potential μ_5. This instability is known as the chiral imbalance instability. The growth rate of this instability will be the maximum at, $k_{max} = 2\mu_5\alpha/3\pi$.

In order to avoid unnecessary complexity, in the next section we consider the case where the right handed particles/antiparticles are much greater than the left handed particles i.e. $\mu_R >> \mu_L$ so that $\mu_5 = \mu_R$.

5.2 Diffusion via nonlinear particle-wave interaction, decorrelation time

We shall use Resonance Broadening theory [290–295]. First we consider the case of high density and low temperature. In this case, it can be shown $\epsilon_{\mathbf{p}} = p - e\left(\frac{\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v}}{2\mu_R}\right) + O(\frac{1}{\mu^2})$[229]. Now, consider the distribution function,

$$n_{\mathbf{p}} = n_{\mathbf{p}}^{0(0)} + en^1_{\mathbf{p}\omega,k}, \tag{5.3}$$

where, $\langle n_{\mathbf{p}}\rangle = \langle n_{\mathbf{p}}^{0(0)}\rangle$, $\langle\rangle$ represents the spatial averaging. $n^1_{\mathbf{p}\omega,k}$ is the coherent response to field fluctuations. Taking the spatial averaging Berry curvature modified kinetic Eq.(4.1) can be written as,

$$\partial_t\langle n_{\mathbf{p}}\rangle = -e^2\left\langle\left(\mathbf{E}_{\omega,k} + \mathbf{v}\times\mathbf{B}_{\omega,k} + i\mathbf{k}\left(\frac{\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v}}{2\mu_R}\right)\right)\cdot\partial_{\mathbf{p}} n^1_{\mathbf{p}\omega,k}\right\rangle \tag{5.4}$$

In the quasilinear theory trajectories of the particles are assumed to be unperturbed irrespective of the presence of the fluctuating fields. As a result, coherent response $n^1_{\mathbf{p}\omega,k}$ has a peak $1/(\omega - \mathbf{k}\cdot\mathbf{v})$. In the resonance broadening theory, one considers the perturbed trajectories of the particles due to the effects of random fields and calculate the approximate coherent response function $n^1_{\mathbf{p}\omega,k}$ as an average over a statistical ensemble or perturbed trajectories. As a result, the peak in the coherent response gets broadened [290, 293]. In the case of resonance broadening theory, response function can be written as [290, 293],

$$n^1_{\mathbf{p},\omega k} = \int_0^\infty dt e^{-i(\omega-\mathbf{k}\cdot\mathbf{v})\mathbf{t}}\langle e^{-ik\delta x(t)}\rangle\left(\mathbf{E}_{\omega,k} + \mathbf{v}\times\mathbf{B}_{\omega,\mathbf{k}} + i\mathbf{k}\left(\frac{\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v}}{2\mu_R}\right)\right)\cdot\partial_{\mathbf{p}}\langle n_{\mathbf{p}}\rangle. \tag{5.5}$$

We take Gaussian probability distribution as,

$$pdf[\delta p] = \frac{1}{\sqrt{\pi D t}} e^{-\frac{(\delta p)^2}{Dt}}. \tag{5.6}$$

With the above probability distribution one can get,

$$\langle e^{-ik\delta x(t)}\rangle_{pdf} \approx e^{-\frac{t^3}{t_c^3}}. \tag{5.7}$$

Here, t_c is given by following equation,

$$t_c^3 = \frac{4\bar{E}_p^2}{k^2 D}, \tag{5.8}$$

where, $\bar{E}_p^2 \equiv \frac{\int d^3\mathbf{p} E_p\langle n_{\mathbf{p}}\rangle}{\int d^3\mathbf{p}\langle n_{\mathbf{p}}\rangle}$.

Substituting Eq.(5.7) in Eq.(5.5) one gets,

$$n^1_{\mathbf{p},\omega k} = \int_0^\infty dt e^{-i(\omega-k\cdot v)t-\frac{t^3}{t_c^3}}\left(\mathbf{E}_{\omega,k} + \mathbf{v}\times\mathbf{B}_{\omega,\mathbf{k}} + i\mathbf{k}\left(\frac{\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v}}{2\mu_R}\right)\right)\cdot\partial_{\mathbf{p}}\langle n_{\mathbf{p}}\rangle. \tag{5.9}$$

Now,

$$\int_0^\infty dte^{-i(\omega-k\cdot v)t-\frac{t^3}{t_c^3}} \simeq -\frac{i}{\omega-\mathbf{k}\cdot\mathbf{v}+i/t_c}. \tag{5.10}$$

Using Eq.(5.9) one can write the following diffusion equation,

$$(\partial_t - \partial_{\mathbf{p}}\cdot\mathbf{D}(\mathbf{p})\cdot\partial_{\mathbf{p}})\langle n_{\mathbf{p}}\rangle = 0, \tag{5.11}$$

where,

$$\mathbf{D}(\mathbf{p}) = -\int d\omega d\mathbf{k}\left(\mathbf{F}_{-\omega,-\mathbf{k}}\frac{\mathbf{i}}{\omega-\mathbf{k}\cdot\mathbf{v}+\mathbf{i}/\mathbf{t_c}}\mathbf{F}_{\omega,\mathbf{k}}\right) \tag{5.12}$$

and

$$\mathbf{F}_{\omega,\mathbf{k}} = e\left(\mathbf{E}_{\omega,\mathbf{k}} + \mathbf{v}\times\mathbf{B}_{\omega,\mathbf{k}} + i\mathbf{k}\left(\frac{\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v}}{2\mu_R}\right)\right). \tag{5.13}$$

In this problem we are interested in the study of diffusion only due to color magnetic excitaions. In this case the diffusion coefficient can be written as,

$$D = ie^2\sum_{\omega,k}\frac{\left(\mathbf{v}\times\delta\mathbf{B}_{\omega,-\mathbf{k}} - i\mathbf{k}\left(\frac{\delta\mathbf{B}_{-\omega,-\mathbf{k}}\cdot\mathbf{v}}{2\mu_R}\right)\right)\left(\mathbf{v}\times\delta\mathbf{B}_{\omega,\mathbf{k}} + i\mathbf{k}\left(\frac{\delta\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v}}{2\mu_R}\right)\right)}{\omega-\mathbf{k}\cdot\mathbf{v}+i/t_c}. \tag{5.14}$$

Now, choosing $\mathbf{k} = k\hat{\mathbf{z}}$, $\delta\mathbf{B}_{\omega,\mathbf{k}} = \delta B_{\omega,k}\hat{\mathbf{y}}$. and considering $\omega = i\gamma$. Then the diffusion coefficient,

$$D = e^2\sum_{\omega,k}\frac{\left(v_z^2|\delta B_{\omega,k}|^2\hat{\mathbf{x}}\hat{\mathbf{x}} + v_xv_z|\delta B_{\omega,k}|^2\hat{\mathbf{x}}\hat{\mathbf{z}} + v_x^2|\delta B_{\omega,k}|^2\hat{\mathbf{z}}\hat{\mathbf{z}} + \frac{v_y^2k^2|\delta B_{\omega,k}|^2}{2\mu_R^2}\hat{\mathbf{z}}\hat{\mathbf{z}}\right)}{(\gamma+1/t_c+ikv_z)} \tag{5.15}$$

For the strong turbulence we can use approximation $(1/t_c)^2 >> (kv_z)^2$ [240]. In this case, at saturation $(\gamma = 0)$ the diffusion coefficient can be written as,

$$D = e^2\sum_{\omega,k}\frac{\left(v_z^2|\delta B_{\omega,k}|^2\hat{\mathbf{x}}\hat{\mathbf{x}} + v_xv_z|\delta B_{\omega,k}|^2\hat{\mathbf{x}}\hat{\mathbf{z}} + v_x^2|\delta B_{\omega,k}|^2\hat{\mathbf{z}}\hat{\mathbf{z}} + \frac{v_y^2k^2|\delta B_{\omega,k}|^2}{2\mu_R^2}\hat{\mathbf{z}}\hat{\mathbf{z}}\right)}{(1/t_c)}. \tag{5.16}$$

Now, taking thermal average of velocities and using Eqs.(5.8, 5.16) one can get the decorrelation time as,

$$\left(\frac{1}{t_c}\right)^4 \sim \frac{e^2k^2}{4\bar{E}_p^{\ 2}}\sum_{\omega',k'}\left(v_T^2|\delta B_{\omega',k'}|^2 + \frac{v_T^2k'^2|\delta B_{\omega',k'}|^2}{2\mu_R^2}\right), \tag{5.17}$$

where, $v_T^2 = \frac{\int d^3\mathbf{p} v_{x,z}^2 \langle n_\mathbf{p}\rangle}{\int d^3\mathbf{p}\langle n_\mathbf{p}\rangle}$. This is the relation between t_c and the intensity of color magnetic excitations.

Now we calculate the decorrelation time for the case of isotropic chiral plasma due to the resonance broadening. We start with the expressions for the self energies for isotropic case, which can be obtained by taking limit, $\xi \to 0$ of Eqs.(4.15,4.16). Due to resonance broadening, self energies acquire a correction as $\omega \to \omega + i/t_c$ and can be written as follows,

$$\Pi_+^{ij}(K) = m_D^2 \int \frac{d\Omega}{4\pi} v^i v^l \left(\delta^{jl} + \frac{v^j k^l}{\mathbf{v}\cdot\mathbf{k} + i/t_c}\right), \tag{5.18}$$

$$\Pi_-^{im}(K) = C_E \int \frac{d\Omega}{4\pi} \left[i\epsilon^{iml} k^l + i\omega \frac{(\epsilon^{jlm} v^i - \epsilon^{ijl} v^m) k^l v^j}{(\mathbf{v}\cdot\mathbf{k} + i/t_c)} \right]. \tag{5.19}$$

It is important to note that here we have considered only the right handed particles so in the Eq.(4.17), m_D^2 and C_E will have contribution from the right handed particles only. Now, using the similar decomposition of self energy as in case of linear stability analysis one can calculate α, and λ to be of the form,

$$\alpha = -\frac{m_D^2\left(\omega + \frac{i}{t_c}\right)}{4k}\left[ln\frac{1-\frac{\omega}{k}+\frac{i}{t_ck}}{1+\frac{\omega}{k}+\frac{i}{t_ck}} \pm i\pi\right]$$
$$+\frac{m_D^2\left(\omega + \frac{i}{t_c}\right)}{4k}\left(\frac{\omega}{k}+\frac{i}{t_ck}\right)\left[2+\left(\frac{\omega}{k}+\frac{i}{t_ck}\right)\left(ln\frac{1-\frac{\omega}{k}+\frac{i}{t_ck}}{1+\frac{\omega}{k}+\frac{i}{t_ck}} \pm i\pi\right)\right],$$

$$\beta = -\frac{m_D^2\left(\omega + \frac{i}{t_c}\right)}{2k}\left(\frac{\omega}{k}+\frac{i}{t_ck}\right)\left[2+\left(\frac{\omega}{k}+\frac{i}{t_ck}\right)\left(ln\frac{1-\frac{\omega}{k}+\frac{i}{t_ck}}{1+\frac{\omega}{k}+\frac{i}{t_ck}} \pm i\pi\right)\right],$$

$$\lambda = kC_E\left[1+\frac{\omega}{2k}\left(ln\frac{1-\frac{\omega}{k}+\frac{i}{t_ck}}{1+\frac{\omega}{k}+\frac{i}{t_ck}} \pm i\pi\right)\right.$$
$$\left.-\frac{\omega}{2}\left(\frac{2}{k}\left(\frac{\omega}{k}+\frac{i}{t_ck}\right)+\frac{1}{k}\left(\frac{\omega}{k}+\frac{i}{t_ck}\right)^2\left(ln\frac{1-\frac{\omega}{k}+\frac{i}{t_ck}}{1+\frac{\omega}{k}+\frac{i}{t_ck}} \pm i\pi\right)\right)\right]. \tag{5.20}$$

Now at saturation $\omega = 0$, therefore,

$$\alpha_{|\omega=0} = -\frac{im_D^2}{4kt_c}\left[-2i\arctan\frac{1}{t_ck} \pm i\pi\right] - \frac{m_D^2}{4k^2t_c^2}\left[2 + \frac{i}{t_ck}\left(-2i\arctan\frac{1}{t_ck} \pm i\pi\right)\right],$$

$$\beta|_{\omega=0} = +\frac{m_D^2}{4k^2t_c^2}\left[2 + \frac{i}{t_ck}\left(-2i\arctan\frac{1}{t_ck} \pm i\pi\right)\right],$$
$$\lambda|_{\omega=0} = kC_E.$$

The decorrelation time can be detrmined by taking the limit $\xi \to 0$ of Eq.(4.28) with $\omega = 0$, and is given by following equation,

$$k^2 - \frac{m_D^2}{2kt_c}\left[\arctan\frac{1}{t_ck} - \frac{\pi}{2}\right] - \frac{m_D^2}{2k^2t_c^2} - \frac{m_D^2}{2k^3t_c^3}\left(\arctan\frac{1}{t_ck} - \frac{\pi}{2}\right) - kC_E = 0. \quad (5.21)$$

This is transcendental equation decorrelation can be obtained by solving this equation.

We consider the case $\mu_R \gg T$, in this case $m_D \sim \left(\frac{2\alpha}{\pi}\right)^{1/2}\mu_R$. Further we consider $k = k_{max} = \frac{2\mu_R\alpha}{3\pi}$ which correspond to the maximum growth rates of chiral instability. In this case, decorrelation time will be dependent on α and μ_R. For $\alpha = 1/137$, the solution for $1/t_c$ of the above equation in terms of μ_R is shown in the following figure.

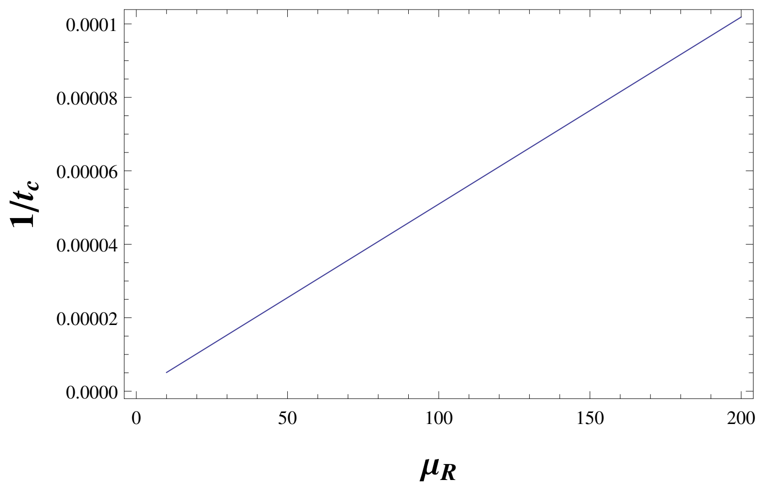


Figure 5.1: Shows plot of decorrelation frequency $1/t_c$ as a function of chiral chemical potential μ_R.

Note that the strong turbulence requires that the condition $\frac{1}{t_c k_{max}} \gg v_z$, which is satisfied in $\mu_R \gg T$ regime. Now if we take $t_c \sim \frac{1}{k}$, $k = k_{max} = \frac{2\mu_R \alpha}{3\pi}$ and $\bar{E}_p \sim \mu_R$ we can determine the saturation level of color magnetic excitations using Eq.(5.17) as,

$$\delta B_{\omega,k} \sim \frac{\mu_R^2}{\sqrt{\alpha}}. \tag{5.22}$$

5.3 Calculation of anomalous viscosity

We follow Ref. [296, 297] to calculate the anomalous viscosity. For simplicity, we make v_x depend on x as,

$$v_x \to v_x - u(x) \tag{5.23}$$

where, $u(x)$ is the mean flow variable.

Now using Eq.(5.16) one can write the diffusion equation (Eq.(5.11)) as,

$$(\partial_t + v \cdot \partial_x)\langle n_{\mathbf{p}}\rangle \simeq e^2 \sum_{\omega,k} \frac{1}{1/t_c}\Bigg(\left(v_T^2 |\delta B_{\omega,k}|^2\right) \partial^2_{\mathbf{p_x}} \langle n_{\mathbf{p}}\rangle + \left(v_T^2 |\delta B_{\omega,k}|^2 + \frac{v_T^2 k^2 |\delta B_{\omega,k}|^2}{4\mu_R^2}\right) \partial^2_{\mathbf{p_z}} \langle n_{\mathbf{p}}\rangle \Bigg). \tag{5.24}$$

The second term can be written as,

$$(v \cdot \partial_x)\langle n_{\mathbf{p}}\rangle \simeq -v_T^2 p \frac{d\langle n_{\mathbf{p}}\rangle}{dp} \partial_x u(x). \tag{5.25}$$

Here, we bring back the term $v \cdot \partial_x$ due to Eq. (5.23). Now, if we consider, $k = k_{max}$, in this case the summation on ω and k can be lifted out and we can write the diffusion equation in terms of mean flow variable as,

$$\partial_t \langle n_{\mathbf{p}}\rangle - v_T^2 p \frac{d\langle n_{\mathbf{p}}\rangle}{dp} \partial_x u(x) \simeq \frac{e^2}{1/t_c}\Bigg(\left(v_T^2 |\delta B_{\omega,k}|^2\right) \partial^2_{\mathbf{p_x}} \langle n_{\mathbf{p}}\rangle + \left(v_T^2 |\delta B_{\omega,k}|^2 + \frac{v_T^2 k^2 |\delta B_{\omega,k}|^2}{4\mu_R^2}\right) \partial^2_{\mathbf{p_z}} \langle n_{\mathbf{p}}\rangle \Bigg). \tag{5.26}$$

Note that in the above equation ω and k respectively correspond to ω_{max} and k_{max}. Now taking moment $\int \frac{d^3p}{(2\pi)^3}\frac{(1+e\delta\mathbf{B}\cdot\mathbf{\Omega_p})}{\epsilon_p}(2p_x^2-p_y^2-p_z^2)$, the left hand side of the above equation will become,

$$LHS = \partial_t(2T^{xx}-T^{yy}-T^{zz}) - \left(\frac{v_T^2}{(2\pi)^3}\int d\Omega(2v_x^2-v_y^2-v_z^2)\times \frac{(e\delta\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v})^2}{4\mu_R^4}\right)\left[\int_0^\infty dpp^4\frac{d\langle n_\mathbf{p}\rangle}{dp}\right]\partial_x u(x). \quad (5.27)$$

Note that in the above expression we have used the definition of energy momentum tensor as,

$$T^{\mu\nu}_{\omega,k} = \int \frac{d^3p}{(2\pi)^3}\frac{(1+e\delta\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{\Omega_p})}{\epsilon_p}p^\mu p^\nu\langle n_\mathbf{p}\rangle. \quad (5.28)$$

Simplifying Eq.(5.27) we can write,

$$LHS = \partial_t(2T^{xx}-T^{yy}-T^{zz}) + \left(\frac{v_T^2}{(2\pi)^3}\int d\Omega(2v_x^2-v_y^2-v_z^2)\times \frac{(e\delta\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v})^2}{4\mu_R^4}\right)\left[\int_0^\infty dp4p^3\langle n_\mathbf{p}\rangle\right]\partial_x u(x). \quad (5.29)$$

Now,

$$RHS = \frac{e^2v_T^2|\delta B_{\omega,k}|^2}{1/t_c}\left(\frac{1}{(2\pi)^3}\int d\Omega(2v_x^2-v_y^2-v_z^2)\frac{(e\delta\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v})^2}{4\mu_R^4}\times \left(\int_0^\infty dpp^3\partial^2_{\mathbf{p_x}}\langle n_\mathbf{p}\rangle + \left(1+\frac{k^2}{4\mu_R^2}\right)\int_0^\infty dpp^3\partial^2_{\mathbf{p_z}}\langle n_\mathbf{p}\rangle\right)\right).$$

Now using,

$$\partial^2_{\mathbf{p_z}}\langle n_\mathbf{p}\rangle = \frac{p_z^2}{p^2}d_p^2\langle n_\mathbf{p}\rangle + \frac{1}{p}d_p\langle n_\mathbf{p}\rangle - \frac{p_z^2}{p^3}d_p\langle n_\mathbf{p}\rangle. \quad (5.30)$$

and

$$\partial^2_{\mathbf{p_x}}\langle n_\mathbf{p}\rangle = \frac{p_x^2}{p^2}d_p^2\langle n_\mathbf{p}\rangle + \frac{1}{p}d_p\langle n_\mathbf{p}\rangle - \frac{p_x^2}{p^3}d_p\langle n_\mathbf{p}\rangle, \quad (5.31)$$

one can get,

$$RHS = \frac{e^2 v_T^2 |\delta B_{\omega,k}|^2}{1/t_c} \left(\frac{1}{(2\pi)^3} \int d\Omega (2v_x^2 - v_y^2 - v_z^2) \frac{(e\delta \mathbf{B}_{\omega,\mathbf{k}} \cdot \mathbf{v})^2}{4\mu_R^4} \times \right.$$
$$\left(\int_0^\infty dpp^3 \left(v_x^2 d_p^2 \langle n_\mathbf{p} \rangle + \frac{1}{p} d_p \langle n_\mathbf{p} \rangle - \frac{v_x^2}{p} d_p \langle n_\mathbf{p} \rangle \right) + \right.$$
$$\left. \left. \left(1 + \frac{k^2}{4\mu_R^2} \right) \int_0^\infty dpp^3 \left(v_z^2 d_p^2 \langle n_\mathbf{p} \rangle + \frac{1}{p} d_p \langle n_\mathbf{p} \rangle - \frac{v_z^2}{p} d_p \langle n_\mathbf{p} \rangle \right) \right) \right).$$

With further simplification, we can write above equation as,

$$RHS = \frac{e^2 v_T^2 |\delta B_{\omega,k}|^2}{1/t_c} \left(\frac{1}{(2\pi)^3} \int d\Omega (2v_x^2 - v_y^2 - v_z^2) \frac{(e\delta \mathbf{B}_{\omega,\mathbf{k}} \cdot \mathbf{v})^2}{4\mu_R^4} \times \right.$$
$$\left(\int_0^\infty dp \left(8 v_x^2 p \langle n_\mathbf{p} \rangle - 2p \langle n_\mathbf{p} \rangle \right) + \right.$$
$$\left. \left. \left(1 + \frac{k^2}{4\mu_R^2} \right) \int_0^\infty dp \left(8 v_z^2 p \langle n_\mathbf{p} \rangle - 2p \langle n_\mathbf{p} \rangle \right) \right) \right).$$

We choose stationary limit in this case $\partial_t (2T^{xx} - T^{yy} - T^{zz}) = 0$, therefore from the diffusion equation (L.H.S=R.H.S) we can get,

$$\partial_x u(x) = \frac{e^2 v_T^2 |\delta B_{\omega,k}|^2}{1/t_c} \frac{\left(8 I_1 J_1 - 4 I_2 J_1 + 8 \left(1 + \frac{k^2}{4\mu_R^2} \right) I_3 J_1 \right)}{5 I_2 J_2} \tag{5.32}$$

where,

$$I_1 = \left(\frac{1}{(2\pi)^3} \int d\Omega (2v_x^2 - v_y^2 - v_z^2) v_x^2 \frac{(e\delta \mathbf{B}_{\omega,\mathbf{k}} \cdot \mathbf{v})^2}{4\mu_R^4} = \frac{e^2 \delta \mathbf{B}^{\mathbf{2}}_{\omega,\mathbf{k}}}{105\pi^2 \mu_R^4} \right.$$
$$I_2 = \left(\frac{1}{(2\pi)^3} \int d\Omega (2v_x^2 - v_y^2 - v_z^2) \frac{(e\delta \mathbf{B}_{\omega,\mathbf{k}} \cdot \mathbf{v})^2}{4\mu_R^4} - \frac{e^2 \delta \mathbf{B}^{\mathbf{2}}_{\omega,\mathbf{k}}}{15\pi^2 \mu_R^4}, \right.$$
$$I_3 = \left(\frac{1}{(2\pi)^3} \int d\Omega (2v_x^2 - v_y^2 - v_z^2) v_z^2 \frac{(e\delta \mathbf{B}_{\omega,\mathbf{k}} \cdot \mathbf{v})^2}{4\mu_R^4} = -\frac{e^2 \delta \mathbf{B}^{\mathbf{2}}_{\omega,\mathbf{k}}}{210\pi^2 \mu_R^4}, \right.$$
$$J_1 = \int_0^\infty dpp \langle n_\mathbf{p} \rangle,$$
$$J_2 = \int_0^\infty dpp^3 \langle n_\mathbf{p} \rangle.$$

Now, using Eq.(2.155) one can write,

$$(2T^{xx} - T^{yy} - T^{zz}) = \int \frac{d\Omega}{(2\pi)^3} \frac{(e\delta\mathbf{B}_{\omega,\mathbf{k}} \cdot \mathbf{v})^2(2v_x^2 - v_y^2 - v_z^2)}{4\mu_R^4} \int_0^\infty dpp^3 \langle n_\mathbf{p} \rangle \quad (5.33)$$

The definition of shear viscosity is,

$$\eta_A = \frac{(2T^{xx} - T^{yy} - T^{zz})}{-4\partial_x u(x)} \quad (5.34)$$

Taking the distribution function of the form, $\langle n_\mathbf{p} \rangle = 1/(\exp{(\mu_R - p)} + 1)$, considering $\mu_R \gg T$ and using Eqs.(5.32,5.33) and $k = k_{max} = \frac{2\mu_R\alpha}{3\pi}$ one can estimate anomalous shear viscosity,

$$\eta_A \sim \frac{\mu_R^2}{t_c} \left(1 + \frac{11\pi^2 T^2}{3\mu_R^2}\right) \quad (5.35)$$

One can notice from Fig. 5.1 that for the case $\mu_R \gg T$ and $k = k_{max}$, $1/t_c$ depends on μ_R in an approximately linear way i.e. $1/t_c \propto \mu_R$. Slope of the curve can be found by a linear fit. For $\alpha_e = 1/137$ the slope is $\sim 5.09 \times 10^-7$. The slope increases by increasing α_e. Thus η scales like as μ_R^3. Note that in this calculation we have assumed that $\mu_R \gg \mu_L$, thus we can approximate μ_R by μ_5.

