TOPICS IN SPIN DYNAMICS OF CHARGED PARTICLES AND ENERGY AND ANGULAR MOMENTUM IN CURVED SPACETIME

K. S. VIRBHADRA

PHYSICAL RESEARCH LABORATORY
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DECLARATION

I hereby declare that the work presented in this thesis is original and has not formed the basis for the award of any degree or diploma by any University or Institution.

Author

K. S. Virbhadra.

K. S. Virbhadra
Theoretical Physics Division
Physical Research Laboratory
Ahmedabad - 380 009
India.

Certified by

Thesis Supervisor

A R Prasanna

A. R. Prasanna
Theoretical Physics Division
Physical Research Laboratory
Ahmedabad - 380 009
India.

Date : August 2, 1991

DEDICATED TO

My Late Mother

and

My Late Younger Sister

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Contents

Chapter 1

Introduction

Noticing that the Newton's law of gravitation is irreconcilable with the special theory of relativity, Einstein postulated a new theory of space, time, and gravitation which is known as the general theory of relativity (GTR). Unlike the case of the special theory of relativity, there have been some objections by few physicists as well as Einstein himself against the new theory of gravitation. Nontheless, it has witnessed better experimental evidences as compared to the Newtonian gravity. Therefore, one has a predilection for the Einstein's theory of gravitation over the Newtonian theory. However, it remained almost in quiesecent stage for a long time, partly due to the reason that it presents an entirely new standpoint which was difficult to understand, and partly because the high energy compact objects were not discovered and the Newtonian theory was sufficient to deal with the the physical systems in hand. However, the discovery of the quassars and compact X-ray sources encouraged many researchers to study general relativity to investigate about various esoteric compact objects. Another striking reason for growing interest in this subject is that a deeper understanding of the classical theory of gravitation could be helpful in the way of achieving a viable theory of quantum gravity. Many researchers have put painstaking efforts for investigating into the various aspects of this subject.

The dynamical features of any field is best understood through the trajectories of the test particles in the representative spacetime. In general theory of relativity, as the gravitational field is represented by the spacetime curvature, the spacetime structure itself dictates the orbit of the test particles which when not subjected to any other interaction, move along geodesics. The trajectories of the test particles in various spacetimes have been studied by many authors which are cited in a paper by Sharp (1979). However, in many of the physical situations there are other fields and interactions, amongst which the electromagnetic field is prevalent one. Charged particles in curved spacetime in the presence of the electromagnetic fields do not follow geodesics and similarly the test particles with spin also deviate from geodesics as shown by Papapetrou (1951). Further Corinaldesi and Papapetrou (1951) discussed the equations of motion for spinning test particles satisfying the condition $S^{i0} = 0$ (where S^{ik} is the spin tensor). One of the important features related to the dynamics of the spin is its precession induced by the interaction with the field. Schiff (1960) studied the geodetic spin precession of a test particle in a free fall about a massive sphere and proposed the result for a plausible test of general relativity. Unfortunately, due to some technical problems, the gyroscope experiment could not be accomplished as yet. However, the study of the spin precession of charged particles in curved spacetime background endowed with electromagnetic fields has not been paid proper attention. A charged spinning particle has its magnetic dipole moment δ, proportional to its spin angular momentum S, through the relation δ = g e S/ 2 m_o, where g the Lande factor has values for example, for electron and proton, g_e = 2.0023 and g_p = 5.59, respectively. In the presence of a magnetic field, the spin angular momentum vector suffers a precession due to the torque $\delta \times B$ acting on it. Anderson (1967) gave a relativistic generalization for a torque acting on a spinning charged particle due to the electromagnetic field. Prasanna and Kumar (1973), using the Anderson's generalization of the torque and Papapetrou's equations of spin and orbit with a Lorentz force term, studied the spin precession of charged particles in Melvin's magnetic universe.

We (Prasanna and Virbhadra 1989; Virbhadra and Prasanna 1989, 1990, 1991) expressed these generalized equations in a rather convenient form and have investigated the same in the following physical situations : (a) a magnetic field dipolar at infinity superposed on Schwarzschild background (Ginzburg- Ozernoi solution), (b) a uniform magnetic field superposed on Schwarzschild background (Wald solution), (c) a Reissner-Nordström source, and (d) a Schwarzschild object embeded in a uniform magnetic field (Ernst solution). Unlike the cae of the Wald solution which is the solution of the Maxwell equations on Schwarzschild background, the Ernst solution being the solution of the Einstein-Maxwell equations incorporates the effects of curvature due to the magnetic field. We confined our attention to the particles in circular motion on the equatorial plane and found the spin precession frequencies for the aforesaid cases. The flat space limit to the expressions obtained for the spin precession frequency yields interesting results. We have found that cases (a) and (b) yield special relativistic contribution to the well known Larmor frequency, whereas case (c) puts a bound ($0 > g$, $g > 2$) on the Lande g-factor. Apart from the above there are other interesting outcomes of these investigations. Thōugh the calculations have been accomplished in a fully covariant prescription, the purely geodetic terms have not appeared in the result. However, when we (Virbhadra and Prasanna 1991) have introduced local Lorentz frames the purely geodetic terms have appeared explicitly along with the other terms.

Another subject which drew our attention is the energy and angular momentum in curved spacetimes. The energy- momentum localization in general relativity is a problematic issue and it has been a subject of extensive research since the outset of the general relativity. While an adequate localization of energy and momentum would have immense benifits, the status is that one does not have that so far. Since Einstein's original pseudotensor (Møller 1958), a fairly large number of expressions for energy and momentum in general relativistic systems have been suggested by many authors, e.g., Tolman (1930), Landau and Lifshitz (1985), Papapetrou (1948), Gupta (1954), Møller (1958), Goldberg (1958), Bergmann (1958), Dirac (1959), Komar (1959, 1962, 1963), Arnowitt et al.(1961), Bondi et al (1962), Hawking (1968), Weinberg (1972), Witten (1981), Penrose (1982), Lynden-Bell and Katz (1985), Nahmad-Achar and Schutz (1987*a*, 1987*b*), Kulkarni et al (1988), Bartnik (1989),Katz and Ori (1990) etc. There are mutually opposing viewpoints that authors share regarding the physical importance of the energy-momenum pseudotensors as well as a possibility of successful localization of energy and momentum in curved spacetime. However, the total energy and momentum in asymptotically flat spacetimes have an unambiguous importance. Weinberg (1972), using his own prescription for energy and angular momentum in asymptotically flat spacetimes, found the total energy and angular momentum associated with the Kerr spacetime to be M and Ma respectively (M stands for the mass parameter whereas a stands for the rotation parameter in Kerr metric). Palmer (1980) discussed the significance of Einstein's energy-momentum pseudotensor in detail. It is well known that the pseudotensors of Einstein, Tolman, and Landau and Lifshitz (LL) can yield sensible result only if the calculations are carried out in quasi-Cartesian coordinates (that in which increase in spatial distance converges the components of the metric tensor to their values of special relativity) which are usually very lengthy to accomplish. Rosen (1956) evaluated energy and momentum of cylindrical gravitational waves in Cylindrical polar coordinats in LL prescription and found that the waves in empty space did not carry energy and momentum. Later he (Rosen 1958) repeated the calculation in quasi-Cartesian coordinates and got the energy and momentum to be finite and reasonable. Møller (1958) constructed a new energy-mometum complex and claimed that one is not anymore bound to the use of the quasi-Cartesian coordinates. Only three years after this work, Møller (1961) realized that the total energy-momentum vector of a closed physical system is not a Lorentz four-vector in his formulation. However, the energy density component of the Møller's complex transforms like a scalar density under purely spatial transforamations.

Cohen and de Felice (1984) calculated the Komar energy in Kerr-Newman spacetime and interpreted that to be effective gravitational mass that a neutral test particle present in the field of the Kerr-Newman object experiences through the gravitational interaction. However, switching off the charge parameter gives no energy to the exterior of the Kerr black hole. Looking into the result of Cohen and de Felice, Kulkarni et al (1988) argued that a modification of the Komar integral was warranted since that did not yield the repulsive effect arising from the rotation. They proposed a new definition of the effective gravitational mass of the Kerr black hole that incorporated the contribution due to the rotation.

