Imprint of Pre-inflation Universe from Signatures in the Cosmic Microwave Background Anisotropy Spectrum

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CERTIFICATE

I feel great pleasure in certifying that the thesis entitled "Imprint of pre-inflation universe from signatures in the cosmic microwave background anisotropy spectrum" embodies a record of the results of investigations carried out by Mr. Akhilesh under my guidance. I am satisfied with the analysis of data, interpretation of results and conclusions drawn.

He has completed the residential requirement as per rules.

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DECLARATION

I hereby declare that the work incorporated in the present thesis entitled "Imprint of pre-inflation universe from signatures in the cosmic microwave background anisotropy spectrum" is my own work and is original. This work (in part or in full) has not been submitted to any University for the award of a Degree or a Diploma.

Akhilesh

Date: Signature of the Candidate

To

Maa And Papa Ji

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Abstract

In this thesis we study the effect of pre-inflationary radiation era and curvature on cosmic microwave background (CMB) anisotropy and polarization. We show that if inflation was preceded by the radiation era, there would exist a decoupled thermal distribution of gravitons at the beginning of inflation that change the nearly scale invariant spectrum of the gravitational waves generated during inflation. Due to this the *B*-mode of the CMB polarization are enhanced at large angles. This enhancement may be observed by future polarization experiments like PLANCK. Observation of this enhancement may help us in testing an important class of models of inflation 'Warm inflation'.

In natural inflation models, the inflaton is pseudo Nambu-Goldstone boson (PNGB). To satisfy observations the spontaneous symmetry breaking (SSB) scale of the PNGB has to be at Planck scale. We show that if one couples this PNGB with radiation bath as in warm inflationary models, the SSB scale can be reduced to the GUT scale. We also show that one can generate spontaneous leptogenesis in this model as PNGB has derivative coupling with the lepton current. Another feature of this model is that it predicts large non-Gaussianity which may be observed in the future PLANCK experiment.

We also show that if the universe had a large curvature before inflation, then there would be a deviation from the scale invariant perturbations of the inflaton at the beginning of inflation. This may affect the large scale CMB anisotropy. We obtain the expression for the power spectrum of comoving curvature perturbation in case of both open and closed universe. We apply the Bunch-Davies boundary condition and evaluate the power spectrum at horizon crossing. We compare the temperature anisotropy generated by using our formula and the Ratra-Peebles formula with the WMAP 5yr data. We find that our power spectrum gives low quadrupole for closed universe but matches with the Ratra-Peebles formula at high l. The dif-

ference in the temperature anisotropy at low l arising due to the different boundary conditions is unobservable because of cosmic variance.

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Chapter 1

Introduction

An important advance in theoretical cosmology is the idea of inflation- a period of rapid accelerated expansion during the early universe before Big-Bang nucleosynthesis [1, 2, 3]. It was introduced to solve certain problems of the standard model of cosmology, and later it was realized [4, 5, 6, 7, 8] that it not only solves the problems of the standard model of cosmology but also provides seeds for the anisotropy in cosmic microwave background (CMB) and structures in the universe. The predictions of inflation that there are super-horizon correlations in the CMB, were first confirmed by Cosmic Background Explorer (COBE) [9]. The precise measurements of CMB anisotropy, being done by Wilkinson Microwave Anisotropy Probe (WMAP) [10] and other ground based, balloon based and satellite based experiments, are also consistent with the early period of inflation.

Inflation sets up the initial conditions for the hot Big-Bang as exponential expansion leaves the universe homogeneous and isotropic at large scales with exponentially small curvature. During inflation the potential energy of a scalar field, called inflaton, dominates the energy density of the universe and inflaton rolls slowly through its potential. When this slow-roll condition breaks down, inflation ends and inflaton decays into other standard model particles (reheating). During slow-roll motion there are quantum fluctuations in the inflaton field and vacuum fluctuations in the transverse traceless tensor part of the metric. The quantum fluctuations in inflaton are coupled to metric perturbations (for e.g gravitational potential) through Einstein's equation. These perturbations, generated during inflation, become super-horizon and they re-enter the horizon during radiation and matter dominated era providing seeds for CMB anisotropy and structures in the universe. Inflation predicts nearly scale invariant and nearly Gaussian density perturbations. It also predicts nearly scale invariant spectrum of gravitational waves. The two-point correlation function of these perturbations in Fourier space is called as power spectrum. The shape of the primordial perturbations is determined in terms of spectral index.

CMB anisotropy and polarization is represented in terms of four two-point correlation functions namely temperature-temperature correlation function TT, polarization correlation functions EE and BB, and temperature-polarization correlation function TE. Here E and B are curl free and divergence-free modes of CMB polarization that are rotationally invariant combinations of Stokes parameters Q and U. The temperature anisotropy and polarization are expanded in terms of spherical harmonics and their two-point correlation functions are given in terms of angular power spectra C_ls. The perturbations responsible for small angle CMB anisotropy entered the horizon earlier to or during recombination, so the primordial perturbations are modified on these scales. These anisotropies show the features of acoustic oscillations set in the electron-baryon plasma before recombination. But the perturbations responsible for the large angle CMB anisotropy entered the horizon after recombination and they contain the signature of primordial perturbations generated during inflation. The metric perturbations generated during inflation give rise to large scale CMB anisotropy via the Scahs-Wolfe effect[11]. We can determine the amplitude and spectral index of the curvature perturbations generated during inflation by CMB anisotropy.

CMB observations give tight constraints on the amplitude and spectral index of primordial perturbations that put an upper bound on the scale of inflation, which in generic models of inflation turns out to be close to the GUT scale ($\sim 10^{16}$GeV). The amplitude and spectral index are determined by the potential of inflaton, so we can constrain the models of inflation by measuring the CMB anisotropy. There are a large number of models of inflation that can generate the required spectrum of the density perturbations, but all these models are not compatible with particle physics as most of them require some unnatural fine tuning of masses and couplings in the potential. For e.g. in $\lambda\phi^4$ model we must have $\lambda \leq 10^{-12}$ to satisfy observations and one can not keep this value small if higher order corrections are included. There are a large number of attempts to realize inflation in particle physics.

One alternative to the standard models of inflation is warm inflation [12], in which inflaton is always coupled with radiation but the latter is sub-dominant in density and does not affect the exponential expansion. In these models inflaton is being dissipated in the radiation and the dissipation along with the Hubble expansion provides slow-roll. In warm inflation the density perturbations are generated due to thermal fluctuations in inflaton field. One can have sufficient duration of slow-roll and can generate nearly scale invariant perturbations in warm inflation models with reasonable values of masses and couplings in the inflaton potential. The another attempt to realize cold inflation is 'Natural inflation' [13, 14, 15] where inflaton is pseudo Nambu Goldstone Boson (PNGB) of some spontaneously broken symmetry at high scale. In these models the flatness of the inflaton potential is protected from higher order corrections due to shift symmetry.

Although the predictions of inflation i.e scale invariant, adiabatic and Gaussian perturbations are almost consistent with CMB observations [16], we lack a unique model of inflation. CMB observations also show some features on large scale anisotropy which are not well understood. The quadrupole anisotropy is suppressed and there are glitches in the low l anisotropy. These features are seen in the super-horizon scales and are generated during the early part of inflation.

To solve the horizon problem we assume that the perturbations corresponding to our horizon size today, were leaving the inflationary horizon at the beginning of inflation and the duration of inflation should be nearly 60 e-folds. It may also happen that inflation could have started earlier and the length scales corresponding to our horizon left the de Sitter horizon later. If this is the case, all information prior to inflation will have no effect on large scales. But if inflation happens for minimal number of e-folds, there is a chance that we can see the signatures of pre-inflation universe. The perturbations corresponding to largest angle CMB anisotropies, that have entered the horizon recently, were generated during the early period of inflation, so they are most likely to reflect the signatures of pre-inflation universe. The conditions prior to inflation are of importance as they determine the large scale (low l) spectrum of the CMB. The conditions prior to inflation can also help us in determining the correct model of inflation.

We have studied the effect of pre-inflationary radiation era and curvature on CMB. We have shown [17] that if there was a radiation era prior to the inflation, then at the time of inflation there will exist a decoupled thermal distribution of gravitons. Gravitational waves generated during inflation will be amplified by the process of stimulated emission into the existing thermal distribution of gravitons. Consequently, the usual zero temperature scale invariant tensor spectrum is modified by a temperature dependent factor. This thermal correction factor amplifies the B-mode polarization of the CMB by an order of magnitude at large angles. The observations of this enhancement can help us in testing warm inflation models.

Radiation era prior to inflation is a natural condition for warm inflation models as in warm inflation models inflaton field is always coupled to the radiation. As mentioned earlier this property of warm inflation can be used to construct a potential of inflaton that has the values of couplings allowed in particle physics. In 'Natural inflation' models, where the inflaton is PNGB, the nearly scale invariant spectrum of density perturbations is attained only when the symmetry breaking scale is of the order of Planck scale. This scale can be reduced to the GUT scale by coupling the PNGB to a thermal bath, as in warm inflation models and one can get the amplitude and spectral index which agrees with the WMAP data [18]. In our work we give a GUT model of PNGB arising out of spontaneously broken lepton number at the GUT scale which gives rise to heavy Majorana masses for the right handed neutrinos which is needed in see-saw models. Since PNGB has a derivative coupling with the lepton current, this model can generate leptogenesis spontaneously.

An interesting observable feature of CMB anisotropy is non-Gaussianity which implies that there may be higher order correlation functions present in the CMB anisotropy. The non-Gaussianity in the CMB anisotropy is generated due to the primordial non-Gaussianity in the perturbations and some other effects. All standard inflationary mechanisms generate very small amount of non-Gaussianity. The observation of non-Gaussian fluctuation will help us testing all the models of inflation. With warm inflation in strong dissipative regime, as in our model, we can have large non-Gaussianity which may be observed in the forthcoming PLANCK experiment.

Anther observable feature of the pre-inflation universe is the curvature at the beginning of inflation. As inflation starts, curvature of the universe goes down very rapidly. According to inflation, curvature today is equal to the curvature during inflation when our horizon left the inflationary horizon. If we observe Ω different than unity today then there was a residual curvature present at the time when our observable universe left the de Sitter horizon. Due to the presence of large curvature at the beginning of inflation there will be deviation from the scale invariant perturbations of the inflaton at the beginning of inflation. This may have some effects on large angle CMB anisotropy. We calculate the density perturbations for both open and closed universe cases using the Bunch-Davies vacuum condition on the initial state [19]. We use our power spectrum to calculate the temperature anisotropy spectrum and compare the results with the WMAP five year data. We find that our power spectrum gives a lower quadrupole anisotropy when $\Omega - 1 > 0$, but matches the temperature anisotropy calculated from the standard Ratra-Peebles power spectrum [20] at large l. The determination of spatial curvature from temperature anisotropy data is not much affected by the different power spectra which arise from the choice of different boundary conditions for the inflaton perturbation.

The outline of the thesis is as follows. In the second chapter we will give a very brief description of the standard theory of density perturbations and theory of CMB anisotropy and polarization. In the third chapter we will describe the effect of pre inflationary radiation era on CMB polarization. In the fourth chapter we will describe warm inflation with PNGB and its phenomenological applications. In the fifth chapter we will describe the effect of curvature on CMB anisotropy and in the last chapter we will conclude.

Chapter 2

Standard theory of density perturbations and CMB anisotropy

In this chapter we will briefly review the theory of inflation and density perturbations in both supercooled and warm inflationary models. We will also summaries the theory of CMB anisotropy and polarization.

2.1 Standard model of cosmology

The standard model of cosmology is based on the assumption that the universe is homogeneous and isotropic at large scales. The geometry of the homogeneous and isotropic space-time is given by Friedmann-Robertson-Walker (FRW) line element

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{(1-Kr^2)} + r^2\, d\theta^2 + r^2\, \sin^2\theta\, d\phi^2\right]. \tag{2.1}$$

Here $a(t)$ is the scale factor which describes the time evolution of the universe, (r, θ, ϕ) are the comoving coordinates and K specifies the spatial 3-curvature of the universe. The signature of the metric is (-,+,+,+). The physical distance can be found by multiplying the comoving distance with scale factor. According to our convention present value is $a = 1$ and it is dimensionless, whereas K has the dimension of $length^{-2}$. The matter content of the universe is described in terms of energy-momentum tensor $T_{\mu\nu}$. For a perfect homogeneous and isotropic fluid it is given as

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p\, g_{\mu\nu}. \tag{2.2}$$

Here ρ and p are the total energy density and pressure of the fluid and u_μ is its four velocity. In the rest frame of the fluid $T_{\mu\nu}$ can be expressed as $diag\{\rho, p, p, p\}$. The energy-momentum tensor satisfies the equation of continuity $\nabla_\mu T_{\mu\nu} = 0$. After solving the Einstein's equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ and the continuity equation for $\mu = 0$ we obtain the following Friedmann equations for the evolution of the scale factor,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}, \tag{2.3}$$

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{2.4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \tag{2.5}$$

here H is the Hubble's constant. The equation of state is defined as $p = w\rho$, where $w = 0$ for matter and $\frac{1}{3}$ for radiation. The behavior of the scale factor is given as $a = t^{\frac{2}{3(1+w)}}$. Radiation and matter densities scale as a^{-4} and a^{-3} respectively. If we divide Eq. (2.3) by H^2, define $\rho_c = \frac{3H^2}{8\pi G}$ and $\Omega = \frac{\rho}{\rho_c}$ we get

$$\Omega - 1 = \frac{K}{a^2 H^2}. \tag{2.6}$$

The curvature of the universe can be described in terms of Ω. Present day observations constrain Ω to be very close to 1 i.e the universe is almost flat.

FRW line element can be written in a different form by using conformal transformation $d\tau = \frac{dt}{a}$ as

$$ds^2 = a^2(\tau)\left[-d\tau^2 + \frac{dr^2}{(1-Kr^2)} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right], \tag{2.7}$$

here τ denotes the conformal time. We can define $\mathcal{H} = \frac{a'}{a} = aH$ as the Hubble constant in conformal time. Since no information can travel faster than light, above theory predicts particle horizon i.e the maximum physical distance traveled by photons from the big bang until time t. It is given by

$$R_H(t) = a(t)\int_0^t \frac{dt'}{a(t')} \sim H^{-1}. \tag{2.8}$$

A physical length scale λ is within the horizon if $\lambda < H^{-1}$. This condition can be expressed in terms of comoving wavenumber $k = \frac{2\pi a}{\lambda}$ as $k \ll aH$ for scales outside the horizon and $k \gg aH$ for scales inside the horizon. There are various shortcomings of the standard model of cosmology. Among them are the horizon or the large scale smoothness problem, the flatness problem, monopole problem, entropy problem and the origin of density perturbation problem.

We will briefly review the flatness problem and the horizon problem.

2.1.1 Flatness problem:

Present observations show that the universe is very nearly spatially flat i.e $\Omega_0 \sim 1$. The curvature of the universe grows as $\Omega - 1 \sim \frac{1}{a^2H^2}$. So to have $\Omega \sim 1$ today $\Omega - 1$ has to be fine tuned up to 1 part in 10^{16} at the time of BBN.

2.1.2 Horizon problem

The angular size of the horizon at the time of recombination was nearly 1° in the sky. During radiation and matter dominated era physical length scales grow as a but the horizon size grows as a^2 and $a^{\frac{3}{2}}$ respectively. So the length scales of angular size larger than 1° (scales corresponding to $l < 200$ for CMB anisotropy) were outside the horizon before and at the time of last scattering. Today we are observing the same average temperature of CMB coming from all regions in the sky. We also observe super-horizon temperature correlations in CMB. There is no way to establish thermal equilibrium if these regions were never in causal contact before last scattering. We also cannot explain the observed super-horizon temperature correlations.

2.2 Introduction to inflation

To solve the horizon problem we should have a period during which the length scales evolve faster than the horizon. As horizon size is nearly H^{-1} and all the physical length scales are proportional to a, this condition can be expressed as

$$\left(\frac{\lambda}{H^{-1}}\right)^{\cdot} = \ddot{a} > 0. \tag{2.9}$$

To achieve this condition the idea of inflation was introduced [1]. During this period universe accelerates i.e $p < -\frac{\rho}{3}$. The length scales that were outside the horizon at the time of recombination were generated inside the horizon during inflation and left the inflationary horizon. This provides a mechanism to establish thermal equilibrium among the regions that were acausal during recombination. It is obvious from Eq. (2.5) that for accelerated expansion $\rho < -3p$. Consider a condition where $p \sim -\rho$, hence energy density remains constant (Eq. 2.4). In this case H remains constant i.e the horizon remains constant. The physical length scales evolve exponentially as $a \sim e^{Ht}$. So the length scales of our horizon size were inside the inflationary horizon and since they grew faster than the horizon during inflation they became super-horizon after the beginning of inflation and re-entered the present horizon during the radiation and matter dominated epoch. So all the length scales we observe today had a chance to be in causal contact during inflation and hence we have the same average temperature of the CMB in all directions. The curvature of the universe is reduced to very small value due to exponential expansion and we have a flat universe at the end of inflation.

2.2.1 Inflaton

During inflation the condition (2.9) must be satisfied, so the pressure has to be negative. Neither radiation nor matter can give negative pressure. This is achieved by a scalar field called inflaton. The action of the scalar field is given by

$$S = -\int d^4x \sqrt{-g}\,\mathcal{L} = \int d^4x \sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi) + V_{interaction}\right]. \qquad (2.10)$$

From this action one can derive the equation of motion of the inflaton field which is given as

$$\ddot{\phi} + (3H+\Gamma)\dot{\phi} - \frac{\nabla^2}{a^2}\phi + V'(\phi) = 0, \qquad (2.11)$$

here $V'(\phi)$ denotes the differentiation of $V(\phi)$ w.r.t ϕ and Γ is due to the interaction of the inflaton field with other particles present in the background. The energy density and the pressure for the scalar field can be found from the energy-momentum tensor of the scalar field

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + V(\phi)\right), \qquad (2.12)$$

hence

$$\rho = T_{00} = \frac{\dot{\phi}^2}{2} + V(\phi) + \frac{(\nabla\phi)^2}{2a^2}, \qquad (2.13)$$

$$p = T_{ij} = \frac{\dot{\phi}^2}{2} - V(\phi) - \frac{(\nabla\phi)^2}{6a^2}. \qquad (2.14)$$

So if we take a homogeneous field with very small kinetic energy we can have $p \sim -\rho$ which can give inflation.

2.2.2 Slow-roll

The inflaton field can be written in terms of time dependent homogeneous part and fluctuations around it as

$$\phi(\mathbf{x},t) = \phi_0(t) + \delta\phi(\mathbf{x},t). \tag{2.15}$$

The equation of motion for the homogeneous part $\phi_0(t)$ (we will denote it as ϕ) can be obtained from Eq. (2.11) i.e

$$\ddot{\phi} + (3H + \Gamma)\dot{\phi} + V'(\phi) = 0. \tag{2.16}$$

In supercooled inflation it is assumed that inflaton is weakly coupled to other fields so Γ is neglected. The condition $\frac{\dot{\phi}^2}{2} \ll V(\phi)$ means that the field rolls down slowly through its potential. To achieve this the potential of inflaton should be very flat. The flatness of the potential is determined by the following parameters called as slow-roll parameters,

$$\varepsilon = -\frac{\dot{H}}{H^2} = \frac{M_P^2}{16\pi}\left(\frac{V'}{V}\right)^2, \tag{2.17}$$

$$\eta = \frac{M_P^2}{8\pi}\left(\frac{V''}{V}\right) = \frac{1}{3}\frac{V''}{H^2}, \tag{2.18}$$

$$\delta = \eta - \varepsilon = -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{2.19}$$

For inflation $\varepsilon \ll 1$, $\eta \ll 1$ and $\delta \ll 1$. The condition $\eta \ll 1$ implies that the mass of the inflaton field should be less than the Hubble constant during inflation. During inflation only the potential energy of the scalar field dominates and all other components are diluted exponentially so Eq. (2.3) becomes

$$H^2 \simeq \frac{8\pi G}{3} V(\phi). \tag{2.20}$$

Under the slow-roll approximation the equation of motion of inflaton becomes

$$3H\dot{\phi} = -V'(\phi). \tag{2.21}$$

The duration of inflation is given in terms of the number of e-foldings defined as

$$N = \int_0^t H dt. \tag{2.22}$$

It can be written in terms of V and V' by using equations (2.20) and (2.21) as

$$N = -\frac{8\pi}{M_P^2}\int_{\phi_i}^{\phi_f}\frac{V}{V'}\,d\phi. \tag{2.23}$$

Here ϕ_i and ϕ_f are the values of the inflaton field at the beginning and the end of inflation respectively. If we assume that the length scales corresponding to our horizon size today were leaving the inflationary horizon at the beginning of inflation, we must have nearly 60 e-foldings. Inflation ends when the slow-roll condition $\varepsilon \ll 1$ breaks down and after that the inflaton field decays into radiation during reheating.

2.3 Perturbations during inflation

A nice feature of inflation is that it generates seeds for CMB anisotropy and structures in the universe. During inflation there are quantum fluctuations in the inflaton field. Since the potential energy of the inflaton field dominates the energy density of the universe, the quantum fluctuations in the inflaton field cause perturbations in the energy-momentum tensor. These perturbations generate the metric perturbations as they are coupled to the geometry through Einstein's equation. A detailed calculations of density perturbations during inflation can be found in [21, 22, 23, 24]. We will follow [21, 24] and briefly describe the generation of scalar and tensor perturbations during inflation.

The perturbations in the metric up to first order can be written as

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}. \tag{2.24}$$

One can split $\delta g_{\mu\nu}$ in scalar, vector and tensor perturbations according to their transformation under 3-rotation. The perturbed line element can be written as

$$\begin{aligned} ds^2 &= a^2(\tau)\left[-(1+2\Phi)\,d\tau^2 + (2B_{,i}+S_i)\,dx^i d\tau + \big((1-2\Psi)\,\delta_{ij}\right. \\ &+ \left. 2E_{,ij} + W_{i,j} + W_{j,i} + h_{ij}\big)\,dx^i dx^j\right]. \end{aligned} \tag{2.25}$$

Here Φ, Ψ, E and B are 3-scalars, S_i and W_i are divergence free 3-vectors, h_{ij} is a traceless and transverse 3-tensor and a comma with a variable means its derivative ∂_i. S_i, W_i and h_{ij}

satisfy the following four constraints,

$$S^i_{,i} = W^i_{,i} = 0, \tag{2.26}$$

$$h^i_i = 0,\ h^i_{j,i} = 0. \tag{2.27}$$

We have four scalar functions, four components for two vectors and two components for tensor i.e ten degrees of freedom to describe perturbed metric. Among these, scalar perturbations are the most important for CMB anisotropy and structure formation. Vector perturbations describes rotational motion of the fluid and they are not generated in standard inflation so we will not consider them. Tensor perturbations h_{ij} describe gravitational waves. These gravitational waves play an important role in CMB polarization. Since scalar, vector and tensor perturbations are decoupled. one can study these perturbations separately.