5.4 Conclusions

We have calculated the coefficient of shear viscosity based on the strong turbulence argument. For the case when $\mu_5/T \gg 1$, the collision rates become insignificant[289] at low temperature, in this regime the decorrelation frequency $1/t_c$ can have a significant contribution in the determination of η. In this low temperature limit the entropy density s scales as $\mu_5^2 T$ and the ratio $\eta/s \propto \mu_5/T$ and it can be a large number. In deriving the above expression of η, we have ignored non-linear wave-wave interaction which can play a role in the case of non-Abelian plasmas. However, to address this question one requires to numerically simulate the chiral plasma instability with the full nonlinearity.

Note that the dimensional argument suggests that for the case when $\mu_5 \ll T$, stress (energy density) $\sim \mu_5^2 T^2$, decorrelation frequency ($1/t_c \sim \omega_{max}$) of the chiral-imbalance instability $\sim \mu_5$ [284] and η scales as $\mu_5 T^2$. Therefore, $\eta/s \propto \mu_5/T$, which could be a small number. We hope that this analytic study will help in the understanding the viscosity due to turbulent transport in parity violating plasma and can be useful in its numerical simulations.

Chapter 6

Summary and Outlook

Matter produced in the heavy ion collisions evolve in several stages (see the section §1.2) and there is no unique theoretical tool which can describe all the stages. Relativistic hydrodynamics can be applied from the stage of local thermodynamical equilibrium until the freeze-out stage. Non-relativistic dissipative hydrodynamics is a very well established theoretical tool. However, the relativistic generalization of the dissipative hydrodynamics is currently an active area of research. By relativistic, we mean that the hydrodynamic equations remain covariant under Lorentz transformations. A complete relativistic hydrodynamic description of a system requires the following additional informations (i) An EoS and (ii) transport coefficients. A straight forward generalization of Navier-Stoke equation has unphysical instabilities and acausal behavior. In this thesis we have focused on causal models of dissipative hydrodynamics. We have developed theory of fluctuations for causal hydrodynamics and calculated viscous correlations which are found to be proportional to the transport coefficients. Another outstanding problem in the context of the heavy ion collision is to understand the required fast thermalization time (< 1 fm/c) of QGP. One of the important mechanism which can be responsible for the fast thermalization, is a collective (plasma) instability, known as Weibel instability[132–141], which occurs when initial parton distribution is considered to be anisotropic in momentum space. The Weibel instability can also give rise to the anomalous (shear) viscosity by turbulent transport which can contribute to the effective shear viscosity of QGP, therefore, may explain the observed low shear viscosity in RHIC experiments. Recently, it has been suggested that the parity violating correlations can be seen in the heavy ion collision experiments [103, 104]. It has also been suggested that the parity violating hydrodynamics or kinetic theory gives a new kind of instability because of finite $(\mu_R - \mu_L)$. This new instability is referred as the chiral-imbalance instability. Based on these facts, we have considered the case of an anisotropic chiral plasma (which is indeed created in the heavy ion collisions) and studied that the two instabilities coexist simultaneously and compete with each other. Further, we have shown that chiral instability may also drive turbulent transport which may lead to the anomalous viscosity by means of an enhanced collisional rate due to turbulence.

In chapter §1, we have briefly discussed the nuclear matter and its phase diagram which suggests us the possibility of producing the strongly interacting matter namely QGP in the laboratories. We have given a brief description of a relativistic heavy ion collision and discussed the space-time evolution of the produced strongly interacting matter in terms of the several stages. We have discussed that the hydrodynamics can be applied to a certain regime. We gave a brief introduction to the relativistic hydrodynamics and discussed why it is important to have viscosity in the hydrodynamic framework, when it is applied for the strongly interacting matter in the context of heavy ion collision. Next, we have discussed the local CP violation and chiral magnetic effect and some experimental results suggesting the possibility of local CP violation.

In chapter §2, we have discussed the necessary theoretical tools required for this thesis, namely the various kinds of hydrodynamic framework derived from the extended irreversible thermodynamics and kinetic theory. We have also discussed the Berry curvature modified kinetic theory framework which describes the CP violation phenomenon and CME.

In chapter §3, we have applied theory of quasi-stationary fluctuations or Onsager theory to calculate the hydrodynamic fluctuations (expressed in terms of correlation functions) in various relativistic hydrodynamic frameworks (causal or acausal) and study their behavior with a simple example of boost invariant

Bjorken Flow [201] which is a good approximation in context of relativistic heavy ion collisions. It can be seen from our results as given in Eqs. (3.19-3.21), (3.33-3.35), (3.55-3.57), (3.50) and (3.60) that the analytic form of the correlation functions remains the same for all the causal hydrodynamics that are considered here and do not depend explicitly on the relaxation time. This indicates a kind of universality of the correlation functions. Also, one can notice from Eqs.(3.19-3.21), (3.33-3.35), (3.55-3.57), (3.50) and (3.60) that the viscous correlation depends on $\epsilon+p-\mu n$ and the ratio of viscous coefficients to the entropy density. The universality of correlation functions is an attribute of the positivity of the four-entropy current given by $T\partial_\mu S^\mu = \Pi^2/\zeta - q^\mu q_\mu/\lambda T + \pi^{\mu\nu}\pi_{\mu\nu}/2\eta \geq 0$, which is used to write the expression for ds/dt by using the constraints $\Delta_{\mu\nu}\pi^{\mu\nu} = 0$, $q_\mu u^\mu = 0$ and $u_\mu \pi^{\mu\nu} = 0$ on dissipative fluxes. These constraints are universal and satisfied in the case of NS as well as for all the causal hydrodynamics, irrespective of the actual form of $\pi_{\mu\nu}$, q^μ and Π. In order to understand the qualitative behavior of the fluctuations, we have numerically investigated the evolution of the correlation function using the one dimensional boost-invariant (Bjorken) flow and found that the qualitative behavior of the correlation functions for the various models of the causal hydrodynamics remain similar to those of the NS theory.

In chapter §4, we have considered the case of anisotropic chiral plasma. We have found that in such a case Weibel and chiral-imbalance instability can occur simultaneously. In a given physical situation, the dominance of either Weibel or the chiral-imbalance instability depends on the three parameters viz. θ_n, ξ and μ_5/T. We have shown that for the values $\theta_n = 0$ and $\mu_5/T \sim 1$, when $\xi \gtrsim 1$ or $\xi_c < \xi \ll 1$, the Weibel modes dominate over the chiral-imbalance instability. It was demonstrated analytically that for $\theta_n = 0$ and $\mu_5/T \sim 1$, when $\xi \sim \xi_c \ll 1$ the growth rates of both the instabilities are comparable. We have also shown numerically that for $\xi < \xi_c$, $\mu_5/T \sim 1$, there exists a critical angle $\theta = \theta_c$ at which the both the instabilities can also have comparable growth rates. We have also shown for the case when $\xi \gg 1$, the chiral-imbalance can dominate over the Weibel modes when $\theta = \theta_{nc}$. A summary of our main results is shown in the following table.

Case	θ_n	μ_5/T	ξ	Dominance of the instability
1	0	~ 1	$\xi \gtrsim 1$	Weibel will dominate
2	0	~ 1	$\xi_c < \xi \ll 1$	Weibel will dominate
3	0	≤ 1	$\xi \sim \xi_c \ll 1$	Both are comparable
4	θ_c	~ 1	$\xi < \xi_c$	Both are comparable
5	θ_{nc}	can have any value	$\xi \gg 1$	Chiral Imbalance

Table 6.1: Summary of results showing the dominance region of the chiral/Weibel instabilities

In chapter §5 we have considered the case of isotopic chiral plasma and discussed that the presence of finite chiral imbalance may lead to chiral imbalance instability. We argued that chiral imbalance instability may drive turbulent transport which may enhance the collisionality. We have calculated the coefficient of shear viscosity based on the strong turbulence argument. For the case $\mu_5/T \gg 1$, if tempreture is low, the collision rate becomes insignificant[289]. In this regime, the decorrelation frequency $1/t_c$ leads to a significant contribution in determining η. We have found that η scales like μ_5^3. Note that in this regime the entropy density s scales as $\mu_5^2 T$ and thus the ratio $\eta/s \propto \mu_5/T$ which could be be a large number. This result may be suitable for neutrons star. For the matter produced in heavy ion collisions one has $\mu_5/T \ll 1$. In this case, on the basis of dimensional arguments, we have shown η/s can be very small. A detailed calculation of this case is a part of our future plan. Here we would like to emphasize that in deriving the expression of η, the non-linear wave-wave interactions have been ignored, which can play a role in case of non-abelian plasmas. Study of such a case requires to numerically simulate the chiral plasma instability with the full nonlinearity.

In this thesis, we have achieved our objectives; (i) to calculate the hydrodynamic fluctuations for the causal hydrodynamics; (ii) to study Weibel and chiral-imbalalance instabilities in an anisotropic chiral plasma such as QGP (created in heavy ion collisions); and (iii) to estimate the anomalous shear viscosity arising because of chiral-imbalance instability. However, it could be more interesting if one could calculate the hydrodynamic fluctuations in the context of parity violating hydrodynamics. It would also be interesting to calculate the plasma electromagnetic (as well as chromodynamic) and charge (electric as well as color) fluctuations in an anisotropic medium where CP is no longer conserved.

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Fluctuations in relativistic causal hydrodynamics

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Abstract

Formalism to calculate the hydrodynamic fluctuations by applying the Onsager theory to the relativistic Navier–Stokes equation is already known. In this work, we calculate hydrodynamic fluctuations within the framework of the second order hydrodynamics of Müller, Israel and Stewart and its generalization to the third order. We have also calculated the fluctuations for several other causal hydrodynamical equations. We show that the form for the Onsager-coefficients and form of the correlation functions remain the same as those obtained by the relativistic Navier–Stokes equation and do not depend on any specific model of hydrodynamics. Further we numerically investigate evolution of the correlation function using the one dimensional boost-invariant (Bjorken) flow. We compare the correlation functions obtained using the causal hydrodynamics with the correlation function for the relativistic Navier–Stokes equation. We find that the qualitative behavior of the correlation functions remains the same for all the models of the causal hydrodynamics.

Keywords: Fluctuations; Onsager coefficients; Causal hydrodynamics

1. Introduction

A study of fluctuations in continuous media is of great interest in physics and it can provide a link between the macroscopic and microscopic points of view. A macroscopic theory such as hydrodynamics provides a simplest possible description of a complicated many-body system in terms of space–time evolution of the mean or averaged quantities like energy density, pressure,

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flow velocity, etc. The fluctuation theory studies small deviations from the mean behavior and it can help in calculating correlation functions for the macroscopic variables [1,2]. In context of relativistic hydrodynamics, results of the fluctuation–dissipation theorem have been studied in Refs. [3,4]. In Ref. [3] the authors have studied the fluctuation in the contexts of general-relativistic Navier–Stokes theory. A more general framework of hydrodynamics described as the divergence type theory (DTT) [5] was considered in Ref. [4]. It ought to be noted that recently in an interesting work in Ref. [6], the authors have applied results of the fluctuation–dissipation theorem to the relativistic *Navier–Stokes* theory of hydrodynamics and calculated the two particles correlators for the one-dimensional hydrodynamics (Bjorken) flow relevant for the relativistic heavy-ion collision experiments at RHIC and LHC. The authors obtained several analytical results for two particle correlation functions. Further, in Ref. [7], the authors have studied the effect of thermal conductivity on the correlation function using the Bjorken-flow. It should be noted here that it is well-known that relativistic Navier–Stokes theory exhibits acausal behavior which can give rise to unphysical instabilities [8]. However the causality can be restored if the terms with higher orders are included in the hydrodynamics as indicated by the Maxwell–Cattaneo law [9]. Indeed these issues do not arise in the second-order causal hydrodynamics theory developed by Müller, Israel and Stewart (MIS) [10]. Form of the Navier–Stokes equations can be determined from the second law of thermodynamics $\partial_\mu S^\mu \geqslant 0$, where S^μ denotes the equilibrium entropy current. However, in general it is not possible for an out-of-equilibrium fluid to have an equilibrium entropy current [9]. In MIS hydrodynamics out-of-equilibrium current can have contributions from dissipative processes like the effect of viscosity and the heat conduction. This has an interesting analogy with the irreversible thermodynamics [11,12]. Further, the MIS hydrodynamics has been extensively applied to study the relativistic heavy-ion collisions [9,13–15] and also in cosmology [16]. Later this formalism was extended to include the effect of third order terms in the gradient expansion [17]. Recently, it has been shown that the derivation of the MIS equations from the underlying kinetic equation may not be unique, there may exist a more general set of hydrodynamic equations which may allow one to obtain MIS equations as a special case [19,18].

Finally, it should be mentioned here that although the divergent type theory (DTT) of relativistic fluid of Geroch–Lindblom [5] allows for a consistent proof of causality and stability of its solutions, it is far from direct thermodynamic intuition. Moreover, the connection between the DTTs and MIS or other causal hydrodynamics theories is not yet clearly established.

In this work we apply the fluctuation–dissipation theorem to MIS equations and also to the hydrodynamics models developed by Denicol, Koide and Rischke (DKR) [19], Jaiswal, Bhalerao and Pal (JBP) [18] and other models based on MIS approach [10,17,20]. Further, we apply these results to study the hydrodynamical evolution using $1+0$ dimensional Bjorken flow. In particular, we calculate the correlators using the Onsager coefficients for various relativistic hydrodynamical theories.

2. Fluctuations and correlations in hydrodynamics

In thermodynamic equilibrium entropy of the system S which is a function of the additive quantities x_k becomes maximum. In equilibrium, S satisfies the condition $X_k = -\frac{\partial S}{\partial x_k} = 0$. However, when the system is slightly away from the equilibrium the generalized forces $X_k \neq 0$ and $\frac{dx_i}{dt} = -\gamma_{ik} X_k + \xi_i$, the summation convention is implied, describes the flux associated with the quantity x_i, here ξ_i are the random forces or the noise term and γ_{ik} are the Onsager coefficients.

The Onsager reciprocity relations imply that $\gamma_{ik} = \gamma_{ki}$. In this phenomenological theory time rate of change of the total entropy $\frac{dS}{dt}$ is given by,

$$\frac{dS}{dt} = -\frac{dx_i}{dt} X_i, \tag{1}$$

which can also be written as,

$$\frac{dS}{dt} = \gamma_{ik} X_k X_i - \xi_i X_i. \tag{2}$$

Correlation between ξ_i is given by the formula,

$$\langle \xi_i(t_1)\xi_k(t_2) \rangle = (\gamma_{ik} + \gamma_{ki})\delta(t_1 - t_2). \tag{3}$$

The correlation functions can be found once γ_{ij} are known [2,6]. In order to find out γ_{ij}, one needs to know the $\frac{dS}{dt}$ for the underlying hydrodynamical theory together with identification of the generalized forces and fluxes. The expression for rate of change of entropy $\frac{dS}{dt}$ can be found either by using equations of hydrodynamics together with the thermodynamic relations or from the kinetic theory [10]. In our work we are using the former approach.

We start with the expressions for the energy–momentum tensor $T^{\mu\nu}$ and the current-density J^μ_B for a viscous fluid,

$$T^{\mu\nu} = T^{\mu\nu}_{id} + \Delta T^{\mu\nu} + S^{\mu\nu}, \tag{4}$$

$$J^\mu_B = n_B u^\mu + \nu^\mu + I^\mu, \tag{5}$$

where, $T^{\mu\nu}_{id} = \epsilon u^\mu u^\nu - p\Delta^{\mu\nu}$ is the ideal part of the energy–momentum tensor with $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$, and ϵ, p, u^μ are the local energy density, pressure and fluid flow four-velocity respectively. It is to be noted that $u^\mu u_\mu = 1$ and $g_{\mu\nu} = \mathrm{diag}(+,-,-,-)$. $\Delta T^{\mu\nu} = \Delta T^{\mu\nu}_{vis} + \Delta T^{\mu\nu}_{heat}$ with $\Delta T^{\mu\nu}_{vis} = \pi^{\mu\nu} - \Delta^{\mu\nu}\Pi$ and $\Delta T^{\mu\nu}_{heat} = W^\mu u^\nu + W^\nu u^\mu$, is the dissipative part of the energy–momentum tensor and $S^{\mu\nu}$ is the stochastic term arising due to the local thermal fluctuations [6]. Similarly, ν^μ and I^μ denote the dissipative (non-equilibrium) and the stochastic terms in baryon current density respectively. n_B denotes the net density of baryon number in local rest frame. $W^\mu = q^\mu + h\nu^\mu$ is the energy flow in local rest frame, $h = (\epsilon + p)/n_B$ is the enthalpy per particle and $\nu^\mu = \Delta^{\mu\nu} J_B \nu$ is the baryon number flow in the local rest frame. For the dissipative fluxes one can always require the following relations to hold $u_\mu \pi^{\mu\nu} = 0$, $\pi^\alpha_\alpha = 0$, $\Delta_{\mu\nu}\pi^{\mu\nu} = 0$, $u_\mu W^\mu = 0$, $u_\mu \nu^\mu = 0$, $u_\mu q^\mu = 0$.

The relevant conservation equations for the hydrodynamics can be written as,

$$\partial_\mu J^\mu_B = Dn_B + n_B \nabla_\mu u^\mu + \partial_\mu \nu^\mu = 0, \tag{6}$$

$$u_\nu \partial_\mu T^{\mu\nu} = D\epsilon + (\epsilon + p + \Pi)\nabla_\mu u^\mu - \pi_{\mu\nu}\nabla^{\langle\mu} u^{\nu\rangle} + \nabla_\mu W^\mu - 2W^\mu D u_\mu = 0, \tag{7}$$

$$\begin{aligned}\Delta^\alpha_\nu \partial_\mu T^{\mu\nu} = {} & (\epsilon + p + \Pi)Du^\alpha - \nabla^\alpha(p + \Pi) + \Delta^{\alpha\nu}\nabla^\sigma \pi_{\nu\sigma} - \pi^{\alpha\nu} Du_\nu \\ & + \Delta^{\alpha\nu} DW_\nu + 2W^{(\alpha}\nabla_\nu u^{\nu)} = 0,\end{aligned} \tag{8}$$

where, $D = u^\mu \partial_\mu$ and $\nabla^\mu = \Delta^{\mu\nu}\partial_\mu$. There are only 5 equations written above and 14 unknowns n_B, ϵ, Π, W^μ, $\pi^{\mu\nu}$ and u^μ. Therefore, 9 additional equations for dissipative fluxes are required to obtain the hydrodynamical solution.

There are two popular choices for u^μ: In Landau–Lifshitz frame u^μ is parallel to the energy-flow and $W^\mu = 0$ which implies that $q^\mu = -h\nu^\mu$. Another choice is the Eckart-frame, where u^μ can be parallel to $J_B\mu$ and $\nu^\mu = 0$ and this would imply $W^\mu = q^\mu$. Now we shall obtain the 9 additional equations and fluctuation correlations in Landau–Lifshitz and Eckart frame respectively.

2.1. Equations for dissipative fluxes in Landau–Lifshitz frame and fluctuation correlations in MIS

In order to derive the 9 additional equations one needs the expression for the out-of-equilibrium entropy four-current. In Landau–Lifshitz frame the expression for the non-equilibrium entropy four-current is given in literature [10,14,15] and is as follows,

$$S^\mu = su^\mu - \frac{\mu_B}{T}\nu^\mu - \left(\beta_0\Pi^2 - \beta_1 q_\nu q^\nu + \beta_2\pi_{\nu\lambda}\pi^{\nu\lambda}\right)\frac{u^\mu}{2T} - \frac{\alpha_0\Pi q^\mu}{T} + \frac{\alpha_1\pi^{\mu\nu}q_\nu}{T}. \tag{9}$$

$\beta_0, \beta_1, \beta_2$ are thermodynamic co-efficients and describe the scalar, vector and tensor contribution to the entropy density respectively. α_0 and α_1 are functions of energy density ϵ and baryon density n_B and they describe viscous and heat coupling.

The divergence of the non-equilibrium entropy four-current (Eq. (9)) using Eqs. (6)–(7) and the thermodynamic relations $d\epsilon = T\,ds - \mu\,dn$ and $\epsilon + p = Ts - \mu n$ can be written as follows,

$$\begin{aligned} T\partial_\mu S^\mu = &-\Pi\left[\partial_\mu u^\mu + \beta_0\dot{\Pi} + \frac{1}{2}T\partial_\mu\left(\frac{\beta_0}{T}u^\mu\right)\Pi + \alpha_0\nabla_\mu q^\mu\right] \\ &- q^\mu\left[-h^{-1}T\nabla_\mu\left(\frac{\mu}{T}\right) - \beta_1\dot{q}_\mu - \frac{1}{2}T\partial_\nu\left(\frac{\beta_1}{T}u^\nu\right)q_\mu - \alpha_1\partial_\nu\pi^\nu_\mu + \alpha_0\partial_\mu\Pi\right] \\ &+ \pi^{\mu\nu}\left[\sigma_{\mu\nu} - \beta_2\dot{\pi}_{\mu\nu} - \frac{1}{2}T\partial_\lambda\left(\frac{\beta_2}{T}u^\lambda\right)\pi_{\mu\nu} + \alpha_1\nabla_{\langle\nu}q_{\mu\rangle}\right], \end{aligned} \tag{10}$$

where we have used the notation $\dot{F} = DF$. According to the second law of thermodynamics we must have $T\partial_\mu S^\mu \geqslant 0$. This inequality will be satisfied if Π, q_μ and $\pi^{\mu\nu}$ satisfy the following equations,

$$\left[\partial_\mu u^\mu + \beta_0\dot{\Pi} + \frac{1}{2}T\partial_\mu\left(\frac{\beta_0}{T}u^\mu\right)\Pi + \alpha_0\nabla_\mu q^\mu\right] = -\frac{\Pi}{\zeta}, \tag{11}$$

$$\left[-h^{-1}T\nabla_\mu\left(\frac{\mu}{T}\right) - \beta_1\dot{q}_\mu - \frac{1}{2}T\partial_\nu\left(\frac{\beta_1}{T}u^\nu\right)q_\mu - \alpha_1\partial_\nu\pi^\nu_\mu + \alpha_0\partial_\mu\Pi\right] = \frac{q_\mu}{\lambda T}, \tag{12}$$

$$\left[\sigma_{\mu\nu} - \beta_2\dot{\pi}_{\mu\nu} - \frac{1}{2}T\partial_\lambda\left(\frac{\beta_2}{T}u^\lambda\right)\pi_{\mu\nu} + \alpha_1\nabla_{\langle\nu}q_{\mu\rangle}\right] = \frac{\pi_{\mu\nu}}{2\eta}. \tag{13}$$

Here we note that sometimes the terms with factor 1/2 on the left hand side of Eqs. (11)–(13) are ignored by arguing that the gradient of thermodynamic quantities is small [21,15]. But in the present case we are retaining these terms. Thus, Eqs. (11)–(13) can be written as,

$$\tau_\Pi\dot{\Pi} + \Pi = -\zeta\nabla_\alpha u^\alpha - l_{\Pi q}\nabla_\mu q^\mu - \left(\frac{1}{2}T\zeta\partial_\mu\left(\frac{\tau_\Pi u^\mu}{\zeta T}\right)\right)\Pi, \tag{14}$$

$$\tau_q\dot{q}_\mu + q_\mu = -\lambda T^2h^{-1}\nabla_\mu\left(\frac{\mu}{T}\right) + l_{q\Pi}\nabla_\mu\Pi - l_{q\pi}\nabla_\nu\pi^\nu_\mu + \frac{1}{2}\lambda T^2\partial_\nu\left(\frac{\tau_\pi u^\nu}{\lambda T^2}\right)q_\mu, \tag{15}$$

$$\tau_\pi\dot{\pi}_{\mu\nu} + \pi_{\mu\nu} = 2\eta\sigma_{\mu\nu} + l_{\pi q}\nabla_{\langle\mu}q_{\nu\rangle} - \eta T\partial_\lambda\left(\frac{\tau_\pi u^\lambda}{2\eta T}\right)\pi_{\mu\nu}. \tag{16}$$

Henceforth, we call Eqs. (14)–(16) as MIS equations. Here, $\tau_\Pi = \zeta\beta_0$, $\tau_q = \lambda T\beta_1$, $\tau_\pi = 2\eta\beta_2$ are identified as the relaxation times and $l_{\Pi q} = \zeta\alpha_0$, $l_{q\Pi} = \lambda T\alpha_0$, $l_{q\pi} = \lambda T\alpha_1$, $l_{\pi q} = 2\eta\alpha_1$ as coupling constants. These 9 equations for the dissipative fluxes together with Eqs. (6)–(8) and equation of state form complete set of the hydrodynamic equations. Note that the limit τ_Π, τ_q, τ_π, $l_{\Pi q}$, $l_{q\Pi}$, $l_{q\pi}$, $l_{\pi q} \to 0$ is the first order limit which correspond to the Navier–Stokes case.