We believe that so long one does not have a successful localization of energy and momentum in general relativistic system, it is desirable that the relative merits and demerits of various definitions to give energy as well as angular momentum be investigated. The energy and momentum pseudotensors of Einstein, Tolman, Landau and Lishitz (LL), and Møller are largely discussed in the literature and we (Virbhadra 1990*a*, 1990*b*, 1990*c*, 1991a, 1991b, 1991c) investigated these complexes for the Kerr-Newman as well as the Vaidya radiating spacetimes. We have evaluated all the components of the energy-momentum pseudotensors of Einstein, Tolman, Landau and Lifshitz, and Møller for the Kerr-Newman (upto the third power of the rotation parameter) as well as the Vaidya radiating spacetimes. We (Virbhadra 1990a, 1990c) have found that the pseudotensors of Einstein, Tolman, and LL give exactly same energy density in Kerr-Newman spacetime, whereas that of Møller gives twice the value obtained using these definitions. All these four pseudotensors yield no energy in the Kerr spacetime and the entire energy in the Kerr-Newman spacetime is due to the electromagnetic field present there. The pseudotensor of LL, being symmetric in indices, can be used to evaluate angular momentum in asymptotically flat spacaetimes. We (Virbhadra 1990c) calculated the same for the K-N spacetime and got sensible result. Unlike the case of the K-N field, we (Virbhadra 1991b) have found that all these comlexes give the same energy density in the Vaidya spacetime. The pseudotensors of Einstein and Tolman gave same result (for all of their components) for the K-N as well as the Vaidya spacetimes. Despite their non-tensor character, the aforementioned pseudotensors are found to be traceless for both spacetimes.

Recently Cooperstock and Richardson (1991) have extended our result for energy in Kerr-Newman field upto the seventh order of the rotation parameter and have found the same relationship that the prescription of Einstein, Tolman, and Landau and Lifshitz give same result whereas that of

Møller yields twice the value. They have also pointed out that the Komar energy for the R-N metric calculated by Cohen and de Felice does not give the correct flat space limit whereas that of Einstein ,Tolman, and Landau and Lifshitz do give.

The energy-momentum complexes discussed above give meaningful result if the calculations are carried out in quasi-Cartesian coordinates (Kerr-Schild Cartesian coordinates satisfy the condition of quasi-Cartesian coordinates). An asymptotically flat metric can always be expressed in quasi-Cartesian coordinates though it may not be in Kerr-Schild form. Using the pseudotensors of Einstein, Landau and Lifshitz, and Møller (Vaidya, 1952, calculated the same in Tolman's prescription), we (Virbhadra 1991*a*) have claculated energy in the Reissner-Nordström metric (in quasi-Cartesian coordinates though the line-element being not in the Kerr-Schild form) and have found that the pseudotensors of Einstein, Tolman, and Møller give respectively same result as we obtained for the same spacetime in Kerr-Schild Cartesian coordinates. However, the pseudotensor of Landau and Lifshitz does not give a consistent result.

The thesis is organised as follows : the chapters two and three contain study of spin dynamics, and the chapters four to six are devoted to the study of energy, momentum and angular momentum in curved spacetimes. We use the geometized units (G = c = 1) and follow the convention that Latin and Greek indices run from 0 to 3 and 1 to 3 respectively. We adopt the Einstein summation convention.

Chapter 2

Spin precession of charged particles in presence of electromagnetic fields on curved spacetime background

2.1 Introduction

Einstein had proposed three classical tests of general relativity, namely the gravitational red-shift of spectral lines, the deflection of light by the sun, and the precession of perihelia of orbits of the inner planets. Various observations have shown excellent agreement with the theoretical predictions bearing out the beauty of the general theory of relativity. In 1964, Shapiro proposed a new test of the theory that there is a general relativistic delay in time for a radar signal to travel to the inner planets and reflected back to the Earth. In 1967, Shapiro, along with some collaborators (Shapiro et al 1968), conducted experiments which yielded remarkable agreement with his theoretical prediction. Schiff (1960) calculated the precession frequency of a gyroscope's spin axis relative to the distant stars when the gyroscope moves in a free fall about the massive sphere, and he proposed this for a test of general relativity. However, due to many technical problems, this plan has not yet succeeded. As the general theory of relativity has witnessed sufficient experimental support, it is accepted as the most satisfactory theory for the description of gravitational phenomena.

An important aspect of the astrophysical phenomena is the role of electromagnetic fields in the dynamics of charged particles which govern the electromagnetic features leading to different kinds of emission mechanisms. Therefore, a detailed knowledge of the dynamical features of charged particles in presence of electromagnetic fields on a curved spacetime background is very much desired. Many authors have studied the orbits of single charged particles (having no spin) in presence of electromagnetic fields, a review of which is presented by Prasanna (1980). However, one knows that many charged particles possess spin and therefore it is also desirable to understand spin precession since this could also have a bearing on certain emission features. Bargmann, Michel and Telegdi (1959) discussed the spin dynamics of charged particles in flat space background. Prasanna and Kumar (1973) did a very limited study of the spin precession of a charged particle in Melvin's magnetic universe. In the context of astrophysical applications it is very desirable to study the orbits as well as spin dynamics of charged particles near objects like neutron stars which are enriched with high magnetic field and reasonably high spacetime curvature. However, one does not have, as yet, an exact solution of Einstein-Maxwell equations describing such objects (there is solution for non-rotating magnetic dipole by Gutsunaev and Manko, 1987, which is however too lenghthy). Therefore, we have considered few cases, though which are not as realistic as in the case of neutron star, gives an understanding of the role of spacetime curvature on the spin dynamics of charged particles in presence of electromagnetic fields. We (Prasanna and Virbhadra 1989; Virbhadra and Prasanna 1989, 1990), using the Papapetrou's equations of spin (Papapetrou 1951) and the Anderson's relativistic generalization (Anderson 1967) for a torque acting on a spinning charged particle due to electromagnetic field, have carried out detailed studies of this subject for various cases and this is the matter of discussions in the following. In the flat spacetime limit, our investigations yield striking results, as it gives special relativistic correction to the well known Larmor precession frequency and further puts a bound to the Lande g-factor.

2.2 Formalism

The magnetic dipole moment δ of a charged spinning particle is proportional to its spin angular momentum S, given by

$$\delta = \frac{geS}{2m_0} \qquad (2.2.1)$$

g,e, and m_0 stand respectively for Lande factor, charge, and rest mass of the test paricle. In presence of a magnetic field B, the spin angular momentum vector suffers a precession due to a torque acting on it, governed through

$$\frac{dS}{dt} = \frac{ge}{2m_0} S \times B \tag{2.2.2}$$

Anderson (1967) gave the relativistic generalization for the torque acting on a charged particle in presence of electromagnetic field:

$$\frac{dS^{ik}}{d\tau} = \frac{ge}{2m_0} \left[S^{kl} F^{i}{}_{l} - S^{il} F^{k}{}_{l} \right] \tag{2.2.3}$$

where S^{ik} is the spin angular momentum tensor and F^{ik} is the electromagnetic field tensor.