Quantum fluctuations in the inflaton field generate perturbations in the energy-momentum tensor which are given as [21]

$$\begin{aligned}\delta T_{\mu\nu} &= \partial_\mu\delta\phi\,\partial_\nu\phi + \partial_\mu\phi\,\partial_\nu\delta\phi - \delta g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi + V(\phi)\right)\\ &- g_{\mu\nu}\left(\frac{1}{2}\delta g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi + g^{\alpha\beta}\partial_\alpha\delta\phi\,\partial_\beta\phi + \frac{\partial V}{\partial\phi}\delta\phi\right).\end{aligned} \tag{2.28}$$

In component form it will be

$$\delta T_{00} = \delta\phi'\,\phi' + 2\Phi V(\phi)\,a^2 + a^2\frac{\partial V}{\partial\phi}\delta\phi\,, \tag{2.29}$$

$$\delta T_{0i} = \partial_i\,\delta\phi\,\phi' + \frac{1}{2}\partial_i B\,\phi'^2 - \partial_i B\,V(\phi)\,a^2\,, \tag{2.30}$$

$$\begin{aligned}\delta T_{ij} &= \left(\delta\phi'\,\phi' - \Phi\phi'^2 - a^2\frac{\partial V}{\partial\phi}\delta^{(1)}\phi - \Psi\phi'^2 + 2\Psi V(\phi)\,a^2\right)\delta_{ij}\\ &+ E_{,ij}\,\phi'^2 - 2E_{,ij}V(\phi)\,a^2\,.\end{aligned} \tag{2.31}$$

Here and afterward a prime on a variable will denote its derivative w.r.t conformal time. Now δT^μ_ν can be obtained by using the relation

$$\begin{aligned}\delta T^\mu_\nu &= \delta\left(g^{\mu\sigma}T_{\sigma\nu}\right)\\ &= \delta g^{\mu\sigma}T_{\sigma\nu} + g^{\mu\sigma}\delta T_{\sigma\nu}.\end{aligned} \tag{2.32}$$

Hence we obtain

$$\delta T^0_0 = \frac{1}{a^2}\left(\Phi\phi'^2 - \delta\phi'\phi' - \delta\phi\frac{\partial V}{\partial\phi}a^2\right), \tag{2.33}$$

$$\delta T^i_0 = \frac{1}{a^2}\left(\partial^i B\phi'^2 + \partial^i\delta\phi\phi'\right), \tag{2.34}$$

$$\delta T^0_i = \frac{-\partial_i\delta\phi\phi'}{a^2}, \tag{2.35}$$

$$\delta T^i_j = \frac{1}{a^2}\left[\left(-\Phi\phi'^2 + \delta\phi'\phi' - \delta\phi\frac{\partial V}{\partial\phi}a^2\right)\delta^i{}_j\right]. \tag{2.36}$$

By using the covariant energy-momentum conservation equation $\nabla_\mu T_{\mu\nu} = 0$ we can calculate the perturbed Klein-Gordon equation describing the evolution of $\delta\phi$,

$$\delta\phi'' + 2\frac{a'}{a}\delta\phi' - \partial^i\partial_i\delta\phi - \Phi'\phi' - 3\Psi'\phi' + \phi'\partial_i\partial^i\left(E' - B\right) + \delta\phi\frac{\partial^2 V}{\partial\phi^2}a^2 + 2\Phi\frac{\partial V}{\partial\phi}a^2 = 0. \tag{2.37}$$

2.3.1 Gauge invariance

An important issue regarding the cosmological perturbation theory is the behavior of the perturbations under coordinate transformation. The coordinate transformations are referred to as the gauge transformations in general relativity. We will briefly review the behavior of scalar perturbations under coordinate transformations and introduce gauge invariant quantities. Consider the following infinitesimal coordinate transformations

$$\tilde{x}^\mu = x^\mu + \xi^\mu, \tag{2.38}$$

where ξ^μ are four infinitesimally small functions of space-time. The metric tensor $g_{\mu\nu}$ in the new coordinates can be found by applying the laws of transformation of a second rank covariant tensor i.e

$$\tilde{g}_{\mu\nu}(\tilde{x}^\mu) = \frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}g_{\alpha\beta}(x^\rho) \tag{2.39}$$

$$= g^{(0)}_{\mu\nu}(x^\mu) + \delta g_{\mu\nu} - g^{(0)}_{\mu\beta}\xi^\beta_{,\nu} - g^{(0)}_{\alpha\nu}\xi^\alpha_{,\mu}. \tag{2.40}$$

Here we have taken only the linear terms in δg and ξ. One can split the metric in the new coordinate as

$$\tilde{g}_{\mu\nu}(x^\mu) = \tilde{g}^{(0)}_{\mu\nu}(\tilde{x}^\mu) + \delta\tilde{g}_{\mu\nu}. \tag{2.41}$$

Using Eq. (2.40) and Eq. (2.41) and taking into account that

$$g^{(0)}_{\mu\nu}(x^\mu) = \tilde{g}^{(0)}_{\mu\nu}(\tilde{x}^\mu) - g^{(0)}_{\mu\nu,\gamma}\xi^\gamma, \tag{2.42}$$

we get

$$\delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} - g^{(0)}_{\mu\nu,\gamma}\xi^\gamma - g^{(0)}_{\mu\beta}\xi^\beta_{,\nu} - g^{(0)}_{\alpha\nu}\xi^\alpha_{,\mu}. \tag{2.43}$$

The spatial part of the infinitesimal vector $\xi^\mu = \left(\xi^0, \xi^i\right)$ can be written as

$$\xi^i = \xi^i_{tr} + \partial^i \xi, \tag{2.44}$$

where ξ^i_{tr} is divergence free vector and ξ represents the real scalar degrees of freedom. One can find the transformation rules for the scalar perturbations Φ, Ψ, E and B using Eq. (2.43)

$$\widetilde{\Phi} = \Phi - \frac{a'}{a}\xi^0 - \xi^{0\prime}, \tag{2.45}$$

$$\widetilde{\Psi} = \Psi + \frac{a'}{a}\xi^0, \tag{2.46}$$

$$\widetilde{E} = E - \xi, \tag{2.47}$$

$$\widetilde{B} = B - \xi' + \xi^0. \tag{2.48}$$

The perturbations in the scalar field transform as

$$\widetilde{\delta\phi}(\tilde{x}^\rho) = \delta\phi(x^\rho) - \phi'\xi^0. \tag{2.49}$$

From these equations one can construct gauge invariant variables, first introduced by Bardeen [25], for Φ and Ψ

$$\Phi^{GI} = -\Phi + \frac{1}{a}\left[\left(-B + E'\right)a\right]', \tag{2.50}$$

$$\Psi^{GI} = -\Psi + \frac{a'}{a}\left(B - \frac{E'}{2}\right). \tag{2.51}$$

and the gauge invariant quantity for the perturbations in the scalar field is

$$\delta\phi^{GI} = -\delta\phi + \phi'\left(E' - B\right). \tag{2.52}$$

We can choose two functions ξ^0 and ξ by imposing the conditions on scalar perturbations. By doing this we are actually fixing a coordinate system or choosing a gauge.

There are a number of gauge choices used in the literature but we use conformal-Newtonian gauge that is defined by the condition $E = B = 0$. In this gauge the metric takes the form

$$ds^2 = -a^2\left[(1+2\Phi)\,d\tau^2 + (1-2\Psi)\,\delta_{ij}dx^i dx^j\right]. \tag{2.53}$$

From equations (2.50) and (2.51) it can be seen that the Bardeen's variables are equal to the conformal-Newtonian gauge variables.

The evolution equations for the conformal-Newtonian gauge variables can be obtained by solving Einstein's equation. With the metric perturbations up to the first order the Einstein tensor can be expressed as $G^{\mu}_{\nu} = G^{\mu(0)}_{\nu} + \delta G^{\mu}_{\nu}$. So Einstein's equation for the perturbations can be written as

$$\delta G^{\mu}_{\nu} = 8\pi G \delta T^{\mu}_{\nu}. \tag{2.54}$$

For the metric (2.53) the perturbed Einstein tensor can be give as [21, 26]

$$\delta G^0_0 = \frac{1}{a^2}\left(6\mathcal{H}^2\Phi + 6\mathcal{H}\Psi' - 2\partial_i\partial^i\Psi\right), \tag{2.55}$$

$$\delta G^0_i = \frac{1}{a^2}\left(-2\,\mathcal{H}\partial_i\Phi - 2\,\partial_i\Psi'\right), \tag{2.56}$$

$$\begin{aligned}\delta G^i{}_j &= \frac{1}{a^2}\Bigg[\left(2\mathcal{H}\Phi' + 4\frac{a''}{a}\Phi - 2\mathcal{H}^2\Phi + \partial_i\partial^i\Phi + 4\mathcal{H}\Psi' + 2\,\Psi'' - \partial_i\partial^i\Psi\right)\delta^i{}_j \\ &- \partial^i\partial_j\Phi + \partial^i\partial_j\Psi\Bigg].\end{aligned} \tag{2.57}$$

From (2.54) we obtain

$$3\mathcal{H}\left(\Psi' + \mathcal{H}\Phi\right) - \nabla^2\Psi = 4\pi G a^2 \delta T^0_0, \tag{2.58}$$

$$-\partial_i\left(\Psi' + \mathcal{H}\Phi\right) = 4\pi G a^2 \delta T^0_i, \tag{2.59}$$

$$\begin{aligned}&\left[\Psi'' + \mathcal{H}(2\Psi+\Phi)' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi + \frac{1}{2}\nabla^2(\Phi-\Psi)\right]\delta^i{}_j \\ &-\frac{1}{2}\partial^i\partial_j(\Phi-\Psi) = 4\pi G a^2 \delta T^i_j.\end{aligned} \tag{2.60}$$

The first two equations are called as Hamiltonian and momentum constraints. If there is no anisotropic stress, δT^i_j will not have any off-diagonal element, so

$$\partial^i\partial_j\,(\Phi-\Psi)=0\Rightarrow\Phi=\Psi. \tag{2.61}$$

Now we can use equations (2.33), (2.35) and (2.36) to get the equations for Ψ in presence of inflaton fluctuations. We also use the fact that

$$\dot{H}=-4\pi G\dot{\phi}^2\Rightarrow\mathcal{H}'-\mathcal{H}^2=-4\pi G\phi'^2. \tag{2.62}$$

Hence we obtain

$$\begin{aligned}
\nabla^2\Psi-3\mathcal{H}\Psi'-\left(\mathcal{H}'+2\mathcal{H}^2\right)\Psi &= 4\pi G\left(\delta\phi\,\phi'+\delta\phi\,\frac{\partial V}{\partial\phi}a^2\right), && (2.63)\\
\Psi'+\mathcal{H}\Psi &= 4\pi G\left(\delta\phi\,\phi'\right), && (2.64)\\
\Psi''+3\mathcal{H}\Psi'+\left(\mathcal{H}'+2\mathcal{H}^2\right)\Psi &= 4\pi G\left(\delta\phi\,\phi'-\delta\phi\,\frac{\partial V}{\partial\phi}a^2\right). && (2.65)
\end{aligned}$$

To get the final equation for Ψ we subtract Eq. (2.63) from Eq. (2.65), use Eq. (2.64) to eliminate $\delta\phi$ and use the Klein-Gordan equation (unperturbed). The final equation in Fourier space becomes

$$\Psi''_{\mathbf{k}}+2\left(\mathcal{H}-\frac{\phi''}{\phi'}\right)\Psi'_{\mathbf{k}}+2\left(\mathcal{H}'-\mathcal{H}\frac{\phi''}{\phi'}\right)\Psi_{\mathbf{k}}+k^2\Psi_{\mathbf{k}}=0. \tag{2.66}$$

In terms of the slow-roll parameters ε and η it becomes

$$\Psi''_{\mathbf{k}}+2\mathcal{H}\,(\eta-\varepsilon)\,\Psi'_{\mathbf{k}}+2\mathcal{H}^2\,(\eta-2\varepsilon)\,\Psi_{\mathbf{k}}+k^2\Psi_{\mathbf{k}}=0. \tag{2.67}$$

This equation implies that Ψ remains nearly constant on super-horizon scales.

2.3.2 Comoving curvature perturbation

Another gauge invariant quantity that is useful for inflationary perturbations is comoving curvature perturbation. It remains constant outside the horizon so it is most widely used in inflation. The intrinsic spatial curvature of constant time hypersurfaces is given as

$$^{(3)}R = \frac{4}{a^2}\nabla^2\Psi. \tag{2.68}$$

Ψ that appears in the spatial part of metric perturbations is called as curvature perturbation. From Eq. (2.46) we know that under the transformation $\tau \to \tau + \xi^0$ on constant time hypersurfaces it transforms as

$$\Psi \to \Psi + \mathcal{H}\xi^0. \tag{2.69}$$

Now we define a hypersurface on which $\delta\phi = 0$. From the transformation rules for $\delta\phi$ (2.49) we see that

$$\delta\phi \to \delta\phi - \phi'\xi^0 = 0 \Longrightarrow \xi^0 = \frac{\delta\phi}{\phi'}.$$

This hypersurface is called as comoving hypersurface and the curvature perturbation in this hypersurface is called as comoving curvature perturbation and is denoted by $\mathcal{R}$ and is given as

$$\begin{aligned}\mathcal{R} &= \Psi_{com} = \Psi + \mathcal{H}\xi^0 = \Psi + \mathcal{H}\frac{\delta\phi}{\phi'} \\ &= \Psi + H\frac{\delta\phi}{\dot{\phi}}. \end{aligned} \tag{2.70}$$

2.3.3 Power spectrum

All the perturbations in Fourier space can be written as

$$f(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} f_{\mathbf{k}}(t). \tag{2.71}$$

The two point correlation of the perturbations in Fourier space is called as power spectrum $P_f(k)$ and is defined as

$$\langle 0|f^*_{\mathbf{k}} f_{\mathbf{k}'}|0\rangle = \delta^{(3)}\left(\mathbf{k} - \mathbf{k}'\right)\frac{2\pi^2}{k^3}P_f(k). \tag{2.72}$$

In conformal Newtonian gauge the relation between the curvature perturbation Ψ and inflaton perturbation $\delta\phi$ on super-horizon scale can be obtained by using Eq. (2.64),

$$\Psi_{\mathbf{k}} \simeq \varepsilon H \frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}}. \tag{2.73}$$

So the gauge-invariant comoving curvature perturbation $\mathcal{R}_{\mathbf{k}}$ will be (using Eq. 2.70)

$$\mathcal{R}_{\mathbf{k}} = \Psi_{\mathbf{k}} + H\frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}} = (1+\varepsilon)H\frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}} \simeq H\frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}}, \tag{2.74}$$

and its power spectrum will be

$$P_{\mathcal{R}} = \frac{k^3}{2\pi^2}\frac{H^2}{\dot{\phi}^2}|\delta\phi_{\mathbf{k}}|^2 = \frac{2k^3}{M_P^2 \varepsilon\pi}|\delta\phi_{\mathbf{k}}|^2. \tag{2.75}$$

Here $|\delta\phi_{\mathbf{k}}|^2 = \langle 0|\delta\phi_{\mathbf{k}}^{\dagger}\delta\phi_{\mathbf{k}}|0\rangle$. To find the expectation value of $\delta\Phi(\mathbf{x},t)$ we can expand it in terms of creation and annihilation operator

$$\delta\phi(\mathbf{x},\tau) = \int \frac{d^3k}{(2\pi)^{3/2}}[a_{\mathbf{k}}\,\delta\phi_k(\tau) + a^{\dagger}_{-\mathbf{k}}\,\delta\phi_k^*(\tau)]\,e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{2.76}$$

The equation of motion for $\delta\phi_k$ can be found from the perturbed Klein-Gordon equation (2.37) in conformal Newtonian gauge

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} + V''\delta\phi_{\mathbf{k}} = -2\Psi_{\mathbf{k}}V' + 4\dot{\Psi}_{\mathbf{k}}\dot{\phi}.$$

The second term on the right hand side can be neglected since $\left|4\dot{\Psi}_{\mathbf{k}}\dot{\phi}\right| \ll |\Psi_{\mathbf{k}}V'|$ on super-horizon scales. Using Eq. (2.73) and the relation $V' = -3H\dot{\phi}$, the perturbed Klein-Gordon equation on super-horizon scales can be rewritten as

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} + \left(V'' - 6\varepsilon H^2\right)\delta\phi_{\mathbf{k}} = 0. \tag{2.77}$$

Introducing another field variable $\delta\sigma_{\mathbf{k}} = \frac{\delta\phi_{\mathbf{k}}}{a}$ the perturbed Klein-Gordon equation in conformal time becomes

$$\delta\sigma''_{\mathbf{k}} + \{k^2 + \left(V'' - 6\varepsilon H^2\right)a^2 - \frac{a''}{a}\} = 0. \tag{2.78}$$

Using the relation between scale factor and conformal time during inflation i.e $a = -\frac{1}{H\tau(1-\varepsilon)}$ and the definition of slow-roll parameter η Eq. (2.78) becomes

$$\delta\sigma_{\mathbf{k}}'' - \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\delta\sigma_{\mathbf{k}} = 0, \tag{2.79}$$

$$\nu^2 = \frac{9}{4} + 9\varepsilon - 3\eta. \tag{2.80}$$

This equation is Bessel equation and the solution can be written in terms of Hankel's functions.

$$\delta\sigma_{\mathbf{k}} = \sqrt{-\tau}\left[c_1(k)H_\nu^{(1)}(-k\tau) + c_2(k)H_\nu^{(2)}(-k\tau)\right]. \tag{2.81}$$

To find the constants $c_1(k)$ and $c_2(k)$ it is assumed that for very short wavelengths the solution matches with the plane wave $\delta\sigma_{\mathbf{k}} \sim e^{-ik\tau}/\sqrt{2k}$. The assumption that in the limit of small wavelengths compared to the horizon size, the modes should behave like canonical plane waves in Minkowski space is called the Bunch Davies boundary condition. For very short wavelengths we have $k \gg aH$ or $(-k\tau \gg 1)$. The asymptotic limit of the Hankel's functions is given as

$$H_\nu^{(1)}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}}e^{i\left(x-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)}, \; H_\nu^{(2)}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}}e^{-i\left(x-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)}, \tag{2.82}$$

Using this we get $c_2(k) = 0$ and $c_1(k) = \frac{\sqrt{\pi}}{2}e^{i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}}$. So Eq. (2.81) becomes

$$\delta\sigma_{\mathbf{k}} = \frac{\sqrt{\pi}}{2}e^{i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}}\sqrt{-\tau}H_\nu^{(1)}(-k\tau). \tag{2.83}$$

Since we are interested in super-horizon scales $-k\tau \ll 1$, we use another asymptotic limit of Hankel's functions i,e $H_\nu^{(1)}(x \ll 1) \sim \sqrt{2/\pi}e^{-i\frac{\pi}{2}}2^{\nu-\frac{3}{2}}\left(\Gamma(\nu)/\Gamma(3/2)\right)x^{-\nu}$ to get the behavior of $\sigma_{\mathbf{k}}$ on super-horizon scales.

$$\delta\sigma_{\mathbf{k}} = e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}}2^{\left(\nu-\frac{3}{2}\right)}\frac{\Gamma(\nu)}{\Gamma(3/2)}\frac{1}{\sqrt{2k}}(-k\tau)^{\frac{1}{2}-\nu}. \tag{2.84}$$

Since ε and η are very small, we can put $\nu \sim \frac{3}{2}$ in the factors, but not in the exponents and we get

$$|\delta\phi_{\mathbf{k}}| \simeq \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu}. \tag{2.85}$$

Hence the power spectrum from Eq. (2.75) becomes

$$P_{\mathcal{R}}(k) = \frac{4\pi}{M_P^2 \varepsilon} \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{n_s - 1} \equiv A_{\mathcal{R}}^2 \left(\frac{k}{aH}\right)^{n_s - 1}, \tag{2.86}$$

where we have defined the *spectral index* n_s of the comoving curvature perturbation as

$$n_s - 1 = \frac{d \ln P_{\mathcal{R}}}{d \ln k} = 3 - 2\nu = 2\tau - 6\varepsilon. \tag{2.87}$$

The above power spectrum can be written as

$$P_{\mathcal{R}}(k) = A_{\mathcal{R}}^2(k_0) \left(\frac{k}{k_0}\right)^{n_s - 1}, \tag{2.88}$$

where $k_0 = a_0 H_0$ is called as pivot point and $A_{\mathcal{R}}^2(k_0)$ is the amplitude of the perturbations corresponding to wave number k_0.