Eq. (10) can be written as,

$$T\partial_\mu S^\mu = \frac{\Pi^2}{\zeta} - \frac{q^\mu q_\mu}{\lambda T} + \frac{\pi^{\mu\nu}\pi_{\mu\nu}}{2\eta} \geqslant 0. \tag{17}$$

Here, $q^\mu q_\mu < 0$ [13]. Now using the identity $\Delta^{\mu\nu}\Delta_{\mu\nu} = 3$ and the condition $\Delta_{\mu\nu}\pi^{\mu\nu} = 0$, one can write Eq. (17) as,

$$\partial_\mu S^\mu = \frac{\Delta T^{\mu\nu}_{vis}}{T}\left(\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta}\right) - \frac{q^\mu q_\mu}{\lambda T^2}. \tag{18}$$

Upon integrating over the whole volume Eq. (18) can be written as,

$$\frac{dS}{dt} = \int d^3x\left[\frac{\Delta T^{\mu\nu}_{vis}}{T}\left(\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta}\right) - \frac{q^\mu q_\mu}{\lambda T^2}\right]. \tag{19}$$

Following identification between the phenomenological variables $(\dot{x}_1, \dot{x}_2)$ and the hydrodynamical variables can be made [6],

$$\dot{x}_1 \to \Delta T^{\mu\nu}_{vis}, \qquad \dot{x}_2 \to q^\mu. \tag{20}$$

A comparison of Eq. (19) with the phenomenological equation (1) will give,

$$X_1 = -\frac{1}{T}\left(\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta}\right)\Delta V,$$
$$X_2 = \frac{q_\mu}{\lambda T^2}\Delta V. \tag{21}$$

Now neglecting the stochastic term in Eq. (2) and comparing it with Eq. (19) one can get,

$$\gamma_{11}X_1 = -\Delta T^{\mu\nu}_{vis}, \tag{22}$$
$$\gamma_{22}X_2 = -q^\mu, \tag{23}$$
$$\gamma_{12} = \gamma_{21} = 0. \tag{24}$$

The coefficients γ_{12} and γ_{21} are zero, because the dissipative fluxes due to heat and viscosity are considered to be independent. Coefficients γ_{11} and γ_{22} are rank-four tensors and they can be parameterized as follows,

$$\gamma_{11} = \left[A\Delta^{\mu\nu\alpha\beta} + B\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\frac{1}{\Delta V}, \qquad \gamma_{22} = \frac{C\Delta^{\mu\nu}}{\Delta V}, \tag{25}$$

where, $\Delta^{\mu\nu\alpha\beta} = \Delta^{\mu\alpha}\Delta^{\nu\beta} - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}$. Now using Eqs. (22), (23) one can determine the coefficients $A = 2\eta T$, $B = \zeta T$ and $C = -\lambda T^2$. Thus one can write,

$$\gamma_{11} = 2T\left[\left(\eta\Delta^{\mu\alpha}\Delta^{\nu\beta} - \frac{1}{3}\eta\Delta^{\mu\nu}\Delta^{\alpha\beta}\right) + \frac{1}{2}\zeta\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\frac{1}{\Delta V}, \qquad \gamma_{22} = -\frac{\lambda T^2\Delta^{\mu\nu}}{\Delta V}. \tag{26}$$

From above expression of γ_{11} one can see that there is an additive contribution of shear and bulk viscosity i.e. one can write it as $\gamma_{11} = (\gamma_{11})_\eta + (\gamma_{11})_\zeta$.

Now following Eq. (2), the correlation functions can be written as,

$$\langle S^{\mu\nu}_{vis}(x_1)S^{\alpha\beta}_{vis}(x_2)\rangle = 2T\left[\eta\left(\Delta^{\mu\alpha}\Delta^{\nu\beta}+\Delta^{\mu\beta}\Delta^{\nu\alpha}\right)+\left(\zeta-\frac{2}{3}\eta\right)\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\delta(x_1-x_2), \tag{27}$$

$$\langle I^\mu(x_1)I^\nu(x_2)\rangle = -2\lambda T^2\Delta^{\mu\nu}\delta(x_1-x_2), \tag{28}$$

$$\langle S^{\mu\nu}_{vis}(x_1)I^\alpha(x_2)\rangle = 0. \tag{29}$$

These are the stochastic or noise auto-correlation functions for the MIS hydrodynamics in the Landau–Lifshitz frame.

2.2. Equations for dissipative fluxes in Eckart frame and fluctuation correlations in MIS

In the Eckart frame expression for the entropy four-current [10,13,14] can be written as,

$$S^\mu = su^\mu + \frac{q_\mu}{T} - \left(\beta_0\Pi^2 - \bar\beta_1 q_\nu q^\nu + \beta_2\pi_{\nu\lambda}\pi^{\nu\lambda}\right)\frac{u^\mu}{2T} - \frac{\bar\alpha_0\Pi q^\mu}{T} + \frac{\bar\alpha_1\pi^{\mu\nu}q_\nu}{T}. \tag{30}$$

Note that here coefficients β_0, β_2 are the same as in Landau–Lifshitz case while the coefficients $\bar\alpha_i$ and $\bar\beta_1$ are given as, $\bar\alpha_i = \alpha_i + \frac{1}{\epsilon+p}$ and $\bar\beta_1 = \beta_1 + \frac{1}{\epsilon+p}$, where α_i, β_1 are the coefficients in Landau–Lifshitz frame. Next, divergence of the non-equilibrium entropy four-current (Eq. (30)) using Eqs. (6)–(7) and the thermodynamic relations $d\epsilon = T\,ds - \mu\,dn$ and $\epsilon + p = Ts - \mu n$ can be written as follows,

$$\begin{aligned}T\partial_\mu S^\mu = {}& -\Pi\left[\partial_\mu u^\mu + \beta_0\dot\Pi + \frac{1}{2}T\partial_\mu\left(\frac{\beta_0}{T}u^\mu\right)\Pi + \bar\alpha_0\nabla_\mu q^\mu\right]\\ & - q^\mu\left[\nabla_\mu \ln T - \dot u_\mu - \bar\beta_1\dot q_\mu - \frac{1}{2}T\partial_\nu\left(\frac{\bar\beta_1}{T}u^\nu\right)q_\mu - \bar\alpha_1\partial_\nu\pi^\nu_\mu + \bar\alpha_0\partial_\mu\Pi\right]\\ & + \pi^{\mu\nu}\left[\sigma_{\mu\nu} - \beta_2\dot\pi_{\mu\nu} - \frac{1}{2}T\partial_\lambda\left(\frac{\beta_2}{T}u^\lambda\right)\pi_{\mu\nu} + \bar\alpha_1\nabla_{\langle\nu}q_{\mu\rangle}\right].\end{aligned} \tag{31}$$

In order to have $T\partial_\mu S^\mu \geqslant 0$ we must have the following equations for the dissipative fluxes,

$$\left[\partial_\mu u^\mu + \beta_0\dot\Pi + \frac{1}{2}T\partial_\mu\left(\frac{\beta_0}{T}u^\mu\right)\Pi + \bar\alpha_0\nabla_\mu q^\mu\right] = -\frac{\Pi}{\zeta}, \tag{32}$$

$$\left[\nabla_\mu \ln T - \dot u_\mu - \bar\beta_1\dot q_\mu - \frac{1}{2}T\partial_\nu\left(\frac{\bar\beta_1}{T}u^\nu\right)q_\mu - \bar\alpha_1\partial_\nu\pi^\nu_\mu + \bar\alpha_0\partial_\mu\Pi\right] = \frac{q_\mu}{\lambda T}, \tag{33}$$

$$\left[\sigma_{\mu\nu} - \beta_2\dot\pi_{\mu\nu} - \frac{1}{2}T\partial_\lambda\left(\frac{\beta_2}{T}u^\lambda\right)\pi_{\mu\nu} + \bar\alpha_1\nabla_{\langle\nu}q_{\mu\rangle}\right] = \frac{\pi_{\mu\nu}}{2\eta}. \tag{34}$$

Thus Eq. (31) can be written as,

$$T\partial_\mu S^\mu = \frac{\Pi^2}{\zeta} - \frac{q^\mu q_\mu}{\lambda T} + \frac{\pi^{\mu\nu}\pi_{\mu\nu}}{2\eta} \geqslant 0, \tag{35}$$

which can easily be casted into the following form,

$$T\partial_\mu S^\mu = \Delta T^{\mu\nu}\left[\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta} - \frac{1}{2\lambda T}(u_\nu q_\mu + u_\mu q_\nu)\right]. \tag{36}$$

Upon integrating over the whole volume Eq. (36) can be written as,

$$\frac{dS}{dt} = \int d^3x \frac{\Delta T^{\mu\nu}}{T}\left[\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta} - \frac{1}{2\lambda T}(u_\nu q_\mu + u_\mu q_\nu)\right], \tag{37}$$

which can be rearranged in the following form,

$$\frac{dS}{dt} = \int d^3x\left[\frac{\Delta T^{\mu\nu}_{vis}}{T}\left(\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta}\right) + \frac{\Delta T^{\mu\nu}_{heat}}{T}\left[\left(\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta}\right) - \frac{1}{2\lambda T}(u_\nu q_\mu + u_\mu q_\nu)\right]\right]. \tag{38}$$

In this case also one can make the identifications as before,

$$\dot{x}_1 \to \Delta T^{\mu\nu}_{vis}, \qquad \dot{x}_2 \to \Delta T^{\mu\nu}_{heat}. \tag{39}$$

The comparison between Eqs. (38) and (1) will give,

$$X_1 = -\frac{1}{T}\left(\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta}\right)\Delta V,$$
$$X_2 = -\frac{1}{T}\left[\left(\frac{\pi_{\mu\nu}}{2\eta} - \frac{\Delta_{\mu\nu}\Pi}{3\zeta}\right) - \frac{1}{2\lambda T}(u_\nu q_\mu + u_\mu q_\nu)\right]\Delta V. \tag{40}$$

Again neglecting the stochastic term in Eq. (2) and comparing it with Eq. (38) one can get,

$$\gamma_{11}X_1 = -\Delta T^{\mu\nu}_{vis}, \tag{41}$$
$$\gamma_{22}X_2 = -\Delta T^{\mu\nu}_{heat}, \tag{42}$$
$$\gamma_{12} = \gamma_{21} = 0. \tag{43}$$

One can use the following parameterization for γ_{11} and γ_{22},

$$\gamma_{11} = \left[A\Delta^{\mu\nu\alpha\beta} + B\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\frac{1}{\Delta V}, \qquad \gamma_{22} = \left[\bar{A}\Delta^{\mu\alpha}u^\nu u^\beta + \bar{B}\Delta^{\nu\beta}u^\mu u^\alpha\right]\frac{1}{\Delta V}. \tag{44}$$

Since we know the forms of (X_1, X_2) and $(\Delta T^{\mu\nu}_{vis}, \Delta T^{\mu\nu}_{heat})$, therefore using Eqs. (44) and Eqs. (41)–(42), one can determine the coefficients $A = 2\eta T$, $B = \zeta T$ and $\bar{A} = \bar{B} = -2\lambda T^2$. Thus γ_{11} and γ_{22} can be written as,

$$\gamma_{11} = 2T\left[\left(\eta\Delta^{\mu\alpha}\Delta^{\nu\beta} - \frac{1}{3}\eta\Delta^{\mu\nu}\Delta^{\alpha\beta}\right) + \frac{1}{2}\zeta\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\frac{1}{\Delta V}, \tag{45}$$
$$\gamma_{22} = -2\lambda T^2\left[\Delta^{\mu\alpha}u^\nu u^\beta + \Delta^{\nu\beta}u^\mu u^\alpha\right]\frac{1}{\Delta V}. \tag{46}$$

Thus one can write the correlation functions using Eq. (3) as,

$$\left\langle S^{\mu\nu}_{vis}(x_1)S^{\alpha\beta}_{vis}(x_2)\right\rangle = 2T\left[\eta\left(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}\right) + \left(\zeta - \frac{2}{3}\eta\right)\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\delta(x_1 - x_2), \tag{47}$$
$$\left\langle S^{\mu\nu}_{heat}(x_1)S^{\alpha\beta}_{heat}(x_2)\right\rangle = -2\lambda T^2\left[\Delta^{\mu\alpha}u^\nu u^\beta + \Delta^{\nu\beta}u^\mu u^\alpha + \Delta^{\mu\beta}u^\nu u^\alpha + \Delta^{\nu\alpha}u^\mu u^\beta\right]\delta(x_1 - x_2), \tag{48}$$
$$\left\langle S^{\mu\nu}_{vis}(x_1)S^{\alpha\beta}_{heat}(x_2)\right\rangle = 0. \tag{49}$$

The form of these correlations is very similar to the correlations obtained for the relativistic Navier–Stokes case [6]. The relaxation time for the dissipative fluxes do not appear explicitly in the expressions for the correlation. However, the evolution of the correlations can be very different as demonstrated later.

2.3. Equations for dissipative fluxes in Landau–Lifshitz frame and fluctuation correlations for other hydrodynamic models

In this section we consider some of the interesting alternate approaches to the causal MIS hydrodynamics and some of its extensions.

2.3.1. Third order hydrodynamics

In Ref. [17] third order corrections to the MIS hydrodynamics were considered when the effect of bulk-viscosity and heat-flux were absent. In this case expression for the non-equilibrium entropy four-current can be written as,

$$S^\mu = su^\mu - \frac{\beta_2}{2T}\pi_{\alpha\beta}\pi^{\alpha\beta}u^\mu + \alpha\frac{\beta_2^2}{T}\pi_{\alpha\beta}\pi^\alpha_\sigma\pi^{\beta\sigma}u^\mu, \tag{50}$$

where, α is a new dimensionless coefficient and it is assumed to be a constant. The last term on the right hand side of the above equation represents the third order correction to the equation of entropy. In order to fulfill the requirement of maximal entropy at equilibrium, the third order term must satisfy the condition $\alpha\frac{\beta_2^2}{T}\pi_{\alpha\beta}\pi^\alpha_\sigma\pi^{\beta\sigma}u^\mu \leqslant 0$. Divergence of the entropy four-current can be written as,

$$\begin{aligned}\partial_\mu S^\mu = {} & \frac{1}{T}\pi_{\alpha\beta}\sigma^{\alpha\beta} - \pi_{\alpha\beta}\pi^{\alpha\beta}\partial_\mu\left(\frac{\beta_2}{2T}u^\mu\right) - \frac{\beta_2}{T}\pi_{\alpha\beta}\dot{\pi}^{\alpha\beta} \\ & + \alpha\partial_\mu\left(\frac{\beta_2^2}{T}u^\mu\right)\pi_{\alpha\beta}\pi^\alpha_\sigma\pi^{\beta\sigma} + 3\tau_\pi\theta\alpha\frac{\beta_2^2}{T}\pi_{\alpha\beta}\pi^\alpha_\sigma\dot{\pi}^{\beta\sigma} \geqslant 0.\end{aligned} \tag{51}$$

Here, the Knudsen number ($= \tau_\pi\theta$) is required to satisfy the condition $\tau_\pi\theta \ll 1$ for the validity of hydrodynamic approach. For the condition $T\partial_\mu S^\mu \geqslant 0$ to be satisfied one must have,

$$\partial_\mu S^\mu = \frac{1}{2\eta T}\pi^{\mu\nu}\pi_{\mu\nu}, \tag{52}$$

which implies that the form of shear viscous tensor $\pi^{\alpha\beta}$ should be given by,

$$\begin{aligned}\pi^{\alpha\beta} = 2\eta T\bigg[& \frac{1}{T}\sigma^{\alpha\beta} - \pi^{\alpha\beta}\partial_\mu\left(\frac{\beta_2}{2T}u^\mu\right) - \frac{\beta_2}{T}\dot{\pi}^{\alpha\beta} \\ & + \alpha\partial_\mu\left(\frac{\beta_2^2}{T}u^\mu\right)\pi^\alpha_\sigma\pi^{\beta\sigma} + 3\tau_\pi\theta\alpha\frac{\beta_2^2}{T}\pi^\alpha_\sigma\dot{\pi}^{\beta\sigma}\bigg].\end{aligned} \tag{53}$$

Since $\tau_\pi\theta \sim \frac{\tau_\pi}{\tau}$ is of the same order as $\frac{\pi^{\alpha\beta}}{T^4}$, therefore, the last term in above equation is a fourth order term [17]. Thus neglecting the last term one can write above equation as,

$$\dot{\pi}^{\alpha\beta} = -\frac{\pi^{\alpha\beta}}{\tau_\pi} + \frac{\sigma^{\alpha\beta}}{\beta_2} - \pi^{\alpha\beta}\frac{T}{\beta_2}\partial_\mu\left(\frac{\beta_2}{2T}u^\mu\right) + \alpha\frac{T}{\beta_2}\partial_\mu\left(\frac{\beta_2^2}{T}u^\mu\right)\pi^\alpha_\sigma\pi^{\beta\sigma}. \tag{54}$$

Here coefficients α and β_2 have respectively the values $\frac{8}{3}$ and $\frac{9}{4e}$ as given in Ref. [17].

Now starting from Eq. (52) and following similar prescription to determine the Onsager coefficient as done in second-order MIS hydrodynamics one can write

$$\dot{x} = \pi^{\mu\nu}, \qquad X = -\frac{1}{2\eta T}\pi^{\mu\nu}\Delta V, \tag{55}$$

and the Onsager coefficient,

$$\gamma = 2\eta T\left[\Delta^{\mu\alpha}\Delta^{\nu\beta} - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\frac{1}{\Delta V}. \tag{56}$$

The viscous correlation function can be written as,

$$\left\langle S_{vis}^{\mu\nu}(x_1)S_{vis}^{\alpha\beta}(x_2)\right\rangle = 2T\left[\eta\left(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}\right) - \frac{2}{3}\eta\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\delta(x_1 - x_2). \tag{57}$$

One can notice that this expression is the same as the one obtained using the second-order theory with $\Pi = 0$.

2.3.2. JBP hydrodynamics

In Ref. [18] the authors have constructed the expression for the entropy four-current S^μ generalized from the Boltzmann H-function and find out the expression for its divergence as,

$$\begin{aligned}\partial_\mu S^\mu = &-\frac{\Pi}{T}\left[\theta + \beta_0\dot{\Pi} + \beta_{\Pi\Pi}\Pi\theta + \alpha_0\nabla_\mu n^\mu + \psi\alpha_{n\Pi}n_\mu\dot{u}^\mu + \psi\alpha_{\Pi n}n_\mu\nabla^\mu\alpha\right]\\ &-\frac{n^\mu}{T}\left[T\nabla_\mu\alpha - \beta_1\dot{n}^\mu - \beta_{nn}n_\mu\theta + \alpha_0\nabla_\mu\Pi + \alpha_1\nabla_\nu\pi_\mu^\nu + \tilde{\psi}\alpha_{n\Pi}\Pi\dot{u}_\mu\right.\\ &\left.+\tilde{\psi}\alpha_{\Pi n}\Pi\nabla_\mu\alpha + \tilde{\chi}\alpha_{\pi n}\pi_\mu^\nu\nabla_\nu\alpha + \tilde{\chi}\alpha_{n\pi}\pi_\mu^\nu\dot{u}_\nu\right]\\ &+\frac{\pi^{\mu\nu}}{T}\left[\sigma_{\mu\nu} - \beta_2\dot{\pi}_{\mu\nu} - \beta_{\pi\pi}\theta\pi_{\mu\nu} - \alpha_1\nabla_{\langle\mu}n_{\nu\rangle} - \chi\alpha_{\pi n}n_{\langle\mu}\nabla_{\nu\rangle}\alpha - \chi\alpha_{n\pi}n_{\langle\mu}\dot{u}_{\nu\rangle}\right]\end{aligned} \tag{58}$$

where $\theta = \partial_\mu u^\mu$. The second law of thermodynamics $T\partial_\mu S^\mu \geqslant 0$ is guaranteed to be satisfied if we have,

$$T\partial_\mu S^\mu = \frac{\Pi^2}{\zeta} - \frac{n^\mu n_\mu}{\lambda} + \frac{\pi^{\mu\nu}\pi_{\mu\nu}}{2\eta}, \tag{59}$$

therefore, π, n^μ and $\pi^{\mu\nu}$ should satisfy the following equations,

$$\left[\theta + \beta_0\dot{\Pi} + \beta_{\Pi\Pi}\Pi\theta + \alpha_0\nabla_\mu n^\mu + \psi\alpha_{n\Pi}n_\mu\dot{u}^\mu + \psi\alpha_{\Pi n}n_\mu\nabla^\mu\alpha\right] = -\frac{\Pi}{\zeta}, \tag{60}$$

$$\begin{aligned}&\left[T\nabla_\mu\alpha - \beta_1\dot{n}^\mu - \beta_{nn}n_\mu\theta + \alpha_0\nabla_\mu\Pi + \alpha_1\nabla_\nu\pi_\mu^\nu + \tilde{\psi}\alpha_{n\Pi}\Pi\dot{u}_\mu\right.\\ &\quad\left.+\tilde{\psi}\alpha_{\Pi n}\Pi\nabla_\mu\alpha + \tilde{\chi}\alpha_{\pi n}\pi_\mu^\nu\nabla_\nu\alpha + \tilde{\chi}\alpha_{n\pi}\pi_\mu^\nu\dot{u}_\nu\right] = \frac{n^\mu}{\lambda},\end{aligned} \tag{61}$$

$$\sigma_{\mu\nu} - \beta_2\dot{\pi}_{\mu\nu} - \beta_{\pi\pi}\theta\pi_{\mu\nu} - \alpha_1\nabla_{\langle\mu}n_{\nu\rangle} - \chi\alpha_{\pi n}n_{\langle\mu}\nabla_{\nu\rangle}\alpha - \chi\alpha_{n\pi}n_{\langle\mu}\dot{u}_{\nu\rangle} = \frac{\pi^{\mu\nu}}{2\eta}, \tag{62}$$

where λ, ζ, $\eta \geqslant 0$ are the coefficients of charge conductivity, bulk viscosity and shear viscosity respectively. Coefficients α_i, β_i, α_{XY}, β_{XX} are the additional transport coefficients and the parameters ψ, χ along with $\tilde{\psi} = 1 - \psi$ and $\tilde{\chi} = 1 - \chi$ describe the contributions due to the cross terms of Π and $\pi^{\mu\nu}$ with n^μ.

The Onsager coefficients in this case too, can be obtained using the parameterization [see Eq. (25)],

$$\gamma_{11} = 2T\left[\eta\Delta^{\mu\alpha}\Delta^{\nu\beta} + \frac{1}{2}\left(\zeta - \frac{2}{3}\eta\right)\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\frac{1}{\Delta V}, \tag{63}$$

$$\gamma_{22} = -\frac{\lambda T\Delta^{\mu\nu}}{\Delta V}. \tag{64}$$

It should be noted that in Ref. [18] the authors have used $\frac{n^\mu}{\lambda}$ in Eq. (59) instead of $\frac{n^\mu}{\lambda T}$ and therefore the Onsager coefficient differs by factor T (see for example, Eqs. (26) and (64)). The correlation functions can be written as,

$$\left\langle S^{\mu\nu}_{vis}(x_1)S^{\alpha\beta}_{vis}(x_2)\right\rangle = 2T\left[\eta\left(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}\right) + \left(\zeta - \frac{2}{3}\eta\right)\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\delta(x_1 - x_2), \tag{65}$$

$$\left\langle I^\mu(x_1)I^\nu(x_2)\right\rangle = -2\lambda T\Delta^{\mu\nu}\delta(x_1 - x_2), \tag{66}$$

$$\left\langle S^{\mu\nu}_{vis}(x_1)I^\alpha(x_2)\right\rangle = 0. \tag{67}$$

2.3.3. DKR hydrodynamics

In Ref. [19], it was demonstrated that derivation of relativistic viscous hydrodynamic equation from the 14-moment method done by Israel and Stewart may not be unique. In Ref. [19], the authors obtained relativistic dissipative hydrodynamic equations for the dissipative fluxes as,

$$\dot{\Pi} = -\frac{\Pi}{\tau_\Pi} - \beta_\Pi\theta - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}, \tag{68}$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\alpha^{\langle\mu}\omega^{\nu\rangle\alpha} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi_\alpha^{\langle\mu}\sigma^{\nu\rangle\alpha} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}, \tag{69}$$

where $\theta = \nabla_\alpha u^\alpha$, and τ's, β's, δ's, λ's are the transport coefficients.

It should be noted that Eq. (69) contains vorticity term $\omega^{\alpha\beta} = \frac{1}{2}(\nabla^\alpha u^\beta - \nabla^\beta u^\alpha)$. Note that in writing the above equations we have considered the fluid with no net baryon number. Thus the Eq. (7) with no net baryon number can be written as,

$$\partial_\mu\left(su^\mu\right) = \frac{\pi^{\mu\nu}\sigma_{\mu\nu}}{T} - \frac{\Pi\nabla_\alpha u^\alpha}{T}. \tag{70}$$

From Eqs. (68) and (69) it is easy to write,

$$\nabla_\alpha u^\alpha = -\frac{\dot{\Pi}}{\beta_\Pi} - \frac{\Pi}{\beta_\Pi\tau_\Pi} - \frac{\delta_{\Pi\Pi}\Pi\nabla_\alpha u^\alpha}{\beta_\Pi} + \frac{\lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}}{\beta_\Pi}, \tag{71}$$

$$\sigma_{\mu\nu} = \frac{\dot{\pi}^{\langle\mu\nu\rangle}}{2\beta_\pi} + \frac{\pi^{\mu\nu}}{2\beta_\pi\tau_\pi} - \frac{\pi_\alpha^{\langle\mu}\omega^{\nu\rangle\alpha}}{\beta_\pi} + \frac{\delta_{\pi\pi}\pi^{\mu\nu}\nabla_\alpha u^\alpha}{2\beta_\pi} + \frac{\tau_{\pi\pi}\pi_\alpha^{\langle\mu}\sigma^{\nu\rangle\alpha}}{2\beta_\pi} - \frac{\lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}}{2\beta_\pi}. \tag{72}$$

Now substituting Eqs. (71) and (72) in Eq. (70) one can write,

$$\begin{aligned}\partial_\mu\left(su^\mu\right) = {} & \frac{\pi_{\mu\nu}}{T}\left[\frac{\dot{\pi}^{\langle\mu\nu\rangle}}{2\beta_\pi} + \frac{\pi^{\mu\nu}}{2\beta_\pi\tau_\pi} - \frac{\pi_\alpha^{\langle\mu}\omega^{\nu\rangle\alpha}}{\beta_\pi}\right.\\ & \left.+ \frac{\delta_{\pi\pi}\pi^{\mu\nu}\nabla_\alpha u^\alpha}{2\beta_\pi} + \frac{\tau_{\pi\pi}\pi_\alpha^{\langle\mu}\sigma^{\nu\rangle\alpha}}{2\beta_\pi} - \frac{\lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}}{2\beta_\pi}\right]\\ & - \frac{\Pi}{T}\left[-\frac{\dot{\Pi}}{\beta_\Pi} - \frac{\Pi}{\beta_\Pi\tau_\Pi} - \frac{\delta_{\Pi\Pi}\Pi\nabla_\alpha u^\alpha}{\beta_\Pi} + \frac{\lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}}{\beta_\Pi}\right].\end{aligned} \tag{73}$$

After substituting back for $\nabla_\alpha u^\alpha$ and $\sigma_{\mu\nu}$ again from Eqs. (71) and (72) into Eq. (73) one can see the terms with the coefficients δ's, τ's, and λ's are of $O(\pi^3)$ or of the higher order, therefore, one can neglect these terms.