On the other hand, even an uncharged particle in a pure gravitational field undergoes precession called geodetic (or de Sitter) precession, governed through the Papapetrou's equations (1951) of spin

$$\frac{DS^{ik}}{D\tau} + \left[U^{i} \frac{DS^{kl}}{D\tau} - U^{k} \frac{DS^{il}}{D\tau} \right] U_{l} = 0 \tag{2.2.4}$$

and of orbit

$$\frac{D}{D\tau} \left[m_0 U^{i} + \frac{DS^{ik}}{D\tau} U_{k} \right] + \frac{1}{2} S^{ab} U^{c} R^{i}{}_{cab} = 0 \tag{2.2.5}$$

where U^{i} is the velocity four-vector and

$$\frac{DU^{i}}{D\tau} \equiv U^{i}{}_{;k} U^{k} \tag{2.2.6}$$

Using the above expressions, one can write the equations of spin and of orbit for charged spinning particles moving in presence of electromagnetic fields in curved spacetime background as follows:

$$\frac{DS^{ik}}{D\tau} + \left[U^i \frac{DS^{kl}}{D\tau} - U^k \frac{DS^{il}}{D\tau}\right] U_l = \frac{ge}{2m_0}\left[S^{kl}F^i{}_l - S^{il}F^k_l\right] \qquad (2.2.7)$$

$$\frac{D}{D\tau}\left[m_0 U^i + \frac{DS^{ik}}{D\tau}U_k\right] + \frac{1}{2}\, S^{ab}U^c R^i{}_{cab} = eF_k{}^i U^k \qquad (2.2.8)$$

where the spin tensor S^{ik} and the velocity four-vector U^i are orthogonal to each other as prescribed by Pirani condition:

$$S^{ik}U_k = 0 \qquad (2.2.9)$$

We (Prasanna and Virbhadra 1989) have expressed the equations of spin and orbit in a rather convenient form:

$$\frac{dS^i}{d\tau} = -\left[\Gamma^i{}_{jk}U^k + U^i\Sigma_j\right] S^j + \frac{e}{2\,m_0}\left[gF_j{}^i - (g-2)\,F_{jk}U^iU^k\right] S^j \quad (2.2.10)$$

$$\frac{dU^i}{d\tau} = -\Gamma^i{}_{jk}U^jU^k - \frac{e}{m_0}F^i{}_kU^k + \Sigma^i \qquad (2.2.11)$$

where S^i is the spin four-vector given by

$$S^i \;=\; \frac{1}{2}\,\epsilon^{ijkl}\, S_{jk}\, U_l \qquad (2.2.12)$$

and

$$\Sigma^i = -\frac{1}{m_0}\left[\frac{D}{D\tau}\left(\frac{DS^{ik}}{D\tau}U_k\right) + \frac{1}{2}S^{ab}U^c R^i{}_{cab}\right] \qquad (2.2.13)$$

As a first approximation, we have neglected the spin curvature interaction terms as well as the second order derivative of the spin terms by ignoring Σ^i and then consider the equations

$$\frac{dS^i}{d\tau} = \alpha^i{}_j \, S^j \qquad (2.2.14)$$

with

$$\alpha^i{}_j = -\Gamma^i_{jk}U^k + \frac{e}{2\,m_0}\left[gF_j{}^i - (g-2)\,F_{jk}U^iU^k\right] \qquad (2.2.15)$$

and

$$\frac{dU^i}{d\tau} = -\left[\Gamma^i{}_{jk}U^j + \frac{e}{m_0}F^i{}_k\right]U^k \qquad (2.2.16)$$

2.3 Static spacetimes endowed with electromagnetic fields

We have studied the spin precession frequency of charged particles in the following situations:

(a) Magnetic field, dipolar at infinity, superposed on a Schwarzschild background. Ginzburg and Ozernoi (1965) found a solution for the Maxwell equations on the Schwarzschild background which is expressed in terms of the vector potential given by (Prasanna and Varma 1977)

$$A_\phi = -\frac{3\mu \sin^2\theta}{8\,M^3}\left[r^2 \ln\left(1-\frac{2M}{r}\right) + 2M\,(r+M)\right] \tag{2.3.17}$$

The Schwarzschild spacetime in Schwarzschild coordinates is expressed by the line-element:

$$d\tau^2 = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.3.18}$$

where μ and M stand for the magnetic dipole moment and the Schwarzschild mass respectively.

(b) A uniform magnetic field, superposed on Schwarzschild background which is given by the Wald solution (Wald 1974; Prasanna and Vishveshwara 1978).

$$A_\phi = \frac{B_0 r^2 \sin^2\theta}{2} \tag{2.3.19}$$

where B_0 stands for the magnetic field parameter.

(c) A spherically symmetric charged massive object. The solution of Einstein-Maxwell equations for this case is the well known Reissner- Nordström solution (Hawking and Ellis 1987), given by the line-element:

$$d\tau^2 = \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2 - \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\ d\phi^2\right) \tag{2.3.20}$$

and the vector potential $A_i = (A_t, 0, 0, 0)$ with

$$A_t = \frac{Q}{r} \tag{2.3.21}$$

where M and Q stand for mass and charge parameters of the central object. It is clear that switching off the charge parameter gives the Schwarzschild solution.

(d) A Schwarzschild object embedded in a uniform magnetic field. Ernst (1974) gave a solution for the corresponding Einstein-Maxwell equations as expressed by the axisymmetric line-element:

$$d\tau^2 = \Lambda^2 \left[\left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 \right] - \Lambda^{-2} r^2 \sin^2\theta d\phi^2 \tag{2.3.22}$$

with the vector potential given by

$$A_\phi = \frac{B r^2 \sin^2\theta}{\Lambda} \tag{2.3.23}$$

where

$$\Lambda = 1 + r^2 B^2 \sin^2\theta \tag{2.3.24}$$

and

$$B \equiv \frac{B_0}{2} \tag{2.3.25}$$

M and B_0 stand respectively for the mass and the magnetic field parameters. Unlike the case of the Wald solution which is the solution of the

Maxwell equations on Schwarzschild backgound, the Ernst solution is the solution of the Einstein-Maxwell equations and this incorporates the effects of the magnetic field on the spacetime curvature.

2.4 Calculations

Confining our attention to the particles in circular motion on the equatorial plane $\left(\theta = \frac{\pi}{2}\right)$ as prescribed by $U^R = U^\theta = 0$ at r = R, the orbit equation along with the orthonormality relation $g_{ij}U^iU^j = 1$, define the orbital frequency U^ϕ. For the physical situations under investigations (mentioned in the last section), one has the orbital equation:

$$\Gamma^R_{\phi\phi}U^{\phi^2} + \Gamma^R_{tt}U^{t^2} = -\frac{e}{m_0}\left[F^R_{\phi}U^\phi + F^R_{t}U^t\right] \tag{2.4.26}$$

with

$$g_{tt}U^{t^2} + g_{\phi\phi}U^{\phi^2} = 1 \tag{2.4.27}$$

where all the functions are evaluated at r = R, the radius of the circular orbit.

For all the cases described in the earlier section, one has the spin equations for particle on the equatorial plane $(\theta = \frac{\pi}{2})$ to be

$$\frac{dS^t}{d\tau} = -\Gamma^t{}_{rt}\left(S^tU^r + S^rU^t\right) + \frac{e}{2m_0}\left[g\ g^{tt}F_{rt}S^r - (g-2)U^t\left\{F_{tr}\left(S^tU^r\right.\right.\right.$$

$$\left.\left.\left. - S^rU^t\right) + F_{r\phi}\left(S^rU^\phi - S^\phi U^r\right)\right\}\right] \quad (2.4.28)$$

$$\frac{dS^r}{d\tau} = - \left(\Gamma^r_{tt}S^tU^t + \Gamma^r_{rr}S^rU^r + \Gamma^r_{\phi\phi}s^\phi U^\phi\right) + \frac{e}{2m_0}\left[g\ g^{rr}\left(F_{tr}S^t + F_{\phi r}S^\phi\right)\right.$$

$$\left. - (g-2)U^r\left\{F_{tr}\left(S^tU^r - S^rU^t\right) + F_{r\phi}\left(S^rU^\phi - S^\phi U^r\right)\right\}\right] \quad (2.4.29)$$

$$\frac{dS^\theta}{d\tau} = -\Gamma^\theta{}_{\theta r}\ S^\theta\ U^r \quad (2.4.30)$$

$$\frac{dS^\phi}{d\tau} = - \Gamma^\phi{}_{r\phi}\left(S^rU^\phi + S^\phi U^r\right) + \frac{e}{2m_0}\left[g\ g^{\phi\phi}\ S^r\ F_{r\phi} - (g-2)U^\phi\right.$$

$$\left.\left\{F_{tr}\left(S^tU^r - S^rU^t\right) + F_{r\phi}\left(S^rU^\phi - S^\phi U^r\right)\right\}\right] \quad (2.4.31)$$