2.3.4 Tensor perturbations

Another important prediction of inflation is generation of tensor perturbations. The tensor perturbations are described by the transverse traceless part of the perturbed metric (2.25) h_{ij}. It can be written as

$$h_{ij} = h^+ \mathbf{e}_{ij}^+ + h^\times \mathbf{e}_{ij}^\times, \tag{2.89}$$

where $\mathbf{e}_{ij}^+ = \hat{\mathbf{e}}_x \otimes \hat{\mathbf{e}}_x - \hat{\mathbf{e}}_y \otimes \hat{\mathbf{e}}_y, \mathbf{e}_{ij}^\times = \hat{\mathbf{e}}_x \otimes \hat{\mathbf{e}}_y + \hat{\mathbf{e}}_y \otimes \hat{\mathbf{e}}_x$ are polarization tensors and $h^{+,\times}$ represents the plus and cross polarizations of gravitational waves respectively. The latter are just like scalar fields and obey Einstein equation,

$$h^{i\prime\prime} + 2\frac{a'}{a} h^{i\prime} + k^2 h^i = 0, \; i = +, \times. \tag{2.90}$$

The above equation is similar to the wave equation for a massless free particle. The power spectrum of tensor perturbations is defined as

$$\frac{k^3}{2\pi^2} \sum_i \langle h^{i\star}(\mathbf{k_1}) h^i(\mathbf{k_2}) \rangle = P_T \delta^{(3)} (\mathbf{k_1} - \mathbf{k_2}) \tag{2.91}$$

We will discuss tensor perturbations in detail in the next chapter. The power spectrum for the tensor perturbations comes out to be

$$P_T = \frac{64\pi}{M_P^2}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{k_0}\right)^{n_T} = A_T^2(k_0)\left(\frac{k}{k_0}\right)^{n_s-1}, \tag{2.92}$$

The amplitude of the tensor perturbations is determined in terms of scalar to tensor ratio r which is defined as

$$r = \frac{P_T}{P_{\mathcal{R}}} = 16\varepsilon. \tag{2.93}$$

2.4 CMB anisotropy and polarization

A major observational support for inflation is CMB anisotropy. The perturbations generated during inflation, that move outside the horizon, re-enter during radiation or matter dominated era and cause CMB anisotropy. In this section we will briefly review the theory of CMB anisotropy and polarization. There are a large number of reviews and books on CMB [26, 27, 28, 29, 30]. CMB radiation can be described in terms of Four Stokes parameters I, Q, U and V. A monochromatic and linearly polarized plane wave propagating in direction $\hat{n}$ can be expressed as

$$\begin{aligned} E(\mathbf{x},t) &= (\hat{\mathbf{e}}_1 E_1 + \hat{\mathbf{e}}_2 E_2)e^{(i\mathbf{k}\cdot\vec{x} - i\omega t)}, && (2.94)\\ E_1 &= a_1 e^{i\delta_1},\ E_2 = a_2 e^{i\delta_2}, && (2.95)\end{aligned}$$

where $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are unit vectors perpendicular to $\hat{n}$. The Stokes parameters are defined as

$$\begin{aligned} I &\equiv \langle a_1^2\rangle + \langle a_2^2\rangle,\\ Q &\equiv \langle a_1^2\rangle - \langle a_2^2\rangle,\\ U &\equiv \langle 2a_1a_2\cos(\delta_2-\delta_1)\rangle,\\ V &\equiv \langle 2a_1a_2\sin(\delta_2-\delta_1)\rangle.\end{aligned}$$

We will not consider the parameter V as it is not generated by Thomson scattering and is absent in standard models of CMB polarization. Consider a right handed rotation by an angle

α in a plane perpendicular to $\hat{n}$, I remains invariant but Q and U changes as

$$Q' = Q\cos 2\alpha + U\sin 2\alpha, \tag{2.96}$$
$$U' = -Q\sin 2\alpha + U\cos 2\alpha. \tag{2.97}$$

We can see that the polarization degree $P = \sqrt{Q^2+U^2}$ is invariant under rotation. Radiation field of CMB can also be characterized in terms of polarization tensor I_{ij}. The Stokes parameters Q and U are defined as $Q = \frac{I_{11}-I_{22}}{2}$ and $U = \frac{I_{12}}{4}$. The temperature anisotropy is given as $T = \frac{I_{11}+I_{22}}{2}$. The polarization tensor is normalized such that it represents the fractional intensity. One can construct two spin weighted quantities for the two Stokes parameters Q and U which have a definite value of spin i.e [31]

$$(Q \pm iU)'(\hat{n}) = e^{\mp 2i\alpha}(Q \pm U)(\hat{n}). \tag{2.98}$$

We can expand them in terms of spin s spherical harmonics as [31]

$$T(\hat{n}) = \sum_{lm} a_{T,lm} Y_{lm}(\hat{n}), \tag{2.99}$$
$$(Q+iU)(\hat{n}) = \sum_{lm} a_{2,lm\,2}Y_{lm}(\hat{n}), \tag{2.100}$$
$$(Q-iU)(\hat{n}) = \sum_{lm} a_{-2,lm\,-2}Y_{lm}(\hat{n}). \tag{2.101}$$

To calculate the power spectra for all these quantities we go to a frame where the wave vector $\mathbf{k}$ is parallel to $\hat{z}$ direction and then sum over all $\mathbf{k}$. This process is complicated as Q and U are not rotationally invariant. One can define two spin raising and lowering operators $\eth$ and $\bar{\eth}$ as [31]

$$\eth\, {}_sY_{lm}(\hat{n}) = [(l-s)(l+s+1)]^{\frac{1}{2}}\, {}_{s+1}Y_{lm}(\hat{n}),$$
$$\bar{\eth}\, {}_sY_{lm}(\hat{n}) = -[(l+s)(l-s+1)]^{\frac{1}{2}}\, {}_{s-1}Y_{lm}(\hat{n}). \tag{2.102}$$

Using equations (2.100) and (2.101) we get

$$\bar{\eth}^{\,2}(Q+iU)(\hat{n}) = \sum_{lm}\left[\frac{(l+2)!}{(l-2)!}\right]^{1/2} a_{2,lm}Y_{lm}(\hat{n}), \tag{2.103}$$

$$\eth^{\,2}(Q-iU)(\hat{n}) = \sum_{lm}\left[\frac{(l+2)!}{(l-2)!}\right]^{1/2} a_{-2,lm}Y_{lm}(\hat{n}). \tag{2.104}$$

$$\tag{2.105}$$

One can introduce linear combinations of $a_{2,lm}$ and $a_{-2,lm}$ as [32, 31]

$$a_{E,lm} = -\frac{(a_{2,lm}+a_{-2,lm})}{2}, \tag{2.106}$$

$$a_{B,lm} = i\frac{(a_{2,lm}-a_{-2,lm})}{2}. \tag{2.107}$$

Now one can construct two rotationally invariant quantities as

$$\begin{aligned} E(\hat{n}) &= -\frac{1}{2}[\bar{\eth}^{\,2}(Q+iU)+\eth^{\,2}(Q-iU)] \\ &= -\frac{1}{2}\sum_{lm}\left[\frac{(l+2)!}{(l-2)!}\right]^{1/2}[a_{2,lm}+a_{-2,lm}]Y_{lm}(\hat{n}) \\ &= \sum_{lm}\left[\frac{(l+2)!}{(l-2)!}\right]^{1/2} a_{E,lm}Y_{lm}(\hat{n}), \end{aligned} \tag{2.108}$$

$$\begin{aligned} B(\hat{n}) &= \frac{i}{2}[\bar{\eth}^{\,2}(Q+iU)-\eth^{\,2}(Q-iU)] \\ &= \frac{1}{2}\sum_{lm}\left[\frac{(l+2)!}{(l-2)!}\right]^{1/2}[a_{2,lm}-a_{-2,lm}]Y_{lm}(\hat{n}) \\ &= \sum_{lm}\left[\frac{(l+2)!}{(l-2)!}\right]^{1/2} a_{B,lm}Y_{lm}(\hat{n}). \end{aligned} \tag{2.109}$$

The expression for the expansion coefficients $a_{T,lm}$, $a_{E,lm}$ and $a_{B,lm}$ can be found by using equation (2.99), (2.108), (2.109) and using the orthogonality conditions of spherical harmonics.

$$a_{T,lm} = \int d\Omega Y^{\star}_{lm}(\hat{n})T(\hat{n}), \tag{2.110}$$

$$a_{E,lm} = \left[\frac{(l+2)!}{(l-2)!}\right]^{-1/2}\int d\Omega Y^{\star}_{lm}(\hat{n})E(\hat{n}), \tag{2.111}$$

$$a_{B,lm} = \left[\frac{(l+2)!}{(l-2)!}\right]^{-1/2}\int d\Omega Y^{\star}_{lm}(\hat{n})B(\hat{n}). \tag{2.112}$$

E and B behave differently under parity transformation. E remains unchanged while B changes sign [32]. The statistics of CMB is described in terms of four angular power spectra that represent the two point correlation between T, E, B and TE. We do not have the correlations between TB and EB as they are generated by parity violating interactions. These two point correlation functions are represented in terms of C_ls given as

$$C_l^{TT} = \frac{1}{2l+1}\sum_{m=-l}^{m=l}\langle a^*_{T,lm}a_{T,lm}\rangle, \tag{2.113}$$

$$C_l^{EE} = \frac{1}{2l+1}\sum_{m=-l}^{m=L}\langle a^*_{E,lm}a_{E,lm}\rangle, \tag{2.114}$$

$$C_l^{BB} = \frac{1}{2l+1}\sum_{m=-l}^{m=l}\langle a^*_{B,lm}a_{B,lm}\rangle, \tag{2.115}$$

$$C_l^{TE} = \frac{1}{2l+1}\sum_{m=-l}^{m=l}\langle a^*_{T,lm}a_{E,lm}\rangle, \tag{2.116}$$

where $\langle a^*_{X,l'm'}, a_{X,lm}\rangle = C_l^{XX}\delta_{l'l}\delta_{m'm}$.

2.4.1 Boltzmann equation

CMB angular power spectra are calculated using Boltzmann equation for photon distribution function f. In case of homogeneous and isotropic universe this function is Bose-Einstein distribution function depending only on the energy of photons. But because of metric perturbations it becomes function of the coordinate x^i, conjugate momenta P^i and conformal time τ. We will describe how one calculates CMB angular power spectra generated due to scalar and tensor perturbations.The Boltzmann equation is given as

$$\frac{df}{d\tau} = C_{collision}. \tag{2.117}$$

The derivative on left hand side is Euler derivative and the right hand side represents the collision term due to Thomson scattering. First we will calculate the left hand side of the equation due to scalar perturbations. The conjugate momenta are defined as $P^\mu = \frac{dx^\mu}{d\lambda}$ where λ is affine parameter. These are related to the proper momenta $p_i = p^i$ measured by an observer at a fixed spatial coordinate by

$$P^0 = \frac{p}{a}(1-\Phi) = \frac{q}{a^2}(1-\Phi), \tag{2.118}$$

$$P_i = a(1-\Psi)\,p_i = (1-\Psi)\,q_i, \tag{2.119}$$

where q_i is comoving three-momenta $q_i = ap_i$. We can write this in terms of its magnitude and direction as $q_i = qn_i$ where $n_i n^i = \delta_{ij} n^i n^j = 1$. So the phase space distribution function for photons can be written as $f\left(x^i, P_j, \tau\right) \rightarrow f\left(x^i, q, n_j, \tau\right)$. We can write photon distribution function as an unperturbed Bose-Einstein distribution $f_0(q)$ and first order perturbations in it i.e $f^{(1)}\left(x^i, q, n_j, \tau\right)$,

$$f\left(x^i, q, n_j, \tau\right) = f_0(q)\left(1 + f^{(1)}\left(x^i, q, n_j, \tau\right)\right). \tag{2.120}$$

So from Eq. (2.117) we get

$$f_0(q)\frac{\partial f^{(1)}}{\partial \tau} + f_0(q)\frac{\partial f^{(1)}}{\partial x^i}\frac{dx^i}{d\tau} + \frac{\partial f_0(q)}{\partial q}\frac{dq}{d\tau} + f_0(q)\frac{\partial f^{(1)}}{\partial n_i}\frac{dn_i}{d\tau} = C_{collision}. \tag{2.121}$$

Here the last term is second order perturbation so it can be neglected, $\frac{dx^i}{d\tau}$ can be written as

$$\frac{dx^i}{d\tau} = \frac{dx^i}{d\lambda}\frac{d\lambda}{d\tau} = \frac{P^i}{P^0} = n^i. \tag{2.122}$$

To calculate $\frac{dq}{d\tau}$ we use geodesic equation for photons i.e

$$\frac{dP^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}P^\alpha P^\beta = 0. \tag{2.123}$$

Using $P^0 = \frac{d\tau}{d\lambda}$ and taking the ith component of momenta above equation becomes

$$P^0\frac{dP^i}{d\tau} + \Gamma^i_{\alpha\beta}P^\alpha P^\beta = 0. \tag{2.124}$$

Here the connections $\Gamma^i_{\alpha\beta}$ include unperturbed part and first order perturbations. Using perturbed affine connections for the given metric perturbations (see [21]) we get

$$\frac{dP^i}{d\tau} = -\partial^i\Phi P^0 - 2\mathcal{H}P^i + 2\Psi' P^i - \frac{P^j P^k}{P^0}\left(\partial^i\Psi\delta_{jk} + \partial_j\Psi\delta^i_k - \partial_k\Psi\delta^i_j\right). \tag{2.125}$$

Using $q = q^i n_i$ and the relation between conjugate momenta P^i (see Eq. (2.119)) and comoving momenta q^i, $\frac{dq}{d\tau}$ can be written as

$$\frac{dq}{d\tau} = \left(a^2(1-\Psi)\frac{\partial P^i}{\partial\tau} + 2\mathcal{H}a^2(1-\Psi)P^i - a^2\Psi' P^i\right)n_i - a^2 P^i \partial_j \Psi^j n_i. \qquad (2.126)$$

Now we can use equations (2.125), (2.126) and (2.119) and take only first order terms to get

$$\frac{dq}{d\tau} = q\Psi' - qn_i\partial_i\Phi. \qquad (2.127)$$

So the Boltzmann equation becomes

$$\frac{\partial f^{(1)}}{\partial\tau} + \frac{\partial f^{(1)}}{\partial x^i}n^i + \frac{\partial \ln f_0}{\partial \ln q}\left(\Psi' - n_i\partial_i\Phi\right) = \frac{1}{f_0}C_{collision}. \qquad (2.128)$$

This equation can be written in Fourier space as

$$\frac{\partial f^{(1)}}{\partial\tau} + ik\mu f^{(1)} + \frac{\partial \ln f_0}{\partial \ln q}\left(\Psi' - ik\mu\Phi\right) = \frac{1}{f_0}C_{collision}. \qquad (2.129)$$

Here $\mu = \hat{k}\cdot\hat{n}$ and k is the wave number of perturbations. Before recombination, photons are tightly coupled to baryon, interacting via Thomson scattering. Due to Thomson scattering photons are polarized in a plane perpendicular to $\hat{n}$. To find the Boltzmann equation for temperature anisotropy and polarization we define reduced phase space density function corresponding to the sum of intensities in two directions $\mathbf{e}_\theta, \mathbf{e}_\phi$ in a plane perpendicular to $\hat{n}$ as

$$F_\gamma = \frac{\int q^2 dq\, q\, f_0(q)\, f^{(1)}(q)}{\int q^2 dq\, q\, f_0(q)}. \qquad (2.130)$$

Similarly one can also define another function G_γ corresponding to difference in the intensities in the two directions $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$. The Boltzmann equations for these two quantities can be obtained from Eq. (2.129) and are given as

$$\frac{\partial F_\gamma}{\partial\tau} + ik\mu F_\gamma - 4\left(\Psi' - ik\mu\Phi\right) = C^I_{collision}, \qquad (2.131)$$

$$\frac{\partial G_\gamma}{\partial\tau} + ik\mu G_\gamma = C^Q_{collision}. \qquad (2.132)$$

The collision terms on right hand side can be found in [33, 34, 35]. We again define temperature anisotropy $\Delta_T = \frac{\delta T}{T}$ and polarization functions Δ_Q and Δ_U as

$$f\left(x^i, q, n_j, \tau\right) = f_0\left(\frac{q}{1+\Delta_T}\right) = f_0(q) + \frac{\partial f_0}{\partial q}\{q(1-\Delta_T) - q\}. \tag{2.133}$$

Δ_T can be written in terms of $f^{(1)}$ using Eq. (2.120) and Eq. (2.133) as

$$f_0(q)\left(1 + f^{(1)}\left(x^i, q, n_j, \tau\right)\right) = f_0(q)\left(1 - \Delta_T \frac{\partial \ln f_0}{\partial \ln q}\right). \tag{2.134}$$

So

$$\Delta_T = -f^{(1)}\left(\frac{\partial \ln f_0}{\partial \ln q}\right)^{-1}. \tag{2.135}$$

Now using the definition of F_γ we get $F_\gamma = 4\Delta_T$. Hence the Boltzmann equations for Δs will be (using (2.131), (2.132))

$$\begin{aligned}
\Delta_T^{(S)\prime} + ik\mu\Delta_T^{(S)} &= \Psi' - ik\mu\Phi \\
&+ \kappa'\{-\Delta_T^{(S)} + \Delta_{T0}^{(S)} + i\mu v_b + \frac{1}{2}P_2(\mu)\Pi\}, && (2.136)\\
\Delta_Q^{(S)\prime} + ik\mu\Delta_Q^{(S)} &= \kappa'\{-\Delta_Q^{(S)} + \frac{1}{2}(1 - P_2(\mu))\Pi\}, && (2.137)\\
\Delta_U^{(S)\prime} + ik\mu\Delta_U^{(S)} &= -\kappa'\Delta_U^{(S)}, && (2.138)
\end{aligned}$$

where S in the superscript represents scalar perturbations, v_b is baryon velocity and $\Pi = \Delta_{T2}^{(S)} + \Delta_{Q0}^{(S)} + \Delta_{Q2}^{(S)}$. κ' denotes the differential optical depth for Thomson scattering and is given by $\kappa' = a n_e x_e \sigma_T$, where $a(\tau)$ is scale factor, n_e is electron number density, x_e is ionization fraction and σ_T is Thomson scattering cross section. One can also define visibility function $g(\tau) = \kappa' e^{(-\kappa)}$. It gives the probability that a photon we observe last scattered at time τ. This function is strongly peaked around recombination. We can add equations (2.137) and (2.138) to find the the Boltzmann equation for the degree of linear polarization P as

$$\Delta_P^{(S)\prime} + ik\mu\Delta_P^{(S)} = \kappa'\{-\Delta_P^{(S)} + \frac{1}{2}(1 - P_2(\mu))\Pi\}. \tag{2.139}$$

Now we will find Boltzmann equation for Δs due to tensor perturbations.The tensor perturbations are described in detail in the last section. Here we work with a different linear combination of $+$ and $\times$ given as following,

$$h^1 = \frac{h^+ - ih^\times}{\sqrt{2}}, \quad (2.140)$$

$$h^2 = \frac{h^+ + ih^\times}{\sqrt{2}}. \quad (2.141)$$

In case of tensor perturbations

$$\frac{dq}{d\tau} = -\frac{1}{2} n^i n^j \frac{\partial h_{ij}}{\partial \tau}. \quad (2.142)$$

Now if we choose $\hat{z}$ axis in the direction of perturbation momentum $\mathbf{k}$, the polarization tensors defined by Eq. (2.89) will be

$$\mathbf{e}^+_{xx} = -\mathbf{e}^+_{yy} = 1, \ \mathbf{e}^\times_{xy} = \mathbf{e}^\times_{yx} = 1. \quad (2.143)$$

So

$$n^i n^j \mathbf{e}^+_{ij} = \sin^2\theta \cos^2\phi - \sin^2\theta \sin^2\phi = \sin^2\theta \cos\phi, \quad (2.144)$$

$$n^i n^j \mathbf{e}^\times_{ij} = 2\sin^2\theta \cos\phi \sin\phi = \sin^2\theta \sin 2\phi. \quad (2.145)$$

Now to describe the Boltzmann equations for two gravitational wave polarizations new variables can be introduced ($\mu = \cos\theta$)[36, 37, 31]

$$\begin{aligned}
\Delta_T^{(T)}(\tau, \hat{n}, \mathbf{k}) &= \left[(1-\mu^2)e^{2i\phi}h^1(\mathbf{k}) + (1-\mu^2)e^{-2i\phi}h^2(\mathbf{k})\right] \tilde{\Delta}_T^{(T)}(\tau, \mu, k), \\
(\Delta_Q^{(T)} + i\Delta_U^{(T)})(\tau, \hat{\hat{n}}, \mathbf{k}) &= \left[(1-\mu)^2 e^{2i\phi}h^1(\mathbf{k}) + (1+\mu)^2 e^{-2i\phi}h^2(\mathbf{k})\right] \tilde{\Delta}_P^{(T)}(\tau, \mu, k), \\
(\Delta_Q^{(T)} - i\Delta_U^{(T)})(\tau, \hat{n}, \mathbf{k}) &= \left[(1+\mu)^2 e^{2i\phi}h^1(\mathbf{k}) + (1-\mu)^2 e^{-2i\phi}h^2(\mathbf{k})\right] \tilde{\Delta}_P^{(T)}(\tau, \mu, k),
\end{aligned} \quad (2.146)$$

where T in the superscript represents tensor perturbations. The Boltzmann equations for these functions can be written as [36, 31]

$$\tilde{\Delta}_T^{(T)\prime} + ik\mu\tilde{\Delta}_T^{(T)} = -h' - \kappa'\left(\tilde{\Delta}_T^{(T)} - \chi\right), \quad (2.147)$$

$$\tilde{\Delta}_P^{(T)\prime} + ik\mu\tilde{\Delta}_P^{(T)} = -\kappa'\left(\tilde{\Delta}_P^{(T)} + \chi\right), \quad (2.148)$$

$$\chi = (\frac{1}{10}\tilde{\Delta}_{T0}^{(T)} + \frac{1}{35}\tilde{\Delta}_{T2}^{(T)} + \frac{1}{210}\tilde{\Delta}_{T4}^{(T)} - \frac{3}{5}\tilde{\Delta}_{P0}^{(T)} + \frac{6}{35}\tilde{\Delta}_{P2}^{(T)} - \frac{1}{210}\tilde{\Delta}_{P4}^{(T)}). \quad (2.149)$$

2.4.2 Calculation of angular power spectra

We can expand $\Delta(\mathbf{k},\hat{n},\tau)$ in terms of Legendre polynomials as

$$\Delta(\mathbf{k},\hat{n},\tau) = \sum_{l=0}^{\infty}(-i)^l(2l+1)\Delta_l(\mathbf{k},\tau)P_l(\mu). \quad (2.150)$$

The quantities $\Delta(\mathbf{k},\hat{n},\tau)$ are random variables and their values depend upon initial perturbations generated during inflation. These can be written as $\Delta(\mathbf{k},\hat{n},\tau) = \Psi(\mathbf{k})\Delta(k,\mu,\tau)$. Let us first calculate the angular power spectra for temperature anisotropy and polarization from scalar perturbations. The present value of temperature anisotropy, $T(\hat{n}) = \Delta_T(\tau_0,\mathbf{x},\mu)$ at $\mathbf{x}=0, \tau=\tau_0$ can be expressed in Fourier modes as

$$T^{(S)}(\hat{n}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\mathbf{k}\Psi(\mathbf{k})\Delta_T^{(S)}(\tau=\tau_0,k,\mu), \quad (2.151)$$

$$(Q^{(S)}+iU^{(S)})(\hat{n}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\mathbf{k}\Psi(\mathbf{k})e^{-2i\delta_{k,n}}\Delta_P^{(S)}(\tau=\tau_0,k,\mu), \quad (2.152)$$

$$(Q^{(S)}-iU^{(S)})(\hat{n}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\mathbf{k}\Psi(\mathbf{k})e^{2i\delta_{k,n}}\Delta_P^{(S)}(\tau=\tau_0,k,\mu). \quad (2.153)$$

Here $\delta_{k,n}$ is the angle needed to rotate the $\mathbf{k}$ and $\hat{n}$ dependent basis to a fixed frame in the sky. The power spectrum for the longitudinal gauge variables $\Psi(\mathbf{k})$ is given as

$$\langle\Psi^\star(\mathbf{k})\Psi(\mathbf{k}')\rangle = \frac{2\pi^2}{k^3}P_\Psi(k)\delta^{(3)}\left(\mathbf{k}-\mathbf{k}'\right). \quad (2.154)$$

One can solve the equations (2.136), (2.139) using line of sight method [38] and the solution will be

$$\begin{aligned}\Delta_T^{(S)} &= \int_0^{\tau_0} d\tau e^{ik\mu(\tau-\tau_0)} S_T^{(S)}(k,\tau), &(2.155)\\ \Delta_P^{(S)} &= \frac{3}{4}\left(1-\mu^2\right)\int_0^{\tau_0} d\tau e^{ik\mu(\tau-\tau_0)} g(\tau)\Pi, &(2.156)\end{aligned}$$

where

$$S_T^{(S)}(k,\tau) = g\left(\Delta_{T0}+\Phi+\frac{v_b'}{k}+\frac{\Pi}{4}+\frac{3\Pi''}{4k^2}\right)+e^{-\kappa}\left(\Phi'+\Psi'\right)+g'\left(\frac{v_b}{k}+\frac{3\Pi'}{4k^2}\right)+\frac{3g''\Pi}{4k^2}. \tag{2.157}$$

Since scalar perturbations do not contribute to B mode so one can find the expression for Δ_E for Eq. (2.156) as [31]

$$\begin{aligned}\Delta_E^{(S)}(\tau_0,k,\mu) &= -\frac{3}{4}\int_0^{\tau_0} d\tau g(\tau)\Pi(\tau,k)\,\partial_\mu^2\left[(1-\mu^2)^2 e^{ik\mu(\tau-\tau_0)}\right]\\ &= \frac{3}{4}\int_0^{\tau_0} d\tau g(\tau)\Pi(\tau,k)\,(1+\partial_\tau^2)^2\left(k^2(\tau-\tau_0)^2 e^{ik\mu(\tau-\tau_0)}\right).\end{aligned} \tag{2.158}$$