One can easily show that $\dot{\pi}^{\langle\mu\nu\rangle} = \dot{\pi}^{\mu\nu} + \pi^{\mu}_{\beta} u^{\nu} Du^{\beta} + \pi^{\nu}_{\alpha} u^{\mu} Du^{\alpha}$. This would imply that,

$$\pi_{\mu\nu}\dot{\pi}^{\langle\mu\nu\rangle} = \pi_{\mu\nu}\dot{\pi}^{\mu\nu}. \tag{74}$$

Now neglecting the terms with the coefficients δ's, τ's, and λ's from Eq. (73) for the reason mentioned above, using Eq. (74) and the identity, $\pi_{\mu\nu}\pi_{\alpha}^{\langle\mu}\omega^{\nu\rangle\alpha} = 0$, one can get,

$$\partial_\mu S^\mu = \left[-\partial_\mu\left(\frac{u^\mu}{4\beta_\pi T}\right) + \frac{1}{2\beta_\pi \tau_\pi T}\right]\pi^{\alpha\beta}\pi_{\alpha\beta} + \left[-\partial_\mu\left(\frac{u^\mu}{2\beta_\Pi T}\right) + \frac{1}{\beta_\Pi \tau_\Pi T}\right]\Pi^2, \tag{75}$$

where, S^μ is the non-equilibrium entropy current for DKR hydrodynamics and has the form,

$$S^\mu = \left(su^\mu - \frac{\pi^{\alpha\beta}\pi_{\alpha\beta}u^\mu}{4\beta_\pi T} - \frac{\Pi^2 u^\mu}{2\beta_\Pi T}\ldots\right). \tag{76}$$

Note that $\beta_{\pi,\Pi} = \frac{\eta}{\tau_{\pi,\Pi}}$. In Eq. (75) the terms with gradients of velocity field can be neglected as $\partial_\mu(\frac{u^\mu}{4\beta_\pi T}) = \frac{\tau_{\pi,\Pi}\theta}{\eta T} \ll \frac{1}{\eta T}$, where $\theta = \partial_\mu u^\mu$ is the inverse of the expansion scale and $\tau_{\pi,\Pi}$ is relaxation time scale. For the system to be in the relaxation regime, one must have $\tau_{\pi,\Pi}\theta \ll 1$ (see Refs. [15,17]). Therefore from Eq. (75) one obtains,

$$\frac{dS}{dt} = \int d^3x \left[\left(\frac{1}{2\beta_\pi \tau_\pi T}\right)\pi^{\alpha\beta}\pi_{\alpha\beta} + \left(\frac{1}{\beta_\Pi \tau_\Pi T}\right)\Pi^2\right]. \tag{77}$$

Further, Eq. (77) can be written in the following form,

$$\frac{dS}{dt} = \int d^3x \left[\frac{\Delta T^{\alpha\beta}_{vis}}{T}\left(\frac{\pi_{\alpha\beta}}{2\beta_\pi \tau_\pi} - \frac{\Delta_{\alpha\beta}\Pi}{3\beta_\Pi \tau_\Pi}\right)\right]. \tag{78}$$

A comparison of the above expression with the phenomenological equation (1) yields,

$$\dot{x} \to \Delta T^{\alpha\beta}_{vis}, \qquad X \to -\frac{1}{T}\left[\left(\frac{\pi_{\alpha\beta}}{2\beta_\pi \tau_\pi} - \frac{\Delta_{\alpha\beta}\Pi}{3\beta_\Pi \tau_\Pi}\right)\right]\Delta V. \tag{79}$$

Again by comparing Eq. (78) with Eq. (2) (when $\xi = 0$) one can get,

$$\gamma X = -\Delta T^{\mu\nu}_{vis}. \tag{80}$$

Where γ is a rank four tensor and can be written as,

$$\gamma = 2T\left[\beta_\pi \tau_\pi \Delta^{\mu\nu\alpha\beta} + \frac{1}{2}\beta_\Pi \tau_\Pi \Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\frac{1}{\Delta V}. \tag{81}$$

Thus the viscous correlations are,

$$\begin{aligned}\left\langle S^{\mu\nu}_{vis}(x_1) S^{\alpha\beta}_{vis}(x_2)\right\rangle = 2T\bigg[&\beta_\pi \tau_\pi\left(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}\right) \\ &+ \left(\beta_\Pi \tau_\Pi - \frac{2}{3}\beta_\pi \tau_\pi\right)\Delta^{\mu\nu}\Delta^{\alpha\beta}\bigg]\delta(x_1 - x_2).\end{aligned} \tag{82}$$

2.3.4. Conformal viscous hydrodynamics

The entropy current for the conformal hydrodynamics [20] can be written as,

$$S^\mu = \left(su^\mu - \frac{\tau_\pi}{4\eta T} \pi_{\alpha\beta}\pi^{\alpha\beta} u^\mu \right). \tag{83}$$

One can easily find the following expression for the Onsager coefficient and the correlation function,

$$\gamma = 2\eta T \left[\Delta^{\mu\alpha}\Delta^{\nu\beta} - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta} \right] \frac{1}{\Delta V}, \tag{84}$$

$$\langle S^{\mu\nu}_{vis}(x_1) S^{\alpha\beta}_{vis}(x_2) \rangle = 2\eta T \left[\left(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha} \right) - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta} \right] \delta(x_1 - x_2). \tag{85}$$

3. Calculation of correlation functions in boost-invariant hydrodynamics

As an example we apply the results obtained in the previous sections to the relativistic heavy-ion collisions for the Bjorken flow. According to the Bjorken scenario in heavy ion collisions, the reaction volume is strongly expanded in the longitudinal direction, i.e. along the collision axis (z-axis). So one can assume that there is no transverse flow. Thus one can describe flow in $1+0$ dimension [22]. It is useful to introduce the light cone variable y and proper time τ which are defined by,

$$\tau = \sqrt{t^2 - z^2} \quad \text{and} \quad y = \operatorname{arc\,tanh}(z/t) = \frac{1}{2}\ln\left(\frac{t+z}{t-z}\right). \tag{86}$$

The partial derivatives in time and space can be expressed as,

$$\begin{bmatrix} \partial_t \\ \partial_z \end{bmatrix} = \begin{bmatrix} \cosh y & -\sinh y \\ -\sinh y & \cosh y \end{bmatrix} = \begin{bmatrix} \partial_\tau \\ \frac{1}{\tau}\partial_y \end{bmatrix}. \tag{87}$$

The flow velocity under the scaling assumption can be written as, $u^\mu = \gamma(1, 0, 0, v_z) = (\frac{t}{\tau}, 0, 0, \frac{z}{\tau}) = (\cosh y, 0, 0, \sinh y)$. We consider only longitudinal flow fluctuations and parameterize the flow velocity [23] as,

$$u^\mu = (\cosh\bar{\theta}, \sinh\bar{\theta}), \tag{88}$$

where $\bar{\theta} = y + \delta\bar{\theta}(y, \tau)$ and $\delta\bar{\theta}(y, \tau)$ are the fluctuations in the longitudinal flow. In scaling limit, $\bar{\theta} = y$. With this parameterization and using the transformation of derivatives one can introduce the operators D, ∇ such that,

$$\begin{bmatrix} D \\ \nabla \end{bmatrix} = \begin{bmatrix} \cosh(\bar{\theta} - y) & \sinh(\bar{\theta} - y) \\ \sinh(\bar{\theta} - y) & \cosh(\bar{\theta} - y) \end{bmatrix} = \begin{bmatrix} \partial_\tau \\ \frac{1}{\tau}\partial_y \end{bmatrix}. \tag{89}$$

In the scaling limit, $D = u^\mu \partial_\mu = \frac{\partial}{\partial\tau} = \partial_\tau$ and $\partial_\mu u^\mu = \nabla\bar{\theta} = \frac{1}{\tau}$.

Since $S^{\mu\nu}$ satisfies the condition,

$$u_\mu S^{\mu\nu} = 0. \tag{90}$$

One can write $S^{\mu\nu}$ as [6],

$$S^{\mu\nu} = w(\tau) f(y, \tau) \Delta^{\mu\nu}, \tag{91}$$

where, $w = \epsilon + p = Ts$ and f is a dimensionless quantity which satisfy $\langle f\rangle = 0$, where $\langle\rangle$ denotes the 'average value'. In heavy-ion collision experiments at LHC or RHIC a baryon free quark–gluon plasma is expected to be produced, therefore $q^\mu = 0$. Thus only viscous-correlations are of interest, which for MIS, JBP and third order (TO) can be written as,

$$\big\langle f(y_1,\tau_1)f(y_2,\tau_2)\big\rangle = \frac{2T(\tau_1)}{A\tau_1 w^2(\tau_1)}\left[\frac{4}{3}\eta(\tau_1)+\zeta(\tau_1)\right]\delta(\tau_1-\tau_2)\delta(y_1-y_2), \tag{92}$$

where $\delta(x_1 - x_2)_{Transverse}$ is replaced by effective transverse area A of colliding nuclei. Note that these correlations are the same as that obtained by authors in Ref. [6] for Navier–Stokes case. Similarly, for DKR case one can write the viscous correlations as,

$$\big\langle f(y_1,\tau_1)f(y_2,\tau_2)\big\rangle = \frac{2T(\tau_1)(\frac{4}{3}\beta_\pi\tau_\pi+\beta_\Pi\tau_\Pi)}{A\tau_1 w^2(\tau_1)}\delta(\tau_1-\tau_2)\delta(y_1-y_2). \tag{93}$$

By defining $\eta = \beta_\pi\tau_\pi$ as in Ref. [19] and neglecting the bulk viscosity for the correlation functions, one can rewrite the correlations for all the models of hydrodynamics that considered here as,

$$\big\langle f(y_1,\tau_1)f(y_2,\tau_2)\big\rangle = \frac{X(\tau_1)_{[E]}}{A}\delta(\tau_1-\tau_2)\delta(y_1-y_2), \tag{94}$$

where,

$$X(\tau_1)_{[E]} = \frac{8}{3\tau_1 w(\tau_1)}\left(\frac{\eta(\tau_1)}{s(\tau_1)}\right)_{[E]}. \tag{95}$$

Here, subscript $[E]$ denotes the particular type of hydrodynamics model considered from the set of hydrodynamics models, for example $[E] = [MIS, JBP, DKR, TO, NS]$.

It is useful to study the correlation function normalized by the initial value of the correlation obtained using the Navier–Stokes theory i.e. $C(\tau)_{[E]} = \frac{w^2(\tau)X(\tau)_{[E]}}{w^2(\tau_0)_{NS}X(\tau_0)_{NS}}$ where, τ_0 is the initial-time for the hydrodynamics. $C(\tau)_{[E]}$ can also be written as,

$$C(\tau)_{[E]} = \frac{(\frac{\tau_0}{\tau})(\frac{\eta(\tau)}{s(\tau)})_{[E]}\frac{w(\tau)}{w(\tau_0)}}{(\frac{\eta(\tau)}{s(\tau)})_{NS}}. \tag{96}$$

Further, we neglect the effect of bulk-viscosity by considering the initial temperature T_i to be much larger than the critical temperature, $T_c = 0.190$ GeV. Now, in the Landau–Lifshitz frame, the energy and the momentum conservation laws are given by,

$$u_\nu\partial_\mu T^{\mu\nu} = D\epsilon + (\epsilon+p)\nabla\bar{\theta} - \pi_{\mu\nu}\nabla^{\langle\mu}u^{\nu\rangle} - S_{\mu\nu}\nabla^{(\mu}u^{\nu)} = 0, \tag{97}$$

$$\Delta^\alpha_\nu\partial_\mu T^{\mu\nu} = (\epsilon+p)Du^\alpha - \nabla^\alpha p + \Delta^{\alpha\nu}\nabla^\sigma\pi_{\nu\sigma} - \pi^{\alpha\nu}Du_\nu + \Delta^{\alpha\nu}\partial^\sigma S_{\sigma\nu} = 0, \tag{98}$$

where, $\pi^{\alpha\beta}$ is the shear stress tensor and the dynamical equation for $\pi^{\alpha\beta}$ can be different for different models of hydrodynamics.

In the scaling limit $\bar{\theta} = y$, $D = u^\mu\partial_\mu = \partial_\tau$, $\partial_\mu u^\mu = \nabla\bar{\theta} = \frac{1}{\tau}$. Using these, one can write the above equations as [19],

$$\partial_\tau\epsilon = -\frac{(\epsilon+p)}{\tau} + \frac{\pi}{\tau}. \tag{99}$$

Here, $\pi = \pi^{00} - \pi^{zz}$, and the noise term is considered to be smaller than the background quantities.

Now equation for π in the scaling limit, for DKR and JBP hydrodynamics can be written as,

$$\partial_\tau \pi + \frac{\pi}{\tau_\pi} = \beta_\pi \frac{4}{3\tau} - \lambda \frac{\pi}{\tau}. \tag{100}$$

For JBP case, coefficients β_π, τ_π and λ are as follows,

$$\beta_\pi = \frac{2p}{3}, \qquad \tau_\pi^{-1} = \frac{5}{9}\frac{\sigma p}{T}, \qquad \lambda = \frac{4}{3}, \tag{101}$$

where, σ is the total cross-section [19] and it is assumed to be independent of energy [24,17]. For DKR hydrodynamics, the parameters β_π, τ_π [19] and λ [25] are,

$$\beta_\pi = \frac{4p}{5}, \qquad \tau_\pi^{-1} = \frac{3}{5}\frac{\sigma p}{T}, \qquad \lambda \equiv \frac{1}{3}\tau_{\pi\pi} + \delta_{\pi\pi} = \frac{38}{21}. \tag{102}$$

Similarly equations for π in the scaling limit MIS and third order hydrodynamics respectively can be written as,

$$\partial_\tau \pi + \frac{\pi}{\tau_\pi} = \frac{\eta}{\tau_\pi}\frac{4}{3\tau} - \frac{1}{2}\pi\left(\frac{1}{\tau} + \frac{\eta T}{\tau_\pi}\frac{\partial}{\partial \tau}\left(\frac{\tau_\pi}{\eta T}\right)\right). \tag{103}$$

$$\partial_\tau \pi + \frac{\pi}{\tau_\pi} = \frac{\eta}{\tau_\pi}\frac{4}{3\tau} - \frac{4}{3}\frac{\pi}{\tau} - \frac{\pi^2}{p\tau}, \tag{104}$$

where $\frac{\eta}{\tau_\pi} = \frac{2p}{3}$ and $\tau_\pi^{-1} = \frac{5}{9}\frac{\sigma p}{T}$.

In what follows we consider the ideal equation of state, $\epsilon = 3p$ with the pressure is given by the bag model, $p = \frac{\pi^2}{30}T^4$. Further, we consider the initial temperature $T_i = 0.310$ GeV and initial viscous stress π either zero or has the Navier–Stokes value that is $\pi = \frac{4}{3}\frac{\eta}{\tau}$ for all the causal hydrodynamics and numerically solve Eqs. (99), (100), Eqs. (99), (103) and Eqs. (99), (104) for evaluating the correlations (96) in case of MIS, JBP, DKR and Third order (TO) hydrodynamics. However, for the Navier–Stokes hydrodynamics one needs to solve only Eq. (99) with the same value of initial temperature and the viscous stress is given by,

$$\pi = \eta \frac{4}{3\tau}. \tag{105}$$

The results of the numerical work are presented in the following section.

4. Results and discussions

We have studied fluctuations in various models of relativistic causal viscous hydrodynamics. Eqs. (27)–(29), (47)–(49), (57), (65)–(67), (82) and (85) represent our main results describing the correlation functions for various models of relativistic causal hydrodynamics. First we should like to note here that the form of the correlation functions given by Eqs. (27)–(29), (47)–(49), (57), (65)–(67), (82) and (85) are strikingly similar to the correlation functions obtained using relativistic Navier–Stokes theory [6,3]. The correlations do not explicitly depend upon the relaxation times that appear in the causal theories of hydrodynamics. This indicates a kind of universality of the correlations given by Eqs. (27)–(29), (47)–(49), (57), (65)–(67), (82) and (85). One can notice from Eq. (27) that the viscous correlation depends on $\epsilon + p - \mu n$ and the ratio of viscous coefficients to the entropy density. The universality can be understood by the positivity argument of four entropy current i.e. $T\partial_\mu S^\mu = \frac{\Pi^2}{\zeta} - \frac{q^\mu q_\mu}{\lambda T} + \frac{\pi^{\mu\nu}\pi_{\mu\nu}}{2\eta} \geqslant 0$. Which is used to write the expression for $\frac{dS}{dt}$ by using the following properties of dissipative flues: $\Delta_{\mu\nu}\pi^{\mu\nu} = 0$,

$q_\mu u^\mu = 0$ and $u_\mu \pi^{\mu\nu} = 0$. These constraints are universal and satisfied in case of Navier–Stokes as well as all causal hydrodynamics no matter what form of $\pi_{\mu\nu}$, q^μ and Π is. The determination of Onsager coefficients [using Eq. (2)] also depends on these constraints leading to the same form for all hydrodynamic theories and consequently the correlation function remains the same for all theories. But in case of DTT kind of hydrodynamics, it is not clear if divergence of the entropy four-current can be expressed directly in terms of scalar product of the viscosity and heat-flux tensors.

In order to understand the evolution of the correlation functions in some details we have calculated the normalized correlation functions given by Eq. (96) for an expanding one-dimensional boost-invariant (Bjorken) flow. In this case all the correlations are proportional to $(\epsilon + p)/\tau$. However, the details of temporal evolution of $\epsilon + p$ vary with the choice of different hydrodynamical models. In Figs. 1–2, we plot the normalized correlation function $C(\tau)_{[E]}$ (Eq. (96)) as a function of time τ, where $[E]$ stands for MIS, JBP, DKR, TO (third order) and NS hydrodynamics. Each figure has five kind of curves: the solid (red) color curve describes the Navier–Stokes case while the dotted–dashed (blue), the dashed (purple), the dotted (green) and large-dashed (black) curves respectively describe MIS, JBP, DKR and TO cases. The left panel shows the case when the initial value for the viscous stress $\pi = 0$, while the right panel represents the case when the initial value of π is the same as the Navier–Stokes case.

There are two possible comparisons between the correlation functions $C(\tau)_{[E]}$. In one such comparison the energy-independent cross-section σ [see Eqs. (101), (102)] is kept the same for all the different versions of the hydrodynamics [19]. Following Refs. [24,19], one can write the viscosity coefficient for the different models of hydrodynamics as $\eta_{DKR} = \frac{4T}{3\sigma}$, $\eta_{MIS} = \frac{6T}{5\sigma} = \eta_{JBP} = \eta_{TO}$ and $\eta_{NS} = 0.8436\frac{3T}{2\sigma}$. Thus the relations between different η are given by the scaling: $\eta_{MIS} = \eta_{JBP} = \eta_{TO} = 9/10\eta_{DKR}$ and $\eta_{NS} = \frac{7.59}{8}\eta_{DKR}$. In the another way of comparing $C(\tau)_{[E]}$, the ratio η/s is kept the same for the different models of the hydrodynamics, while the σ is varied for the different models.

Fig. 1 shows the case when the transport cross-section is kept the same for all the models of hydrodynamics. The inset figure in all the diagrams shows the plots of correlation functions with better resolution in τ range between 3 fm/c to 6 fm/c. Cases (a–b), (c–d) and (e–f) correspond to $\frac{\eta_{DKR}}{s} = 0.08$, 0.56 and 1.60. Values of η/s for other models can be found using the scaling relation discussed above. The initial temperature T_i and initial time τ_i are respectively chosen to be 310 MeV and 0.5 fm/c.

One can notice for Figs. 1(a–b) that when η_{DKR}/s is close to the minimum possible value $(1/4\pi)$, all the correlations overlap with each other. This is expected as all the viscous hydrodynamics models should approach the ideal hydrodynamics limit when $\eta/s \approx 1/4\pi$. Figs. 1(c–d) correspond to the case when $\frac{\eta_{DKR}}{s} = 0.56$, i.e. almost seven times larger than the most minimum value, the correlations only marginally differ from each other. Overall difference between the correlation functions obtained using initial condition $\pi = 0$ and $\pi \neq 0$ is not significant. However, when $\pi = 0$ case Navier–Stokes correlation slightly dominates over the correlation functions obtained using the causal models. While for the case when the initial value of π is the same as Navier–Stokes value, it is the MIS correlation function dominates over the other correlation functions. Figs. 1(e–f), correspond to the case when η/s almost twenty times larger than the minimum value. In Fig. 1(e) the Navier–Stokes correlation first increases with time and then decreases. However, all the causal models correlation decreases with time. Rise in the Navier–Stokes correlation can be attributed to the unphysical behavior noted in Ref. [26]. In this case it may be possible to distinguish between the correlation function from the Navier–Stokes theory from the causal hydrodynamics models. However, the correlation function of the

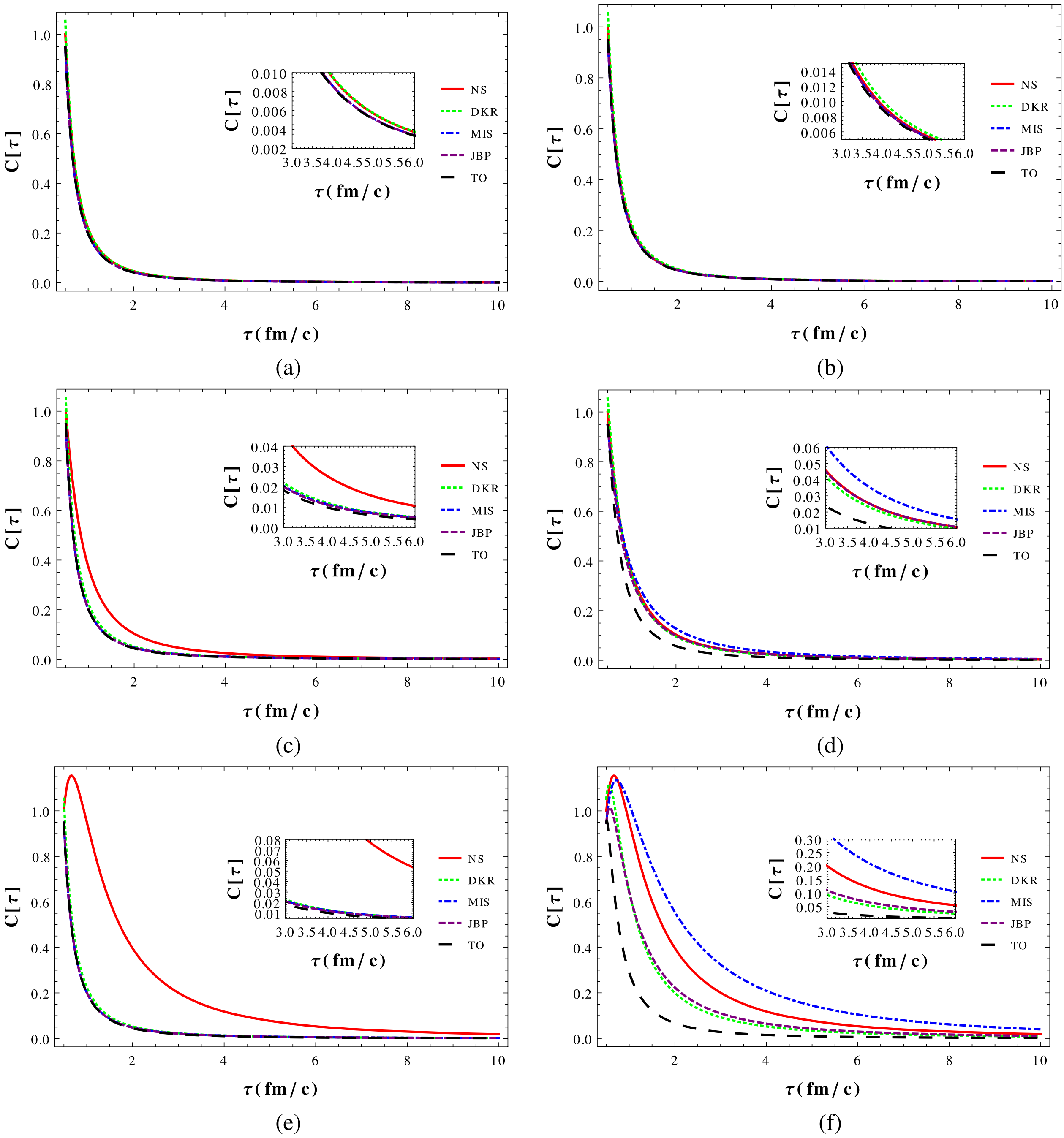

Fig. 1. (a), (b), (c), (d), (e) and (f) show time evolution of the function $C(\tau)_{[E]}$ [see Eq. (96)] with the same initial temperature $T_i = 310$ MeV. Where $[E]$ corresponds to NS, MIS, DKR, JBP and TO hydrodynamics. The coefficient of viscosity is calculated using $\eta_{DKR} = \frac{4T}{3\sigma}$. The scaling $\eta_{MIS} = \eta_{JBP} = \eta_{TO} = 9/10\eta_{DKR}$ and $\eta_{NS} = \frac{7.59}{8}\eta_{DKR}$ ensure that the cross-section remains same in the comparison between the models of hydrodynamics. Cases (a), (c) and (e) correspond to $\frac{\eta_{DKR}}{s} = 0.08, 0.56, 1.60$ respectively with initial time $\tau_0 = 0.5$ fm/c and $\pi_0 = 0.0$ for all causal approaches. While the cases (b), (d), and (f) correspond to the same $\frac{\eta_{DKR}}{s}$ and τ_0 as in the former cases, but with π_0 equal to Navier–Stokes initial value for all the hydrodynamic approaches. (For interpretation of the colours in this figure, the reader is referred to the web version of this article.)

causal models overlaps with each other. But when the Navier–Stokes value for the initial stress π_0 is chosen for the causal models, all the correlation functions first increase with time and later the plummet with time. This case can be considered to be unphysical as for all the hydrodynamics models initially $\epsilon + p < \pi$. The condition $\epsilon + p < \pi$ violates the validity of the second order hydrodynamics.