As S^θ does not appear in other equations except in (2.4.30), one can without loss of generality consider it to be zero so that $dS^\theta/d\tau = 0$. We confine our attention to particles in circular orbits. The relevant spin equations for the physical situations under investigation are

$$\frac{dS^R}{d\tau} = L_1 \; S^\phi \tag{2.4.32}$$

$$\frac{dS^\phi}{d\tau} = L_2 \; S^R \tag{2.4.33}$$

with

$$L_1 = \left[\Gamma^R_{\ tt} g^{tt} g_{\phi\phi} - \Gamma^R_{\ \phi\phi}\right] U^\phi - \frac{ge}{2m_0} \; g^{RR} \; \left[F_{R\phi} - F_{Rt} \frac{g_{\phi\phi} g^{tt} U^\phi}{U^t}\right] \tag{2.4.34}$$

$$L_2 = - \; \Gamma^\phi_{\ R\phi} U^\phi + \frac{ge}{2m_0} g^{\phi\phi} F_{R\phi} - \frac{(g-2)\, e}{2m_0} U^\phi \left[F_{Rt} U^t + F_{R\phi} U^\phi\right] \tag{2.4.35}$$

The polar coordinates have an inherent rotation with respect to Cartesian coordinates. Since we are looking for the effective spin precession, it is best to get rid of this inherent rotation of the coordinate system by going over to the Cartesian coordinates. Therefore, we have rewritten the spin equations in terms of the components of the spin vector in Cartesian coordinates:

$$S^x = S^R \cos\phi - R S^\phi \sin\phi,$$

$$S^y = S^R \sin\phi + R S^\phi \cos\phi,$$

$$S^z = 0 \tag{2.4.36}$$

The relevant spin equations in terms of Cartesian components are:

$$\frac{dS^x}{d\tau} = \frac{1}{R}\left[-S^x\ cos\phi\ sin\phi\ \beta + S^y\left\{cos^2\phi\ \beta - R\left(U^\phi + L_2 R\right)\right\}\right] \quad (2.4.37)$$

$$\frac{dS^y}{d\tau} = \frac{1}{R}\left[S^x\left\{cos^2\phi\ \beta + \left(RU^\phi - L_1\right)\right\} + S^y cos\phi\ sin\phi\ \beta\right] \quad (2.4.38)$$

with

$$\beta = L_1\ +\ L_2\ R^2 \quad (2.4.39)$$

Further solving them, we have found the spin precession frequency ω_s which is given by

$$\omega_s^2 = \left(U^\phi - \vartheta_1\right)\left(U^\phi - \vartheta_2\right) \quad (2.4.40)$$

where

$$\vartheta_1 = \frac{L_1}{R},$$
$$\vartheta_2 = R\ L_2 \quad (2.4.41)$$

Using the above, we have further evaluated the orbital as well as spin precession frequencies for the cases described in the last section.

(a) Schwarzschild background with superposed magnetic field (which is dipolar at infinity).

The orbital frequency Ω ($\Omega = cU^{\phi}$) is given by

$$(R-3M)\,\Omega^2 - \frac{3e\mu}{2M^2 m_0} F_R \Omega - \frac{M}{R^2} = 0 \qquad (2.4.42)$$

and the spin precession frequency ω_s is given by

$$\left(\omega_s^2\right)_{global} = \frac{9ge\mu F_R}{4m_0 M}\left(1-\frac{2M}{R}\right)^{-1}\left[\frac{ge\mu F_R}{4m_0 M^3 R^2} + \frac{U^{\phi}}{R^2} + \frac{(g-2)\,e\mu F_R}{4m_0 M^3} U^{\phi 2} + \left(1-\frac{2}{g}\right) U^{\phi 3}\right] \qquad (2.4.43)$$

with

$$F_R = \left(\frac{R}{2M}-1\right)\ln\left(1-\frac{2M}{R}\right) + \left(1-\frac{M}{R}\right) \qquad (2.4.44)$$

As in the subsequent chapter, we will discuss the spin precession in local Lorentz frame, we are using the word *global* here for distinction.

(b) Schwarzschild background with superposed uniform magnetic field:

The orbital frequency Ω is given by

$$(R-3M)\Omega^2 + \frac{eBR}{m_0}\left(1-\frac{2M}{R}\right)\Omega - \frac{M}{R^2} = 0 \tag{2.4.45}$$

and the spin precession frequency is given by

$$(\omega_s)^2_{global} = \left(\frac{geB_0}{2m_0}\right)^2 \left[\left(1-\frac{2M}{R}\right) - \frac{6M}{R}U^\phi \frac{m_0}{geB_0} - \left(1-\frac{2}{g}\right)\left(1-\frac{2M}{R}\right) R^2 U^{\phi 2} - 6MR\left(1-\frac{2}{g}\right)\frac{m_0}{geB_0}U^{\phi 3}\right] \tag{2.4.46}$$

(c) Reissner-Nordström field:

The orbital frequency Ω is given by

$$\left[\left(M-\frac{Q^2}{R}\right) + \frac{\sigma^{1/2}eQ}{2m_0} - R\sigma\right]\Omega^2 + \left[\left(\frac{M}{R^2}-\frac{Q^2}{R^3}\right) + \frac{eQ\sigma^{1/2}}{m_0 R^2}\right] \tag{2.4.47}$$

with

$$\sigma = 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \tag{2.4.48}$$

and the spin precession frequency ω_s is given by

$$(\omega_s)^2_{global} = g(g-2)\left[\frac{eQU^\phi}{2m_0 R}\right]^2 \left[1 + \frac{2m_0 R}{geQ}U^t\left(\frac{3M}{R} - \frac{2Q^2}{R^2}\right)\right] \tag{2.4.49}$$

(d) Ernst field:

The orbital frequency Ω is given by

$$[(R-3M)-(\Lambda-1)(3R-5M)]\,\Omega^2+\frac{2\Lambda eB}{m_0}(R-2M)\;\Omega$$
$$-\Lambda^2\left[\frac{M}{R^2}+\frac{2(\Lambda-1)}{R}\left(1-\frac{3M}{2R}\right)\right]=0 \qquad (2.4.50)$$

and the spin precession frequency is given by

$$(\omega_s)^2_{global}=\left(\frac{geB}{m_0\Lambda^4}\right)\left[\frac{geB}{m_0}\quad\left(1-\frac{2M}{R}\right)+\frac{U^\phi}{\Lambda}\left\{\left(6-5\Lambda-\Lambda^5\right)+(3\Lambda-4)\,\frac{3M}{R}\right\}\right.$$
$$+\quad\frac{U^{\phi^2}(\Lambda-1)}{\Lambda^2}\frac{e}{m_0B}\left(1-\frac{2M}{R}\right)+\left(1-\frac{2}{g}\right)\frac{U^{\phi^3}R^2}{\Lambda^3}$$
$$\left.\left\{\left(4-3\Lambda-\Lambda^5\right)+\frac{M}{R}\,(5\Lambda-8)\right\}\right]$$
$$+\quad\left({\omega_s}^2\right)^{geod}_{global} \qquad (2.4.51)$$

with

$$\left(\omega_s{}^2\right)^{geod}_{global} = \frac{2(\Lambda - 1)U^{\phi^2}}{\Lambda^6}\left[(\Lambda - 1)(\Lambda + 4) + (8 - 5\Lambda)\frac{M}{R}\right] \qquad (2.4.52)$$

2.5 Discussion

It is well known that the non-relativistic value for the ratio of the spin precession frequency to the orbital frequency of a charged spinning particle, moving in a circular orbit in presence of a magnetic field, is $g/2$ (g stands for the Lande $g-$factor). Expressed through (2.4.42) to (2.4.52) are the spin precession frequency and the orbital frequency for a charged spinning particle moving in a circular orbit on equatorial plane of a compact object endowed with electromagnetic fields. Cases (a) and (b) deal with dipole and uniform magnetic fields respectively, wherein the spacetime curvature due to the compact massive object modifies the magnetic field but the magnetic fields are too weak to contribute to the spacetime curvature. The expressions obtained for the spin precession frequency ω_s, and the orbital frequency Ω contain terms due to the gravitaional as well as the purely special relativistic effects. The case (d) again deals with a uniform magnetic field wherein, however, the magnetic field is strong enough to contribute to the spacetime curvature.