We can expand plane waves $e^{ik\mu(\tau-\tau_0)}$ in terms of spherical Bessel functions as (let $x = k(\tau_0-\tau)$)

$$e^{ik\mu(\tau-\tau_0)} = e^{-ix\mu} = \sum_l (-i)^l (2l+1)\, j_l(x) P_l(\mu). \tag{2.159}$$

Using equations (2.151), (2.110) and (2.155) we get

$$a_{T,lm} = \int d\Omega Y_{lm}^\star(\hat{n})\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\mathbf{k}\Psi(\mathbf{k})\int_0^{\tau_0} e^{-ix\mu} d\tau S_T^{(S)}(k,\tau). \tag{2.160}$$

Putting this value of $a_{T,lm}$ in equation (2.113), using (2.154) and (2.159) we get the angular power spectrum of TT as [31]

$$C_l^{(S),TT} = 4\pi\int\frac{dk}{k}P_\Psi(k)\left|\int_0^{\tau_0} d\tau S_T^{(S)}(k,\tau)\, j_l(x)\right|^2. \tag{2.161}$$

Similarly the power spectrum for EE can be calculated as

$$C_l^{(S),EE} = (4\pi)\frac{(l+2)!}{(l-2)!}\int k^2 dk P_\Psi(k)\left[\frac{3}{4}\int_0^{\tau_0} d\tau g(\tau)\Pi(\tau,k)\frac{j_l(x)}{x^2}\right]^2. \tag{2.162}$$

To get the cross-correlation between T and E we can use the differential equation satisfied by the spherical Bessel functions, $j_l'' + 2j_l'/x + [1 - l(l+1)/x^2]j_l = 0$. Introducing

$$\begin{aligned}
\Delta_{Tl}^{(S)}(k) &= \int_0^{\tau_0} d\tau S_T^{(S)}(k,\tau) j_l(x), \\
\Delta_{El}^{(S)}(k) &= \sqrt{\frac{(l+2)!}{(l-2)!}}\int_0^{\tau_0} d\tau S_E^{(S)}(k,\tau) j_l(x), \\
S_E^{(S)}(k\tau) &= \frac{3g(\tau)\Pi(\tau,k)}{4x^2},
\end{aligned} \tag{2.163}$$

we get

$$\begin{aligned}
C_l^{(S),(T,E)} &= (4\pi)\int \frac{dk}{k} P_\phi(k)\left[\Delta_{T,El}^{(S)}(k)\right]^2, \\
C_l^{(S),(TE)} &= (4\pi)\int \frac{dk}{k} P_\phi(k)\Delta_{Tl}^{(S)}(k)\Delta_{El}^{(S)}(k).
\end{aligned} \tag{2.164}$$

Now we will calculate the angular power spectra due to tensor perturbations. The solution of the equations (2.147) and (2.148) can be obtained by line of sight method as [38]

$$\begin{aligned}
\Delta_T^{(T)}(\tau_0,\hat{n},\mathbf{k}) &= \left[(1-\mu^2)e^{2i\phi}h^1(\mathbf{k}) + (1-\mu^2)e^{-2i\phi}h^2(\mathbf{k})\right]\int_0^{\tau_0} d\tau e^{ix\mu} S_T^{(T)}(k,\tau), \\
(\Delta_Q^{(T)} + i\Delta_U^{(T)})(\tau_0,\hat{n},\mathbf{k}) &= \left[(1-\mu)^2 e^{2i\phi}h^1(\mathbf{k}) + (1+\mu)^2 e^{-2i\phi}h^2(\mathbf{k})\right]\int_0^{\tau_0} d\tau e^{ix\mu} S_P^{(T)}(k,\tau), \\
(\Delta_Q^{(T)} - i\Delta_U^{(T)})(\tau_0,\hat{n},\mathbf{k}) &= \left[(1+\mu)^2 e^{2i\phi}h^1(\mathbf{k}) + (1-\mu)^2 e^{-2i\phi}h^2(\mathbf{k})\right]\int_0^{\tau_0} d\tau e^{ix\mu} S_P^{(T)}(k,\tau),
\end{aligned} \tag{2.165}$$

where

$$\begin{aligned}
S_T^{(T)}(k,\tau) &= -\dot{h}e^{-\kappa} + g\chi, \\
S_P^{(T)}(k,\tau) &= -g\chi.
\end{aligned} \tag{2.166}$$

For tensor modes all three quantities $\Delta_T^{(T)}$, $\Delta_{\tilde{E}}^{(T)}$ and $\Delta_{\tilde{B}}^{(T)}$ are non-vanishing and one can calculate them by acting with the spin raising and lowering operators on the above equations.

$$\begin{aligned}
\Delta_T^{(T)}(\tau_0,\hat{n},\mathbf{k}) &= \left[(1-\mu^2)e^{2i\phi}h^1(\mathbf{k})+(1-\mu^2)e^{-2i\phi}h^2(\mathbf{k})\right]\int_0^{\tau_0} d\tau S_T^{(T)}(\tau,k)\, e^{-ix\mu}, \\
\Delta_E^{(T)}(\tau_0,\hat{n},\mathbf{k}) &= \left[(1-\mu^2)e^{2i\phi}h^1(\mathbf{k})+(1-\mu^2)e^{-2i\phi}h^2(\mathbf{k})\right]\hat{\mathcal{E}}(x)\int_0^{\tau_0} d\tau S_P^{(T)}(\tau,k)\, e^{-ix\mu}, \\
\Delta_B^{(T)}(\tau_0,\hat{n},\mathbf{k}) &= \left[(1-\mu^2)e^{2i\phi}h^1(\mathbf{k})-(1-\mu^2)e^{-2i\phi}h^2(\mathbf{k})\right]\hat{\mathcal{B}}(x)\int_0^{\tau_0} d\tau S_P^{(T)}(\tau,k)\, e^{-ix\mu}.
\end{aligned} \tag{2.167}$$

where $\hat{\mathcal{E}}(x) = -12 + x^2[1-\partial_x^2] - 8x\partial_x$ and $\hat{\mathcal{B}}(x) = 8x + 2x^2\partial_x$. Now angular power spectra for all these modes can be obtained as in the case of scalar perturbations. They will be

$$C_l^{(T),TT} = (4\pi)\frac{(l+2)!}{(l-2)!}\int\frac{dk}{k}P_T(k)\left|\int_0^{\tau_0} d\tau S_T^{(T)}(k,\tau)\frac{j_l(x)}{x^2}\right|^2, \tag{2.168}$$

$$C_l^{(T),EE} = (4\pi)\int\frac{dk}{k}P_T(k)\left(\int_0^{\tau_0} d\tau S_P^{(T)}(k,\tau)\left[-j_l(x)+j_l''(x)+\frac{2j_l(x)}{x^2}+\frac{4j_l'(x)}{x}\right]\right)^2 \tag{2.169}$$

,

$$C_l^{BB} = (4\pi)\int\frac{dk}{k}P_T(k)\left(\int_0^{\tau_0} d\tau S_P^{(T)}(k,\tau)\left[2j_l'(x)+\frac{4j_l}{x}\right]\right)^2. \tag{2.170}$$

Here we have used the definition of tensor power spectrum $P_T(k)$ given in (2.91) One can further simplify these expressions by integrating by parts the derivatives $j_l'(x)$ and $j_l''(x)$. This finally leads to

$$\begin{aligned}
\Delta_{Tl}^{(T)} &= \sqrt{\frac{(l+2)!}{(l-2)!}}\int_0^{\tau_0} d\tau S_T^{(T)}(k,\tau)\frac{j_l(x)}{x^2}, \\
\Delta_{E,Bl}^{(T)} &= \int_0^{\tau_0} d\tau S_{E,B}^{(T)}(k,\tau)\, j_l(x), \\
S_E^{(T)}(k,\tau) &= g\left(\chi - \frac{\ddot{\chi}}{k^2}+\frac{2\chi}{x^2}-\frac{\dot{\chi}}{kx}\right) - \dot{g}\left(\frac{2\dot{\chi}}{k^2}+\frac{4\chi}{kx}\right) - 2\ddot{g}\frac{\chi}{k^2}, \\
S_B^{(T)}(k,\tau) &= g\left(\frac{4\chi}{x}+\frac{2\dot{\chi}}{k}\right) + 2\dot{g}\frac{\chi}{k}.
\end{aligned} \tag{2.171}$$

The power spectra are given by

$$\begin{aligned}
C_l^{(X)} &= (4\pi)\int\frac{dk}{k}P_T(k)\left[\Delta_{Xl}^{(T)}(k)\right]^2, \\
C_l^{(TE)} &= (4\pi)\int\frac{dk}{k}P_T(k)\Delta_{Tl}^{(T)}(k)\Delta_{El}^{(T)}(k),
\end{aligned} \tag{2.172}$$

where X stands for T, E or B.

These C_ls are calculated using public domain codes CAMB [39] or CMBFAST [38]. It is clear from above formula that these C_ls are related to the primordial power spectra of scalar and tensor perturbations. The perturbations that entered the horizon before recombinations are modified due to acoustic oscillations in the electron baryon plasma so they do not retain information of primordial perturbations. The perturbations that entered the horizon after recombination are responsible for large angle CMB anisotropy and their effect on CMB anisotropy is known as Sachs Wolfe effect [11].

2.4.3 Large scale CMB anisotropy

To calculate large scale CMB anisotropy let us write the temperature anisotropy as from Eq. (2.151)

$$T^{(S)}(\hat{n}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3\mathbf{k}\Delta_T^{(S)}(\tau = \tau_0, \mathbf{k}, \mu). \tag{2.173}$$

The integral solution for $\Delta_T^{(S)}(\tau = \tau_0, \mathbf{k}, \mu)$ is given by Eq. (2.155). For large angle anisotropy we can neglect the Doppler term and higher multipoles in the source term (2.157). Using Eq. (2.155) and Eq. (2.157) we get

$$\Delta_T^{(S)}(\tau = \tau_0, \mathbf{k}, \mu) = \int_0^{\tau_0} d\tau e^{ik\mu(\tau-\tau_0)} \left[g\left(\Delta_{T0} + \Phi\right) + e^{-\kappa}\left(\Phi' + \psi'\right)\right]. \tag{2.174}$$

The visibility function is peaked around decoupling time τ_{rec} so we can take sudden decoupling limit in which the visibility function behaves as Dirac delta function around τ_{rec} and its integral can be approximated as step function i.e

$$\begin{aligned} g(\tau) &= \delta(\tau - \tau_{rec}), & (2.175) \\ e^{-\kappa(\tau,\tau_0)} &= \theta(\tau - \tau_{rec}). & (2.176) \end{aligned}$$

Putting these in Eq. (2.174) we get

$$\Delta_T^{(S)}(\tau = \tau_0, \mathbf{k}, \mu) = e^{ik\mu(\tau_{rec}-\tau_0)} \left|(\Delta_{T0} + \Phi)\right|_{\tau_{rec}} + \int_{\tau_{rec}}^{\tau_0} d\tau e^{ik\mu(\tau-\tau_0)}\left(\Phi' + \psi'\right). \tag{2.177}$$

The first term in Eq. (2.177) represents the ordinary SW effect and the second term corresponds to the integrated SW effect. If we take adiabatic perturbations and no anisotropic stress ($\Psi = \Phi$), we get $|(\Delta_{T0} + \Phi)|_{\tau_{rec}} = \frac{\Psi}{3}$ and we can neglect the integrated SW effect. So

$$T^{(S)}(\hat{n}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3\mathbf{k} e^{ik\mu(\tau_{rec}-\tau_0)} \frac{\Psi(\mathbf{k})}{3}. \tag{2.178}$$

We can neglect τ_{rec} in the exponential. Using the definition of C_ls and expanding the plane wave in terms of spherical Bessel functions (see Eq. (2.159)) we get

$$C_l^{TT} = \frac{4\pi}{9} \int \frac{dk}{k} P_\Psi(k) j_l^2(k\tau_0). \tag{2.179}$$

Assuming that the power spectrum has the form $P_\Psi(k) = A^2 k^{n_s-1}$ we get

$$C_l^{TT} = \frac{4\pi A^2}{9} \int dk k^{n_s-2} j_l^2(k\tau_0). \tag{2.180}$$

After integration it gives

$$C_l^{TT} = \frac{4\pi A^2}{9\,\tau_0^{n_s-1}} 2^{n_s-4} \frac{\Gamma\left(l + \frac{n_s}{2} - \frac{1}{2}\right)\Gamma(3-n_s)}{\Gamma\left(l - \frac{n_s}{2} + \frac{5}{2}\right)\Gamma^2\left(2 - \frac{n}{2}\right)}. \tag{2.181}$$

For scale invariant spectrum ($n_s = 1$) the above equation becomes

$$\frac{l(l+1)}{2\pi} C_l^{TT} = \frac{A^2}{9}. \tag{2.182}$$

If we take the power spectrum of comoving curvature perturbations Eq. (2.86) we get ($\Phi_\mathbf{k} = \frac{3}{5}\mathcal{R}_\mathbf{k}$)

$$\frac{l(l+1)}{2\pi} C_l^{TT} = \frac{A_\mathcal{R}^2}{25}. \tag{2.183}$$

From CMB observations we can determine the amplitude and spectral index of primordial perturbations. WMAP 5-year data gives the values of these parameters to be $A_\mathcal{R}^2 = (2.41 \pm 0.11) \times 10^{-9}$ and $n_s = 0.963_{0.015}^{0.014}$. From the amplitude of the power spectrum we get using Eq. (2.86)

$$A_\mathcal{R}^2 = \frac{4\pi}{M_P^2 \varepsilon} \left(\frac{H}{2\pi}\right)^2 \sim 10^{-9}. \tag{2.184}$$

Writing H in terms of potential of the scalar field, we get from above equation that $\left(\frac{V}{\varepsilon}\right)^{\frac{1}{4}} \sim 10^{16}$GeV. This gives an upper bound on the scale of inflation. If we take $m^2\phi^2$ or $\lambda\phi^4$ potentials, we get that for 60 e-foldings $\phi > M_P$, $m \sim 10^{13}$GeV and $\lambda \sim 10^{-12}$. It implies that inflaton should be weakly coupled and there is huge difference between its mass and value. Such values of field and couplings are not natural in particle physics. There are a large number of attempts to construct a model of inflation. Among one of these attempts is natural inflation [13, 14, 40]. We will discuss these models in details in the subsequent chapters.

2.5 Warm inflation

There is another class of models of inflation called warm inflation [12]. In these models inflaton is interacting with other fields during inflation. Radiation is also present sub-dominantly and radiation density is nearly constant. The energy of inflaton is constantly being transferred to the radiation by dissipation. The damping of inflaton is provided both by dissipation and Hubble constant. The evolution equation for the inflaton is given by [41]

$$\ddot{\phi} + (3H + \Gamma)\dot{\phi} + V'(\phi) = 0, \tag{2.185}$$

where $\Gamma(\phi, T)$ is damping term and $V(\phi, T)$ is thermodynamic potential. We can define a parameter $Q = \frac{\Gamma}{3H}$ that describes the strength of the thermal damping compared to the expansion damping. For warm inflation in strong dissipative regime $Q \gg 1$. The equation for radiation energy density is given by

$$\dot{\rho}_r + 4H\rho_r = \Gamma\dot{\phi}^2. \tag{2.186}$$

Here the second term on the left hand side describes the decrease in the radiation density due to Hubble expansion and the term of the right hand side describes the production of radiation due to dissipation. Because of slow-roll the right hand side is nearly constant. So $\dot{\rho}_r \sim 0$. Since the temperature of the universe remains nearly constant no reheating is required. Inflation ends when potential energy of the inflaton field falls bellow the radiation energy density and the universe becomes radiation dominated. Because of the damping term Γ warm inflation has extra slow-roll parameters which are given by

$$\beta = \frac{M_P}{8\pi\left(\frac{\Gamma' V'}{\Gamma V}\right)}, \tag{2.187}$$

$$\delta_T = \frac{TV'^{\phi,T}}{V'}. \tag{2.188}$$

Under the slow-roll approximation we get from Eq. (2.185) that

$$\dot{\phi} = -\frac{V'}{3H(1+Q)}. \tag{2.189}$$

The slow-roll conditions $\dot{\phi}^2 \ll V(\phi)$ and $\ddot{\phi} \ll 3H(1+Q)$ here imply that all of the slow-roll parameters are smaller than $1+Q$. So the slow-roll conditions here are relaxed compared to the supercooled inflation.

The density perturbations in warm inflation models are generated by thermal fluctuations in the inflaton field. These fluctuations are calculated using Langevin equation. A detailed derivation in the strong coupling regime can be found in [41]. The power spectrum for the comoving curvature perturbations in this regime is given by [41]

$$P_{\mathcal{R}} = \left(\frac{\pi}{4}\right)^{\frac{1}{2}} \frac{H^{\frac{5}{2}}\Gamma^{\frac{1}{2}}T}{\dot{\phi}^2}. \tag{2.190}$$

The expression for spectral index can be obtained by using the definition (2.87) and it is given as

$$n_s - 1 = \frac{1}{Q}\left(-\frac{9}{4}\varepsilon + \frac{3}{2}\eta - \frac{9}{4}\beta\right). \tag{2.191}$$

Warm inflation models predict nearly scale invariant spectrum of scalar perturbations with natural values of masses and couplings in the inflaton potential.

As we have seen in this chapter that the large scale CMB anisotropy and polarization depends on the parameters of inflation, we can study imprint of pre-inflation universe by observing large scale CMB anisotropy and polarization.

Chapter 3

Effect of pre-inflation thermal era on CMB polarization

3.1 Introduction

In this chapter we describe the effect of pre-inflationary radiation era on B-mode of polarization and its implications for warm inflation models. Pre-inflation radiation era has been studied in [42, 17, 43, 44]. Bhattacharya et all [42] have studied the effect of pre-inflation radiation era on the scalar perturbations generated during inflation. They have considered that the fluctuations corresponding to our length scales were generated during inflation by stimulated emission in the existing background of the thermal inflatons decoupled before inflation. They assume that inflaton was in thermal equilibrium before inflation and it was decoupled from radiation at the beginning of inflation but it retained its thermal equilibrium. Conventionally the power spectrum is calculated assuming Bunch Davies boundary condition. They have calculated the inflaton power spectrum by assuming the initial state of the inflaton to have thermal distribution. Because of this the curvature power spectrum is modified at large scales and CMB anisotropy at large angles is enhanced. Comparison of this modified CMB anisotropy with WMAP data can give an upper bound on the temperature of inflaton at the time when length scales corresponding to our horizon were leaving the de Sitter horizon.

Similar approach can be used in case of tensor perturbations generated during inflation [17]. Inflationary models predict a nearly scale invariant spectrum of gravitational waves [45, 46] that has not been observed till now. The temperature anisotropy generated from tensor perturbations is very small compared to the scalar once and is only significant at large l. The amplitude of tensor perturbations is determined in terms of tensor to scalar ratio defined in equation (2.93). It depends on the slow-roll parameter ε. Observations of tensor perturbations will help us in determining the scale of inflation. The definitive test of the existence of these cosmological gravitational waves would be the observation of B mode polarization in the CMB as they are generated only by gravitational waves [31, 47]. The recent WMAP three year results [48] give only an upper bound on the B mode polarization, $\frac{(l+1)l}{2\pi}C^{BB}_{l=(2-6)} < 0.05(\mu K)^2$.

Because of a radiation era prior to inflation there would be a thermal background of gravitons at the time of inflation. This thermal distribution of gravitons would have decoupled close to Plank era. As in the case of scalar perturbations, the generation of tensor perturbation during inflation would be by *stimulated emission* into this existing thermal background of gravitational waves. This may change the scale invariant power spectrum of the primordial gravitational waves [49].

3.2 Tensor perturbations during inflation with a prior radiation era

As mentioned earlier, the tensor perturbations have two independent degrees of freedom which can be chosen as h^+ and $h^\times$ polarization modes. To compute the spectrum of gravitational waves $h(\mathbf{x},\tau)$ during inflation, we express $h^{(+)}$ and $h^{(\times)}$ in terms of the creation- annihilation operator as [49]:

$$\begin{aligned} h^{(i)}(\mathbf{x},\tau) &= \frac{\sqrt{16\pi}}{a(\tau)M_P}\int\frac{d^3k}{(2\pi)^{3/2}}[a_k f_k(\tau) \\ &+ a^\dagger_{-k} f^*_k(\tau)]\, e^{i\mathbf{k}\cdot\mathbf{x}} \\ &\equiv \int\frac{d^3k}{(2\pi)^{3/2}} h^{(i)}(\mathbf{k},\tau)\, e^{i\mathbf{k}\cdot\mathbf{x}}, \end{aligned} \tag{3.1}$$

here h^i obeys the Einstein's equation (2.90). Here we define the power spectrum of the tensor perturbations $h(\mathbf{k})$

$$\langle h^\star(\mathbf{k})h(\mathbf{k}')\rangle \equiv \frac{2\pi^2}{k^3}P_h\,\delta^3(\mathbf{k}-\mathbf{k}')\,. \tag{3.2}$$

This is equivalent to the definition (2.91) with $P_T = 4P_h$. The usual quantization condition between the fields and their canonical momenta yields $[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta^3(\mathbf{k}-\mathbf{k}')$ and the vacuum satisfies $a_{\mathbf{k}}|0\rangle = 0$. If the graviton field had zero occupation prior to inflation then the vacuum expectation value of the number operator i.e $\langle a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}}\rangle = 0$ and we would obtain a correlation function $\sim |f_k(\tau)|^2$. However if the graviton field was in thermal equilibrium at some earlier epoch it will retain its thermal distribution even after decoupling from the other radiation fields and its occupation number will be given by [49]:

$$\langle a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}'}\rangle = \left(\frac{1}{e^{\hbar k/K_B T} - 1}\right) \delta^3(\mathbf{k} - \mathbf{k}')\,. \tag{3.3}$$

Using Eq. (3.1) and Eq. (3.3) it can be seen that (putting $\hbar = K_B = 1$):

$$\begin{aligned} \langle h^{\star}(\mathbf{k}) h(\mathbf{k}')\rangle &= \frac{16\pi |f_k(\tau)|^2}{a^2(\tau) M_P^2}\left(1 + \frac{2}{e^{\frac{k}{T}} - 1}\right)\delta^3(\mathbf{k} - \mathbf{k}')\,, \\ &= \frac{16\pi |f_k(\tau)|^2}{a^2(\tau) M_p^2} \coth\left[\frac{k}{2T}\right] \delta^3(\mathbf{k} - \mathbf{k}')\,. \end{aligned} \tag{3.4}$$

From the defining relation, Eq. (3.2), for the tensor power spectrum and Eq. (3.4) we find that the power spectrum for the thermal gravitons can be expressed in terms of the mode functions $f_k(\tau)$ as:

$$P_h(k) = \frac{8k^3}{\pi M_P^2}\frac{|f_k|^2}{a^2(\tau)} \coth\left[\frac{k}{2T}\right]\,. \tag{3.5}$$

The equation for the mode functions $f_k(\tau)$ can be obtained from Eq. (2.90) i.e

$$f_k'' + \left(k^2 - \frac{a''}{a}\right) f_k = 0\,. \tag{3.6}$$

Above equation can be solved for a quasi de Sitter space as done for scalar perturbations in previous chapter (see Eq. (2.78) and (2.79)) In quasi de Sitter space conformal time τ and the scale factor $a(\tau)$ are related by $a(\tau) = -1/H\tau(1-\varepsilon)$

So the equation (3.6) becomes,

$$f_k'' + \left[k^2 - \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\right] f_k = 0\,, \tag{3.7}$$

where $k = |\mathbf{k}|$ and $\nu = \frac{3}{2} + \varepsilon$. Eq. (3.7) has the general solution given by,

$$f_k(\tau) = \sqrt{-\tau}\left[c_1(k) H_\nu^{(1)}(-k\tau) + c_2(k) H_\nu^{(2)}(-k\tau)\right]\,. \tag{3.8}$$

As done for the scalar perturbations in the previous chapter we assume that when the modes are well within the horizon they can be approximated by flat space-time solutions ${f_k}^0(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau}, (k \gg aH)$. Matching the general solution in Eq. (3.8) with the solution in the high frequency (flat space-time) limit gives the value of the constants of integration $c_1(k) = \frac{\sqrt{\pi}}{2} e^{i(\nu+\frac{1}{2})\frac{\pi}{2}}$ and $c_2(k) = 0$. Eq. (3.8) then implies that for $-k\tau \gg 1$ or $k \ll aH$,

$$f_k(\tau) = e^{i(\nu-\frac{1}{2})\frac{\pi}{2}} 2^{\nu-\frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} \frac{1}{\sqrt{2k}} (-k\tau)^{\frac{1}{2}-\nu}. \tag{3.9}$$

Substituting the solution as given in Eq. (3.9) for the super-horizon modes ($k \ll aH$) in Eq. (3.5) for the tensor power spectrum, we obtain:

$$P_h(k) = \frac{16\pi}{M_P^2} \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{n_T} \coth\left[\frac{k}{2T}\right], \tag{3.10}$$

so

$$P_T = \frac{64\pi}{M_P^2} \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{n_T} \coth\left[\frac{k}{2T}\right], \tag{3.11}$$

with $n_T = 3 - 2\nu = -2\varepsilon$ and it is called as spectral index. We can now rewrite the power spectrum as,

$$P_T(k) = A_T(k_0) \left(\frac{k}{k_0}\right)^{n_T} \coth\left[\frac{k}{2T}\right], \tag{3.12}$$

where k_0 is referred to as the pivot point and $A(k_0)$ is the normalization constant, $A_T(k_0) = \frac{64\pi}{M_P^2} \left(\frac{H_{k_0}}{2\pi}\right)^2$ where H_{k_0} is the Hubble parameter evaluated when $aH = k_0$ during inflation.