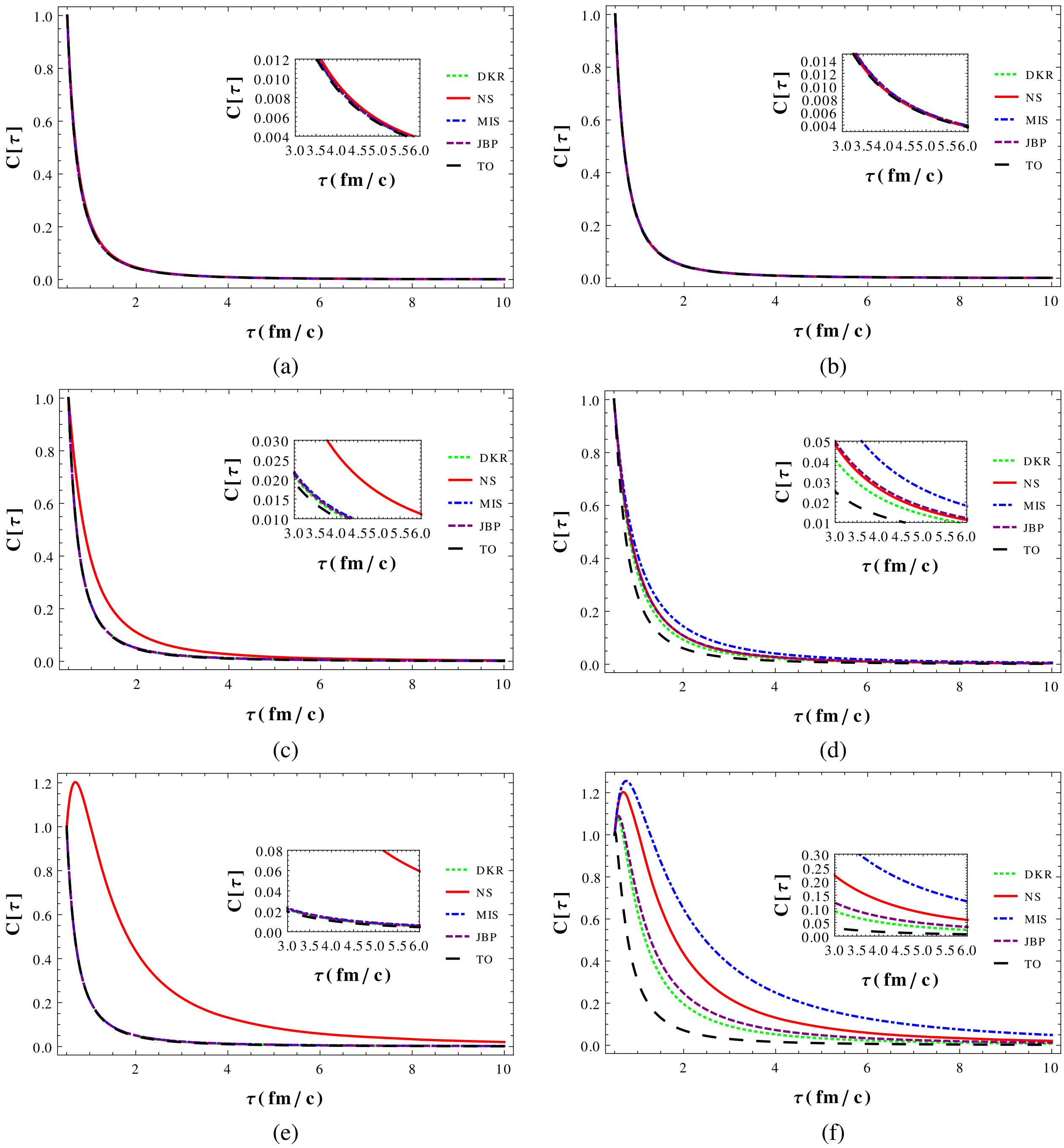

Fig. 2. (a), (c), (b), (d), (e) and (f) show the time evolution of the function $C(\tau)_{[E]}$ [see Eq. (96)] with the same initial temperature $T_i = 310$ MeV. Where $[E]$ corresponds to NS, MIS, DKR, JBP and TO hydrodynamics. Note that in all the figures the ratio of the viscosity to entropy density is kept the same for all the hydrodynamic approaches. Figs. 2(a), 2(c) and 2(e) correspond to $\frac{\eta}{s} = 0.08$, 0.56, 1.60 respectively with initial time $\tau_0 = 0.5$ fm/c and $\pi_0 = 0.0$ for all causal approaches. While Figs. 2(b), 2(d) and 2(f) correspond to the same $\frac{\eta}{s}$ and τ_0 as in the former cases but with π_0 equal to Navier–Stokes initial value for all the hydrodynamic approaches. (For interpretation of the colours in this figure, the reader is referred to the web version of this article.)

Fig. 2 corresponds to the case when the ratio of the viscosity coefficient to the entropy density is kept the same for all the five models of hydrodynamics. Cases (2a–2b), (2c–2d) and (2e–2f) respectively correspond to the situation when $\frac{\eta}{s}$ equal to 0.08, 0.56 and 1.60. The initial temperature and the initial times are kept the same as in the case for Fig. 1. One can notice that as $C(\tau)$ in Eq. (96) remains the same for all the hydrodynamical models, all the correlation functions, start at the same initial value. This was not the case in Fig. 1. Otherwise the general features about the correlation function remain the same as in Fig. 1. Moreover, we have changed the values of initial temperature and initial time. In these cases also the general features of the correlation function remains similar to those discussed in Fig. 1.

Finally we would like to discuss the importance our results. We first like to note that in the present work we have extended the formalism to calculate hydrodynamic fluctuations given in Ref. [2] to the relativistic causal theories. We have demonstrated that the form of the correlation functions in causal hydrodynamic theories remains the same as in the relativistic Navier–Stokes case [6]. This result is not expected apriori, as the underlying hydrodynamic equations for the causal theories [9,10,17–19] are very different than the Navier–Stokes equation. Eqs. (27)–(29), (47)–(49), (57), (65)–(67), (82) and (85) can be employed to calculate the two particle correlators [see Ref. [6]], which can be compared with the experimental data. However, this would require the solution of inhomogeneous (with noise term) hydrodynamical equations (of different types) in 3-dimension. Further, in the present example we have dealt with boost invariant one dimensional flow. However, for a non-central heavy-ion collision, the vorticity can play a significant role [27]. The presence of finite vorticity can cause the difference in the evolution in the correlation function for the different models of hydrodynamics remains to be seen. One can notice from Eq. (69) that vorticity can drive dynamics of the viscous stress in DKR hydrodynamics. However this will require to solve hydrodynamical equation in $2+1$ or $3+1$ dimensions. This is at present, beyond the scope of this work. Finally the numerical example that we have considered here, we plot the correlation function vs time. However, this numerical result cannot be compared with the experimental data. But, this can give us some idea about how the correlation function of different hydrodynamics compare with each other. We find that the correlation functions obtained using various causal theories do not significantly differ from each other for a variety of values of initial conditions and η/s. However, the correlation function obtained using NS-theory can have unphysical behavior for higher values of η/s and the NS-correlation function differs from the correlation functions obtained using the causal hydrodynamics.

5. Conclusions

We have studied fluctuations in various models of relativistic causal hydrodynamics. We have found that the general properties of the dissipative part of the energy–momentum tensor due to the viscosity and heat-flux play an important role in determining the Onsager coefficients and the correlation functions. We find that the analytic form of the correlation functions remains same for all the causal hydrodynamics that considered here and do not depend explicitly on the relaxation time. Further our numerical investigations also suggest that the qualitative behavior of the correlation functions for the various models of the causal hydrodynamics remains similar to those of the Navier–Stokes theory at least for a one dimensional boost-invariant flow.

Note added in proof

After this manuscript was prepared, we have found in Ref. [28] on arXiv that the authors have applied the fluctuation–dissipation relation to the relativistic viscous hydrodynamics with the memory effects. We have also found that in Ref. [29] the authors have calculated hydrodynamic fluctuation for MIS Hydrodynamics. Ours and their results match with each other. In Ref. [29] the author has obtained dynamics of the noise-function, while in our approach the noise-function is assumed to be given. However, we believe that one can obtain the noise function dynamics from the arguments similar to the one given in Ref. [13] to obtain the dynamics of dissipative fluxes.

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On the chiral imbalance and Weibel instabilities

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ABSTRACT

We study the chiral-imbalance and the Weibel instabilities in presence of the quantum anomaly using the Berry-curvature modified kinetic equation. We argue that in many realistic situations, e.g. relativistic heavy-ion collisions, both the instabilities can occur simultaneously. The Weibel instability depends on the momentum anisotropy parameter ξ and the angle (θ_n) between the propagation vector and the anisotropy direction. It has maximum growth rate at $\theta_n = 0$ while $\theta_n = \pi/2$ corresponds to a damping. On the other hand the pure chiral-imbalance instability occurs in an isotropic plasma and depends on difference between the chiral chemical potentials of right and left-handed particles. It is shown that when $\theta_n = 0$, only for a very small values of the anisotropic parameter $\xi \sim \xi_c$, growth rates of the both instabilities are comparable. For the cases $\xi_c < \xi \ll 1$ or $\xi \gtrsim 1$ at $\theta_n = 0$, the Weibel modes dominate over the chiral-imbalance instability if $\mu_5/T \leq 1$. However, when $\mu_5/T \geq 1$, it is possible to have dominance of the chiral-imbalance modes at certain values of θ_n for an arbitrary ξ.

The scope of applying kinetic theory to understand a variety of many-body problems arising in various branches of physics is truly enormous [1]. The conventional Boltzmann or Vlasov equations imply that the vector current associated with the gauge charges is conserved. But till recently a very important class of physical phenomena associated with the CP-violation or the triangle-anomaly were left out from the purview of a kinetic theory. In such phenomena the axial current is not conserved. It should be noted that there also exist several parity-violating hydrodynamics in literature [2–5]. But a hydrodynamical approach requires that the system under consideration remains in a thermal and chemical equilibrium. However, many applications of the chiral (CP-violating) physics may involve a non-equilibrium situation e.g. during the early stages of relativistic heavy-ion collisions. Therefore it is highly desirable to have a proper kinetic theory framework to tackle the CP-violating effect. Recently there has been a lot of progress in developing such a kinetic theory. In Refs. [6–11] it was shown that if the Berry curvature [12] has nonzero flux across the Fermi-surface then the particles on the surface can exhibit a chiral anomaly in presence of an external electromagnetic field. In this case the non-conservation of the chiral current J^μ can be attributed to Adler–Bell–Jackiw anomaly [13–15]. It can be shown that if a system of charged fermions does not conserve parity, it can develop an equilibrium electric current along the direction of the applied magnetic field [16]. This is so called the chiral-magnetic effect (CME). It has been suggested that a strong magnetic field created in relativistic-heavy-ion experiments can lead to CME in the quark–gluon plasma [17–19]. Indeed the recent experiments with STAR detector at Relativistic Heavy Ion Collider (RHIC) qualitatively agree with a local parity violation. However, more investigations are required to attribute this charge asymmetry with the CME [20,21]. The idea that a Berry-phase can influence the electronic properties [e.g. [22] and references cited therein] is well-known in condensed matter literature and it has applications in Weyl semimetal [23], graphene [24] etc. There exists a deep connection between a CP-violating quantum field theory and the kinetic theory with the Berry curvature corrections. In Ref. [25] it was shown that the parity-odd and parity-even correlations calculated using the modified kinetic theory are identical with the perturbative results obtained in next-to-leading order hard dense loop approximation.

In this work we aim to apply the kinetic theory with the Berry curvature corrections to some non-equilibrium situations. We first note that the results obtained in Refs. [6,25] are limited to low temperature regime $T \ll \mu_5$, where μ_5 is chiral chemical potential, when the Fermi surface is well-defined. Recently in Ref. [26] it was argued that the domain of validity of the modified kinetic theory can be extended beyond the Fermi surface to include the effect of finite temperature. As expected from the considerations of quantum-field theoretic approach [27–29] the parity-odd contribution remains temperature independent. Using the modified-

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kinetic theory [25] in presence of the chiral-imbalance the collective modes in electromagnetic or quark–gluon plasmas were analyzed [30]. In such a system CP-violating effect can split transverse waves into two branches [31]. It was shown in Ref. [30] that one of the transverse branches can become unstable in a quasi-static limit i.e. $\omega \ll k$ where, ω and k respectively denote frequency and wave-number of the transverse wave The instability can lead to the growth of Chern–Simons number (or magnetic-helicity in plasma physics parlance) at expense of the chiral-imbalance. Similar kinds of instabilities were found in Refs. [32–36] in different contexts.

It may be possible to observe the instability reported in Ref. [30] in the relativistic heavy-ion collisions. But in a realistic scenario the initial distribution function $n^0_{\mathbf{p}}$ for the strongly interacting matter formed during the collision can be anisotropic in the momentum space. This kind of initial distribution known to lead to the Weibel instability of the transverse modes. In the context of relativistic heavy-ion collision experiments Weibel instability has been extensively studied [37–41]. The Weibel instability is also well-known in the condensed matter [42,43] and plasma physics literatures [44–46] and it can generate magnetic fields in the plasma. Further it should be emphasized that both the chiral-imbalance and the Weibel instability can operate in the quasi-static regime. Therefore in the present work we aim to analyze the collective modes in an anisotropic chiral plasma and study how the chiral-imbalance and Weibel instabilities can influence each other. We believe that the results presented here will be useful in studying Weyl metals and the quark–gluon plasma created in relativistic heavy-ion collisions. We consider weak gauge field limit and assume the following power counting scheme: $\partial_x = O(\delta)$ and $A^\mu = O(\epsilon)$ where, ϵ and δ are small independent parameters. In this scenario the Berry curvature modified collisionless kinetic (Vlasov) equation at the leading order in A^μ is given by [25]:

$$(\partial_t + \mathbf{v}\cdot\partial_{\mathbf{x}})n_{\mathbf{p}} + (e\mathbf{E} + e\mathbf{v}\times\mathbf{B} - \partial_{\mathbf{x}}\epsilon_{\mathbf{p}})\cdot\partial_{\mathbf{p}}n_{\mathbf{p}} = 0 \tag{1}$$

where, $\mathbf{v} = \frac{\mathbf{p}}{p}$, $\epsilon_{\mathbf{p}} = p(1 - e\mathbf{B}\cdot\Omega_{\mathbf{p}})$ and $\Omega_{\mathbf{p}} = \pm\mathbf{p}/2p^3$. Here $\pm$ sign corresponds to right and left handed fermions respectively. In absence of the Berry curvature term (i.e. $\Omega_{\mathbf{p}} = 0$) $\epsilon_{\mathbf{p}}$ is independent of x, Eq. (1) reduces to the standard Vlasov equation.

The current density $\mathbf{j}$ is defined as:

$$\mathbf{j} = -e\int\frac{d^3p}{(2\pi)^3}\Big[\epsilon_{\mathbf{p}}\partial_{\mathbf{p}}n_{\mathbf{p}} + e\left(\Omega_{\mathbf{p}}\cdot\partial_{\mathbf{p}}n_{\mathbf{p}}\right)\epsilon_{\mathbf{p}}\mathbf{B} + \epsilon_{\mathbf{p}}\Omega_{\mathbf{p}}\times\partial_{\mathbf{x}}n_{\mathbf{p}}\Big] + e\mathbf{E}\times\sigma, \tag{2}$$

where, $\partial_{\mathbf{p}} = \frac{\partial}{\partial\mathbf{p}}$ and $\partial_{\mathbf{x}} = \frac{\partial}{\partial\mathbf{x}}$. The last term on the right hand side of the above equation represents the anomalous Hall current with σ given as follows:

$$\sigma = e\int\frac{d^3p}{(2\pi)^3}\Omega_{\mathbf{p}}n_{\mathbf{p}}. \tag{3}$$

Using Maxwell's equations and the linear response theory it is easy to write down the expression for the inverse of the propagator in temporal gauge $A_0 = 0$ as follows,

$$[(k^2-\omega^2)\delta^{ij} - k^ik^j + \Pi^{ij}(K)]E^j = [\Delta^{-1}(K)]^{ij}E^j = i\omega j^i_{ext}(k). \tag{4}$$

Here, $\Pi^{ij}(K)$ is the retarded self energy which follows from the expression of the induced current $j^\mu_{ind} = \Pi^{\mu\nu}(K)A_\nu(K)$ and $[\Delta^{-1}(K)]^{ij}$ is the inverse of the propagator. Dispersion relation can be obtained by finding the poles of the propagator $[\Delta(K)]^{ij}$.

Let us first concentrate on right handed fermions with chemical potential μ_R. We consider the background distribution is of the form $n^0_{\mathbf{p}} = 1/[e^{(\epsilon_{\mathbf{p}}-\mu_R)/T}+1]$. In linear response theory one is interested in the induced current upto a linear-order deviation in the gauge field. We follow the power counting scheme for gauge field A_μ and derivatives ∂_x as discussed earlier, and consider deviations in the current and the distribution function up to $O(\epsilon\delta)$. Thus we can write the distribution in Eq. (1) as follows,

$$n_{\mathbf{p}} = n^0_{\mathbf{p}} + e(n^{(\epsilon)}_{\mathbf{p}} + n^{(\epsilon\delta)}_{\mathbf{p}}) \tag{5}$$

where $n^0_{\mathbf{p}}$ is the background distribution function in presence of Berry curvature, while $n^{(\epsilon)}_{\mathbf{p}}$ and $n^{(\epsilon\delta)}_{\mathbf{p}}$ are the perturbations of order $O(\epsilon)$ and $O(\epsilon\delta)$ around $n^0_{\mathbf{p}}$. Since $n^0_{\mathbf{p}}$ contains the Berry curvature contribution (due to $\epsilon_{\mathbf{p}}$) therefore it can also be split into order $O(0)$ and $O(\epsilon\delta)$ i.e., $n^0_{\mathbf{p}} = n^{0(0)}_{\mathbf{p}} + en^{0(\epsilon\delta)}_{\mathbf{p}}$, where $n^{0(0)}_{\mathbf{p}} = \frac{1}{[e^{(p-\mu_R)/T}+1]}$ is the part of background distribution function without Berry curvature correction, while $n^{0(\epsilon\delta)}_{\mathbf{p}} = \left(\frac{\mathbf{B}\cdot\mathbf{v}}{2pT}\right)\frac{e^{(p-\mu_R)/T}}{[e^{(p-\mu_R)/T}+1]^2}$ is the part of background distribution with Berry curvature correction. In order to bring in effect of anisotropy we follow the arguments of Ref. [41]. It is assumed that the anisotropic equilibrium distribution function can be obtained from a spherically symmetric distribution function by rescaling of one direction in the momentum space. We consider that there is a momentum anisotropy in direction of a unit vector $\hat{\mathbf{n}}$. Noting that $p = |\mathbf{p}|$, we replace $p \to \sqrt{\mathbf{p}^2 + \xi(\mathbf{p}\cdot\hat{\mathbf{n}})^2}$ in the expression of $n^0_{\mathbf{p}}$ to get anisotropic distribution function. Here ξ is an adjustable anisotropy parameter satisfying a condition $\xi > -1$. It is convenient to define a new variable $\tilde{p}$ such that $\tilde{p} = p\sqrt{1+\xi(\mathbf{v}\cdot\hat{\mathbf{n}})^2}$. Using this new variable one can write $n^{0(0)}_{\mathbf{p}} = \frac{1}{[e^{(\tilde{p}-\mu_R)/T}+1]}$ and $n^{0(\epsilon\delta)}_{\mathbf{p}} = \left(\frac{\mathbf{B}\cdot\mathbf{v}}{2\tilde{p}T}\right)\frac{e^{(\tilde{p}-\mu_R)/T}}{[e^{(\tilde{p}-\mu_R)/T}+1]^2}$.

The anomalous Hall current term in Eq. (2) will vanish if the distribution function is spherically symmetric in the momentum space. However, for an anisotropic distribution function this may not be true in general. Since the Hall-current term depends on electric field, it can be of order $O(\epsilon\delta)$ or higher. As we are interested in finding deviations in current and distribution function up to order $O(\epsilon\delta)$, only $n^{0(0)}_{\mathbf{p}}$ would contribute to the Hall current term. Next, we consider σ from Eq. (3) which can be written as

$$\sigma = \frac{e}{2}\int d\Omega d\tilde{p}\frac{\mathbf{v}}{[1+\xi(\mathbf{v}\cdot\hat{\mathbf{n}})]^{1/2}}\frac{1}{(1+e^{(\tilde{p}-\mu_R)/T})}. \tag{6}$$

Since $\mathbf{v}$ is a unit vector one can express $\mathbf{v} = (sin\theta cos\phi, sin\theta sin\phi, cos\theta)$ in spherical coordinates. By choosing $\hat{\mathbf{n}}$ in z-direction, without any loss of generality, one has $\mathbf{v}\cdot\hat{\mathbf{n}} = cos\theta$. Thus the angular integral in the above equation becomes $\int d(cos\theta)d\phi\frac{\mathbf{v}}{(1+\xi cos^2\theta)^{1/2}}$. Therefore σ_x and σ_y components of Eq. (6) will vanish as $\int_0^{2\pi} sin\phi d\phi = 0$ and $\int_0^{2\pi} cos\phi d\phi = 0$. While σ_z will vanish because of the integration with respect to $\cos\theta$ variable. Consequently, the anomalous Hall current term will not contribute for the problem at the hand.

Now the kinetic equation (1) can be split into two equations valid at $O(\epsilon)$ and $O(\epsilon\delta)$ scales of distribution function as written below,

$$(\partial_t + \mathbf{v}\cdot\partial_{\mathbf{x}})n^{(\epsilon)}_{\mathbf{p}} = -(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\partial_{\mathbf{p}}n^{0(0)}_{\mathbf{p}} \tag{7}$$

$$(\partial_t + \mathbf{v}\cdot\partial_{\mathbf{x}})(n^{0(\epsilon\delta)}_{\mathbf{p}} + n^{(\epsilon\delta)}_{\mathbf{p}}) = -\frac{1}{e}\partial_{\mathbf{x}}\epsilon_{\mathbf{p}}\cdot\partial_{\mathbf{p}}n^{0(0)}_{\mathbf{p}} \tag{8}$$

Similarly, the current defined in Eq. (2) can also split into $O(\epsilon)$ and $O(\epsilon\delta)$ scales as given below,

$$\mathbf{j}^{\mu(\epsilon)} = e^2\int\frac{d^3p}{(2\pi)^3}v^\mu n^{(\epsilon)}_{\mathbf{p}} \tag{9}$$

$$\mathbf{j}^{i(\epsilon\delta)} = e^2\int\frac{d^3p}{(2\pi)^3}\left[v^in^{(\epsilon\delta)}_{\mathbf{p}} - \left(\frac{v^j}{2p}\frac{\partial n^{0(0)}_{\mathbf{p}}}{\partial p^j}\right)B^i - \epsilon^{ijk}\frac{v^j}{2p}\frac{\partial n^{(\epsilon)}_{\mathbf{p}}}{\partial x^k}\right] \tag{10}$$

After adding the contribution from all type of species i.e. right/left handed fermions with charge e and chemical potential μ_R/μ_L as well as right/left handed antifermions with charge $-e$ and chemical potential $-\mu_R/\mu_L$, using the expression $j^{\mu}_{ind} = \Pi^{\mu\nu}(K)A_\nu(K)$ and Eqs. (7)–(10) one can obtain the expression for self energy, $\Pi^{ij} = \Pi^{ij}_+ + \Pi^{ij}_-$. Here we would like to mention that Π^{ij}_+ and Π^{ij}_- respectively denote parity-even and parity-odd parts of the self-energy given by following equations,

$$\Pi^{ij}_+(K) = m_D^2 \int \frac{d\Omega}{4\pi} \frac{v^i(v^l + \xi(\mathbf{v}\cdot\hat{\mathbf{n}})\hat{n}^l)}{(1+\xi(\mathbf{v}\cdot\hat{\mathbf{n}})^2)^2} \left(\delta^{jl} + \frac{v^j k^l}{\mathbf{v}\cdot\mathbf{k} + i\epsilon}\right), \tag{11}$$

$$\begin{aligned}\Pi^{im}_-(K) = C_E \int \frac{d\Omega}{4\pi} \Bigg[& \frac{i\epsilon^{jlm}k^l v^j v^i(\omega + \xi(\mathbf{v}\cdot\hat{\mathbf{n}})(\mathbf{k}\cdot\hat{\mathbf{n}}))}{(\mathbf{v}\cdot\mathbf{k}+i\epsilon)(1+\xi(\mathbf{v}\cdot\hat{\mathbf{n}})^2)^{3/2}} \\ & + \left(\frac{v^j + \xi(\mathbf{v}\cdot\hat{\mathbf{n}})\hat{n}^j}{(1+\xi(\mathbf{v}\cdot\hat{\mathbf{n}})^2)^{3/2}}\right) i\epsilon^{iml}k^l v^j \\ & - i\epsilon^{ijl}k^l v^j \left(\delta^{mn} + \frac{v^m k^n}{\mathbf{v}\cdot\mathbf{k}+i\epsilon}\right)\left(\frac{v^n + \xi(\mathbf{v}\cdot\hat{\mathbf{n}})\hat{n}^n}{(1+\xi(\mathbf{v}\cdot\hat{\mathbf{n}})^2)^{3/2}}\right)\Bigg]\end{aligned} \tag{12}$$

where,

$$\begin{aligned} m_D^2 = & -\frac{e^2}{2\pi^2}\int_0^\infty d\tilde{p}\,\tilde{p}^2 \Bigg[\frac{\partial n^{0(0)}_{\tilde{\mathbf{p}}}(\tilde{p}-\mu_R)}{\partial\tilde{p}} + \frac{\partial n^{0(0)}_{\tilde{\mathbf{p}}}(\tilde{p}+\mu_R)}{\partial\tilde{p}} \\ & + \frac{\partial n^{0(0)}_{\tilde{\mathbf{p}}}(\tilde{p}-\mu_L)}{\partial\tilde{p}} + \frac{\partial n^{0(0)}_{\tilde{\mathbf{p}}}(\tilde{p}-\mu_L)}{\partial\tilde{p}}\Bigg] \\ C_E = & -\frac{e^2}{4\pi^2}\int_0^\infty d\tilde{p}\,\tilde{p} \Bigg[\frac{\partial n^{0(0)}_{\tilde{\mathbf{p}}}(\tilde{p}-\mu_R)}{\partial\tilde{p}} - \frac{\partial n^{0(0)}_{\tilde{\mathbf{p}}}(\tilde{p}+\mu_R)}{\partial\tilde{p}} \\ & - \frac{\partial n^{0(0)}_{\tilde{\mathbf{p}}}(\tilde{p}-\mu_L)}{\partial\tilde{p}} + \frac{\partial n^{0(0)}_{\tilde{\mathbf{p}}}(\tilde{p}-\mu_L)}{\partial\tilde{p}}\Bigg]. \end{aligned} \tag{13}$$

We would like to mention that the total induced current is, $\mathbf{j_{ind}} = \mathbf{j}^\epsilon + \mathbf{j}^{\epsilon\delta}$, where $\mathbf{j}^\epsilon$ gives contribution of the order of the square of plasma frequency or m_D^2. The plasma frequency contains additive contribution from the densities of all species i.e. right-handed particle/antiparticles and left-handed particles/antiparticles. The current $\mathbf{j}^{\epsilon\delta}$ arises due to chiral-imbalance. Its contribution from each plasma specie, depends upon $e\vec{\Omega}_p$. Since $e\vec{\Omega}_p$ can change sign depending on the plasma specie, C_E will contain both positive and negative signs. Consequently a relative signs of fermion and anti-fermion are different in m_D^2 and C_E. After performing above integrations one can get $m_D^2 = e^2\left(\frac{\mu_R^2+\mu_L^2}{2\pi^2} + \frac{T^2}{3}\right)$ and $C_E = \frac{e^2\mu_5}{4\pi^2}$, where $\mu_5 = \mu_R - \mu_L$. It should be emphasized here that $C_E = 0$ when there is no chiral-imbalance whereas $m_D^2 \neq 0$. It should also be noted that the terms with anisotropy parameter ξ are contributing in the parity-odd part of the self-energy given by Eq. (12). Introduction of chemical potential μ_5 for chiral fermions requires some qualification. Physically a chiral chemical potential implies an imbalance between the right handed and left handed fermion. This in turn related to the topological charge [17,32]. It should be noted here that due to the axial anomaly chiral chemical potential is not associated with any conserved charge. It can still be regarded as 'chemical potential' if its variation is sufficiently slow [30].