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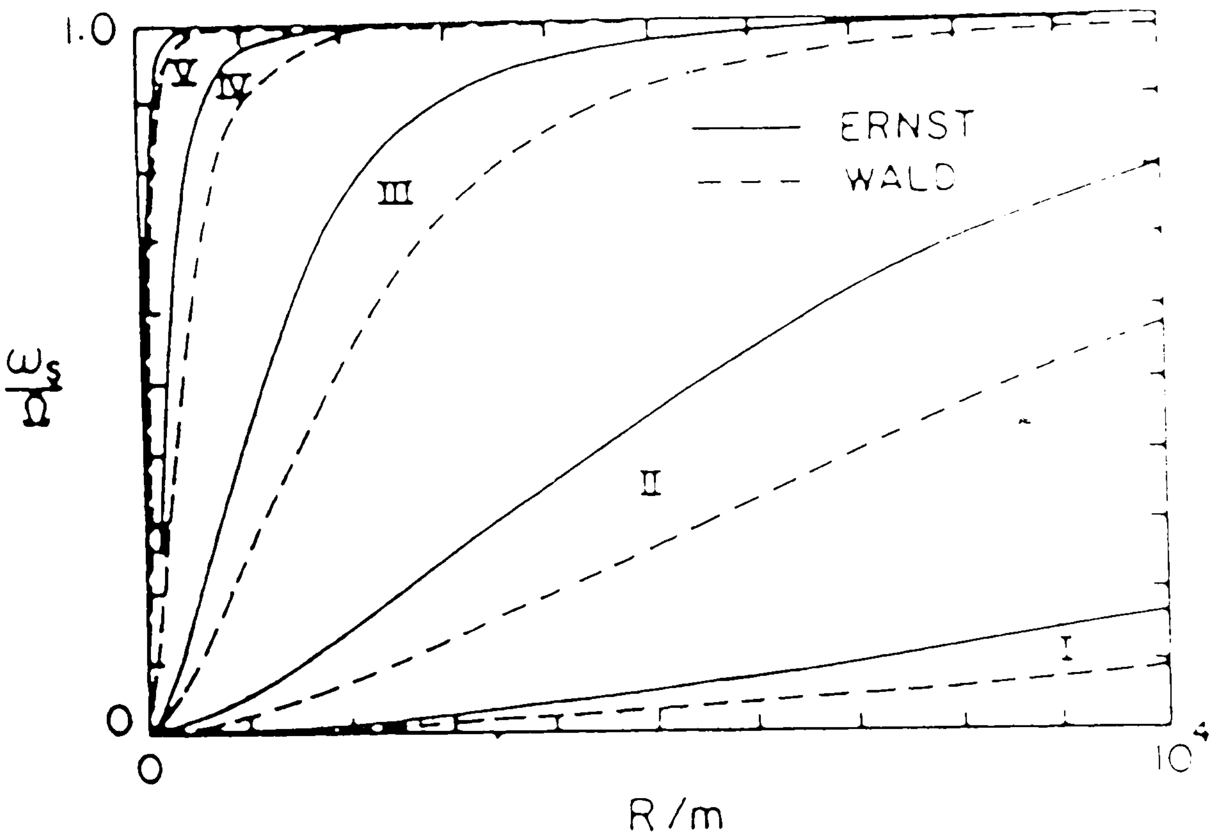

Fig. 1 : The ratio of the spin precession frequency to the orbital frequency versus the radial distance of the test particle (electron) from the central object of one solar mass is plotted for magnetic fields $B = 10^{-9}$(I), 10^{-8}(II), 10^{-7}(III), 10^{-6}(IV), 10^{-5} (V) Gauss for the Wald and the Ernst cases.

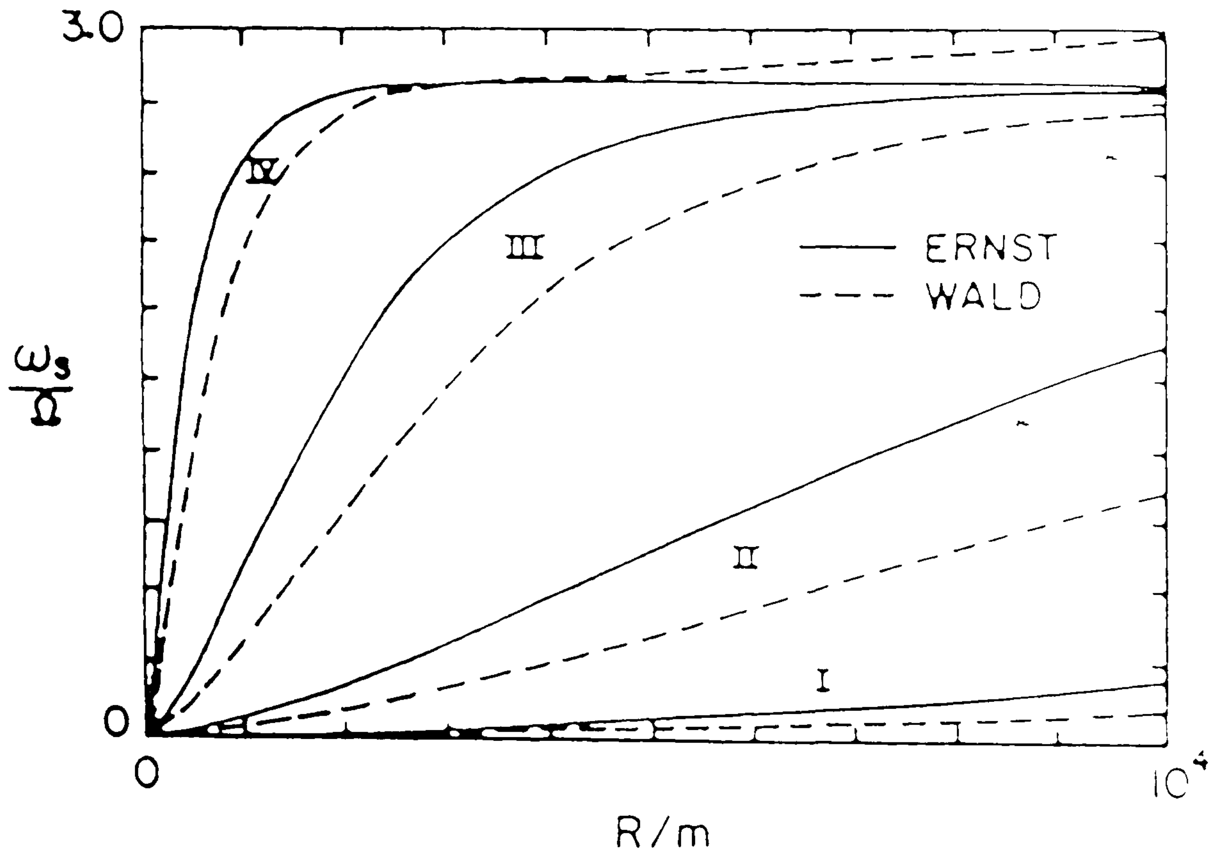

Fig. 2 : The ratio of the spin precession frequency to the orbital frequency versus the radial distance of the test particle (proton) from the central object of one solar mass is plotted for magnetic fields $B = 10^{-6}$(I), 10^{-5}(II), 10^{-4}(III), 10^{-3}(IV) Gauss for the Wald and the Ernst cases.