3.3 Effect of pre-inflationary radiation era on CMB polarization

The modification in the tensor power spectrum due to pre-inflationary radiation era may affect the large scale CMB anisotropy and polarization. Since temperature anisotropy gets very small contribution from tensor perturbations, above effect can be neglected for TT correlations. The EE polarization signal also gets a contribution from the tensor perturbations but it is mainly dominated by the scalar perturbations. So the best signal for gravitational waves is the BB polarization angular spectrum which is generated by the primordial tensor perturbations only. The angular power spectrum of the BB polarization modes generated by the gravitational waves is given by Eq. (2.170) [47],

$$\begin{aligned} C_l^{BB} &= (4\pi)\int \frac{dk}{k} P_h(k) \\ \times & \left| \int d\tau g(\tau)\chi(k,\tau)\left[2j_l''(x)+\frac{4j_l(x)}{x}\right]\right|^2 , \end{aligned} \tag{3.13}$$

where $g(\tau)=\kappa' e^{-\kappa}$ is the visibility function and κ' is the differential optical depth for Thomson scattering.

The temperature dependent factor becomes important when the ratio $k/(2T)$ is less than unity. The co-moving wave-number k and the co-moving temperature T can be related to the physical parameters at the time of inflation as follows. Taking the largest measurable perturbation scale $k_{now}/a_{now} \simeq R_h^{-1}$ (where $R_h = 4000$Mpc is the size of the present horizon), and assuming that perturbations of the present horizon scale were just leaving the inflationary horizon H^{-1} at the beginning of inflation we see that the temperature at the beginning of inflation T_i/a_i must be,

$$\frac{k}{2T} = \frac{Ha_i}{2T_i} < 1 , \tag{3.14}$$

in order to have a significant effect on the tensor power spectrum. $T_i/a_i \sim (30V/g_*\pi^2)^{1/4}$, V being the inflaton potential which is related to the curvature at the time of inflation, $H = (8\pi/3)^{1/2}V^{1/2}/M_P$ and $g_* \sim 100$ is the effective number of spin/polarization degrees of freedom of relativistic particles. Therefore inflation is expected to start at a temperature $T_i/a_i = 0.24(HM_P)^{1/2}$. Actually the gravitons which are decoupled will have a temperature slightly below the radiation temperature because of the particles (like the inflaton itself) which have annihilated into radiation prior to inflation. But as the effective number of degrees of freedom, $g_* \sim 100$, is large this difference of temperature is not significant. So for inflation at the GUT scale, $V^{1/4} \sim 10^{15}$GeV, we have $H \sim 10^{11}$GeV and the temperature at the start of inflation $T_i/a_i \sim 10^{14-15}$GeV. So the enhancement of the graviton power spectrum by the factor $\coth(\frac{k}{2T}) = \coth(\frac{Ha_i}{2T_i})$ could be by as large as a factor of 10^{4-5} at low k due to thermal gravitons.

In Fig. 3.1 we show the angular correlations of CMBR temperature and polarization assuming a thermal graviton spectrum (along with the WMAP three years data [48]). The plots for TT, TE and BB correspond to co-moving graviton temperature $T = 0.001\text{Mpc}^{-1}$. For comparison we have plotted the BB angular correlations at $T = 0$. We see that with a tem-

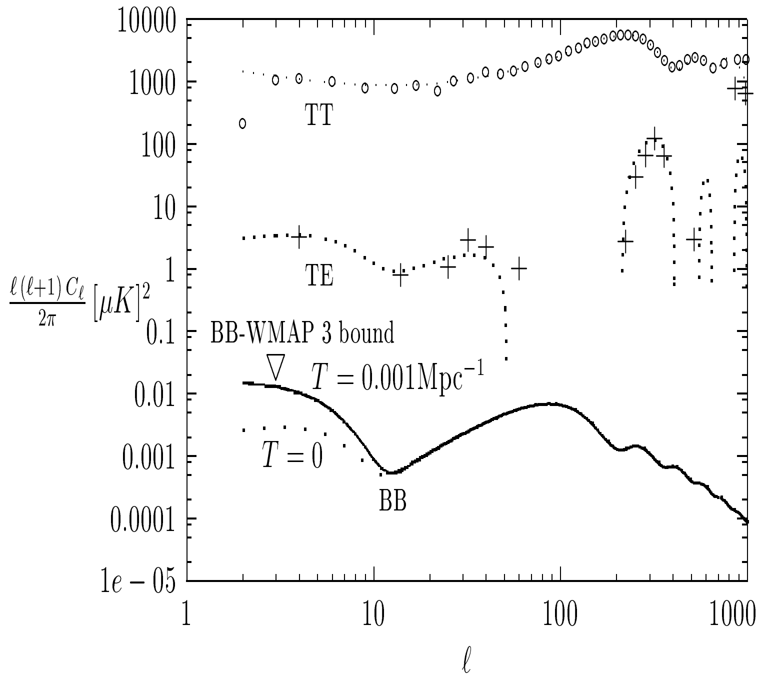

Figure 3.1: The TT, TE and the BB correlations with thermal graviton spectrum along with the WMAP three years data [48]. The plots for TT , TE and BB correspond to co-moving graviton temperature $T = .001\mathrm{Mpc}^{-1}$. For comparison we have plotted the BB angular correlations at $T = 0$. We see that with a graviton temperature $T = .001\mathrm{Mpc}^{-1}$ the BB correlations are amplified at $l < 30$ [17].

perature $T = 0.001\mathrm{Mpc}^{-1}$ the BB correlations are amplified at $l < 30$. We see that only the BB correlation is enhanced by the correction to the tensor power spectrum as expected. The contribution of tensors to the TT angular spectrum is comparable at low l to the contribution from the scalars and there exists the possibility that this large tensor contribution at low l may be detected from the analysis of the TT angular spectrum alone.

We have added the unlensed scalar and tensor contributions to generate the TT, EE, TE and BB correlations. The plots were obtained by running CMBFAST [38], with the following parameters $\Omega_b = 0.05$, $\Omega_c = 0.25$ and $\Omega_v = 0.70$. Optical depth $\tau = 0.08$ and Hubble parameter $h = 0.7$. The value of scalar spectral index $n_s = 0.97$ and the value of tensor spectral index is taken $n_T = -0.01$. Tensor to scalar ratio is taken to be $r(k_0) = 0.1$ at $k_0 = 0.002\mathrm{Mpc}^{-1}$. The output of the CMBFAST was normalized to the WMAP values at $k = 0.002\mathrm{Mpc}^{-1}$ (i.e $l = 30$). For the curves shown in Fig. 3.1 the tensor power spectra is modified due to thermal effects with $\frac{k}{2T} = 500k$. At $k = 0.0002\mathrm{Mpc}^{-1}$, $\coth(500k) = 10$ so there is a large enhancement of the BB polarization at $l = 2-6$, while at $k_0 = 0.002\mathrm{Mpc}^{-1}$, $\coth(500k_0) \sim 1.3$ and there is hardly any enhancement of the BB signal (or in the value of $r(k_0)$ in keeping with the observational constraints from WMAP+SDSS [50]). The magnitude of the co-moving graviton temperature needed to produce this effect is $T/a_{now} = 10^{-3}\text{Mpc}^{-1}$. This corresponds to a temperature of $T_i/a_i \simeq 4 \times R_h^{-1} \times a_{now}/a_i = 4H$ (where $R_h \sim 4000$Mpc is the size of the present horizon). As we have seen inflation can start as soon as the temperature T_i/a_i falls below $V^{1/4} \sim 10^4 H$. Consequently a temperature larger than $4H$ at the beginning of inflation is not unreasonably high.

In standard inflation models the vacuum fluctuations of the inflaton field give the density perturbations. As mentioned earlier because of radiation era prior to inflation the scalar power spectrum will be modified due to thermal distribution of inflaton that were decoupled at the beginning of inflation [42]. The power spectrum for the curvature perturbations will be,

$$P_{\mathcal{R}}(k) = \frac{H^4}{4\pi^2 \dot{\phi}^2} \left(\frac{k}{aH}\right)^{n_s - 1} \coth\left[\frac{k}{2T}\right] . \tag{3.15}$$

The extra temperature dependent term implies that there should be an up-turn of the TT anisotropy spectrum at low l . This expected up-turn in $l(l+1)C_l$ is not seen in the WMAP one-year TT spectrum [42]. This means that there is no significant number density of background density of inflatons at the time when the modes which are currently entering our horizon, were exiting the horizon during inflation. This could happen for two reasons. The background density of inflatons may have decayed or annihilated into lighter particles by this time or the inflaton was cooled from the expected temperature of $0.24(HM_P)^{1/2}$ to below H by the time the modes corresponding to our present horizon were leaving the De-Sitter horizon. This implies that there were an extra $\Delta N = \ln(0.24(M_P/H)^{1/2})$ e-foldings (which has the value $\Delta N \sim 10$ for GUT scale inflation) than what is needed to solve the horizon problem. In the case of gravitons the first condition does not apply as they decouple at the Planck scale and if the expected upturn in the BB mode spectrum is not seen that would imply that the duration of inflation was longer than what is needed to solve the horizon problem.

3.4 Implications for warm inflation models

In warm inflation models [12, 51] where the inflaton is in thermal equilibrium with the radiation bath and the scalar curvature perturbations are generated by thermal fluctuations instead of by quantum fluctuations, there is no $\coth(k/2T)$ correction in the inflaton power spectrum due to stimulated emission. However this correction factor will be present in the graviton spectrum since gravitons are still produced by quantum fluctuations. The temperature of the thermal bath remains constant during warm inflation but the graviton temperature will decrease exponentially. The scalar curvature perturbation in warm inflation is given as Eq. (2.190) [41]:

$$P_{\mathcal{R}}^{(\mathrm{warm})} = \left(\frac{\pi}{4}\right)^{1/2} \frac{H^{5/2}\Gamma^{1/2}T_r}{\dot{\phi}^2}, \tag{3.16}$$

where Γ designates the decay width of the inflaton field and T_r is the temperature of the radiation bath.

There are observational constraints on the tensor scalar ratio defined as:

$$r(k_0) = \frac{P_T(k_0)}{P_{\mathcal{R}}(k_0)}. \tag{3.17}$$

From the combination of WMAP three year data [50] and SDSS large scale structure surveys [52] we have the bound $r(k_0 = 0.002\mathrm{Mpc}^{-1}) < 0.28(95\% CL)$ where $k_0 = 0.002\mathrm{Mpc}^{-1}$ corresponds to $l = \tau_0 k_0 \simeq 30$ with the distance to the decoupling surface $\tau_0 = 14,400\mathrm{Mpc}$. SDSS measures galaxy distributions at red-shifts $z \sim 0.1$ and probes k in the range $0.016h\mathrm{Mpc}^{-1} < k < 0.11h\mathrm{Mpc}^{-1}$. From the expressions of $P_{\mathcal{R}}$ in warm inflation, Eq. (3.16), and P_T we see that the scalar-tensor ratio in warm inflation models (assuming a nearly scale invariant tensor power spectrum) has a scale dependence at large angles given by:

$$r(k) \simeq r(k_0)\frac{\coth[\frac{k}{2T}]}{\coth[\frac{k_0}{2T}]} \simeq r(k_0)\left(\frac{k_0}{k}\right). \tag{3.18}$$

We see that $r(k)$ has a spectral index $n_T \sim -1$ for large scale perturbations. If we consider $k \sim 0.0002\mathrm{Mpc}^{-1}$ which corresponds to $l \sim 3$ then the value of $r(k) = 10r(k_0)$. So even with $r(k_0) \sim 0.1$ as constrained by galaxy surveys, we can have $r(k) \simeq 1$ at the quadrupole anisotropy. The B mode polarization at $l = 3$ is enhanced from its value at $l = 30$ by a corresponding factor of 10. This is true as long as the temperature $T_i/a_i \leq 10^4 H$ which as we have seen in the earlier discussion is expected if there is a thermal era prior to inflation.

For example taking the inflaton potential to be $V = (1/2)m^2\phi^2$, we have the scalar power:

$$P_{\mathcal{R}}^{(\mathrm{warm})}(k_0) = 5.3\frac{\Gamma^{5/2}\phi_0^{1/2}T_r}{M_p^{5/2}m^{3/2}}, \tag{3.19}$$

and the tensor power,

$$P_T(k_0) = \frac{128\pi}{9}\frac{m^2\phi_0^2}{M_P^4}\coth\left[\frac{k_0}{2T}\right], \tag{3.20}$$

and the scalar-tensor ratio,

$$r(k_0) = 8.413\left(\frac{m^{7/2}\phi_0^{3/2}}{M_P^{3/2}}\right)\frac{1}{\Gamma^{5/2}T_r}\coth\left[\frac{k_0}{2T}\right], \tag{3.21}$$

where ϕ_0 is the value of the inflaton field when the scale $k_0 = 0.002\text{Mpc}^{-1}$ was leaving the inflaton horizon. By choosing the parameters $m = 1.4\times10^{12}$GeV, $\Gamma = 0.5\times10^{13}$GeV, $T \simeq T_r = 0.24\times10^{16}$GeV, $\phi_0 \simeq 0.8\times10^{19}$GeV we have $P_R \simeq 2.3\times10^{-9}$ as required by WMAP three year data and $r(k_0) = 0.095$. The value of r is larger at $k = 0.0002\text{Mpc}^{-1}$ by a factor of ~ 10 and the B-modes are magnified at $l = 3$ compared to their value at $l = 30$ by a factor 10, also in warm inflation scenarios.

3.5 Conclusions

Direct observation of gravitational waves would nail the last still unconfirmed prediction of inflation. The amplitude of gravitational waves gives the Hubble curvature during inflation and would tell us the value of the inflation potential [53, 54]. In addition gravitational waves produced during inflation can have several applications like leptogenesis by the gravitational spin-coupling to neutrinos [55] or by a gravitational Chern-Simon coupling of the lepton number current [56]. Observation of the B-mode polarization in the CMB would confirm the existence of primordial super-horizon gravitational waves. Observationally, the three year WMAP data only gives an upper bound on C_l^{BB} with $l = (2-6)$ [48]. The error bars on the C_l^{BB} are presently a factor of five larger than the predictions from standard inflation theory with scalar tensor ratio as large as 0.3, which is close to the observational upper bound $r_{0.002} < 0.28(95\%CL)$.

In this chapter we have studied the effects of pre-inflationary radiation era on B-modes. Pre-inflation radiation era has also been studied by Powell and Kinney [43] and Wang et all [44]. They study the dynamics of phase transition between radiation era and inflation. If inflation lasted for nearly 60 e-foldings, it is possible that the length scales corresponding to the large scale anisotropy were generated in the radiation era. They apply the boundary conditions in the radiation dominated era and show that there will be suppression at low l.

But in our study we have considered that all the perturbations are generated during inflation as stimulated emission in the existing background of thermal gravitons.

Due to thermal gravitons the C_l^{BB} at low $l \simeq (2-6)$ could be larger by a factor of 10 compared to what would be expected from the observational constraint on r and could be within the range of observability of WMAP. The upcoming Planck experiment [57] will measure $C_{l=(1-10)}^{BB}$ at the level of $10^{-4}(\mu K)^2$. Ground based polarization experiments [58] like QUaD, QUIET, Clover and PolarBear measure anisotropies at small angular scales only (at $l > 100$ where thermal effects discussed here are negligible) and can observe C_l^{BB} at the level $10^{-2}(\mu K)^2$. These experiments can probe r in the range $0.05-0.1$ independent of thermal effects. A combination of data from WMAP/Plank at large angles and ground based polarization experiments at small angles will therefore either observe or definitely rule out the thermal enhancement effect.

If WMAP or Planck rule out a spectral index of $n_T \sim -1$ at low l, which is the prediction from thermal gravitons, then for the standard inflationary models it would mean that the duration of inflation has to be longer by $\Delta N = \ln(0.24(M_P/H)^{1/2})$ e-foldings than what is needed to solve the horizon problem. Warm inflation models[51, 59] cannot evade this constraints by supercooling during inflation. If B-modes are observed and the tensor spectral index at low l is not close to -1, then warm inflation models can be ruled out.

Chapter 4

Natural inflation at the GUT scale

4.1 Introduction

It is well known that for single slow-rolling field to satisfy observations the ratio of the height of the potential to the $(width)^4$ should be [60]

$$\frac{\Delta V}{(\Delta\phi)^4} \leq O\left(10^{-6} - 10^{-8}\right) \tag{4.1}$$

where ΔV is the change in the potential and $\Delta\phi$ is the change in the field ϕ during slow-roll. This condition means that inflaton must be extremely weakly self-coupled, with effective quartic self-coupling constant (in realistic models, $\lambda < 10^{-12}$). The small ratio of mass scales required by above condition quantifies the flatness of the potential. This is known as the fine-tuning problem in inflation. To realize inflationary models in particle physics some times such a small coupling is postulated and is fine tuned to be small due to radiative corrections. In supersymmetric models such small coupling arise due to small ratio of mass scales but these models have limitations.

There is another model of inflation to explain this small mass ratio that is called as 'Natural Inflation' [13, 14, 40]. In this model inflaton is pseudo Nambu-Goldstone Boson (PNGB). Its potential is flat due to shift symmetry. Nambu-Goldstone bosons (NGB) arise due to spontaneous breaking of a global symmetry and their potential is exactly flat as long as the shift symmetry is exact. Since we need slow-rolling, there should be explicit symmetry breaking to make the potential nearly flat. Due to explicit symmetry breaking NGB acquires mass and it is called as pseudo Nambu-Goldstone boson (PMGB). It has the potential of the form

$$V = \Lambda^4 \left(1 + \cos\left(\frac{\phi}{f}\right)\right). \tag{4.2}$$

Here one has two scales, f is the spontaneous symmetry breaking scale and and Λ is explicit symmetry breaking scale. The small mass ratio required to satisfy observations (Eq. (4.1)) can be easily accommodated in case of PNGB. For example, if we take Λ at the GUT scale and f larger than GUT scale, we can have $\frac{\Delta V}{(\delta\phi)^4} \sim \frac{\Lambda}{f^4}$ to be very small.

As we have seen earlier that the parameters of the potential can be determined by amplitude and spectral index of primordial curvature power spectrum, in the natural inflation models the symmetry breaking scale f is related to the spectral index as $n_s = 1 - M_p^2/(8\pi f^2)$ and CMB observations constrain n_s to be $0.948 < n_s < 0.977$ [16]. This implies that the symmetry breaking scale f has to be close to the Planck scale [15]. As discussed in Banks et al [61] a symmetry breaking scale larger than M_P makes the theory susceptible to large quantum corrections which can destabilize the flat PNGB potential. There have been several attempts at solving this large f problem in natural inflation. Arkadi-Hamed et al [62] invoke extra dimensions with the Wilson loop of a gauge field in the extra dimension to explain why $f \sim M_P$. Similar arguments are also given by Kaplan and Weiner [63]. Kim et al [64] invoke two field natural inflation to bring down the symmetry breaking scale below Planck scale. Kinney and Mahanthappa [65] show that in some special symmetry breaking schemes the quadratic term in the PNGB field is subdominant compared to the higher order terms and in these models the symmetry breaking scales can be lower than the Planck scale.

In this chapter we show that if the PNGB inflaton is coupled to a radiation bath (with a sub-dominant energy density) as in warm inflation models [12] the symmetry breaking scale f can be in the GUT scale and be consistent with the observations of the temperature anisotropy spectrum observed by WMAP [16]. In this model since the dissipative coupling of the PNGB inflaton makes it roll slowly even in a steep potential, f is lowered from M_P to $M_{GUT} \sim 10^{16} GeV$.

4.2 The potential for PNGB

As a specific model let us consider the SU(5) model where the right handed neutrino N is a singlet. In the see-saw mechanism [66] one generates a heavy Majorana mass by coupling this right handed neutrino to a SU(5) singlet Higgs,

$$\mathcal{L}_\nu = gH\left(\overline{(N_R)^C}N_R + H.C\right). \tag{4.3}$$

This Lagrangian obeys U(1) lepton number symmetry i.e $N \to Ne^{-i\Lambda}$, $H \to He^{2i\Lambda}$. In order to break lepton number spontaneously we have a potential for the Higgs

$$-\mathcal{L}_H = \frac{\lambda}{8}(H^\dagger H - \frac{f^2}{2})^2. \tag{4.4}$$

Here f is the spontaneous symmetry breaking scale.