In order to get the expression for the propagator Δ^{ij} it is necessary to write Π^{ij} in a tensor decomposition. For the present problem we need six independent projectors. For an isotropic parity-even plasmas one may need the transverse $P_T^{ij} = \delta^{ij} - k^i k^j/k^2$ and the longitudinal $P_L^{ij} = k^i k^j/k^2$ tensor projectors. Due to the presence of *anisotropy* vector $\hat{\mathbf{n}}$, one needs two more projectors $P_n^{ij} = \tilde{n}^i\tilde{n}^j/\tilde{n}^2$ and $P_{kn}^{ij} = k^i\tilde{n}^j + k^j\tilde{n}^i$ [47]. To account for parity odd effect we have included two anti-symmetric operators $P_A^{ij} = i\epsilon^{ijk}\hat{k}^k$ and $P_{An}^{ij} = i\epsilon^{ijk}\tilde{n}^k$ where, $\tilde{n}^i = (\delta^{ij} - \frac{k^i k^j}{k^2})\hat{n}^j$. Thus we write Π^{ij} into the basis spanned by the above six operators as:

$$\Pi^{ij} = \alpha P_T^{ij} + \beta P_L^{ij} + \gamma P_n^{ij} + \delta P_{kn}^{ij} + \lambda P_A^{ij} + \chi P_{An}^{ij} \tag{14}$$

where, α, β, γ, δ λ and χ are some scalar functions of k and ω and are yet to be determined. Similarly we can write $[\Delta^{-1}(k)]^{ij}$ appearing in Eq. (4) as

$$[\Delta^{-1}(K)]^{ij} = C_T P_T^{ij} + C_L P_L^{ij} + C_n P_n^{ij} + C_{kn}P_{kn}^{ij} + C_A P_A^{ij} + C_{An}P_{An}^{ij}. \tag{15}$$

Using Eqs. (4), (14), (15), one can find relationship between C's and the scalar functions appearing in Eq. (14) as:

$$\begin{aligned} & C_T = k^2 - \omega^2 + \alpha,\, C_L = -\omega^2 + \beta,\, C_n = \gamma,\, C_{kn} = \delta, \\ & C_A = \lambda,\, C_{An} = \chi. \end{aligned} \tag{16}$$

For $\xi \to 0$, using Eqs. (11)–(12), one finds $\alpha_{|\xi=0} = \Pi_T$, $\beta_{|\xi=0} = \frac{\omega^2}{k^2}\Pi_L$, $\gamma_{|\xi=0} = 0$, $\delta_{|\xi=0} = 0$, $\lambda_{|\xi=0} = -\frac{\Pi_A}{2}$ and $\chi_{|\xi=0} = 0$, where

$$\begin{aligned} \Pi_T &= m_D^2 \frac{\omega^2}{2k^2}\left[1 + \frac{k^2-\omega^2}{2\omega k}\ln\frac{\omega+k}{\omega-k}\right], \\ \Pi_L &= m_D^2\left[\frac{\omega}{2k}\ln\frac{\omega+k}{\omega-k} - 1\right], \\ \Pi_A &= -2kC_E\left(1 - \frac{\omega^2}{k^2}\right)\left[1 - \frac{\omega}{2k}\ln\frac{\omega+k}{\omega-k}\right]. \end{aligned} \tag{17}$$

Scalar functions Π_T, Π_L and Π_A respectively describe the transverse, longitudinal and the axial parts of the self-energy decomposition when $\xi = 0$ [30].

Using the orthogonality condition, $[\Delta^{-1}(K)]^{ij}[\Delta(K)]^{jl} = \delta^{il}$, $[\Delta(K)]^{jl}$ can be determined. Poles of $[\Delta(K)]^{jl}$ are given by following equation.

$$\begin{aligned} & 2k\tilde{n}^2 C_A C_{An} C_{kn} + C_A^2 C_L + \tilde{n}^2 C_{An}^2(C_n + C_T) \\ & + C_T(k^2\tilde{n}^2 C_{kn}^2 - C_L(C_n + C_T)) = 0. \end{aligned} \tag{18}$$

Eq. (18) is the general dispersion relation and it is quite complicated to solve analytically or numerically. Here we would like to ascertain that α, β, γ and δ appearing in C's are same as those given in Ref. [41]. The new contributions come in terms of λ and χ which contain the effect of parity violation. The standard criterion for the Weibel instability [39] is not applicable here due to the parity violating effect. First we note that the chiral instability occurs in the quasi-stationary regime i.e. $|\omega| \ll k$ and if the initial distribution function of the plasma is isotropic then the chiral-imbalance modes have an isotropic dispersion relation. While the Weibel instability occurs due to an anisotropy in the initial momentum distribution in the plasma and the instability can be present in the quasi-stationary regime. We study numerical solutions of Eq. (18) in quasi-stationary limit. Further we note that when $C_A, C_{An} = 0$, there is no chiral-imbalance and one can get the pure Weibel modes from Eq. (18). The pure chiral-imbalance modes can be obtained by setting $C_n, C_{kn}, C_{An} = 0$ in Eq. (18). In order to obtain the growth-rates for the instabilities, one needs to solve Eq. (18) for ω. By setting $\frac{\partial\omega}{\partial k} = 0$ one can find k_{max} for which the instability can grow maximally. Upon substituting k_{max} in the expression for ω and using $\omega = i\Gamma$, one can find the growth rate Γ for the instability.

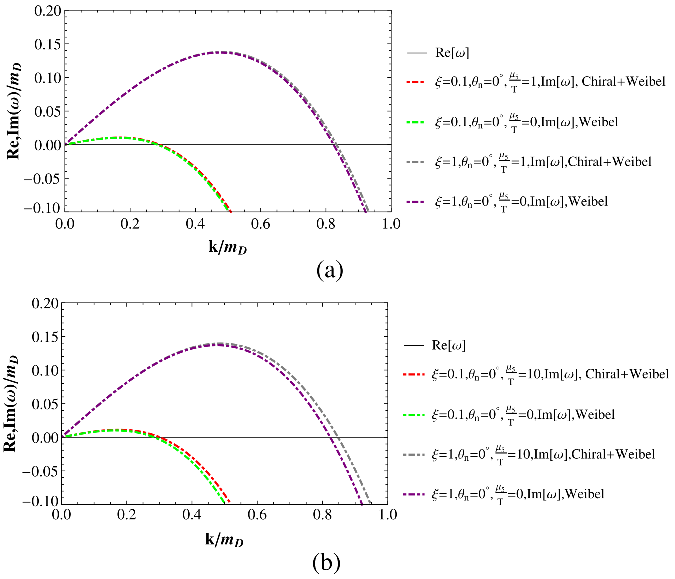

Fig. 1. Shows plots of real and imaginary part of the transverse dispersion relation for the case when the angle θ_n between the propagation vector $\mathbf{k}$ of the perturbation and the anisotropy direction is zero. The modes are purely imaginary and the real part of frequency $\omega = 0$. (a) shows comparison between pure Weibel modes ($\mu_5 = 0$) with the cases when both the Weibel and chiral-imbalance instabilities are present when $\mu_5/T = 1$ and $\xi = 0.1, 1$. (b) depicts the similar comparison when $\mu_5/T = 10$. It shows that by increasing μ_5/T the chiral-imbalance instability become stronger. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

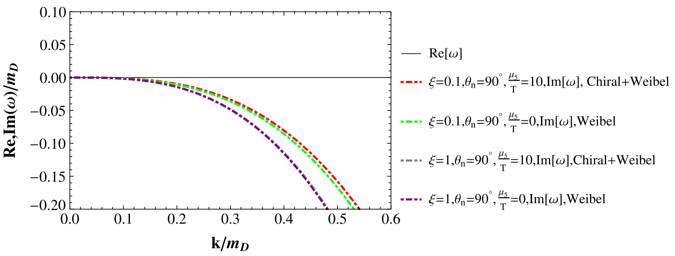

Fig. 2. Shows plots of the dispersion relation when $\theta_n = \pi/2$. The pure Weibel modes are known to give damping when $\theta_n = \pi/2$. For the instances when both the chiral-imbalance and Weibel instabilities are present ($\mu_5/T = 10$ and $\xi = 0.1, 1$) the damping can become weaker. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Figs. 1 and 2 depict a comparison between the pure Weibel modes (i.e. $\mu_5 = 0$) with the mixed modes i.e. when both chiral-imbalance and momentum-anisotropy are present. Before we discus the result, it should be noted that direction between the propagation vector $\mathbf{k}$ and the anisotropy vector $\hat{\mathbf{n}}$ is quantified by angle θ_n i.e. $\mathbf{k}\cdot\hat{\mathbf{n}} = k\cos\theta_n$ where, k is magnitude of vector $\mathbf{k}$. In Figs. 1(a)–1(b) we have considered the case $\theta_n = 0$, where the values $\mu_5/T = 1$ and $\mu_5/T = 10$ correspond to the mixed modes while $\mu_5/T = 0$ represents the pure Weibel modes. These figures show that the Weibel modes become strong with increasing values of anisotropy parameter ξ. It can also be seen that by increasing μ_5/T the chiral-imbalance modes become stronger, leading to enhancement of mixed modes. In the discussion below we have obtained analytic results for $\xi \ll 1$ and found a critical value ξ_c at $\theta_n = 0$ such that for $\xi < \xi_c$ the chiral-imbalance modes will dominate while for $\xi > \xi_c$ the Weibel instability can dominate. Fig. 2 depicts the case when $\theta_n = \pi/2$. Here pure Weibel modes are damped which is a well known result. The damping is increasing with increasing ξ but it can become weaker by increasing μ_5/T.

It is important to note that there also exists a situation $\xi \gg 1$ when the chiral-imbalance instability can play a dominant role in anisotropic plasma. This is because the Weibel instability growth rate is dependent on θ_n and it is possible to find a particular value of $\theta_n = \theta_{nc}$ when the growth rate of the pure-Weibel mode is close to zero. By setting $\omega = 0$ in the pure Weibel dispersion relation, one can find for $\xi \gg 1$, $\theta_{nc} \sim \left(\frac{\pi m_D^2}{2k^2}\right)^{1/2} \frac{1}{\xi^{1/4}}$. In the regime $\xi < 1$ but closer to unity at $\theta_n = 0$, a comparison between the growth rates of the chiral-imbalance (Γ_{ch}) and Weibel (Γ_w) instabilities is given in the following table:

ξ	0.6	0.7	0.8	0.9
$\frac{\Gamma_{ch}}{\Gamma_w}$	$\frac{0.0088\alpha_e^{3/2}\mu_5^3}{T^3}$	$\frac{0.0076\alpha_e^{3/2}\mu_5^3}{T^3}$	$\frac{0.0067\alpha_e^{3/2}\mu_5^3}{T^3}$	$\frac{0.0060\alpha_e^{3/2}\mu_5^3}{T^3}$

Thus the ratio $\frac{\Gamma_{ch}}{\Gamma_w}$ decreases by increasing values ξ while keeping μ_5/T fixed. This is because Γ_w increases by increasing ξ. For $\alpha_e = \frac{1}{137}$ and $\mu_5/T \leq 1$ one can clearly see from the table that the ratio $\frac{\Gamma_{ch}}{\Gamma_w} \ll 1$. Thus Weibel modes dominate in this case. However when $\mu_5/T \gg 1$ chiral-imbalance modes can also dominate.

Now we consider the case $\xi \ll 1$. This approximation is valid when the initial momentum anisotropy is weak or the Weibel instability has already nearly thermalized (or isotropized) the plasma. This may not be an unlikely scenario in the heavy-ion collisions as the growth rates for the Weibel instabilities can be much larger than the chiral instability. In this case it is possible to evaluate all the integrals in the dispersion relation analytically and one can express α, β, γ, δ, λ and χ up to linear order in ξ as follows,

$$\begin{aligned}
\alpha &= \Pi_T + \xi\Big[\frac{z^2}{12}(3+5\cos 2\theta_n)m_D^2 - \frac{1}{6}(1+\cos 2\theta_n)m_D^2 \\
&\quad + \frac{1}{4}\Pi_T\Big((1+3\cos 2\theta_n) - z^2(3+5\cos 2\theta_n)\Big)\Big]; \\
z^{-2}\beta &= \Pi_L + \xi\Big[\frac{1}{6}(1+3\cos 2\theta_n)m_D^2 + \Pi_L\Big(\cos 2\theta_n \\
&\quad - \frac{z^2}{2}(1+3\cos 2\theta_n)\Big)\Big]; \\
\gamma &= \frac{\xi}{3}(3\Pi_T - m_D^2)(z^2-1)\sin^2\theta_n; \\
\delta &= \frac{\xi}{3k}(4z^2 m_D^2 + 3\Pi_T(1-4z^2))\cos\theta_n; \\
\lambda &= -\frac{\mu_5 k e^2}{4\pi^2}\Big[(1-z^2)\frac{\Pi_L}{m_D^2}\Big] - \xi\frac{\mu_5 k e^2}{32\pi^2}\Big[(1-z^2)\frac{\Pi_L}{m_D^2} \\
&\quad \times\Big((1+7\cos 2\theta_n) - 3z^2(1+3\cos 2\theta_n)\Big) \\
&\quad + \frac{1}{3}(1+11\cos 2\theta_n) - z^2(3+5\cos 2\theta_n)\Big]; \\
\chi &= \xi\left[f(\omega,k)\right],
\end{aligned} \tag{19}$$

where, $z = \frac{\omega}{k}$ and $f(\omega,k)$ is some function k and ω. But in the present analysis exact form of $f(\omega,k)$ is not required. Using the above equations with Eqs. (16), (17) one can finally express Eq. (18) in terms of k and ω. One can notice from Eq. (19) that the most significant contribution for γ, δ, λ and χ is $O(\xi)$. Thus in the present scheme of approximation one can write Eq. (18) up to $O(\xi)$ as:

$$C_A^2 C_L - C_T C_L (C_n + C_T) = 0, \tag{20}$$

which in turn can give following two branches of the dispersion relation,

$$C_A^2 - C_T^2 - C_n C_T = 0, \tag{21}$$

$$C_L = 0. \tag{22}$$

Note that when $C_A = 0$, Eqs. (21)–(22) reduce to exactly the same dispersion relation discussed in Ref. [41] for the Weibel instability in an anisotropic plasma when there is no parity violating effect. Let us consider Eq. (21), it can be written as:

$$(k^2 - \omega^2)^2 + (k^2 - \omega^2)(2\alpha + \gamma) + \alpha^2 + \alpha\gamma - \lambda^2 = 0. \tag{23}$$

This equation is a quadratic equation in $(k^2 - \omega^2)$ with the solution,

$$(k^2 - \omega^2) = \frac{-(2\alpha + \gamma) \pm 2\lambda}{2}. \tag{24}$$

Now, it is of particular interest to consider the quasi-static limit $|\omega| \ll k$. In this limit $\alpha \sim \Pi_T$, $\beta \sim \frac{\omega^2}{k^2}\Pi_L$ and $\lambda \sim -\frac{\Pi_A}{2}$. Π_L, Π_T and Π_A can be obtained by expanding Eq. (17) in the quasi static limit as:

$$\Pi_{T\,|_{|\omega| \ll k}} = \left(\mp i \frac{\pi}{4}\frac{\omega}{k}\right) m_D^2;$$

$$\Pi_{L\,|_{|\omega| \ll k}} = m_D^2 \left[\mp i \frac{\pi}{2}\frac{\omega}{k} - 1\right]$$

$$\Pi_{A\,|_{|\omega| \ll k}} = \frac{\mu_5 k e^2}{2\pi^2}\left(\frac{\Pi_{L\,|_{|\omega| \ll k}}}{m_D^2}\right) \tag{25}$$

Thus in the quasi-stationary limit one can write positive branch of the transverse modes given by Eq. (24) as:

$$\rho(k) = \frac{\left(\frac{4\alpha_e \mu_5}{\pi^2 m_D^2}\right) k^2 \left[1 - \frac{\pi k}{\mu_5 \alpha_e} + \frac{\xi(1+5\cos 2\theta_n)}{12} + \frac{\xi(1+3\cos 2\theta_n)}{12}\frac{\pi m_D^2}{\mu_5 \alpha_e k}\right]}{\left[1 + \frac{2\mu_5 \alpha_e k}{\pi m_D^2}(1 - \frac{\xi}{4}) + \xi\cos 2\theta_n \left(1 - \frac{7\mu_5\alpha_e k}{2\pi m_D^2}\right)\right]} \tag{26}$$

Here we have used $\omega = i\rho(k)$ and defined $\alpha_e = \frac{e^2}{4\pi}$ as the electromagnetic coupling. It is clear from Eq. (26) that ω is purely an imaginary number and its real-part is zero i.e. $Re(\omega) = 0$. Positive $\rho(k) > 0$ implies an instability as $e^{-i(i\rho(k))t} \sim e^{+\rho(k)t}$. From Eq. (26), in the limit $\xi \to 0$ and $\mu_5 \to 0$ one gets $\rho(k) = -\frac{4k^3}{\pi m_D^2}$. Thus for an isotropic plasma (of massless particles) without any chiral-imbalance there is no unstable propagating mode when $\omega \ll k$. This is consistent with fact that without any source of free energy there should not be any unstable mode.

Now let us first consider that the quasi-static limit, $|\omega| \ll k$, is indeed satisfies for Eq. (26). Since we have already assumed that $\xi \ll 1$ and $\alpha_e < 1$, also for $\mu_5 \ll T$ one has $\frac{\mu_5}{m_D} \approx \frac{1}{2\alpha_e^{1/2}}\left(\frac{\mu_5}{T}\right)$. It is then rather easy to show that $\rho/k \ll 1$, if the condition $\frac{k^2}{m_D^2} \ll 1$ is satisfied. In this case denominator of Eq. (26) can be approximated to unity. Now we write the above equation as:

$$\rho(k) = \frac{4}{\pi}\frac{k^2}{m_D^2}\left[\frac{\alpha_e \mu_5}{\pi} - k + \frac{\alpha_e \xi \mu_5}{12\pi}(1 + 5\cos 2\theta_n) + \frac{\xi}{12}\frac{m_D^2}{k}(1 + 3\cos 2\theta_n)\right]. \tag{27}$$

Here we emphasize that when $\xi = 0$, first two terms in the square bracket survive and Eq. (27) matches with the dispersion relation of the chiral instability given in Ref. [30]. When $\mu_5 = 0$, the second and the last term survives to give the Weibel modes considered in Ref. [41]. Term with $\alpha_e \xi \mu_5$ factor arises due to the interaction between the Weibel and chiral-imbalance modes.

Before we analyze the interplay between the chiral-imbalance and the Weibel instabilities, it is instructive to qualitatively understand their origin. First consider the chiral-imbalance instability. For a such a plasma 'chiral-charge' density n is given by $\partial_t n + \nabla \cdot \mathbf{j} = \frac{2\alpha_e}{\pi}\mathbf{E}\cdot\mathbf{B}$. From this one can estimate the axial charge density $n \sim \alpha_e k A^2$ where A is the gauge-field. Assuming that there are only right handed particles i.e. ($\mu_5 \sim \mu_R$), the number and energy densities of the plasma are respectively given by $\mu_5 T^2$ and $\mu_5{}^2 T^2$. The fermionic number density associated with the gauge field can be estimated from the Chern–Simon term to be $\alpha_e k A^2$. The number densities associated with the fields and particles have same value for $k_1 \sim \frac{\mu_5 T^2}{\alpha_e A^2}$. The typical energy for the gauge field is $\epsilon_A \sim k^2 A^2$. For this particular value of k_1 it can be seen that $\epsilon_A = \mu_5{}^2 T^2 \frac{T^2}{\alpha_e^2 A^2}$. Thus there exists a state satisfying the condition $\frac{T^2}{\alpha_e^2} < A^2$ for which energy in the gauge field is lower than particle energy. This leads to the chiral-imbalance instability [30, 34]. The Weibel instability arises when the equilibrium distribution function of the plasma has anisotropy in the momentum space [44,45]. The anisotropy in the momentum space can be regarded as anisotropy in temperature. Suppose there is plasma which is hotter in y-direction than x or z direction one may write the distribution function $n_p^0 = \frac{1}{1+e^{-(\sqrt{p_x^2+(1+\xi)p_y^2+p_z^2})/T}}$. If in this situation a disturbance with a magnetic-field $B = B_0 cos(kx)$ arises, say from noise, one can write the Lorentz force term in the kinetic equation as $e(v \times B)\cdot\partial_p n_p^0 = e[\xi(v_z B_x - v_x B_z)\frac{p_y}{T}]\left(\frac{-e^{-(\sqrt{p_x^2+(1+\xi)p_y^2+p_z^2})/T}}{1+e^{-(\sqrt{p_x^2+(1+\xi)p_y^2+p_z^2})/T}}\right)$. This Lorentz-force can produce current-sheets where the magnetic field changes its sign. The current-sheet in turn enhances the original magnetic field [44,45].

The Weibel instability is known to grow maximally for $\theta_n = 0$. In the quasi-static limit the instability has maximum growth rate $\Gamma_w \sim \frac{8\xi^{3/2}}{27\pi}m_D$ for $k = \frac{\sqrt{\xi}}{3}m_D$. For the chiral imbalance instability the maximum growth rates $\Gamma_{ch} \sim \frac{16\alpha_e^3}{27\pi^4}\left(\frac{\mu_5}{m_D}\right)^2\mu_5$, occur at $k \sim \frac{2\alpha_e}{3\pi}\mu_5$ [30]. Thus the ratio $\frac{\Gamma_{ch}}{\Gamma_w} \sim \frac{2}{\pi^3}\left(\frac{\alpha_e}{\xi^{1/2}}\right)^3\left(\frac{\mu_5}{m_D}\right)^3 \sim \frac{1}{4\pi^3}\left(\frac{\alpha_e}{\xi}\right)^{3/2}\left(\frac{\mu_5}{T}\right)^3$, where we have used $\frac{\mu_5}{m_D} \approx \frac{1}{2\alpha_e^{1/2}}\left(\frac{\mu_5}{T}\right)$. The ratio $\frac{\Gamma_{ch}}{\Gamma_w}$ becomes unity when $\xi_c \approx 2^{2/3}\left(\frac{\alpha_e}{4\pi^2}\right)\left(\frac{\mu_5}{T}\right)^2$. When $\mu_5 \sim T$ and $\alpha_e = 1/137$ (QED), one can estimate $\xi_c < 10^{-3}$. ξ_c will change if coupling varies (QCD case). Thus for $\xi_c < \xi \ll 1$ the Weibel instability can dominates over the chiral imbalance modes. However, it may be still possible to see the chiral-imbalance modes if we consider θ_n-dependence of the instability described by Eq. (27). In Eq. (27) the Weibel instability term vanishes if $\theta_n \sim \frac{1}{2}\cos^{-1}(1/3) \sim 55°$. For this value of θ_n the interaction term between the Weibel and the chiral modes becomes negative and tries to suppress the unstable mode. However this term is very small in comparison to the pure chiral term.