The figures (1) and (2) present the variation of ω_s/Ω with respect to R/M for various magnetic field strengths for electron ($g = 2.0023$) and proton ($g = 5.59$) for the Wald as well as the Ernst cases. The curves reach asymptotically their approximate value $g/2$ for $v \ll c$ for the Wald case (asymptotic value differs from $g/2$ due to special relativistic effect). Near the central object, due to the spacetime curvature, the value for ω_s/Ω differs appreciably from its flat space value. A decrease in magnetic field decreases the spin precession frequency more than it decreases the orbital frequency, so that their ratio ω_s/Ω decreases with the decreases in the magnetic field. One can see clearly that for the Wald case the asymptotic value for ω_s/Ω goes beyond the non-relativistic value $g/2$ due to special relativistic effect. However, it is interestinig to note that for the Ernst case (where the magnetic field contributes to the spacetime curvature and therefore the spacetime is not asymptotically flat), it seems to saturate near $g/2$ value. Near the central object, the ratio of the frequencies for the Ernst case is more than that of the Wald case for the same magnetic field strength and the radial distance. The plots (1) and (2) are for the electron and proton respectively, which are meant to study the behaviour of test particles of different specific charge (e/m_0) and Lande g-values. For the same magnetic field and the radial distance (near the central object), ω_s/Ω for proton is much less than that of electron. The role of the spacetime curvature to decrease the value of ω_s/Ω from its non-relativistic value is more for the proton than for the electron.

By the expression obtained for the spin precession frequency of the test particle in the Reissner-Nordström field (case c), it is clear that the frequency decreases rapidly with the increase in the radial distance of the test particle from the central object. One gets two terms for the spin precession frequency, the first term is purely special relativistic whereas the second term is general relativistic. The special relativistic contribution to the spin precession frequency is inversely proportional to the rest mas of the test particle. Though the spacetime is not flat at $r = 2Q^2/3M$, it is curious to note that the general relativistic effect on the spin precession frequency vanishes dramatically at this radial distance.

In flat space limit, for cases (a) and (b), one gets

$$\left(\frac{\omega_s}{\Omega}\right)_{MS} = \frac{g}{2}\left[1 + \left(1 - \frac{2}{g}\right) V^2 \gamma^2\right]^{\frac{1}{2}} \tag{2.5.53}$$

MS stands for the magnetostatic case and γ is the Lorentz factor $(1 - V^2)^{-1/2}$. It is clear from the expression (2.4.43) and (2.4.46) that one gets a special relativistic correction to the well known Larmor precession frequency given by

$$\omega_s = \frac{ge\mu}{2m_0 R^3}\left[1 + \left(1 - \frac{2}{g}\right) V^2 \gamma^2\right]^{\frac{1}{2}} \tag{2.5.54}$$

for the magnetic dipole field (case a)

and

$$\omega_s = \frac{geB}{2m_0}\left[1 + \left(1 - \frac{2}{g}\right) V^2 \gamma^2\right]^{\frac{1}{2}} \tag{2.5.55}$$

for the uniform magnetic field (case b).

The special relativistic term is proportional to a factor which depends on the kinetic energy of the test particle. For particle having $g = 2$, the special relativistic term dramatically vanishes. For $g < 2$, the special relativistic term decreases the spin precession frequency whereas for particles with $0 > g$ or $g > 2$, it increases the frequency. However, as all the known charged particles have $0 > g$ or $g > 2$, the special relativistic effect is to increase the Larmor frequency.

The spin precession in the Reissner-Nordström field (case c) is a purely relativistic effect. In the flat space limit,

$$\left(\frac{\omega_s}{\Omega}\right)_{ES} = \frac{g}{2}\left(1 - \frac{2}{g}\right)^{1/2} V^2 \gamma \tag{2.5.56}$$

ES stands for the electrostatic case. It is clear that this result is applicable only to particles having $g < 0$ or $g > 2$. However, one does not have, as yet, any experimentally observed particle which lies in the range $0 < g < 2$. In passing, it is worth citing a discussion by Cohen and Mustafa (1986) that the Dirac quantum theory predicted $g = 2$ for the electron though it was not clear if the result were special relativistic or quantum mechanical. They further cited (Sakurai 1967) that Feynman had shown that one can get for the electron $g = 2$ from the non-relativistic quantum mechanics alone. Therefore, it appears from our investigation that special relativity puts a bound to the Lande g-value in a natural way.

Though the calculations have been carried out in a fully general relativistic prescription, it is clear from the expressions for the spin precession frequency that the purely geodetic terms have not appeared explicitly in this treatment (excluding the Ernst case which is not an asymptotically flat metric). However, when we have introduced local Lorentz frame, the purely geodetic terms have appeared explicitly along with the other terms (to be discussed in the subsequent chapter).

Chapter 3

Spin precession of charged particles in local Lorentz frame

3.1 Formalism

In the previous chapter, we have seen that the purely geodetic effects do not appear in the spin precession frequency of a charged particle moving in presence of electromagnetic field in curved spacetime background (except in the case of the Ernst spacetime). Therefore, it is very much desirable to see if the purely geodetic terms appear when the present investigation is extended to local Lorentz frame. This is the subject of discussion in this chapter. We begin with a discussion of the local Lorentz tetrad as discussed by Weinberg (1972).

The principle of equivalence allows one to erect, at every point X, a set of coordinates ${\psi_X}^{(a)}$ which are locally inertial at that point. In any general non-inertial coordinate system, a metric is

$$g_{ik} = \lambda^{(a)}_{i} \lambda^{(b)}_{k} \eta_{(ab)} \tag{3.1.1}$$

where

$$\lambda^{(a)}_{i}(X) \equiv \left\{ \frac{\partial \psi_X^{(a)}(x)}{\partial x^i} \right\}_{x=X} \tag{3.1.2}$$

$\lambda^{(a)}_{i}$ is a set of four linearly independent covariant vector fields which is called a tetrad. The index in the bracket is the tetrad index which runs from 0 to 3. These four-vectors, so long they are linearly independent, can be of any length and can have any angle among themselves.

Under a general non-inertial coordinate transformation, a tetrad components transform as following:

$$\lambda^{(a)}_{i}(x) \to \lambda'^{(a)}_{i} = \frac{\partial x^k}{\partial x'^i} \lambda^{(a)}_{k} \tag{3.1.3}$$

Consider a contravariant vector field V^i, then the corresponding tetrad components are given by

$$V^{(a)} = \lambda^{(a)}_{i} V^i \tag{3.1.4}$$

V^i is a single four-vector whereas $V^{(a)}$ are four scalars in a locally inertial frame. Similarly, the tetrad components of a covariant vector V_i are given by

$$V_{(b)} = \lambda_{(b)}{}^{i} \, V_i \tag{3.1.5}$$

where

$$\lambda_{(b)}^{\;i} = \eta_{(ab)} g^{ik} \lambda^{(a)}_{\;k} \tag{3.1.6}$$

with

$$\lambda^{(a)}_{\;i} \, \lambda_{(a)}^{\;k} = \delta_i^{\;k}$$

$$\lambda^{(a)}_{\;i} \, \lambda_{(b)}^{\;i} = \delta^{(a)}_{(b)} \tag{3.1.7}$$

and

$$\lambda_{(a)}^{\;i} \, \lambda_{(b)}^{\;k} \, g_{ik} = \eta_{(ab)} \tag{3.1.8}$$

Similarly, the tetrad components of a general tensor can be obtained by

$$T^{(a)(b)....}{}_{(m)(n)....} = T^{ij...}{}_{kl..} \, \lambda^{(a)}_{\;i} \, \lambda^{(b)}_{\;j} \, \lambda_{(m)}^{\;k} \, \lambda_{(n)}^{\;l} .... \tag{3.1.9}$$

3.2 Calculations

We are interested in calculating the spin precession frequency for a charged particle in presence of electromagnetic field on curved spacetime background. We have considered a test particle to be moving in a circular orbit on the equatorial plane of the compact object. Further without loss of generality, we have taken the θ component of the spin vector to be zero (see equations 2.4.28-31 and the discussion following this). As we are interested in finding the spin precession frequency in local Lorentz frame, we obtain local tetrad components of the spin vector through

$$S^{(R)} = \lambda^{(R)}{}_{R} \quad S^{R},$$
$$S^{(\phi)} = \lambda^{(\phi)}{}_{\phi} \quad S^{\phi} \tag{3.2.10}$$