At the minima of the potential the Higgs is given by $H = \frac{1}{\sqrt{2}} f e^{i\frac{\phi}{f}}$. Here the radial mode of H is superheavy and is frozen out, so it can be neglected. The angular variable ϕ is the Goldstone boson of the spontaneously broken lepton number symmetry i,e U(1) symmetry. So the Lagrangian becomes

$$\mathcal{L}_{eff} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \left(\frac{i}{2}\overline{N_R}\gamma^\mu\partial_\mu N_R - \frac{i}{2}(\partial_\mu\overline{N_R}\gamma^\mu N_R)\right) - g\frac{f}{\sqrt{2}}e^{i\frac{\phi}{f}}\left(\overline{(N_R)^C}N_R + H.C\right) \tag{4.5}$$

This Lagrangian is symmetric under the transformations $\phi \to \phi + 2\Lambda$ and $N \to Ne^{-i\Lambda}$. At this stage ϕ is massless. Quantum gravity effects are expected to break global symmetries at the Planck scale. If there is an explicit symmetry breaking due to gravity the Goldstone boson acquires mass. The explicit symmetry breaking term can be of the form

$$-\mathcal{L} = \frac{M^2}{M_P}\left(\overline{(N_R)^C}N_R + H.C\right) + O(\frac{1}{M_P^2}). \tag{4.6}$$

Because of this explicit symmetry breaking the Goldstone boson acquire mass and its potential can be given as [67]

$$V(\phi) = \Lambda^4\left(1 + \cos\left(\frac{\phi}{f}\right)\right). \tag{4.7}$$

Λ is related to explicit symmetry breaking scale $\mu = \frac{M^2}{M_P}$. The mass of the PNGB is given by $m_\phi = \frac{\mu^2}{f} = \frac{\Lambda^2}{f}$. This implies that $\Lambda = \mu = \frac{M^2}{M_P}$. Now if we take $M \sim M_{GUT} \sim 10^{16} - 10^{17}$GeV then we have $\Lambda \sim 10^{13} - 10^{14}$GeV which is the allowed range by WMAP data.

4.3 Warm natural inflation

In warm inflation the equation of motion of inflaton field is given by Eq. (2.185)

$$\ddot{\phi} + (3H + \Gamma)\dot{\phi} + V'(\phi, T) = 0. \tag{4.8}$$

In slow roll approximation we neglect $\ddot{\phi}$ in the Eq. (4.8). During inflation the potential energy of the inflaton field dominates over radiation density. So the dynamics of ϕ field is governed by

$$\dot{\phi} = -\frac{V'}{3H + \Gamma}, \tag{4.9}$$

$$H^2 = \frac{8\pi}{3M_p^2} V. \tag{4.10}$$

Rewriting the slow role parameters

$$\begin{aligned} \varepsilon &= \frac{M_p^2}{16\pi}\left(\frac{V'}{V}\right)^2, \ \eta = \frac{M_p^2}{8\pi}\frac{V''}{V}, \\ \beta &= \frac{M_p^2}{8\pi}\frac{\Gamma' V'}{\Gamma V}, \ \delta_T = \frac{M_p^2}{8\pi}\frac{T V'_{,T}}{V'}. \end{aligned} \tag{4.11}$$

As mentioned earlier the density perturbations during warm inflation are generated by thermal fluctuations. The power spectrum for the density perturbations given in [41] is

$$P_{\mathcal{R}} = \left(\frac{\pi}{4}\right)^{\frac{1}{2}} \frac{H^{\frac{5}{2}}\Gamma^{\frac{1}{2}} T}{\dot{\phi}^2}, \tag{4.12}$$

which can be written in terms of potential and its derivative using Eq. (4.10) and Eq. (4.9) as

$$P_{\mathcal{R}} = \left(\frac{\pi}{4}\right)^{1/2} \left(\frac{8\pi}{3M_p^2}\right)^{5/4} \frac{V^{5/4}\Gamma^{5/2} T}{V'^2}. \tag{4.13}$$

Using the natural inflation potential (4.7) we get for the power spectrum,

$$P_{\mathcal{R}} = \left(\frac{\pi}{4}\right)^{1/2} \left(\frac{8\pi}{3M_p^2}\right)^{5/4} \frac{\Gamma^{5/2}\, T\, f^2}{\Lambda^3} \frac{\left(1+\cos\frac{\phi}{f}\right)^{(5/4)}}{\sin^2\frac{\phi}{f}}. \tag{4.14}$$

The spectral index is defined as (2.87)

$$n_s - 1 = \frac{\partial \ln P_{\mathcal{R}}}{\partial \ln k}. \tag{4.15}$$

In terms of the slow roll parameters this can be written as (2.191

$$n_s - 1 = \frac{3H}{\Gamma}\left(-\frac{9}{4}\varepsilon + \frac{3}{2}\eta - \frac{9}{4}\beta\right). \tag{4.16}$$

For the given potential (4.7) the spectral index will be

$$n_s - 1 = -\frac{3H}{\Gamma}\frac{3M_p^2}{64\pi f^2}\frac{\left(3+\cos\frac{\phi}{f}\right)}{\left(1+\cos\frac{\phi}{f}\right)}. \tag{4.17}$$

The observational constraint on n_s from WMAP 5-year data [16] is $0.948 < n_s < 0.977$. So it is obvious from above Eq. that if we take warm inflation in strong dissipative regime i.e Γ is very large compared to H, we can have small value of f (fig. 4.1). But the cold natural inflation models on the other hand the spectral index $n_s = 1 - M_p^2/(8\pi f^2)$. This implies that in the cold natural inflation models WMAP data gives a strong constrain $f > 0.7 M_P$ [15].

The slow roll parameter ε for this model is

$$\varepsilon = \frac{M_p^2}{16\pi f^2}\frac{\sin^2\frac{\phi}{f}}{\left(1+\cos\frac{\phi}{f}\right)^2}. \tag{4.18}$$

At the end of inflation $\varepsilon = 1 + Q$, where $Q = \frac{\Gamma}{3H}$. This will give ϕ_f as

$$\cos\frac{\phi_f}{f} = \left(\frac{1-(1+r)\frac{16\pi f^2}{M_p^2}}{1+(1+r)\frac{16\pi f^2}{M_p^2}}\right). \tag{4.19}$$

Putting $Q = 3.9 \times 10^4$ and $f = 8 \times 10^{16}$GeV we get $\phi_f = 2.9 f$. One can calculate the value of ϕ at the time when length scales corresponding to our horizon were leaving the inflationary

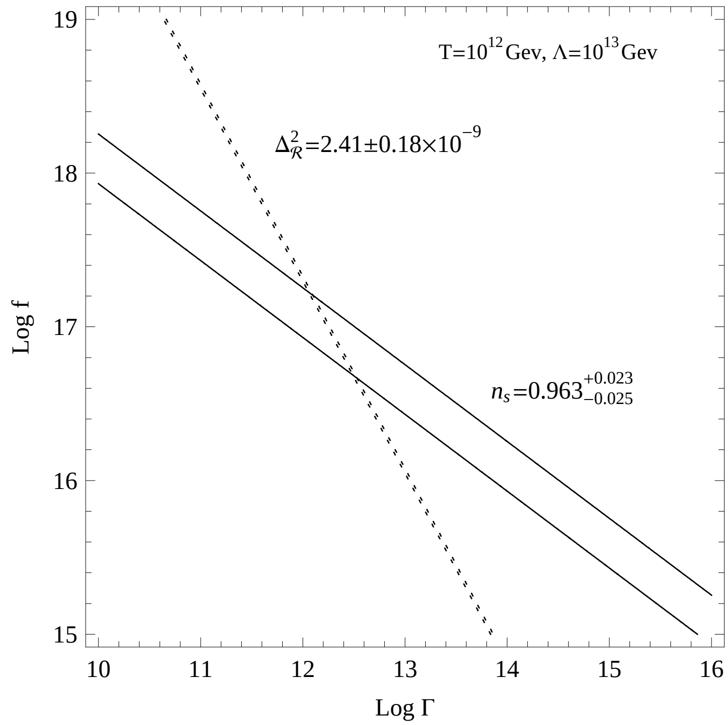

Figure 4.1: The allowed range of f(GeV) and Γ(GeV) from the range of spectral index n_s and the amplitude of curvature perturbations $\Delta^2_{\mathcal{R}}$ from WMAP. Fig. from [18].

horizon. The e-foldings may be calculated as

$$\begin{aligned} N &= \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}} d\phi = \frac{8\pi\Gamma}{3HM_p^2} \int_{\phi_f}^{\phi_i} \frac{V}{V'} d\phi \\ &= \frac{16\pi\Gamma f^2}{3HM_p^2} \left(\log \frac{\sin\left(\frac{\phi_f}{2f}\right)}{\sin\left(\frac{\phi_i}{2f}\right)} \right). \end{aligned} \tag{4.20}$$

The scalar field lies between πf and 0. For $N = 60$ we get $\phi_i = 1.02 f$. The value of the scalar field remains in the GUT regime and still gives adequate e-foldings to solve the horizon and curvature problems.

4.3.1 Microphysical model for large dissipation

As shown in the last section we need a large dissipation coefficient to satisfy observations. The dissipation mechanism to realize warm inflation was studied in [68, 69, 51]. In [68] the effective evolution equation of motion of an overdamped field interacting with other fields was studied. It was shown that large dissipation during warm inflation is possible if we consider a large number of fields. Later in [69, 51] a different approach was developed in which dissipation was achieved by indirect coupling of the inflaton to the radiation. In these models the mass of some heavy scalar fields coupled to inflaton changes due to the evolution the background inflaton field and it excites heavy scalar fields to decay into lighter fields. The coupling of the heavy scalar field χ with inflaton can be described by another explicit symmetry breaking term

$$\mathcal{L}_\chi = 2g^2\phi^2\chi^2. \tag{4.21}$$

This field χ is again coupled to the radiation field σ as

$$\mathcal{L}_{\chi\sigma} = \frac{1}{\sqrt{2}} h f \left(\sigma^2\chi^\star + \chi^2\sigma^\star\right). \tag{4.22}$$

Another feature of this two stage coupling is that it generates a large dissipation without destabilizing the inflaton potential by loop corrections [70]. The dissipation coefficient Γ for this model has been calculated by Berera et al [70],

$$\Gamma = \frac{16}{\pi}\frac{g^2}{h^2} T \ln\frac{T}{m_\chi} \tag{4.23}$$

The interaction terms in the Lagrangian (4.21) can generate one loop corrections to the inflaton mass that can destabilize the flatness of the potential (4.7). For the potential to remain flat the mass correction $g^2 f^2$ should be smaller than $\frac{\Lambda^4}{f^2}$. If we take $\Lambda \sim 10^{13}$GeV and $f \sim 10^{16}$GeV then $g \leq 10^{-6}$. For the validity of above expression (4.23), the mass of χ field should be smaller than T. So if we take one loop correction to the mass of χ field ($T \sim 10^{12}$GeV) because of σ field h should be smaller than 10^{-4}. If we take g and h of the same order we can have $\Gamma \sim 10^{12}$Gev.

4.3.2 Predictions for non-Gaussianity

We infer physics during inflation by determining the amplitude and spectral index of the scalar perturbations and tensor to scalar ratio. These parameters are not enough to constrain the inflationary models as there are various mechanism to produce the desired spectrum of primordial perturbations during inflation. There is another observable that can help us in determining the correct model of inflation. It is the deviation from the pure Gaussian statistics i.e presence of higher order correlation functions of CMB temperature anisotropy. The statistical properties of the fluctuations on the sky are determined by the n-point correlation function i.e

$$\langle f(\hat{n}_1) f(\hat{n}_2) f(\hat{n}_3)_{....} f(\hat{n}_N)\rangle. \tag{4.24}$$

If the perturbations are Gaussian, all odd correlation functions are zero and all higher order correlation functions can be expressed in terms of two point correlation functions. So tow point correlation function is the only parameter required for the Gaussian distribution. But if the distribution is not Gaussian one needs higher order correlation functions. The three point correlation function of scalar curvature perturbations in uniform energy density hypersurface ζ is called as bispectrum defined as [71]

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle = (2\pi)^3 B_\zeta(k_1,k_2,k_3)\,\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\,. \tag{4.25}$$

We use ζ instead of $\mathcal{R}$ here as it is used most commonly in the literature of non-Gaussianity [72, 73, 74]. One can define a non-linearity parameter f_{NL} which is observationally important for non-Gaussianity as

$$f_{NL} = \frac{5}{6}\frac{B(k_1,k_2,k_3)}{\mathcal{P}(k_1)\mathcal{P}(k_2)+\mathcal{P}(k_2)\mathcal{P}(k_3)+\mathcal{P}(k_3)\mathcal{P}(k_1)}. \tag{4.26}$$

Here we use the different definition of the power spectrum i.e

$$\langle\zeta^\star(\mathbf{k}_1)\zeta(\mathbf{k}_2)\rangle = \left(2\pi^3\right)\mathcal{P}(\mathbf{k}_1)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)\,. \tag{4.27}$$

In simple inflationary models non-Gaussianity arises due to higher order interaction terms in the potential and non-linearity in the gravitational potential. In these models f_{NL} is given as

$$f_{NL} = \frac{5}{12}(n_s-1)\,. \tag{4.28}$$

So in simple inflationary models one gets very small deviation from Gaussian fluctuations.

In warm inflation non-Gaussianity was studied by Gupta et al [75, 76] and Moss et al [71]. Gupta et al have considered the non-Gaussianity due to third order term in the inflaton potential and have shown that its magnitude is comparable to the case of supercooled inflation. Moss et al have taken into account the non-linear coupling between radiation and inflaton fluctuations on sub-horizon scales also and has shown that large amount of non-Gaussianity can be generated during warm inflation. The f_{NL} for warm inflation models is given by [71]

$$f_{NL} = -15\ln\left(1+\frac{\Gamma}{42H}\right)-\frac{5}{2}. \tag{4.29}$$

Taking the allowed range of Γ from the fig (4.1) i.e $1\times 10^{12} < \Gamma < 3\times 10^{12}$ we get $-122.6 < f_{NL} < -106.2$ which is allowed by WMAP-5 data [16] ($-151 < f_{Nl} < 253$).

4.3.3 Leptogenesis

Big-Bang Nucleosynthesis calculations show that the present day baryon to photon ratio is $\eta_B = 6.1^{+0.3}_{-0.2}\times 10^{-10}$. This corresponds to the baryon asymmetry of the order of one part in 30 million during early universe. It was pointed out in [77] that the asymmetric decay of heavy neutrino into charged leptons and charged anti-leptons due to CP violation creates a lepton asymmetry in the early universe. This lepton asymmetry is converted to baryon asymmetry by sphaleron processes at the electroweak scale. In this section we discuss how one can generate lepton asymmetry from warm natural inflation.

The PNGB coupling to lepton current is obtained from (4.3) as

$$\mathcal{L}_{int} = \frac{1}{f}\partial_\mu\phi\, j^\mu_L \tag{4.30}$$

For the homogeneous inflaton this will be

$$\mathcal{L}_{int} = \frac{\dot\phi}{f}n_L, \tag{4.31}$$

here n_L is lepton number. Therefore $\frac{\dot\phi}{f}$ is like a chemical potential for the lepton number, $\mu_L = \frac{\dot\phi}{f}$. At equilibrium the lepton number is given by

$$\begin{aligned} n_L &= g_\nu \frac{T^3}{6}\left(\frac{\mu_L}{T}\right) \\ &= g_\nu \frac{\dot\phi T^2}{6f}. \end{aligned} \tag{4.32}$$

So the lepton to entropy ratio will be

$$\eta_L = \frac{n_L}{s} = \frac{15}{4\pi^2}\frac{g_\nu\dot\phi}{g_\star f T}. \tag{4.33}$$

Using slow roll approximation $\dot{\phi} = -\frac{V'}{\Gamma}$. For this model we get

$$\eta_L = \frac{15}{4\pi^2}\frac{g_\nu \Lambda^4}{g_\star f^2 \Gamma T}. \tag{4.34}$$

If we take $\Lambda \sim 10^{13}$GeV, $f \sim 10^{17}$GeV, $\Gamma \sim 10^{12}$ GeV and $T \sim 10^{12}$GeV, we get from (4.34) $\eta_L \sim 10^{-10}$ (fig. 4.2).

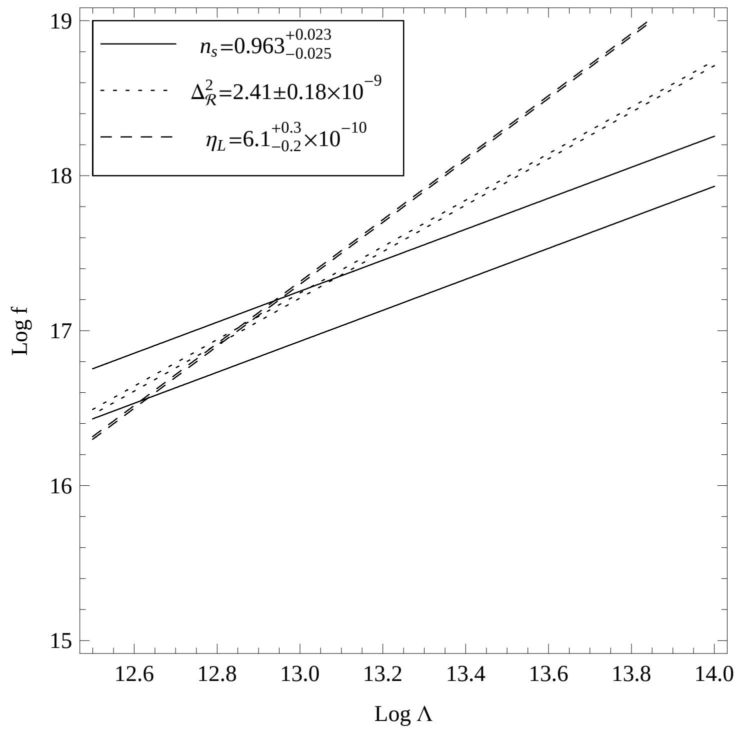

Figure 4.2: The allowed range of f(GeV) and Λ(GeV) using spectral index n_s curvature perturbations $\Delta^2_{\mathcal{R}}$ and lepton to entropy ratio $\eta_L$for $T = 10^{12}$GeV and $\Gamma = 10^{12}$GeV. Fig. from [18].

If the lepton number is violated spontaneously at scale f then there is an effective lepton number violating dimension five operator [78]

$$\mathcal{L}_{\not{L}} = \frac{2}{f} hhll + hc \tag{4.35}$$

where l is the lepton doublet and h is the Higgs doublet of the standard model. When the electroweak symmetry is broken by the Higgs acquiring a vev v then it generates a light neutrino mass $m_\nu = 4\frac{v^2}{f}$. The operator (4.35) can wipe out any generated lepton number at high temperature by the lepton number violating interactions $l + h \rightarrow l^c + h^\dagger$. The interaction rate of this lepton number violating reaction is [79]

$$\Gamma_{\not{L}} = 0.04\frac{T^3}{f^2}. \tag{4.36}$$

These lepton number violating interactions will decouple at a temperature

$$T_d = 4.16\left(\frac{f^4\Lambda^4}{M_P^2}\right)^{\frac{1}{6}}. \tag{4.37}$$

For $\Lambda \sim 10^{13}$GeV and $f \sim 10^{17}$GeV this temperature is $T \sim 10^{14}$GeV. Since the temperature of the radiation bath is $T < 10^{13}$GeV the lepton asymmetry generated by the rolling PNGB field will not be washed out by lepton number violating interactions with the light Higgs.

The fact that PNGB's coupling to the lepton/baryon current is of the derivative coupling form which gives rise to spontaneous leptogenesis of Cohen and Kaplan [80] was first recognized by Dolgov et al [81, 82]. In [81, 82] a natural inflation without damping was examined for generation of baryon/lepton number. It was found that oscillations of the inflaton at the end of inflation wipes out the baryon/lepton asymmetry so the PNGB model of creating B/L asymmetry during natural inflation was considered unfeasible [81, 82]. In [83] it was shown that if one assumes the chaotic inflation potential $m^2\phi^2$ and couples the inflaton to radiation as in warm inflation and in addition assumes a $\partial_\mu\phi\, j^\mu_{B,L}$ coupling of the inflaton then one can get the required baryon asymmetry with a suitable choice of parameters.

4.4 Conclusions

There has been a long standing problem with utilizing the flat potential of PNGB's for inflation as the nearly scale invariant power spectrum which is consistent with observations generated only when the symmetry breaking scale $f \sim M_P$ [13, 14, 40, 61, 62, 63, 64]. In this chapter we have shown that by coupling the inflaton to a radiation bath (as in warm inflation models [12]) can reduce f to the GUT scale. The value of the inflaton field $\phi \sim f \sim M_{GUT}$ which makes the inflaton potential stable against Planck scale radiative corrections. We give a model of inflation where the inflaton is the PNGB arising from spontaneous breaking of lepton number which also gives a large Majorana mass for the right handed neutrinos as required in see-saw models [66]. Since the PNGB's have a derivative coupling to the lepton current this model also generates a lepton asymmetry spontaneously [80] during inflation. We show that with the parameters of the inflation model which give the correct amplitude and spectral index of CMBR also give the required lepton asymmetry of $\eta_L \sim 10^{-10}$ which can be converted to a baryon asymmetry of the same order by sphaleron processes in the electro-weak era [77].

Chapter 5

Effect of spatial curvature on CMB anisotropy

5.1 Introduction

As we have mentioned earlier that the era prior to inflation [1] is expected to leave some imprint on the perturbation modes which leave the horizon earlier and are the last to re-enter our horizon and these effects would be observable if the duration of inflation is nearly 60 e-folds. In the previous chapters we have shown how the radiation era prior to inflation affects large scale CMB anisotropy and polarization. In this chapter we will describe how large scale CMB anisotropy are affected if we had a large curvature before inflation.

There are well motivated cosmological models where the universe could have a non-zero curvature when inflation started. The modes which exited the horizon at that time will carry an imprint of the curvature in the spectrum of the density perturbations. As the curvature of the universe decreases exponentially after the beginning of inflation, there may be a residual curvature still present by the time the scales which are entering our horizon at present were leaving the inflationary horizon. Because of this there will be deviation from the scale invariant perturbations due to non-zero curvature. We calculate the density perturbations in both open and closed universe and show that we get a low quadrupole if universe was closed before inflation.

A calculation of the density perturbations generated during inflation in a universe with a non-zero spatial curvature was first performed by Abbott and Scheafer [84]. They performed the calculation of density perturbation by assuming scale invariant perturbations. Lyth and

Stewart [85] and Ratra and Peebles [20] have studied quasi-de-Sitter models . They performed the calculation using conformal boundary conditions for the mode functions. A calculation for open universe inflation and assuming the Bunch-Davies initial conditions for the mode function was done by Sasaki et al [86] and Bucher et al [87]. They have considered inflation in two stages. In one stage inflaton is stuck to a flase vacuum and in the next stage inflaton rolls slowly towards its true minima and the open universe arises by the nucleation of a single bubble. To calculate the density perturbations they have also considered Bunch Davies vacuum modes. In our study we obtain the same solutions for the mode functions as [85, 20, 86, 87] but the main difference is that we evaluate the power spectrum at horizon crossing. We assume adiabatic perturbations which are frozen after the modes exit the horizon. The horizon crossing condition also involves the curvature and that accounts for the main difference between our result and earlier work [85, 20, 86, 87].