In Fig. 3 we plot the dispersion relation given by Eq. (26) as function of $k_N = \frac{\pi}{\alpha_e \mu_5}k$ for various values of ξ which is given in units of ξ_c and the propagation angle θ_n. y-axis shows the $Re[\omega]$ and $Im[\omega]/\left(\frac{4\alpha_e^3 \mu_5{}^3}{\pi^4 m_D^2}\right)$. Note that the real part of the frequency $Re[\omega]$ is zero. For the case when $\xi = 0$ there is no Weibel mode and the only the chiral-imbalance can give the instability. Whereas

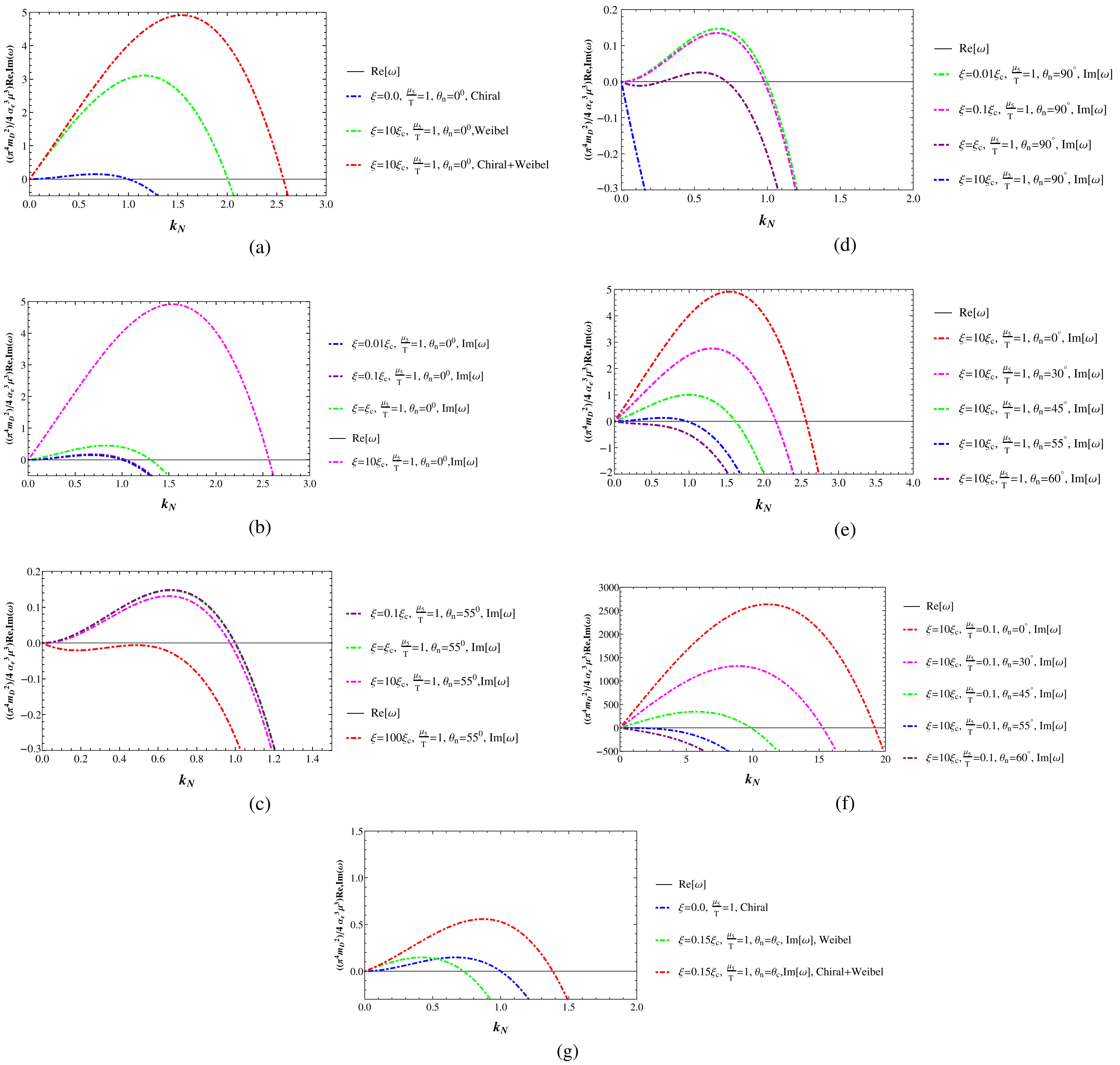

Fig. 3. Shows plots of real and imaginary part of the dispersion relation. Here θ_n is the angle between the wave vector k and the anisotropy vector. Real part of dispersion relation is zero. (a) show plots for three cases: (i) pure chiral (no anisotropy), (ii) pure Weibel (chiral chemical potential = 0) and (iii) when both chiral and Weibel instabilities are present. (b)–(d) represent the case when both the instabilities are present but the anisotropy parameter varies at different values of θ_n for fixed $\mu_5/T = 1$. (e)–(f) represent the case when both instabilities are present for a fixed anisotropy parameter at different values of θ_n when $\mu_5/T = 1$ and $\mu_5/T = 0.1$ respectively. (g) represents the case when for a particular value of $\theta_n \sim \theta_c$ both the instabilities have equal growth rates. Here frequency is normalized in unit of $\omega/\left(\frac{4\alpha_e^3\mu_5{}^3}{\pi^4 m_D^2}\right)$ and wave-number k by $k_N = \frac{\pi}{\mu_5\alpha_e}k$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

when $\mu_5 = 0$, only Weibel instability will contribute. From the condition $\rho(k) > 0$, one can obtain the range of the instability which can be stated as:

$$k_N = 1 + \frac{\xi\,(1+cos2\theta_n)}{12} + \left[\left(1 + \frac{\xi\,(1+cos2\theta_n)}{12}\right)^2 + \frac{\pi^2\xi\,(1+3cos2\theta_n)}{3\alpha_e}\right]^{1/2} \tag{28}$$

In Fig. 3(a) we have shown for $\theta_n = 0$, the pure Weibel case ($\xi = 10\xi_c$ and $\mu_5 = 0$) and the pure chiral-imbalance case ($\xi = 0$ and $\mu_5 \neq 0$) along with the case when both the instabilities are present (i.e. $\xi = 10\xi_c$ and $\mu_5 \neq 0$). The plot shows that the pure Weibel modes dominating over the pure chiral-imbalance case. But the combined effect of both the instabilities is much more pronounced. The maximum growth rate and the range of the instability are altered significantly for the combined case. In Figs. 3(b)–3(d) we study the cases where both the instabilities are

Table 1
Summary of results.

Case	θ_n	μ_5/T	ξ	Dominance of the instability
1	0	~ 1	$\xi \gtrsim 1$	Weibel will dominate
2	0	~ 1	$\xi_c < \xi \ll 1$	Weibel will dominate
3	0	≤ 1	$\xi \sim \xi_c \ll 1$	Both are comparable
4	θ_c	~ 1	$\xi < \xi_c$	Both are comparable
5	θ_{nc}	can have any value	$\xi \gg 1$	Chiral imbalance

present and ξ and θ_n vary when $\mu_5/T = 1$. *It is important to note that in this analysis we are showing the plots of the dispersion relation following the same normalization as used in Ref. [30] so that we can compare our results. Due to the normalization of dispersion relation Weibel term picks up factor μ_5/T. Therefore, Weibel instability appears to be also dependent on μ_5/T, apart from the variables ξ and θ_n. However, in order to take limit $\mu_5 \to 0$ one need to undo the normalization in terms of Im$[\omega]$ and k.* Fig. 3(b) shows clearly shows, for $\theta_n = 0$ when condition $\xi \ll \xi_c$ is satisfied, the chiral-imbalance instability dominates over the Weibel modes. However, such values of ξ are extremely small. For the cases when $\xi \geq \xi_c$ the Weibel modes are dominating. Contribution from the Weibel modes is maximum for $\theta_n = 0$ and the modes are strongly damped at $\theta_n = \pi/2$. Angular part in the dispersion relation for the pure Weibel modes becomes zero when $\theta_n \approx 55^0$. In this case one can see that chiral-imbalance modes can remain dominant. This case is shown in Fig. 3(c). It should be noted that for the case when $\xi \gg \xi_c$ the contribution from the coupling term between the Weibel and chiral-imbalance modes become sufficiently strong and it can again suppress the instability. In Fig. 3(d) we have shown the case when $\theta_n = \pi/2$. The modes with $\xi \geq \xi_c$ are strongly damped and there is no instability. Here the coupling term between the two modes also contribute in the damping of the instability. In Figs. 3(e)–3(f) we have plotted the unstable modes for $\xi = 10\xi_c$ for different values of θ_n, when $\mu_5/T = 1$ and 0.1 respectively. When $\mu_5/T = 0.1$ (i.e. $\mu_5 \ll T$) the instability increases enormously. Now by comparing the growth rates of pure-Weibel and pure chiral imbalance modes, when $\mu_5/T = 1$, one can find that they become equal at $\theta_c = \frac{1}{2}\cos^{-1}\left[\left(\frac{2}{27}\right)^{2/3}\frac{3\alpha_e}{\xi\pi^2} - \frac{1}{3}\right]$. Fig. 3(g) represents this case where we have shown that the growth rate of pure Weibel case at $\xi = 0.15\xi_c$ becomes comparable to pure chiral-imbalance mode with $\xi = 0$. The topmost (red) curve in this figure shows the case when both the modes operate together. This case shows that the combined effect of the instability can significantly alter the range and the growth rate of the instability.

In conclusion, we have studied collective modes in an anisotropic chiral plasma where the both Weibel and chiral-imbalance instabilities are present. Out of these two instabilities which one will dominate in a given physical situation depends upon three parameters, θ_n, ξ and μ_5/T. We have demonstrated that for the values $\theta_n = 0$ and $\mu_5/T \sim 1$, when $\xi \geq 1$, $\xi < 1$ but closer to unity or $\xi_c < \xi \ll 1$, the Weibel modes dominate over the chiral-imbalance instability. It was shown analytically that for $\theta_n = 0$ and $\mu_5/T \sim 1$, only for a very small values of the anisotropic parameter $\xi \sim \xi_c \ll 1$ growth rates of the both instabilities are comparable. It was also demonstrated numerically that for $\xi < \xi_c$, $\mu_5/T \sim 1$, there exist a critical angle $\theta = \theta_c$ at which the growth rates of two instabilities can also be comparable. We have also shown for the case when $\xi \gg 1$, the chiral-imbalance can dominate over the Weibel modes when $\theta = \theta_{nc}$. A summary of our main results is shown in Table 1.

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Shear Viscosity of Turbulent Chiral Plasma

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Abstract

It is well known that the difference between the chemical potentials of left-handed and right-handed particles in a parity violating (chiral) plasma can lead to an instability. We show that the chiral instability may drive turbulent transport. Further we estimate the anomalous viscosity of chiral plasma arising from the enhanced collisionality due to turbulence.

Keywords: Chiral Imbalance, Berry curvature, anomalous viscosity

1. Introduction

The suggestion that the strongly interacting matter created in the relativistic heavy-ion collision experiments can have local P and CP violations has created a lot of excitement. According to Refs. [1, 2, 3, 4] the proposed P and CP violations in QCD can be due to finite nonzero topological charges present at high-temperature and density. In presence of a very strong magnetic field (which can be created during the heavy-ion collision) the nonzero topological charge can induce a net chiral imbalance. As a result particles with positive and negative charges will traverse in opposite directions along the magnetic field and thus a net charge separation can occur. This phenomenon is called 'chiral magnetic effect' (CME)[3, 5, 6]. In a different context, this phenomenon has also been considered in the field of cosmology [7, 8, 9]. Recently an experiment with the STAR detector at RHIC has been performed to observe the CME by measuring the three particle azimuthal correlators sensitive to the charge separation. It has been found that in case of Au-Au and Cu-Cu collisions at $\sqrt{s} = 200$ GeV correlation of opposite charges separates out [10, 11] which can be an indication of CME or P and CP violation. These developments have created a lot of interests in this field. Theoretical models that study these aspects of strongly interacting matter consider a plasma of massless fermions which interact with each other in chiral invariant way. There exists both hydrodynamical and kinetic theory based models describing such a plasma in which the quantum mechanical nature of the chiral anomaly can have a macroscopic consequences. In this paper we shall focus on the kinetic theory approach. Recently it was shown that the CME and other CP violating effects can be incorporated within a kinetic theory framework [12, 13, 14, 15] by using the Berry curvature [16] corrections. The kinetic theory approach is more general in comparison with a hydrodynamical framework and can be applied to study various equilibrium and nonequilibrium situations.

It should be noted here that the effect of parity violation because of weak-interaction considered to be important in the context of core collapsing supernova and the formation of neutron stars[17, 18] e.g. the peculiar velocity of pulsar [19] or in the generation of magnetic field during the core collapsing neutron star [20, 21, 22]. However, the role of parity violating effects due to the strong sector in a quark star is not fully explored. In the present work we consider the chiral-plasma instability (CPI) which may arise either in core collapsing supernova due to weak process[23] or in a quark matter in the interior of a neutron star due to strong process. Such instabilities have

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been studied in the context of electromagnetic and quark-gluon plasma at finite temperature using the Berry-curvature modified kinetic equation[24, 25]. A similar kind of instability can exist in case of a electroweak plasma and early universe [21]. In Ref. [24] it was argued that the chiral-imbalance instability can lead to the growth of Chern-Simons number (or magnetic-helicity in plasma physics context) at expense of the chiral imbalance. Subsequently in Refs.[21, 26] it was shown that the generation of magnetic helicity in presence of chiral instability may lead to a huge magnetic field of the order of 10^{16} G in core of a compact star. Such kind of instabilities was mentioned in Refs. [27, 28, 29, 30, 31] in different context and may be seen in heavy ion collisions.

In this paper we calculate the coefficient of shear viscosity arising due to the CPI generated turbulence transport in a chiral plasma. By definition, η measures ratio of stress to velocity gradient. Stress in a medium arises because of momentum-transfer/diffusion generated by a velocity gradient[32]. The momentum transfer in a medium is usually governed by collision. But in case of turbulence interaction between particles and field can enhance the decorrelation frequency and the effective viscosity can be written as,

$$\eta \sim \frac{Stress}{\nu_{collision} + \nu_{decorrelation}}, \tag{1}$$

where, $\nu_{collision}$ and $\nu_{decorrelation}$ respectively denote the collision and decorrelation frequencies. In the case of a neutron star collision frequency can become very small as temperature T become small [33] and thus the decorrelation frequency can have dominant contribution in determining η.

2. Linear Response Analysis and Chiral Instability

We start with the Berry-curvature modified collisionless kinetic (Vlasov) equation at the leading order in A^μ [15] given as:

$$(\partial_t + \mathbf{v}\cdot\partial_{\mathbf{x}})n_{\mathbf{p}} + (e\mathbf{E} + e\mathbf{v}\times\mathbf{B} - \partial_{\mathbf{x}}\epsilon_{\mathbf{p}})\cdot\partial_{\mathbf{p}}n_{\mathbf{p}} = 0 \tag{2}$$

where $\mathbf{v} = \frac{\mathbf{p}}{p}$, $\epsilon_{\mathbf{p}} = p(1 - e\mathbf{B}\cdot\mathbf{\Omega}_{\mathbf{p}})$ and $\mathbf{\Omega}_{\mathbf{p}} = \pm\mathbf{p}/2p^3$ is the Berry curvature. $\pm$ sign corresponds to right and left-handed fermions respectively. Note that when $\mathbf{\Omega}_{\mathbf{p}}$=0, energy of a chiral fermion $\epsilon_{\mathbf{p}}$ is independent of x, Eq.(2) reduces to the standard Vlasov equation.

Current density $\mathbf{j}$ is defined as:

$$\mathbf{j} = -e\int\frac{d^3p}{(2\pi)^3}\Big[\epsilon_{\mathbf{p}}\partial_{\mathbf{p}}n_{\mathbf{p}} + e\left(\mathbf{\Omega}_{\mathbf{p}}\cdot\partial_{\mathbf{p}}n_{\mathbf{p}}\right)\epsilon_{\mathbf{p}}\mathbf{B} + \epsilon_{\mathbf{p}}\mathbf{\Omega}_{\mathbf{p}}\times\partial_{\mathbf{x}}n_{\mathbf{p}}\Big] + e\mathbf{E}\times\sigma, \tag{3}$$

where, $\partial_{\mathbf{p}} = \frac{\partial}{\partial\mathbf{p}}$ and $\partial_{\mathbf{x}} = \frac{\partial}{\partial\mathbf{x}}$. The $e\mathbf{E}\times\sigma$ of the above equation represents the anomalous Hall current. Where σ is as follows:

$$\sigma = e\int\frac{d^3p}{(2\pi)^3}\mathbf{\Omega}_{\mathbf{p}}n_{\mathbf{p}}. \tag{4}$$

Let us first consider right handed fermions with chemical potential μ_R. In this case we can take equilibrium distribution function of the form $n^0_{\mathbf{p}} = 1/[e^{(\epsilon_{\mathbf{p}}-\mu_R)/T} + 1]$.

Now for a linear response analysis we express Eq.(2) and Eq.(3) by a linear-order deviation in the gauge field. We consider the power counting scheme [15] for gauge field $A_\mu = O(\epsilon)$ and derivatives $O(\delta)$, where ϵ and δ are small and independent parameters. In this scheme one can write the distribution function in Eq.(2) as follows,

$$n_{\mathbf{p}} = n^0_{\mathbf{p}} + e(n^{(\epsilon)}_{\mathbf{p}} + n^{(\epsilon\delta)}_{\mathbf{p}}), \tag{5}$$

where, $n^0_{\mathbf{p}} = n^{0(0)}_{\mathbf{p}} + e n^{0(\epsilon\delta)}_{\mathbf{p}}$ with $n^{0(0)}_{\mathbf{p}} = \frac{1}{[e^{(p-\mu_R)/T}+1]}$ and $n^{0(\epsilon\delta)}_{\mathbf{p}} = \left(\frac{\mathbf{B}\cdot\mathbf{v}}{2pT}\right)\frac{e^{(p-\mu_R)/T}}{[e^{(p-\mu_R)/T}+1]^2}$.

Now from Eq.(3), the anomalous Hall-current term $e\mathbf{E}\times\sigma$, can be of order $O(\epsilon\delta)$ or higher. Here we are interested in finding deviations in current up to order $O(\epsilon\delta)$ therefore, only $n^{0(0)}_{\mathbf{p}}$ should contribute to σ in the anomalous Hall term. Hence σ from Eq.(4) will be

$$\sigma = \frac{e}{2}\int d\Omega dp\frac{\mathbf{v}}{(1 + e^{(p-\mu_R)/T})} = 0. \tag{6}$$

Thus the anomalous Hall current term will not contribute.

Now the kinetic equation (2) at $O(\epsilon)$ and $O(\epsilon\delta)$ scales of distribution function can be written as,

$$(\partial_t + \mathbf{v}\cdot\partial_{\mathbf{x}})n^{(\epsilon)}_{\mathbf{p}} = -(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\partial_{\mathbf{p}}n^{0(0)}_{\mathbf{p}} \tag{7}$$

$$(\partial_t + \mathbf{v}\cdot\partial_{\mathbf{x}})(n^{0(\epsilon\delta)}_{\mathbf{p}} + n^{(\epsilon\delta)}_{\mathbf{p}}) = -\frac{1}{e}\partial_{\mathbf{x}}\epsilon_{\mathbf{p}}\cdot\partial_{\mathbf{p}}n^{0(0)}_{\mathbf{p}} \tag{8}$$

Similarly equation for the current defined in Eq.(3) at $O(\epsilon)$ and $O(\epsilon\delta)$ can be written as,

$$\mathbf{j}^{\mu(\epsilon)} = e^2 \int \frac{d^3p}{(2\pi)^3} v^\mu n_{\mathbf{p}}^{(\epsilon)} \tag{9}$$

$$\mathbf{j}^{\mathbf{i}(\epsilon\delta)} = e^2 \int \frac{d^3p}{(2\pi)^3} \left[v^i n_{\mathbf{p}}^{(\epsilon\delta)} - \left(\frac{v^j}{2p} \frac{\partial n_{\mathbf{p}}^{0(0)}}{\partial p^j} \right) B^i - \epsilon^{ijk} \frac{v^j}{2p} \frac{\partial n_{\mathbf{p}}^{(\epsilon)}}{\partial x^k} \right] \tag{10}$$

Using Eqs.(7, 8, 9, 10) and the expression $j^\mu_{ind} = \Pi^{\mu\nu}(K)A_\nu(K)$ one can obtain the expression for the spatial part of self energy, $\Pi^{ij} = \Pi^{ij}_+ + \Pi^{ij}_-$ for right handed particles. If we have contribution from all type of species i.e. right/left fermions with charge e and chemical potential μ_R/μ_L as well as right/left handed antifermions with charge $-e$ and chemical potential $-\mu_R/\mu_L$, then, Π^{ij}_+ (parity even part of polarization tensor) and Π^{ij}_- (parity-odd part) can be written as,

$$\Pi^{ij}_+(K) = m_D^2 \int \frac{d\Omega}{4\pi} v^i v^l \left(\delta^{jl} + \frac{v^j k^l}{\mathbf{v}\cdot\mathbf{k} + i\epsilon} \right), \tag{11}$$

$$\Pi^{im}_-(K) = C_E \int \frac{d\Omega}{4\pi} \left[i\epsilon^{iml} k^l + i\omega \frac{(\epsilon^{jlm} v^i - \epsilon^{ijl} v^m) k^l v^j}{(\mathbf{v}\cdot\mathbf{k} + i\epsilon)} \right] \tag{12}$$

where,

$$\begin{aligned} m_D^2 &= -\frac{e^2}{2\pi^2} \int_0^\infty dp p^2 \Bigg[\frac{\partial n_{\mathbf{p}}^{0(0)}(p-\mu_R)}{\partial p} + \frac{\partial n_{\mathbf{p}}^{0(0)}(p+\mu_R)}{\partial p} \\ &\qquad + \frac{\partial n_{\mathbf{p}}^{0(0)}(p-\mu_L)}{\partial p} + \frac{\partial n_{\mathbf{p}}^{0(0)}(p+\mu_L)}{\partial p} \Bigg] \\ C_E &= -\frac{e^2}{4\pi^2} \int_0^\infty dp p \Bigg[\frac{\partial n_{\mathbf{p}}^{0(0)}(p-\mu_R)}{\partial p} - \frac{\partial n_{\mathbf{p}}^{0(0)}(p+\mu_R)}{\partial p} \\ &\qquad - \frac{\partial n_{\mathbf{p}}^{0(0)}(p-\mu_L)}{\partial p} + \frac{\partial n_{\mathbf{p}}^{0(0)}(p+\mu_L)}{\partial p} \Bigg]. \end{aligned} \tag{13}$$

Note that while deriving these expression we have chosen the temporal gauge i.e. $A_0 = 0$. It is easy to perform above integartions and get $m_D^2 = e^2 \left(\frac{\mu_R^2 + \mu_L^2}{2\pi^2} + \frac{T^2}{3} \right)$ and $C_E = \frac{e^2 \mu_5}{4\pi^2}$, where $\mu_5 = \mu_R - \mu_L$. From here it is clear that that when there is no chiral imbalance $C_E = 0$ whereas $m_D^2 \neq 0$. Introduction of chemical chemical potential μ_5 for chiral fermions requires some clarification. Physically it can be interpreted as the imbalance between the right handed and left handed fermion and is arises beacause of the topological charge[5, 27].

Now Maxwell's equation is

$$\partial_\nu F^{\mu\nu} = J^\mu_{ind} + J^\mu_{ext} \tag{14}$$

Taking the fourier transform and using the expression of the induced current $j^\mu_{ind} = \Pi^{\mu\nu}(K)A_\nu(K)$ and choosing temporal gauge $A_0 = 0$ as one can get,

$$[(k^2 - \omega^2)\delta^{ij} - k^i k^j + \Pi^{ij}(K)]E^j = i\omega j^i_{ext}(k). \tag{15}$$

One can define inverse of the propagator as,

$$[\Delta^{-1}(K)]^{ij} = (k^2 - \omega^2)\delta^{ij} - k^i k^j + \Pi^{ij}(K). \tag{16}$$

Dispersion relation can be obtained by finding the poles of $[\Delta(K)]^{ij}$. In order to find the poles of the propagator Δ^{ij} we write Π^{ij} in a tensor decomposition. For the current problem we need three independent projectors, transverse $P_T^{ij} = \delta^{ij} - k^i k^j / k^2$, longitudinal $P_L^{ij} = k^i k^j / k^2$ and a parity odd tensor projector $P_A^{ij} = i\epsilon^{ijk}\hat{k}^k$. Thus we write Π^{ij} as:

$$\Pi^{ij} = \Pi_T P_T^{ij} + \Pi_L P_L^{ij} + \Pi_A P_A^{ij} \tag{17}$$

where, Π_T, Π_L and Π_A are some scalar functions of k and ω and need to be determined.

Following the decomposition of Π^{ij}, one can also decompose $[\Delta^{-1}(k)]^{ij}$ appearing in Eq.(16) as

$$[\Delta^{-1}(K)]^{ij} = C_T P_T^{ij} + C_L P_L^{ij} + C_A P_A^{ij}. \tag{18}$$

where coefficient C's are related to the scalar functions defined in Eq.(17) by following equation:

$$C_T = k^2 - \omega^2 + \Pi_T, C_L = -\omega^2 + \Pi_L, C_A = \Pi_A.$$

Now using Eq.(17) one can write $\Pi_T = \frac{1}{2} P_T^{ij} \Pi^{ij}$, $\Pi_L = P_L^{ij} \Pi^{ij}$ and $\Pi_A = -\frac{1}{2} P_A^{ij} \Pi^{ij}$ and then using the Eqs.(11-12) for Π^{ij} one can obtain,

$$\begin{aligned} \Pi_T &= m_D^2 \frac{\omega^2}{2k^2} \left[1 + \frac{k^2 - \omega^2}{2\omega k} \ln \frac{\omega + k}{\omega - k} \right], \\ \Pi_L &= m_D^2 \frac{\omega^2}{k^2} \left[\frac{\omega}{2k} \ln \frac{\omega + k}{\omega - k} - 1 \right], \\ \Pi_A &= k C_E \left(1 - \frac{\omega^2}{k^2} \right) \left[1 - \frac{\omega}{2k} \ln \frac{\omega + k}{\omega - k} \right]. \end{aligned} \tag{19}$$

Now using the fact that a vector and its inverse exists in same space, we can expand $[\Delta(K)]^{ij}$ in the tensor projector basis as:

$$[\Delta(K)]^{ij} = a P_L^{ij} + b P_T^{ij} + c P_A^{ij}. \tag{20}$$

Now using the relation $[\Delta^{-1}(K)]^{ij}[\Delta(K)]^{jl} = \delta^{il}$ one can obtain the coefficients a, b, c in terms of the coefficients C's appearing in Eq.(18). Poles of the $[\Delta^{-1}(K)]^{ij}$ can be obtained by equating denominators of the expressions for a, b, c with zero. In the present case we have same denominator for b and c while it is different for a therefore the dispersion relation:

$$C_A^2 - C_T^2 = 0, \tag{21}$$

$$C_L = 0. \tag{22}$$

Here we would like to note that the dispersion relation given by Eq.(22) gives only oscillations and do not have instability therefore, it is not of our interest. Dispersion relation given by Eq.(21) can be written as:

$$\omega^2 = k^2 + \Pi_T \pm \Pi A \tag{23}$$

In the quasi-static limit i.e. $|\omega| << k$, one can write Π_T Π_L and Π_A as:

$$\Pi_{T|_{|\omega|<<k}} = \left(\mp i\frac{\pi}{4}\frac{\omega}{k}\right)m_D^2;$$
$$\Pi_{L|_{|\omega|<<k}} = O(\omega^2/k^2) + ...$$
$$\Pi_{A|_{|\omega|<<k}} = -\frac{\mu_5 k e^2}{4\pi^2}\left(\mp i\frac{\pi}{2}\frac{\omega}{k} - 1\right). \tag{24}$$

In this limit Eq.(23) with the minus sign will give the dispersion relation $\omega = i\rho(k)$ where $\rho(k)$ is given by,

$$\rho(k) = \left(\frac{4\alpha\mu_5}{\pi^2 m_D^2}\right)k^2\left[1 - \frac{\pi k}{\mu_5\alpha_e}\right] \tag{25}$$

Here we have used and defined $\alpha = \frac{e^2}{4\pi}$ as the electromagnetic coupling. It is clear from Eq.(25) that ω is purely an imaginary number and its real-part is zero i.e. $Re(\omega) = 0$. Positive $\rho(k) > 0$ implies an instability as $e^{-i(i\rho(k))t} \sim e^{+\rho(k)t}$ due net chiral chemical potential μ_5. Thus plasma has exponential instability that can drive turbulence. Instability will be maximum at $k_{max} = \frac{2\mu_5\alpha}{3\pi}$. For simplicity, in the next section we consider the case of right handed particles only.