S^{θ} component has been considered to be zero. The tetrad components $\lambda^{(a)}_{i}$ are obtained through the relation

$$g_{ij}\ \lambda^{i}{}_{(a)}\ \lambda^{j}{}_{(b)} = \ \eta_{(ab)} \tag{3.2.11}$$

As we have already discussed in the last chapter that the polar coordinates have inherent rotation with respect to the Cartesian coordinates and therefore we rewrite the tetrad componets in terms of their Cartesian components, which are given below:

$$S^{(x)} = S^{(R)} cos\phi - RS^{(\phi)} sin\phi,$$
$$S^{(y)} = S^{(R)} sin\phi + RS^{(\phi)} cos\phi \tag{3.2.12}$$

Using (2.4.32-33), (3.2.9), and (3.2.11), one rewrites the spin equations in local Lorentz frame as:

$$\frac{dS^{(x)}}{d\tau} = \frac{1}{R\,\lambda^{(R)}_{R}\,\lambda^{(\phi)}_{\phi}}\Big[\;-\; S^{(x)}\,cos\phi\;sin\phi\;\Delta + S^{(y)}\Big\{\cos^2\phi\;\Delta$$
$$-\; R\,\lambda^{(\phi)}_{\phi}\left(L_2\,R\,\lambda^{(\phi)}_{\phi} + U^{(\phi)}\lambda^{(R)}_{R}\right)\Big\}\Big] \quad (3.2.13)$$

and

$$\frac{dS^{(y)}}{d\tau} = \frac{1}{R\,\lambda^{(R)}_{R}\,\lambda^{(\phi)}_{\phi}}\Big[S^{(x)}\Big\{cos^2\phi\;\Delta \;+\; \lambda^{(R)}_{R}\;\left(R\,U^{(\phi)}\,\lambda^{(\phi)}_{\phi} - L_1\,\lambda^{(R)}_{R}\right)\Big\}$$
$$+\; S^{(y)}cos\phi\;sin\phi\;\Delta\Big] \qquad (3.2.14)$$

with

$$\Delta \;=\; L_1\left(\lambda^{(R)}_{R}\right)^2 +\; L_2\,R^2\;\left(\lambda^{(\phi)}_{\phi}\right)^2 \qquad (3.2.15)$$

Solving them one gets the spin precession frequency in local Lorentz frame, which is given by

$$\omega_s^2 \;=\; \left(U^{(\phi)} - \chi_1\right)\left(U^{(\phi)} - \chi_2\right) \qquad (3.2.16)$$

with

$$\chi_1 = \frac{\lambda^{(R)}_{\;\;R}}{R\lambda^{(\phi)}_{\;\;\phi}}\;L_1,$$

$$\chi_2 = \frac{R\lambda^{(\phi)}_{\ \phi}}{\lambda^{(R)}_{\ R}} L_2 \tag{3.2.17}$$

where L_1 and L_2 are given by (2.4.34) and (2.4.35) respectively. Using the above, the spin precession frequency in local Lorentz frame for the physical situations discussed in section (2.3) have been evaluated (Virbhadra and Prasanna 1991) which are given in the following.

(a) Schwarzschild background with superposed magnetic field dipolar at infinity:

$$\begin{aligned}\left[\omega_s^2\right]_{local} = \ & \left(\omega_s{}^2\right)_{global} + \left\{\omega_s{}^2\right\}^{geod}_{sch} + \frac{3e\mu F_R}{4m_0M^2R}\left(1-\frac{2M}{R}\right)^{-1} U^\phi \\ & \left[2g\left\{\left(1-\frac{2M}{R}\right)^{-1/2} - \left(1-\frac{M}{R}\right)\right\}\right. \\ & \left. + \ (g-2)\left(RU^\phi\right)^2\left\{\left(1-\frac{2M}{R}\right)^{1/2} - 1\right\}\right]\end{aligned} \tag{3.2.18}$$

where $(\omega_s)_{global}$ and F_R are respectively given by (2.4.43) and (2.4.44). $\{\omega_s\}^{geod}_{sch}$ stands for the purely geodetic spin precession frequency in Schwarzschild field, which is given as following:

$$\left\{\omega_s^2\right\}_{Sch}^{geod} = \left[\left(2 - \frac{3M}{R}\right) - \left(1 - \frac{2M}{R}\right)^{-1/2}\left(2 - \frac{5M}{R}\right)\right] U^{\phi 2} \quad (3.2.19)$$

(b) Schwarzschild background with superposed uniform magnetic field:

The spin precession frequency of a charged particle in local Lorentz frame for the Wald case is given by

$$\left[\omega_s^2\right]_{local} = \left(\omega_s^2\right)_{global} + \left\{\omega_s^2\right\}_{Sch}^{geod} + \frac{geB_0}{m_0} U^{\phi} \times \left[\left\{1 - \left(1 - \frac{2M}{R}\right)^{1/2}\right\}\left\{1 + \left(1 - \frac{2}{g}\right)\frac{R^2 U^{\phi 2}}{2}\right\} - \frac{M}{R}\right] \quad (3.2.20)$$

$(\omega_s)_{global}$ for the Wald case is given by (2.4.46) and $\{\omega_s\}_{Sch}^{geod}$ is given by (3.2.19).

(c) Reissner-Nordström field:

$$\left[\omega_s^2\right]_{local} = \left(\omega_s^2\right)_{global}^{R-N} + \left\{\omega_s^2\right\}_{R-N}^{geod} + \frac{geQU^t}{2m_0 R} U^{\phi 2} \times$$

$$\left[\left(1-\frac{2}{g}\right)\left\{\left(1-\frac{2M}{R}+\frac{Q^2}{R^2}\right)^{1/2}-1\right\}\right.$$

$$\left.+\ \frac{1}{U^{t2}}\left\{\left(1-\frac{2M}{R}+\frac{Q^2}{R^2}\right)^{-1/2}-1\right\}\right] \quad (3.2.21)$$

where $(\omega_s)^{R-N}_{global}$ is given by (2.4.49) and $\{\omega_s^2\}^{geod}_{R-N}$ is the purely geodetic spin precession frequency in the R-N field, which is given by

$$\left\{\omega_s^2\right\}^{geod}_{R-N}=\left[\left(2-\frac{3M}{R}+\frac{2Q^2}{R^2}\right)-\left(1-\frac{2M}{R}+\frac{Q^2}{R^2}\right)^{-1/2}\left(2-\frac{5M}{R}+\frac{3Q^2}{R^2}\right)\right]U^{\phi 2} \quad (3.2.22)$$

(d) Ernst field:

The spin precession frequency in the local Lorentz frame is given by

$$\begin{aligned}\left[\omega_s^2\right]_{local} = & \left(\frac{geB}{m_0\Lambda^4}\right)\left[\frac{geB}{m_0}\left(1-\frac{2M}{R}\right)+\frac{U^{(\phi)}}{\Lambda}\left(1-\frac{2M}{R}\right)^{1/2}\right. \\ & \left\{2(3-2\Lambda)\left(1-\frac{2M}{R}\right)^{1/2}-\Lambda\left(2\Lambda^2+\frac{M}{R}\left(1-\frac{2M}{R}\right)^{-1/2}\right)\right\} \\ & +\ (g-2)\frac{eB}{m_0\Lambda^2}R^2U^{(\phi)^2}\left(1-\frac{2M}{R}\right)+\left(1-\frac{2}{g}\right)\frac{R^2U^{(\phi)^3}}{\Lambda^3}\end{aligned}$$

$$\times \left[(4-3\Lambda)\left(1-\frac{2M}{R}\right)-\Lambda\left\{\Lambda^2\left(1-\frac{2M}{R}\right)^{1/2}+\frac{M}{R}\right\}\right]\bigg]$$

$$+\left\{\omega_s^2\right\}^{geod}_{Ernst\ local} \qquad (3.2.23)$$

where the last term $\{\omega_s^2\}^{geod}_{Ernst\ local}$ stands for the purely geodetic precession frequency in the local Lorentz frame which is given by

$$\begin{aligned}\left\{\omega_s^2\right\}^{geod}_{Ernst\ local} &= \frac{{U^{(\phi)}}^2}{\Lambda^3}\left[\frac{(\Lambda-2)(3\Lambda-4)}{\Lambda^3}\left(1-\frac{2M}{R}\right)+\frac{M(\Lambda-2)}{R\Lambda^5}+\Lambda^3\right.\\ &+\left.\left(1-\frac{2M}{R}\right)^{-1/2}\left\{2(2\Lambda-3)\left(1-\frac{2M}{R}\right)+\frac{M\Lambda}{R}\right\}\right]\end{aligned}$$