The corrections to the power spectrum at horizon scales are multiplicative powers of $(1 \pm K/\beta^2)$, where the curvature $K = (\Omega_0 - 1)(a_0 H_0)^2$ and β is the comoving canonical wavenumber. We calculate the primordial power spectrum for the case closed and open universe at the time of inflation. We choose the Bunch-Davies boundary condition to normalize the wave-functions. For the case of closed universe we obtain the following expression for the power spectrum

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2}\frac{1}{\left(1+\frac{K}{\beta^2}\right)^2}, \qquad \frac{\beta}{\sqrt{K}} = 3,4,5\cdots \text{ (for } K>0) \tag{5.1}$$

and for the case of inflation in an open universe

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2}\frac{1}{\left(1-\frac{|K|}{\beta^2}\right)^2\left(1+\frac{|K|}{\beta^2}\right)}, \qquad \frac{\beta}{\sqrt{|K|}} > 1, \text{ (for } K<0) \tag{5.2}$$

where ϕ is the inflaton field. In the case of closed universe β takes discrete values in units of $\sqrt{K} = R_c^{-1}$ (R_c being the curvature radius), the modes corresponding to $\beta/\sqrt{K} = 1,2$ can be eliminated by gauge transformations [84] so there is a large-wavelength cutoff at $\beta_c^{-1} = R_c/3$. This large wavelength cut-off in a closed universe has been used to explain the observed low CMB anisotropy at low multi-poles [88] and [89]. In the case of open universe only modes with $\beta > \sqrt{|K|}$ cross the inflationary horizon.

Our result for the power spectrum in the closed and open universe cases differs from the phenomenological power spectrum [90],

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2}\,\frac{1}{1+\frac{K}{\beta^2}} \tag{5.3}$$

used in the calculation of CMB anisotropies in both the closed and open cases. Our results agree qualitatively with (5.3) in that the power at small β is suppressed in the closed universe inflation (5.1) and enhanced in the open universe (5.2).

According to inflation [1], the curvature of the present universe $\Omega_0 - 1$ is related to the curvature at any time during inflation $\Omega_i - 1$ as

$$\frac{\Omega_0 - 1}{\Omega_i - 1} = \left(\frac{a_i H_i}{a_0 H_0}\right)^2. \tag{5.4}$$

If a_i is the scale factor at the time during inflation when scales of the size of our present horizon were exiting the inflationary horizon then $a_0 H_0 = a_i H_i$ and $\Omega_0 = \Omega_i$. If at the beginning of inflation $(\Omega_{start} - 1) = O(1)$ then in order to have a deviation of say one-percent from unity in the present curvature, the number of e-foldings prior to the a_i must be small. Putting an upper bound on the present curvature $(\Omega_0 - 1)$ from observations also puts a lower bound on the number of extra e-foldings necessary in inflation in addition to the minimum number needed to solve the horizon problem [91].

The geometry of the universe can be determined from the CMB anisotropy from the angular size of the acoustic peak. The curvature density is defined as $\Omega_K = 1 - \Omega = -\frac{K}{a^2 H^2}$ (see Eq. (2.6)). The angular diameter distance of an object in case of non-flat universe is given as

$$d_A = \frac{a}{H_0\sqrt{|\Omega_K|}} \begin{cases} \sinh\left[\sqrt{\Omega_K} H_0 s\right] & \Omega_K > 0 \\ \sin\left[\sqrt{-\Omega_K} H_0 s\right] & \Omega_K < 0 \end{cases} \tag{5.5}$$

There is degeneracy in the parameters so we need other data sets to constrain the curvature of the universe. The constraints on the density of the universe Ω depend upon priors like the value H_0 and Ω_λ. The constraint on curvature from WMAP five year data [16] for the ΛCDM model is $(\Omega_0 - 1) = 0.099 \pm 0.1$ and for the $w-CDM$ model is $(\Omega_0 - 1) = 0.122 \pm 0.1$. Combining other data sets like LSS [92] and HST [93] constraints the curvature more tightly. For example WMAP5+HST data constraints the $(\Omega_0 - 1) = 0.017 \pm 0.02$ for the $w-CDM$ model. However these constraints are loosened again if the assumption of adiabatic perturbation is relaxed. For example the if the perturbation is assumed to be isocurvature then combination of WMAP [16] , LSS [92] and HST [93] and supernovae observations gives a constraint on the density of the universe as $(\Omega_0 - 1) = 0.06 \pm 0.02$ [94] which means that the curvature at one-σ could be as large as $K/(a_0H_0)^2 = 0.08$. In the case of the closed universe the power spectrum (5.1) $P_{\mathcal{R}} \propto (1 + K/\beta^2)^{-2}$ at the scale of the horizon $\beta = a_0H_0$ would be suppressed by about 16% compared to the power for the flat universe.

We use our power spectrum to calculate the temperature anisotropy spectrum and compare the results with the WMAP five year data assuming adiabatic perturbations. We find that our power spectrum gives a lower quadrupole anisotropy when $\Omega_0 - 1 > 0$, but matches the temperature anisotropy calculated from the standard Ratra-Peebles power spectrum at large l. We also find that using the closed universe power spectrum (5.44) for larger values of Ω_0 the quadrupole anisotropy is suppressed more. However the WMAP observation of a strong suppression of the quadrupole temperature anisotropy cannot be explained by the modified power spectrum for a closed universe as suggested by [88] for realistic values of other parameters (like H_0) .

5.2 Scalar power spectrum

We expand the inflaton field $\phi(\mathbf{x},t) \equiv \phi(t) + \delta\phi(\mathbf{x},t)$, where the perturbations $\delta\phi$ around the constant background $\phi(t)$ obey the minimally coupled KG equation (in the spatially flat gauge)

$$\delta\ddot{\phi} + 3\frac{\dot{a}}{a}\delta\dot{\phi} - \frac{1}{a^2}\nabla^2\delta\phi = 0. \tag{5.6}$$

With the separation of variables

$$\delta\phi(\mathbf{x},t) = \sum_k \delta\phi_k(t)Q(\mathbf{x},k) \tag{5.7}$$

the KG equation can be split as

$$\delta\ddot{\phi}_k + 3\frac{\dot{a}}{a}\delta\dot{\phi}_k + \frac{k^2}{a^2}\delta\phi_k = 0, \tag{5.8}$$

$$\nabla^2 Q(\mathbf{x},k) = -k^2 Q(\mathbf{x},k), \tag{5.9}$$

where ∇^2 is the Laplacian operator for the spatial part. Making the transformation $d\tau = dt/a$ and $\sigma(\tau,k) = a(\tau)\delta\phi_k(\tau)$ we get the KG equation for $\sigma(\tau,k)$

$$\sigma'' + (k^2 - \frac{a''}{a})\sigma = 0, \tag{5.10}$$

where primes denote derivatives w.r.t conformal time τ.

The Friedman equations in conformal time are

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\rho a^2 - K, \tag{5.11}$$

$$\left(\frac{a'}{a}\right)' = -\frac{4\pi G}{3}(\rho + 3p)a^2. \tag{5.12}$$

Consider the universe with cosmological constant and curvature, then $\rho = \rho_\lambda$ and $p = -\rho_\lambda$ and we get using the Friedman equations,

$$\frac{a''}{a} = \frac{16\pi G}{3}\rho_\lambda a^2 - K \equiv 2a^2 H_\lambda^2 - K, \tag{5.13}$$

where $H_\lambda = (\frac{8\pi G}{3}\rho_\lambda)^{1/2}$ is the Hubble parameter during pure inflation. Substituting (5.13) in the KG equation (5.10) we obtain,

$$\sigma'' + (k^2 - 2a^2 H_\lambda^2 + K)\sigma = 0. \tag{5.14}$$

The curvature affects the wave equation of $\sigma(\tau)$ in the explicit dependence K and also in the changed dynamics of $\tau-$dependence of the scale factor a which is important in the early stages of inflation.

The scalar field perturbation can be written as

$$\delta\phi(\mathbf{x},\tau) = \sum_k \frac{\sigma(\tau,k)}{a(\tau)} Q(\mathbf{x},k), \tag{5.15}$$

where $\sigma(\tau)$ is the solution of equation (5.14) and the spatial harmonics $Q(\mathbf{x},k)$ are solutions of equation (5.9) [84]. One can separate the radial and angular modes of $Q(\mathbf{x},\mathbf{k})$ as

$$Q(\mathbf{x},k) = \Phi_\beta^l(r) Y_l^m(\theta,\phi), \tag{5.16}$$

where $\beta = (k^2 + K)^{1/2}$ are the eigenvalues of the radial-part of the Laplacian with eigenfunctions given by the hyperspherical Bessel functions $\Phi^l_\beta(r)$ which are listed in [84]. In the limit $K \to 0$, the radial eigenfunctions $\Phi^l_\beta(r) \to j_l(kr)$. For a closed universe $\Phi^l_\beta(r)$ is given as [84]

$$\Phi^l_\beta(r) = \frac{Norm.Constant}{(\sin y)^{\frac{1}{2}}} P^{-\frac{1}{2}-l}_{-\frac{1}{2}+\beta}(\cos y)\,, \tag{5.17}$$

where $y = 1 + \frac{Kr^2}{4}$. For the closed universe Φ^l_β must satisfy periodic boundary condition to be single valued i.e

$$\Phi^l_\beta(-\cos y) = \cos\left[(\beta - 1 - l)\,\pi\right] \Phi^l_\beta(\cos y). \tag{5.18}$$

So β must be an integer. The values $\beta = 1$ and $\beta = 2$ correspond to the modes that can be removed by gauge transformations [25].

For open universe the radial part of the eigen functions will be

$$\Phi^l_\beta(r) = \frac{Norm.Constant}{(\sinh y)^{\frac{1}{2}}} P^{-\frac{1}{2}-l}_{-\frac{1}{2}+i\beta}(\cosh y)\,. \tag{5.19}$$

Here β can take any positive real value as there are no periodic boundary conditions to satisfy. The orthogonality property of these functions is given as

$$\int \gamma r^2 dr d\Omega Q^{lm}_\beta(r,\theta,\phi) Q^{*\,l'm'}_{\beta'}(r,\theta,\phi) = \frac{1}{\beta^2}\delta_{ll'}\delta_{mm'}\delta_{\beta\beta'}, \tag{5.20}$$

where $\gamma = (1 + \frac{Kr^2}{4})^{-3}$ is the determinant of the spatial metric, and completeness is

$$\sum_{l,m}\int \beta^2 d\beta Q^{lm}_\beta(r,\theta,\phi) Q^{*\,lm}_\beta(r',\theta',\phi') = \gamma^{-1}\frac{1}{r^2}\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi'). \tag{5.21}$$

In case of closed universe the integral over β is replaced by sum over the integers $\beta/\sqrt{K} = 3, 4, 5 \ldots$.

The gauge invariant perturbations are a combination of metric and inflaton perturbations. The comoving curvature perturbation is gauge invariant and it is related to the inflaton perturbations in the spatially flat gauge as

$$\mathcal{R}(\mathbf{x},\tau) = \frac{H}{\dot{\phi}}\delta\phi(\mathbf{x},\tau). \tag{5.22}$$

Comoving curvature perturbation is frozen outside the horizon till it re-enters in the radiation or matter era. CMB anisotropies at large angles are caused by curvature perturbations in the surface of last scattering which enter in the matter era. The Sachs-Wolfe effect at large angles, relates the temperature perturbation in the direction $\hat{n}$ observed by the observer located at the point $(\mathbf{x_0}, \tau_0)$ to the curvature perturbation at the point $(\mathbf{x}_{LS}, \tau_{LS})$ in the LSS,

$$\frac{\delta T(\mathbf{x_0}, \hat{\mathbf{n}}, \tau_0)}{T} = \frac{1}{5}\mathcal{R}(\mathbf{x}_{LS}, \tau_{LS}), \tag{5.23}$$

where $\mathbf{x}_{LS} = \hat{n}(\tau_{LS} - \tau_0)$. Using the completeness of $Q^{lm}_{\beta}(r, \theta, \phi)$ we can expand $\mathcal{R}$ as a sum-over the eigenmodes,

$$\mathcal{R}(\mathbf{x}_{LS}, \tau_{LS}) = \sum_{lm} \int \beta^2 d\beta \left[\frac{H}{\dot{\phi}}\delta\phi_{\beta}(\tau)\right]_{\tau=\tau_*} Q^{lm}_{\beta}(\mathbf{x}_{LS}). \tag{5.24}$$

Here we have used the fact that $\mathcal{R}$ does not change after exiting the horizon during inflation (at a conformal time which we shall denote by τ_*) till it re-enters the horizon close to the LS era. Using the Sachs-Wolfe relation (5.23) and the mode expansion of the curvature perturbation (5.24) and using the orthogonality (5.20) of Q^{lm}_{β}, we obtain

$$\left\langle \frac{\delta T(\hat{n}_1)}{T}\frac{\delta T(\hat{n}_2)}{T} \right\rangle = \sum_{l} \frac{2l+1}{4\pi} P_l(\hat{n}_1 \cdot \hat{n}_2) \int \beta^2 d\beta \frac{1}{25} |\mathcal{R}(\beta, \tau_*)|^2 \, |\Phi^l_{\beta}(\tau_0 - \tau_{LS})|^2. \tag{5.25}$$

The angular spectrum C_l of temperature anisotropy defined by

$$\left\langle \frac{\delta T(\hat{n}_1)}{T}\frac{\delta T(\hat{n}_2)}{T} \right\rangle = \sum_{l} \frac{2l+1}{4\pi} P_l(\hat{n}_1 \cdot \hat{n}_2) C_l \tag{5.26}$$

can be written in terms of the power spectrum of curvature perturbations by comparing (5.26) with (5.25),

$$C_l = 4\pi \int \frac{d\beta}{\beta} \frac{1}{25} |P_{\mathcal{R}}(\beta)|^2 \, |\Phi^l_{\beta}(\tau_0 - \tau_{LS})|^2, \tag{5.27}$$

where the power spectrum of curvature perturbations is defined as

$$P_{\mathcal{R}}(\beta) = \frac{\beta^3}{2\pi^2} \left[\left(\frac{H}{\dot{\phi}}\right)^2 |\delta\phi_{\beta}(\tau)|^2 \right]_{\tau=\tau_*}. \tag{5.28}$$

We shall now derive the power spectrum for the open and closed inflation universes.

5.3 Closed universe inflation

For a closed universe, $K > 0$, from the Friedman equation (5.11) we get

$$\dot{a} = H_\lambda a \sqrt{1 - \frac{K}{H_\lambda^2 a^2}}, \tag{5.29}$$

which can be integrated to give

$$a(t) = \frac{\sqrt{K}}{H_\lambda} \cosh H_\lambda t. \tag{5.30}$$

The solution (5.30) represents a bounce solution where there is a contracting phase for $t < 0$ and a bounce at $a(t = 0) = \frac{\sqrt{K}}{H_\lambda}$ and an expanding phase for $t > 0$. We shall choose the expanding phase when the cosmological constant starts dominating over the curvature energy as the start of inflation. It is during the expanding phase that the modes exit the horizon (to re-enter later during radiation and or matter era). Our resu lts do not depend on the history of the universe prior to $t = 0$ i.e whether there was a contracting phase and a bounce at $t = 0$ or a closed universe began directly in the expanding phase after quantum tunneling as in [95].

The conformal time is given by

$$\tau(a) = \int \frac{da}{H_\lambda a^2 \sqrt{1 - \frac{K}{H_\lambda^2 a^2}}} = \frac{-1}{\sqrt{K}} \arcsin \frac{\sqrt{K}}{a H_\lambda}. \tag{5.31}$$

The conformal time spans the interval $\tau = (-\frac{\pi}{2\sqrt{K}}, 0)$ as the scale factor a varies between $(\frac{\sqrt{K}}{H_\lambda}, \infty)$, so for $K > 0$ our initial conditions are different from the standard inflation case. The dependence of the scale factor on the conformal time is obtained from (5.31)

$$a(\tau) = -\frac{\sqrt{K}}{H_\lambda} \frac{1}{\sin \sqrt{K}\tau}. \tag{5.32}$$

The conformal time KG equation (5.14) for the closed-inflationary universe is of the form

$$\sigma''(\tau) + \left[k^2 - K\left(2\mathrm{cosec}^2 \sqrt{K}\tau - 1\right)\right] \sigma(\tau) = 0. \tag{5.33}$$

This equation can be solved exactly and the solutions are

$$\begin{aligned}\sigma(\tau,k) &= c_1\left(-\sqrt{K}\cot\sqrt{K}\tau + i\sqrt{k^2+K}\right)e^{i\sqrt{k^2+K}\tau} \\ &+ c_2\left(-\sqrt{K}\cot\sqrt{K}\tau - i\sqrt{k^2+K}\right)e^{-i\sqrt{k^2+K}\tau}.\end{aligned} \tag{5.34}$$

As done in the case of the flat universe the normalization constants c_1 and c_2 are determined by imposing the Bunch-Davies initial condition which states that modes which are deep inside the horizon in the past should behave like positive frequency plane waves,

$$\sigma(\tau \to \frac{-\pi}{2\sqrt{K}}, k) = \frac{1}{\sqrt{2\beta}}\, e^{i\beta\tau}, \tag{5.35}$$

where $\beta = (k^2+K)^{1/2}$. This implies that $c_2 = 0$ and

$$|c_1| = \frac{1}{\sqrt{2}(\beta)^{3/2}}. \tag{5.36}$$

The quantum field $\sigma(\mathbf{x},\tau)$ can be expanded in terms of the mode functions (5.34) as

$$\sigma(\mathbf{x},\tau) = \sum_{lm}\int \beta^2 d\beta \left(a_{\beta lm}\, Q_\beta^{lm}(\mathbf{x})\,\sigma(\tau,\beta) \;+\; a^\dagger_{\beta lm}\, Q^{*lm}_{\beta}(\mathbf{x})\,\sigma^*(\tau,\beta)\right). \tag{5.37}$$

Using the commutation relation of $\sigma(\mathbf{x},\tau)$ and the orthogonality of Q_β^{lm} (5.20) we see that the creation and annihilation operators obey the canonical commutation relations

$$[a_{\beta lm}, a^\dagger_{\beta' l' m'}] = \frac{1}{\beta^2}\delta(\beta-\beta')\delta_{ll'}\delta_{mm'}. \tag{5.38}$$

From the foregoing discussion it is clear that β is the radial canonical momentum. The quantum fluctuations become classical when $\beta = aH$. We shall evaluate the power spectrum at horizon crossing, as the modes do not change after exiting the inflation horizon till they re-enter the horizon in the radiation or matter era.

Substituting the constants c_1 and c_2 in the general solution (5.34) and going back to the $\delta\phi$, we find that

$$\langle 0|\delta\phi_\beta(\tau)^2|0\rangle = \frac{1}{a(\tau)^2}\left[\frac{\beta^2 + K\cot^2\sqrt{K}\tau}{2\beta^3}\right]. \tag{5.39}$$

We want to evaluate the spectrum of perturbations at horizon crossing as adiabatic perturbations do not change after horizon crossing. The horizon crossing condition is given by

$$\beta = a_* H(a_*) = a_* \left(H_\lambda^2 - \frac{K}{a_*^2} \right)^{1/2} \tag{5.40}$$

from which we obtain the values of the scale factor

$$a_* = \frac{(\beta^2 + K)^{1/2}}{H_\lambda} \tag{5.41}$$

and of the conformal time

$$\tau_* = -\frac{1}{\sqrt{K}} \arctan \frac{\sqrt{K}}{\beta} \tag{5.42}$$

at horizon crossing. The corresponding value of the Hubble parameter is

$$H(a_*) = H_\lambda \frac{\beta}{(\beta^2 + K)^{1/2}}. \tag{5.43}$$

The power spectrum $\mathcal{P}(\beta)$ in this case is given by

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2 \dot{\phi}^2} \frac{1}{\left(1 + \frac{K}{\beta^2}\right)^2}. \tag{5.44}$$

5.4 Open universe inflation

Now we consider the case of an open universe with $K < 0$. From the Friedman equation (5.11) we have

$$\dot{a} = H_\lambda a \sqrt{1 + \frac{|K|}{H_\lambda^2 a^2}}, \tag{5.45}$$

where we work with the absolute value of the curvature, taking into account that $|K| = -K$ in this case. The above expression can be integrated to give

$$a(t) = \frac{\sqrt{|K|}}{H_\lambda} \sinh H_\lambda t \tag{5.46}$$

with initial condition $a(t=0) = 0$. In the case of open universe there is no classical contracting phase or bounce. The universe begins in the expanding phase and the curvature decreases in time compared to the cosmological constant which at large t gives a pure de-Sitter expansion. Models of open universe inflation where the universe arises at $t = 0$ by quantum tunneling have been studied in [96, 87, 86].

The conformal time is

$$\tau(a) = \int \frac{da}{H_\lambda a^2\sqrt{1+\frac{|K|}{H_\lambda^2 a^2}}} = \frac{-1}{\sqrt{|K|}}\operatorname{arcsinh}\frac{\sqrt{|K|}}{H_\lambda a}. \tag{5.47}$$

The conformal time spans the interval $\tau = (-\infty, 0)$ as the scale factor varies in the interval $a = (0, \infty)$. We can solve for $a(\tau)$ and obtain

$$a(\tau) = -\frac{\sqrt{|K|}}{H_\lambda}\frac{1}{\sinh\sqrt{|K|}\tau}. \tag{5.48}$$

The conformal time KG equation for the open-inflationary universe is of the form

$$\sigma''(\tau) + \left[k^2 - |K|\left(2\,\mathrm{cosech}^2\sqrt{|K|}\tau + 1\right)\right]\sigma(\tau) = 0. \tag{5.49}$$

This equation has exact solutions

$$\begin{aligned}\sigma(\tau) &= c_1\left(-\sqrt{|K|}\coth\sqrt{|K|}\tau + i\sqrt{k^2-|K|}\right)e^{i\sqrt{k^2-|K|}\tau} \\ &+ c_2\left(\sqrt{|K|}\coth\sqrt{|K|}\tau + i\sqrt{k^2-|K|}\right)e^{-i\sqrt{k^2-|K|}\tau}.\end{aligned} \tag{5.50}$$

The normalization constants c_1 and c_2 are chosen by imposing the Bunch-Davies initial condition, that in the infinite past $\tau \to -\infty$ the modes were well within the inflation horizon and were positives frequency plane waves,

$$\sigma(k, \tau \to -\infty) = \frac{1}{\sqrt{2\beta}}e^{i\beta\tau}, \tag{5.51}$$

where for the open universe $\beta = \sqrt{k^2 - |K|}$. Imposing this condition on (5.50) we obtain the integration constants, $c_2 = 0$ and

$$|c_1| = \frac{1}{\sqrt{2\beta}k}. \tag{5.52}$$

We then obtain for the magnitude of $\delta\phi_\beta(\tau) = \sigma(\tau)/a(\tau)$ the expression,

$$|\delta\phi_\beta(\tau)|^2 = \frac{1}{a(\tau)^2}\left[\frac{\beta^2 + |K|\coth^2\sqrt{|K|}\tau}{2(\beta^2+|K|)\,\beta}\right]. \tag{5.53}$$

Since the adiabatic modes freeze after Hubble crossing, the power spectrum is evaluated at horizon crossing. The horizon crossing condition is given by

$$\beta = a_* H(a_*) = a_* \left(H_\lambda^2 + \frac{|K|}{a_*^2} \right)^{1/2}, \tag{5.54}$$

and we obtain for the scale factor at Hubble crossing

$$a_* = \frac{(\beta^2 - |K|)^{1/2}}{H_\lambda}, \tag{5.55}$$

and the corresponding conformal time is given by

$$\tau_* = -\frac{1}{\sqrt{|K|}} \operatorname{arctanh} \frac{\sqrt{|K|}}{\beta}. \tag{5.56}$$

The Hubble parameter at horizon crossing is

$$H(a_*) = H_\lambda \frac{\beta}{\sqrt{\beta^2 - |K|}}. \tag{5.57}$$

We notice that in an open-universe stage of inflation, only the modes satisfying the condition $\beta^2 > |K|$ will cross the Hubble radius.