3. Diffusion via nonlinear particle-wave interaction, decorrelation time

We shall use Resonance Broadening theory [34, 35, 36, 37, 38, 39]. First we consider the case of high density and low temperature, it can shown $\epsilon_{\mathbf{p}} = p - e\left(\frac{\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v}}{2\mu_R}\right) + O(\frac{1}{\mu^2})$[15]. Now consider the distribution function,

$$n_{\mathbf{p}} = n_{\mathbf{p}}^{0(0)} + en^1_{\mathbf{p}\omega,k}. \tag{26}$$

where $\langle n_{\mathbf{p}}\rangle = \langle n_{\mathbf{p}}^{0(0)}\rangle$, $\langle\rangle$ represents the spatial averaging. $n^1_{\mathbf{p}\omega,k}$ is the coherent response to field fluctuations. Taking the spatial averaging Berry curvature modified kinetic Eq.(2) can be written as,

$$\partial_t\langle n_{\mathbf{p}}\rangle = -e^2\left\langle\left(\mathbf{E}_{\omega,k} + \mathbf{v}\times\mathbf{B}_{\omega,k} + i\mathbf{k}\left(\frac{\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v}}{2\mu_R}\right)\right)\cdot\partial_{\mathbf{p}}n^1_{\mathbf{p}\omega,k}\right\rangle \tag{27}$$

In the quasilinear theory trajectories of the particles are assumed to be unperturbed irrespective of the presence of fluctuating fields. As a result coherent response $n^1_{\mathbf{p}\omega,k}$ has a peak $1/(\omega - \mathbf{k}\cdot\mathbf{v})$. In the resonance broadening theory one considers the perturbed trajectories of the particles due to effects of random fields and calculate the approximate coherent response function $n^1_{\mathbf{p}\omega,k}$ as an average over a statistical ensemble or perturbed trajectories. As a results the peak in the coherent response gets broadened[34, 37]. In the case of resonance broadening theory, response function can be written as[34, 37];

$$n^1_{\mathbf{p},\omega k} = \int_0^\infty dte^{-i(\omega-\mathbf{k}\cdot\mathbf{v})\mathbf{t}}\langle e^{-ik\delta x(t)}\rangle\left(\mathbf{E}_{\omega,k} + \mathbf{v}\times\mathbf{B}_{\omega,\mathbf{k}} + i\mathbf{k}\left(\frac{\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v}}{2\mu_R}\right)\right)\cdot\partial_{\mathbf{p}}\langle n_{\mathbf{p}}\rangle \tag{28}$$

We take Gaussian probability distribution as,

$$pdf[\delta p] = \frac{1}{\sqrt{\pi Dt}}e^{-\frac{(\delta p)^2}{Dt}}. \tag{29}$$

With the above probability distribution one can get,

$$\langle e^{-ik\delta x(t)}\rangle_{pdf} \approx e^{-\frac{t^3}{t_c^3}}. \tag{30}$$

Here, t_c is given by following equation,

$$t_c^3 = \frac{4\bar{E}_p^2}{k^2 D}, \tag{31}$$

where, $\bar{E}_p^2 \equiv \frac{\int d^3\mathbf{p}E_p\langle n_{\mathbf{p}}\rangle}{\int d^3\mathbf{p}\langle n_{\mathbf{p}}\rangle}$.

Substituting Eq.(30) in Eq.(28) one gets,

$$n^1_{\mathbf{p},\omega k} = \int_0^\infty dte^{-i(\omega-k\cdot v)t-\frac{t^3}{t_c^3}}\left(\mathbf{E}_{\omega,k} + \mathbf{v}\times\mathbf{B}_{\omega,\mathbf{k}} + i\mathbf{k}\left(\frac{\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v}}{2\mu_R}\right)\right)\cdot\partial_{\mathbf{p}}\langle n_{\mathbf{p}}\rangle \tag{32}$$

Now,

$$\int_0^\infty dte^{-i(\omega-k\cdot v)t-\frac{t^3}{t_c^3}} \simeq -\frac{i}{\omega - \mathbf{k}\cdot\mathbf{v} + i/t_c}. \tag{33}$$

Using Eq.(32) one can write the following Diffusion equation,

$$(\partial_t - \partial_{\mathbf{p}}\cdot\mathbf{D}(\mathbf{p})\cdot\partial_{\mathbf{p}})\langle n_{\mathbf{p}}\rangle = 0, \tag{34}$$

where,

$$\mathbf{D}(\mathbf{p}) = -\int d\omega d\mathbf{k}\left(\mathbf{F}_{-\omega,-\mathbf{k}}\frac{\mathbf{i}}{\omega - \mathbf{k}\cdot\mathbf{v} + \mathbf{i}/\mathbf{t_c}}\mathbf{F}_{\omega,\mathbf{k}}\right) \tag{35}$$

and

$$\mathbf{F}_{\omega,\mathbf{k}} = e\left(\mathbf{E}_{\omega,\mathbf{k}} + \mathbf{v}\times\mathbf{B}_{\omega,\mathbf{k}} + i\mathbf{k}\left(\frac{\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v}}{2\mu_R}\right)\right). \tag{36}$$

In this problem we are interested in the studying diffusion only due to color magnetic excitaions. In this case the Diffusion coefficient can be written as,

$$D = ie^2\sum_{\omega,k}\frac{\left(\mathbf{v}\times\delta\mathbf{B}_{\omega,-\mathbf{k}} - i\mathbf{k}\left(\frac{\delta\mathbf{B}_{-\omega,-\mathbf{k}}\cdot\mathbf{v}}{2\mu_R}\right)\right)\left(\mathbf{v}\times\delta\mathbf{B}_{\omega,\mathbf{k}} + i\mathbf{k}\left(\frac{\delta\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v}}{2\mu_R}\right)\right)}{\omega - \mathbf{k}\cdot\mathbf{v} + i/t_c}. \tag{37}$$

Now, choosing $\mathbf{k} = k\hat{\mathbf{z}}$, $\delta\mathbf{B}_{\omega,\mathbf{k}} = \delta B_{\omega,k}\hat{\mathbf{y}}$. and considering $\omega = i\gamma$. Then the diffusion coefficient,

$$D = e^2\sum_{\omega,k}\frac{\left(v_z^2|\delta B_{\omega,k}|^2\hat{\mathbf{x}}\hat{\mathbf{x}} + v_xv_z|\delta B_{\omega,k}|^2\hat{\mathbf{x}}\hat{\mathbf{z}} + v_x^2|\delta B_{\omega,k}|^2\hat{\mathbf{z}}\hat{\mathbf{z}} + \frac{v_y^2k^2|\delta B_{\omega,k}|^2}{2\mu_R^2}\hat{\mathbf{z}}\hat{\mathbf{z}}\right)}{(\gamma + 1/t_c + ikv_z)} \tag{38}$$

For strong turbulence we can use approximation $(1/t_c)^2 >> (kv_z)^2$[40]. In this case, at saturation ($\gamma = 0$) the diffusion coefficient can be written as,

$$D = e^2\sum_{\omega,k}\frac{\left(v_z^2|\delta B_{\omega,k}|^2\hat{\mathbf{x}}\hat{\mathbf{x}} + v_xv_z|\delta B_{\omega,k}|^2\hat{\mathbf{x}}\hat{\mathbf{z}} + v_x^2|\delta B_{\omega,k}|^2\hat{\mathbf{z}}\hat{\mathbf{z}} + \frac{v_y^2k^2|\delta B_{\omega,k}|^2}{2\mu_R^2}\hat{\mathbf{z}}\hat{\mathbf{z}}\right)}{(1/t_c)}. \tag{39}$$

Now, taking thermal average of velocities and using Eqs.(31, 39) one can get the decorrelation time as,

$$\left(\frac{1}{t_c}\right)^4 \sim \frac{e^2k^2}{4\bar{E_p}^2}\sum_{\omega',k'}\left(v_T^2|\delta B_{\omega',k'}|^2 + \frac{v_T^2k'^2|\delta B_{\omega',k'}|^2}{2\mu_R^2}\right), \tag{40}$$

where, $v_T^2 = \frac{\int d^3\mathbf{p}v_{x,z}^2\langle n_\mathbf{p}\rangle}{\int d^3\mathbf{p}\langle n_\mathbf{p}\rangle}$. This is the relation between t_c and the intensity of color magnetic excitations. Now we calculate the decorrelation time by incorporating the non-linear corrections in the self energy due to resonance broadening.

Thus due to non-linear wave particle interactions self energy calculated in Eqs.(11,12) acquires a corrections as $\omega \to \omega + i/t_c$.

$$\Pi_+^{ij}(K) = m_D^2\int\frac{d\Omega}{4\pi}v^iv^l\left(\delta^{jl} + \frac{v^jk^l}{\mathbf{v}\cdot\mathbf{k} + i/t_c}\right), \tag{41}$$

$$\Pi_-^{im}(K) = C_E\int\frac{d\Omega}{4\pi}\left[i\epsilon^{iml}k^l + i\omega\frac{(\epsilon^{jlm}v^i - \epsilon^{ijl}v^m)k^lv^j}{(\mathbf{v}\cdot\mathbf{k} + i/t_c)}\right]. \tag{42}$$

It is important to note that here we have considered only right handed particles so in the Eq.13, m_D^2 and C_E will have contribution from right handed particles only. Now, using the similar decomposition of self energy as in case of linear stability analysis one can calculate Π_T, and Π_A to be of the form,

$$\Pi_T = -\frac{m_D^2\left(\omega + \frac{i}{t_c}\right)}{4k}\left[ln\frac{1 - \frac{\omega}{k} + \frac{i}{t_ck}}{1 + \frac{\omega}{k} + \frac{i}{t_ck}} \pm i\pi\right]$$
$$+\frac{m_D^2\left(\omega + \frac{i}{t_c}\right)}{4k}\left(\frac{\omega}{k} + \frac{i}{t_ck}\right)\left[2 + \left(\frac{\omega}{k} + \frac{i}{t_ck}\right)\left(ln\frac{1 - \frac{\omega}{k} + \frac{i}{t_ck}}{1 + \frac{\omega}{k} + \frac{i}{t_ck}} \pm i\pi\right)\right],$$

$$\Pi_L = -\frac{m_D^2\left(\omega + \frac{i}{t_c}\right)}{2k}\left(\frac{\omega}{k} + \frac{i}{t_ck}\right)\left[2 + \left(\frac{\omega}{k} + \frac{i}{t_ck}\right)\left(ln\frac{1 - \frac{\omega}{k} + \frac{i}{t_ck}}{1 + \frac{\omega}{k} + \frac{i}{t_ck}} \pm i\pi\right)\right],$$

$$\Pi_A = kC_E\left[1 + \frac{\omega}{2k}\left(ln\frac{1 - \frac{\omega}{k} + \frac{i}{t_ck}}{1 + \frac{\omega}{k} + \frac{i}{t_ck}} \pm i\pi\right)\right.$$
$$\left.-\frac{\omega}{2}\left(\frac{2}{k}\left(\frac{\omega}{k} + \frac{i}{t_ck}\right) + \frac{1}{k}\left(\frac{\omega}{k} + \frac{i}{t_ck}\right)^2\left(ln\frac{1 - \frac{\omega}{k} + \frac{i}{t_ck}}{1 + \frac{\omega}{k} + \frac{i}{t_ck}} \pm i\pi\right)\right)\right]. \tag{43}$$

Now at saturation $\omega = 0$, therefore

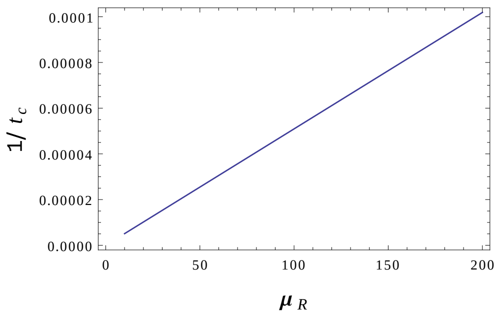
$$\Pi_T = -\frac{im_D^2}{4kt_c}\left[-2i\arctan\frac{1}{t_ck} \pm i\pi\right] - \frac{m_D^2}{4k^2t_c^2}\left[2 + \frac{i}{t_ck}\left(-2i\arctan\frac{1}{t_ck} \pm i\pi\right)\right],$$
$$\Pi_L = +\frac{m_D^2}{4k^2t_c^2}\left[2 + \frac{i}{t_ck}\left(-2i\arctan\frac{1}{t_ck} \pm i\pi\right)\right],$$
$$\Pi_A = kC_E.$$

We determine the decorrelation time from the dispersion relation given in Eq.(23) with $\omega = 0$.

$$k^2 - \frac{m_D^2}{2kt_c}\left[\arctan\frac{1}{t_ck} - \frac{\pi}{2}\right] - \frac{m_D^2}{2k^2t_c^2} - \frac{m_D^2}{2k^3t_c^3}\left(\arctan\frac{1}{t_ck} - \frac{\pi}{2}\right) - kC_E = 0 \tag{44}$$

This is transcendental equation decorrelation can be obtained by solving this equation.

We consider the case $\mu_R \gg T$, in this case $m_D \sim \left(\frac{2\alpha}{\pi}\right)^{1/2}\mu_R$. Further we consider $k = k_{max} = \frac{2\mu_R\alpha}{3\pi}$ which correspond to maximum growth rates of chiral instability. In this case decorrelation time will be dependent on α and μ_R. For $\alpha = 1/137$ the solution for $1/t_c$ of above equation in terms of μ_R is shown in the following figure.



(a)

Figure 1: Shows plot of decorrelation frequency $1/t_c$ as a function of chiral chemical potential μ_5.

Note that the strong turbulence require that the condition $\frac{1}{t_c k_{max}} \gg v_z$ is satisfied in $\mu_R \gg T$ regime. Now if we take $t_c \sim \frac{1}{k}$, $k = k_{max} = \frac{2\mu_R\alpha}{3\pi}$ and $\bar{E}_p \sim \mu_R$ we can determine the saturation level of color magnetic excitations using Eq.(40) as,

$$\delta B_{\omega,k} \sim \frac{\mu_R^2}{\sqrt{\alpha}}. \tag{45}$$

4. Calculation of anomalous viscosity

We follow Ref.[41, 42] to calculate the anomalous viscosity. For Simplicity we make v_x depend on x as,

$$v_x \rightarrow v_x - u(x) \tag{46}$$

where, $u(x)$ is the mean flow variable.

Now using Eq.(39) one can write the diffusion equation (Eq.(34)) as,

$$(\partial_t + v\cdot\partial_x)\langle n_{\mathbf{p}}\rangle \simeq e^2 \sum_{\omega,k} \frac{1}{1/t_c}\Bigg(\left(v_T^2|\delta B_{\omega,k}|^2\right)\partial^2_{\mathbf{p_x}}\langle n_{\mathbf{p}}\rangle + \left(v_T^2|\delta B_{\omega,k}|^2 + \frac{v_T^2 k^2|\delta B_{\omega,k}|^2}{4\mu_R^2}\right)\partial^2_{\mathbf{p_z}}\langle n_{\mathbf{p}}\rangle\Bigg). \tag{47}$$

Second term can be written as;

$$(v\cdot\partial_x)\langle n_{\mathbf{p}}\rangle \simeq -v_T^2 p\frac{d\langle n_{\mathbf{p}}\rangle}{dp}\partial_x u(x). \tag{48}$$

Here, we bring back the term $v\cdot\partial_x$. Now, if we consider, $k = k_{max}$, in this case the summation on ω and k can be lifted out and we can write the diffusion equation in terms of mean flow variable as,

$$\partial_t\langle n_{\mathbf{p}}\rangle - v_T^2 p\frac{d\langle n_{\mathbf{p}}\rangle}{dp}\partial_x u(x) \simeq \frac{e^2}{1/t_c}\Bigg(\left(v_T^2|\delta B_{\omega,k}|^2\right)\partial^2_{\mathbf{p_x}}\langle n_{\mathbf{p}}\rangle + \left(v_T^2|\delta B_{\omega,k}|^2 + \frac{v_T^2 k^2|\delta B_{\omega,k}|^2}{4\mu_R^2}\right)\partial^2_{\mathbf{p_z}}\langle n_{\mathbf{p}}\rangle\Bigg). \tag{49}$$

Note that in the above equation ω and k respectively corresponds to ω_{max} and k_{max}. Now taking moment $\int \frac{d^3p}{(2\pi)^3}\frac{(1+e\delta\mathbf{B}\cdot\mathbf{\Omega_p})}{\epsilon_p}(2p_x^2 - p_y^2 - p_z^2)$, the left hand side of above equation will become,

$$LHS = \partial_t(2T^{xx} - T^{yy} - T^{zz}) - \Big(\frac{v_T^2}{(2\pi)^3}\int d\Omega(2v_x^2 - v_y^2 - v_z^2)\times \frac{(e\delta\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v})^2}{4\mu_R^4}\Big)\left[\int_0^\infty dpp^4\frac{d\langle n_{\mathbf{p}}\rangle}{dp}\right]\partial_x u(x) \tag{50}$$

Note that in the above expression we have used the definition of energy momentum tensor as,

$$T^{\mu\nu}_{\omega,k} = \int\frac{d^3p}{(2\pi)^3}\frac{(1+e\delta\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{\Omega_p})}{\epsilon_p}p^\mu p^\nu\langle n_{\mathbf{p}}\rangle. \tag{51}$$

Simplifying above Eq.(50) we can write,

$$LHS = \partial_t(2T^{xx} - T^{yy} - T^{zz}) + \Big(\frac{v_T^2}{(2\pi)^3}\int d\Omega(2v_x^2 - v_y^2 - v_z^2)\times \frac{(e\delta\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v})^2}{4\mu_R^4}\Big)\left[\int_0^\infty dp4p^3\langle n_{\mathbf{p}}\rangle\right]\partial_x u(x). \tag{52}$$

Now,

$$RHS = \frac{e^2 v_T^2|\delta B_{\omega,k}|^2}{1/t_c}\Bigg(\frac{1}{(2\pi)^3}\int d\Omega(2v_x^2 - v_y^2 - v_z^2)\frac{(e\delta\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v})^2}{4\mu_R^4}\times \left(\int_0^\infty dpp^3\partial^2_{\mathbf{p_x}}\langle n_{\mathbf{p}}\rangle + \left(1+\frac{k^2}{4\mu_R^2}\right)\int_0^\infty dpp^3\partial^2_{\mathbf{p_z}}\langle n_{\mathbf{p}}\rangle\right)\Bigg).$$

Now using,

$$\partial^2_{\mathbf{p_z}}\langle n_{\mathbf{p}}\rangle = \frac{p_z^2}{p^2}d_p^2\langle n_{\mathbf{p}}\rangle + \frac{1}{p}d_p\langle n_{\mathbf{p}}\rangle - \frac{p_z^2}{p^3}d_p\langle n_{\mathbf{p}}\rangle. \tag{53}$$

and writing $\partial^2_{\mathbf{p_x}}\langle n_{\mathbf{p}}\rangle$ in a similar fashion, One can get;

$$RHS = \frac{e^2 v_T^2|\delta B_{\omega,k}|^2}{1/t_c}\Bigg(\frac{1}{(2\pi)^3}\int d\Omega(2v_x^2 - v_y^2 - v_z^2)\frac{(e\delta\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v})^2}{4\mu_R^4}\times \left(\int_0^\infty dpp^3\left(v_x^2 d_p^2\langle n_{\mathbf{p}}\rangle + \frac{1}{p}d_p\langle n_{\mathbf{p}}\rangle - \frac{v_x^2}{p}d_p\langle n_{\mathbf{p}}\rangle\right) + \left(1+\frac{k^2}{4\mu_R^2}\right)\int_0^\infty dpp^3\left(v_z^2 d_p^2\langle n_{\mathbf{p}}\rangle + \frac{1}{p}d_p\langle n_{\mathbf{p}}\rangle - \frac{v_z^2}{p}d_p\langle n_{\mathbf{p}}\rangle\right)\right)\Bigg).$$

With further simplification we can write above equation as,

$$RHS = \frac{e^2 v_T^2|\delta B_{\omega,k}|^2}{1/t_c}\Bigg(\frac{1}{(2\pi)^3}\int d\Omega(2v_x^2 - v_y^2 - v_z^2)\frac{(e\delta\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v})^2}{4\mu_R^4}\times \left(\int_0^\infty dp\left(8v_x^2 p\langle n_{\mathbf{p}}\rangle - 2p\langle n_{\mathbf{p}}\rangle\right) + \left(1+\frac{k^2}{4\mu_R^2}\right)\int_0^\infty dp\left(8v_z^2 p\langle n_{\mathbf{p}}\rangle - 2p\langle n_{\mathbf{p}}\rangle\right)\right)\Bigg).$$

We choose stationary limit in this case $\partial_t(2T^{xx} - T^{yy} - T^{zz}) = 0$, therefore from the diffusion equation (L.H.S=R.H.S) we can get,

$$\partial_x u(x) = \frac{e^2 v_T^2|\delta B_{\omega,k}|^2}{1/t_c}\frac{\left(8I_1J_1 - 4I_2J_1 + 8\left(1+\frac{k^2}{4\mu_R^2}\right)I_3J_1\right)}{5I_2J_2} \tag{54}$$

where,

$$I_1 = \left(\frac{1}{(2\pi)^3}\int d\Omega(2v_x^2 - v_y^2 - v_z^2)v_x^2\frac{(e\delta\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v})^2}{4\mu_R^4} = \frac{e^2\delta\mathbf{B}^2_{\omega,\mathbf{k}}}{105\pi^2\mu_R^4}\right.$$

$$I_2 = \left(\frac{1}{(2\pi)^3}\int d\Omega(2v_x^2 - v_y^2 - v_z^2)\frac{(e\delta\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v})^2}{4\mu_R^4} - \frac{e^2\delta\mathbf{B}^2_{\omega,\mathbf{k}}}{15\pi^2\mu_R^4},\right.$$

$$I_3 = \left(\frac{1}{(2\pi)^3}\int d\Omega(2v_x^2 - v_y^2 - v_z^2)v_z^2\frac{(e\delta\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v})^2}{4\mu_R^4} = -\frac{e^2\delta\mathbf{B}^2_{\omega,\mathbf{k}}}{210\pi^2\mu_R^4},\right.$$

$$J_1 = \int_0^\infty dpp\langle n_\mathbf{p}\rangle,$$

$$J_2 = \int_0^\infty dpp^3\langle n_\mathbf{p}\rangle.$$

Now, using Eq.(51) one can write,

$$(2T^{xx} - T^{yy} - T^{zz}) = \int\frac{d\Omega}{(2\pi)^3}\frac{(e\delta\mathbf{B}_{\omega,\mathbf{k}}\cdot\mathbf{v})^2(2v_x^2 - v_y^2 - v_z^2)}{4\mu_R^4}\int_0^\infty dpp^3\langle n_\mathbf{p}\rangle \tag{55}$$

The definition of shear viscosity is,

$$\eta_A = \frac{(2T^{xx} - T^{yy} - T^{zz})}{-4\partial_x u(x)} \tag{56}$$

Taking the distribution function of the form, $\langle n_\mathbf{p}\rangle = \frac{1}{\exp(\mu_R - p) + 1}$, considering $\mu_R \gg T$ and using Eqs.(54,55) and $k = k_{max} = \frac{2\mu_R\alpha}{3\pi}$ one can estimate anomalous shear viscosity,

$$\eta_A \sim \frac{\mu_R^2}{t_c}\left(1 + \frac{11\pi^2T^2}{3\mu_R^2}\right) \tag{57}$$

One can notice from Fig.(1) that that for the case $\mu_R \gg T$ and $k = k_{max}$, $1/t_c$ depends on μ_R in an approximately linear way i.e. $1/t_c \propto \mu_R$. Slope of the curve can be found by a linear fit. For $\alpha = 1/137$ the slope is $\sim 5.09 * 10^-7$. The slope increases by increasing α. Thus η scales like μ_R^3.

5. Conclusion

We have calculated the coefficient of shear viscosity based on the strong turbulence argument. For the case when $\mu_R/T \gg 1$, the collision rates becomes insignificant[33] at low temperature, in this regime the decorrelation frequency $1/t_c$ can have a significant contribution in determining η. In this low temperature limit the entropy density s scales as $\mu_R^2 T$ and the ratio $\eta/s \propto \mu_R/T$ and it can be a large number. In deriving the above expression of η we have ignored non-linear wave-wave interaction which can play a role in case of non-Abelian plasmas. However to address this question one require to numerically simulate the chiral plasma instability with the full nonlinearity.

Note that dimensional argument suggests that for the case when $\mu_R \ll T$, stress (energy density) $\sim \mu_R^2 T^2$, decorrelation frequency ($1/t_c \sim \omega_{max}$) of CPI $\sim \mu_R$ [25] and η scales as $\mu_R T^2$. Therefore, $\eta/s \propto \mu_R/T$, which could be a small number. We hope that this analytic study will help in understanding the viscosity due to turbulent transport in parity violating plasma and can be useful in it numerical simulations.

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