With this, we obtain the following expression for the curvature power spectrum at Hubble crossing

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2 \dot{\phi}^2} \frac{1}{\left(1 - \frac{|K|}{\beta^2}\right)^2 \left(1 + \frac{|K|}{\beta^2}\right)}. \tag{5.58}$$

5.5 Effect of curvature on temperature anisotropy spectrum

There are first principle calculations of power spectrum in a non-flat inflationary universe [20], [85], [97], [98], [86] and [87] . Our results for the primordial power spectra for both the closed and open pre-inflation universe cases differ in some details from these earlier papers because of differences in the way we have implemented the initial conditions. Our results of the primordial power spectrum have been derived assuming that the vacuum state in the infinite past was the Bunch-Davies vacuum and we have evaluated the primordial power spectrum at horizon crossing of the perturbation modes.

The power spectrum obtained by Ratra and Peebles [20] and Lyth and Stewart [85] for the open universe case, obtained by assuming conformal boundary condition for the initial state at $\tau \to -\infty$ is

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2}\frac{1}{\left(1+\frac{|K|}{\beta^2}\right)}. \tag{5.59}$$

This is sometime written in the form

$$\beta^{-3}P_{\mathcal{R}}(\beta) \propto \frac{1}{\beta(\beta^2+K)} \equiv \frac{1}{\beta(\beta^2+1)}. \tag{5.60}$$

Bucher et al [87] and Sasaki et al [86] consider an open universe with a tunneling solution and assume that the initial states annihilate the Bunch-Davies vacuum and obtain a power spectrum ,

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2}\frac{1}{\left(1+\frac{|K|}{\beta^2}\right)}\coth\left[\frac{\pi\beta}{\sqrt{|K|}}\right]. \tag{5.61}$$

In our study we also assume a Bunch-Davies vacuum but we consider the standard slow roll inflation model, where the expansion was dominated by the curvature term prior to inflation, and evaluate the power spectrum at the horizon exit $a_* H(a_*) = \beta$. In our solution for the power spectrum of the open universe case (5.2) we have a factor of $1/(1-|K|/\beta^2)$ instead of $\coth(\pi\beta/\sqrt{|K|})$ of (5.61). All three solutions for the power spectrum (5.2), (5.59) and (5.61) agree in the limit of small curvature $|K|/\beta^2 \to 0$.

For the closed universe case, the curvature power spectrum as a function of β is obtained in [89] numerically and they find that curvature causes a suppression of the power spectrum at low β which agrees with our result. Analytic expressions for the power spectrum for closed universe inflation is also given by Starobinsky [98] where it is seen that the power spectrum is enhanced at low β for inflation with positive curvature. We find that for the case of closed universe the power spectrum is slightly suppressed at low β and our result agrees qualitatively with that of [89] but disagrees with [98].

As an example the experimental bound on the total density of the universe from a combination of WMAP and HST supernovae observations is $0.98 < \Omega_0 < 1.06$ [10, 16] in the w-cdm models. If one uses the Ratra-Peebles form of the power spectrum for the closed universe

$$P_{\mathcal{R}}(\beta) = \frac{H_\lambda^4}{2\pi^2\dot{\phi}^2}\frac{1}{\left(1-\frac{K}{\beta^2}\right)}, \tag{5.62}$$

we see that for perturbations of the horizon size $\beta \simeq H_0 a_0$, the power spectrum is suppressed by up to 6% (compared to the flat universe). On the other hand if one uses the power spectrum (5.1) for the closed universe the suppression of large scale power can be as large as 12%. Although the Ratra-Peebles formula for the power spectrum was derived for open universe inflation ($K < 0$) it is used in numerical programs like CAMB [39] and CMBFAST [38] also for the closed universe case with $K > 0$ when deriving the temperature anisotropy spectrum.

In principle the choice of power spectrum used as an input (as in CAMB and CMBFAST) will affect the determination of cosmological parameters like Ω_0, H_0, n_s etc from the CMB data. In Fig. 5.1 we show the temperature anisotropy for a closed universe with $\Omega_0 = 1.06$ calculated using the power spectrum (5.44) (dashed line) and the temperature anisotropy calculated using the Ratra-Peebles power spectrum (5.60) (solid line).

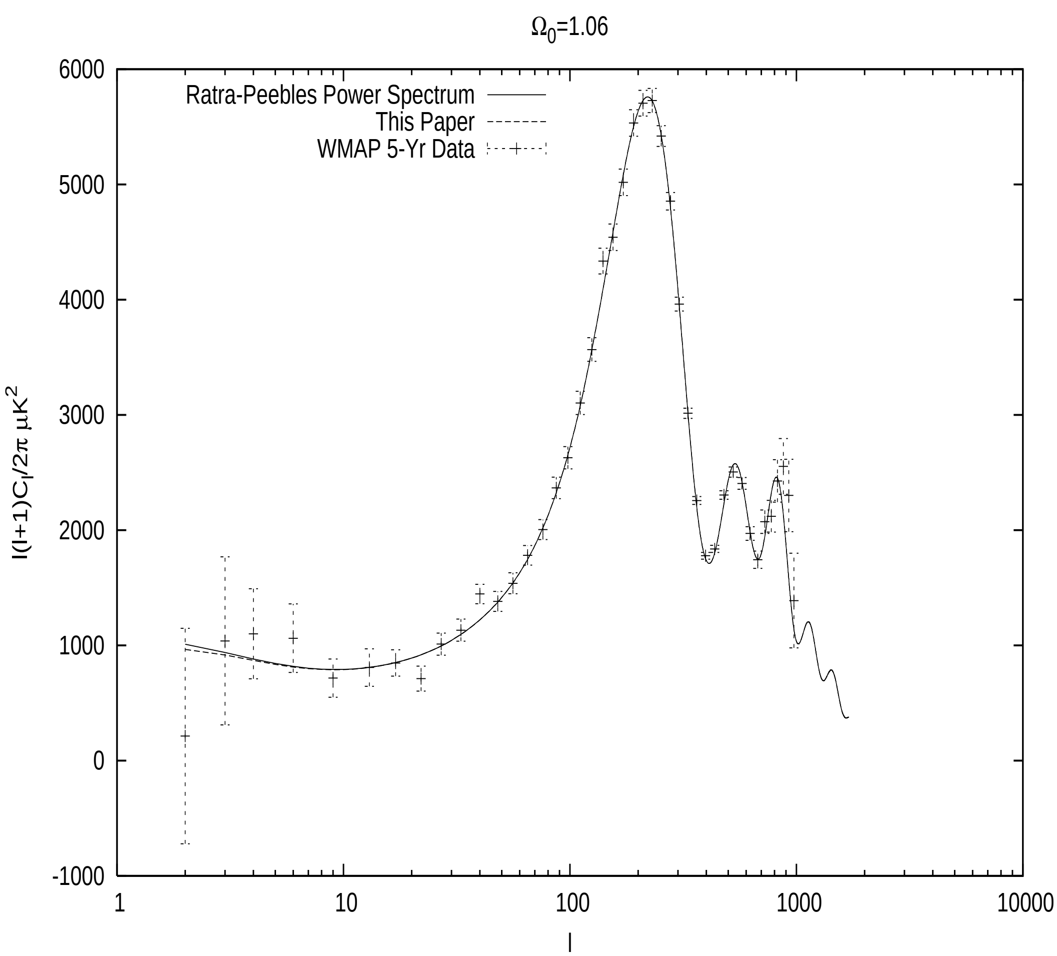

Figure 5.1: Comparison of temperature anisotropy with the Ratra-Peebles power spectrum (5.60) and the power spectrum (5.44) derived assuming a Bunch-Davies vacuum. The temperature anisotropy has been calculated for a closed universe with $\Omega_0 = 1.06$. Fig. from [19].

We modified the CAMB program to determine the temperature anisotropy spectrum and we have taken the best fit values of all other parameters like n_s, h, τ etc. We find that there is some difference between the two close to $l = 2$ but essentially no difference at large l. The difference at lower l is highlighted in Fig. 5.2 where we have shown the same plot as in Fig. 5.1, but only for the low l values. We see that the temperature anisotropy calculated using (5.44) shows a suppression at low l compared to the Ratra-Peebles form. However owing to the large cosmic variance statistical error at low l the difference is statistically insignificant.

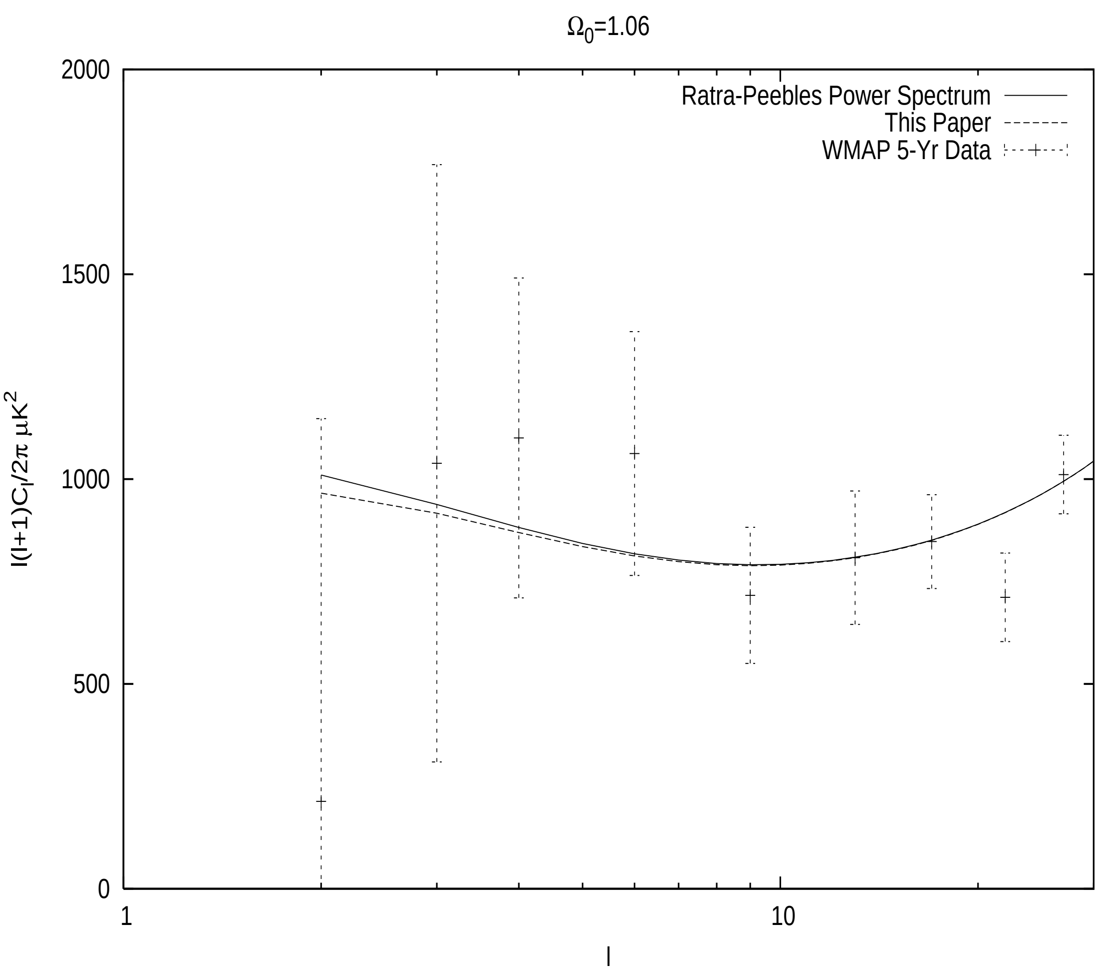

Figure 5.2: Comparison of temperature anisotropy with the Ratra-Peebles power spectrum (5.60) and the power spectrum (5.44) at low values of l for a closed universe with $\Omega_0 = 1.06$. Fig. from [19].

In Fig. 5.3 we show that in the case of closed universe for larger values of Ω_0 the quadrupole anisotropy is even more suppressed and fits the WMAP data better using the closed universe power spectrum (5.44). This qualitatively supports the idea proposed in [88] that a positive spatial curvature should suppress the power at low l but the magnitude of the suppression for realistic values of parameters is not enough to explain the quadrupole suppression observed by WMAP.

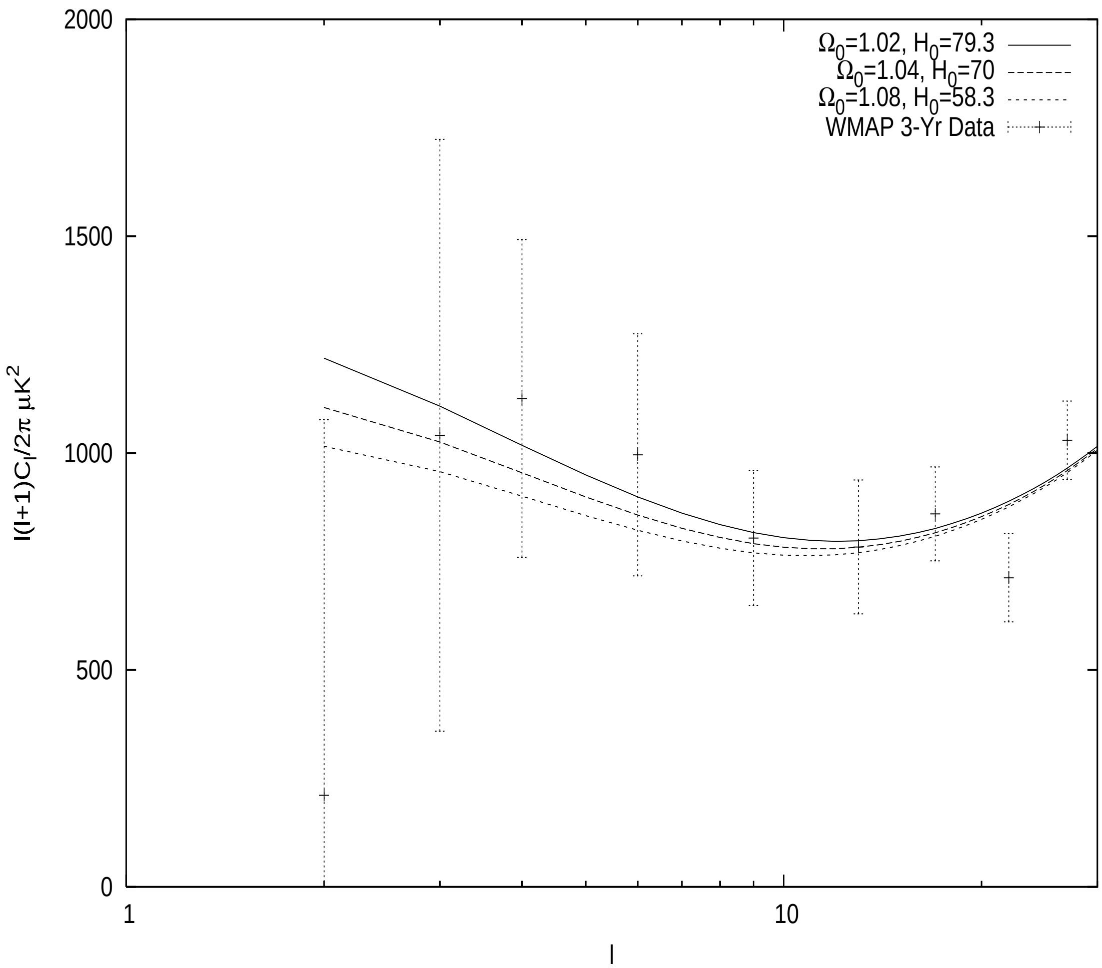

Figure 5.3: Suppression of quadrupole temperature anisotropy with increasing spatial curvature from the power spectrum (5.44). Fig. from [19].

In Fig. 5.4 we show the allowed parameter space of the Hubble parameter and curvature from the WMAP-5 data for the $\Omega wCDM$ model. We have used the power spectrum (5.44) and the Ratra-Peebles form (5.60) to calculate the theoretical prediction for the temperature anisotropy using CAMB. Marginalizing all other parameters we plot the allowed values of H_0 and Ω_0 at 90%$C.L.$ Since the theoretical prediction from the two power spectra match closely except at low l, the chi-square from the two differs only in the second decimal place and the allowed parameter space from the two power spectra are identical as shown in Fig. 5.4.

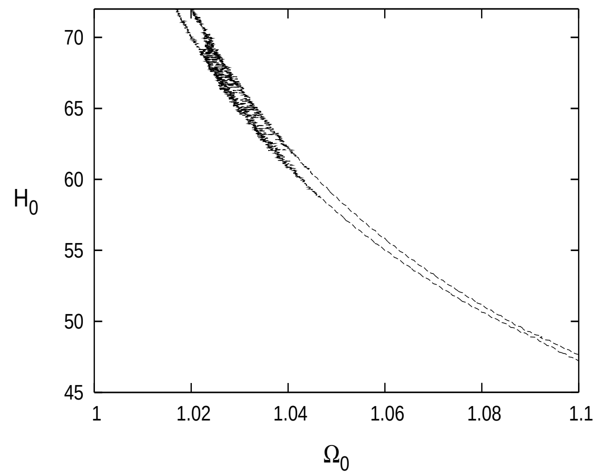

Figure 5.4: The allowed parameter space for Ω_0 and Hubble parameter H_0 (in units $Km/sec/Mpc$) at 90%$C.L$ from WMAP5 data. There is no discernible difference in the parameter space when one assumes the Ratra-Peebles form of power spectrum (5.60) or the form (5.44) calculated in this chapter. Fig from [19].

5.6 Conclusion

At the beginning of inflation the curvature $\Omega - 1$ is expected to be of order one. By the time perturbations of our horizon size exit the inflation horizon, the curvature drops to $\Omega_0 - 1$ which is the present value. A non-zero observation of the curvature will tell us whether the universe prior to inflation was open or closed (even though it is almost flat now) and put constraints on the number of extra e-foldings that must have occurred beyond the minimum number needed to solve the horizon problem. Spatial curvature is a threshold effect which can give us information on the pre-inflation universe from observations of the CMB anisotropy at large angles, similar to the effect of a possible pre-inflation thermal era [42, 17]. From the power spectrum of the closed (5.1) and open inflation (5.2) cases we see that if $K > 0$, power is suppressed at large angles and if $K < 0$ power is enhanced at large angles. The WMAP observation of a strong suppression of the quadrupole temperature anisotropy cannot be explained by the modified power spectrum for a closed universe as suggested by [88] for realistic values of other parameters (like H_0) . The determination of the spatial curvature from the WMAP data is not observably affected by the choice of the boundary condition used for the determination of the primordial power spectrum in a curved inflationary universe. The difference in the anisotropy at low l from different calculations of the power spectrum are smaller than the cosmic variance and therefore are indistinguishable even in principle.

Chapter 6

Conclusion and discussions

Inflation [1] solves the major problems of the standard model of Big-Bang cosmology and it also provides seeds for the CMB anisotropy and structures in the universe. It predicts nearly scale invariant and Gaussian density perturbations that are consistent with the CMB observations [16]. To satisfy CMB observations ther must be nearly 60 efoldings from the end of inflation to the time when the length scales corresponding to our horizon leaves the inflationary horizon. The amplitude and the spectral index of the scalar perturbations must be [16] $A_{\mathcal{R}} = 2.41 \pm 0.11 \times 10^{-9}$ and $n_s = 0.963^{+0.014}_{-0.015}$ respectively. These observational constraints put an upper bound on the scale of inflation that turns out to be at the GUT scale ($(V)^{\frac{1}{4}} \leq 10^{16}$GeV). The lower bound on the scale of inflation comes from the observations of tensor perturbations which will be confirmed by the detection of B-mode of CMB polarization. The parameters of the inflaton potential are determined by the amplitude and the spectral index of the density perturbations. As in $m^2\phi^2$ or $\lambda\phi^4$ type of potentials the value of ϕ must be greater than M_P and $\lambda \leq 10^{-12}$. These models are not natural from particle physics point of view.

There are large number of attempts to come up with a model that can make sense from observational and theoretical point of view. One of them is warm inflation [12] in which radiation is also present during inflation but it remains subdominant. Inflaton is being dissipated into the radiation, so the temperature remains nearly constant and there is no phase of reheating. In these models density perturbations are generated by thermal fluctuations in the inflaton field. Since thermal fluctuations are classical so this model also provide a solution to the quantum to classical transition problem of cold inflation. In cold inflation models there is a model of inflation called as, ”Natural inflation” in which inflaton is PNGB of some spontaneously broken symmetry. This model can accomodate small scales as $\lambda \sim \frac{\Lambda^4}{f^4}$ where Λ is explicit symmetry breaking scale and f is spontaneous symmetry breaking (SSB) scale. The flatness of the potential in this model is protected from higher order corrections due to shift symmetry. Observational constraints on n_s require that the SSB scale f should be $f \sim M_P$ [15].

Although inflation seems to be a best model to understand the origin of CMB anisotropy, there are still some unexplained features in the large scale CMB anisotropy like low quadrupole and glitches at low l. As these anisotropies were generated during the very early stages of inflation, they can have imprint of pre-inflationary universe. Study of pre-inflation universe can give us information about the correct model of inflation.

In this thesis we have presented the effect of pre-inflationary radiation era and effect of curvature before inflation on CMB anisotropy and polarization. We have shown [17] that due to the decoupled thermal distribution of gravitons B-modes of CMB polarization will be enhanced at low l. As WMAP gives an upper bound on the B-modes but the standard theory predicts very low tensor to scalar ratio which is below the sensitivity of WMAP. Due to this enhancement WMAP in future may see a signal of B-modes and it will put an upper bound on the graviton temperature when our observable universe was leaving the inflationary horizon. If this temperature is less than the temperature at the beginning of inflation we will have extra e-foldings than what is required to solve horizon problem. In warm inflation the temperature remains nearly constant so the predictions of thermal enhancement of tensor modes cannot be evaded (if not seen in B-modes) by assuming extra e-foldings as in cold inflation with a pre-inflation thermal era.

Warm inflationary models require radiation as sub-dominant component to provide damping during inflation. We have studied warm inflation with PNGB potential [18] and we have shown that the spontaneous symmetry breaking scale can be reduced to the GUT scale. Another feature of this model is that it can generate large non-Guassianity that can be observed in future experiments. Since PNGB has derivative coupling to the lepton current, this model can automatically generated lepton asymmetry during inflation.

Another important question related to the pre-inflation universe is whether the universe was closed or open before inflation. Inflation tells that the curvature of the universe at the time when our horizon left the de Sitter horizon should be equal to the curvature at present. If we observe a non-zero value of the curvature at present, we can say that inflation lasted only for minimum number of e-foldings and the universe was not flat before inflation. A large curvature before inflation may affect large scale CMB anisotropy. We have calculated power spectrum for curvature perturbations in both closed and open universe during inflation [19]. We have used Bunch-Davies boundary condition for the mode functions and have evaluated the power spectrum at horizon crossing. It is observed that there is suppression at low l for closed universe. But this suppression is not sufficient to explain the low value of quadrupole. The effect at low l temperature anisotropy due to different boundary conditions that are used to calculate different power spectra, is not observable because of cosmic variance.

List of publications

- Kaushik Bhattacharya, Subhendra Mohanty and Akhilesh Nautiyal,
  Enhanced polarization of CMB from thermal gravitational waves,
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