(3.2.24)

with

$$U^{(\phi)}=\frac{U^\phi}{\Lambda} \qquad (3.2.25)$$

3.3 Discussion

Expressed through equations (3.2.18) to (3.2.25) are spin precession frequencies in local Lorentz frames for different cases. Switching off the charge parameter of the test particle, one had noted in the previous chapter that the expressions for the spin precession frequencies in the global frame for the first three cases vanish whereas for the case of the Ernst spacetime, one gets non-vanishing terms given by (2.4.52). However, in local Lorentz frames one gets purely geodetic terms (along with other terms) in all these cases given by (3.2.19), (3.2.22)(3.2.24). It is clear that $Q = 0$ in (3.2.22) as well as $\Lambda = 1$ in (3.2.24) give (3.2.19) as expected. Both in Schwarzschild and R-N geometries, which are asymptotically flat, the geodetic terms explicitly manifest only in local frames whereas in the Ernst geometry (which is not asymptotically flat), the purely geodetic term appears even in global frame. In the Reissner-Nordström and the Ernst spacetimes, the electrostatic and the magnetostatic fields respectively contribute to the curvature of the spacetime which are clearly reflected in these results. It is interesting to note that the purely geodetic spin precession frequencies are found to be proportional to the respective orbital frequencies of the test particles for all the cases we have investigated.

It would indeed be interesting to extend the calculations to the case of non-circular orbits as well as for orbits off the equatorial plane. As we have neglected the non-linear spin-orbit coupling terms, it could be interesting to see the effects on the spin precession when these are included. Moreover, it is not clear to us why the purely geodetic terms do not appear in the expressions for the spin precession frequency in global frame except in the case of the Ernst field. This requires further serious investigations.

Chapter 4

Energy and angular momentum in Kerr-Newman spacetime

4.1 A brief history of energy-momentum localization in general relativity

It is well known that the conserved quantities such as energy, momentum, and angular momentum have their significant role in physics as they provide the first integral of equations of motions. However, their localization in a field have been a problematic issue. The localization of energy-momentum in curved spacetime has been a subject of extensive research since the outset of general relativity. Though an adequate localization would have immense advantages, the status is that one does not have that as yet. There are, however, mutually opposing views that authors share regarding a possibility of successful localization of energy and momentum in general relativistic systems. Misner, Thorne, and Wheeler (1973, p467) argued that to look for a local gravitational energy-momentum is looking for the right answer to the wrong question. Many resarchers believe that only the total energy and mometum of asymptotically flat spacetimes make physical sense. Misner, Thorne, and Wheeler, however, concede that for spherical systems, the gravitational potential energy is correct and meaningful (Lynden-Bell and Katz 1985; Misner Thorne and Wheeler 1973, p603).However, Cooperstock and Sarracino (1978) believe that if the energy localization is meaningful in spherically symmetric systems, it is certainly meaningful for systems which are not spherically symmetric.

Bondi (1990) has written, " In the newtonian theory the notion of gravitational potential energy is used with ease to describe exchanges between gravitational and other forms of energy, though its non-localizability occasionally causes difficulties and it is not adequate to describe certain transfer of energy". He has further added, "In newtonian theory, the essential characteristic of energy is that it is conserved. Its location does not directly reveal itself because it contributes neither to inertia nor to gravitation. It is therefore entirely acceptable to use the device of potential energy, which ensures conservation at the price of being non-localizable. In relativity, a non-localizable form of energy is inadmissible, because any form of energy contributes to gravitation and so its location can in principle be found".

In flat spacetime, the energy and momentum conservation law is given by

$$\frac{\partial T^{ik}}{\partial x^k} = 0 \tag{4.1.1}$$

where T^{ik} stands for the energy-momentum tensor appearing as source term in Einstein-Maxwell equations. However, in the presence of gravitational field the above is generalized as following:

$$T^{k}_{\ i;k} = 0 \tag{4.1.2}$$

Bondi (1990) has stated that nevertheless the vanishing of covariant diveregence of energy-momentum tensor is often called a conservation law, in fact it is a law of non-conservation. The reason is that in presence of gravitational field, merely the four momentum of matter cannot be conserved. What can be conserved is the four momentum of matter plus the gravitational field. Therefore, what one requires is

$$\frac{\partial}{\partial x^k}\left[\sqrt{-g}\left(T_i^{\ k} + \mathcal{F}_i^{\ k}\right)\right] = 0 \tag{4.1.3}$$

T_i^k stands for the matter energy-momentum tensor whereas $\mathcal{F}_i^{\ k}$ is given by the usual Lagrangian density $\mathcal{L}$ as following:

$$\sqrt{-g}\ \mathcal{F}_i^{\ k} = \frac{1}{16\pi}\left[\frac{\partial \mathcal{L}}{\partial g^{pq}_{\ \ ,k}}\ g^{pq}_{\ \ ,i} \ - \ g_i^{\ k}\mathcal{L}\right] \tag{4.1.4}$$

Einstein (Møller 1958) gave an expression for the total energy-momentum (matter plus field) complex which is however a pseudotensor. Due to the non-tensorial character of the energy-momentum pseudotensor, it was not taken seriously. Schrödinger could find a coordinate system in which all the components of the pseudotensor vanished for the Schwarzschild spacetime (Goldberg 1958). However, Einstein defended his pseudotensor by showing that the total energy-momentum transforms like a free four-vector under linear coordinate transformation (Goldberg 1958). Eddington showed that there is an agreement between the pseudotensorial energy flux and the radiation damping of a radiation source which again favors importance of the pseudotensor (Eddington 1965, Cooperstock and Lim 1987). Following a hypothetical discussion between Sagredus (who presents the orthodox view) and Salvaties (who attempts to show that the orthodoxy should reexamine its case), Palmer (1980) discussed the importance of the Einstein's energy-momentum pseudotensor in detail. After Einstein's energy-mometum pseudotensor, a fairly large number of prescriptions for energy in a general relativistic system have been proposed by many authors, e.g; Tolman (1930), Landau and Lifshitz (1985), Papapetrou (1948), Gupta (1954), Møller (1958), Goldberg (1958), Bergmann (1958), Dirac (1959), Komar (1959, 1962, 1963), Arnowitt et al.(1961), Bondi et al (1962), Hawking (1968), Weinberg (1972), Witten (1981), Penrose (1982), Lynden-Bell and Katz (1985), Nahmad-Achar and Schutz (1987*a*, 1987*b*), Kulkarni et al (1988), Bartnik (1989), and Katz and Ori (1990) etc. Using Tolman's definition of energy, Vaidya (1952), after he found solution of Einstein's equations for a spherically symmetric radiating star, calculated energy associated with a general non-static spherically symmetric spacetime. This gives energy in the Reissner-Nordström as well as the Vaidya radiating spacetimes. The result shows that there is no energy associated with Schwarzschild field and therefore the entire energy associated with the Schwarschild black hole is confined to its interior only. However, in the Reissner-Nordström case, the energy is shared by its interior as well as its exterior.

The energy-mometum pseudotensors of Einstein as well as Tolman have mixed indices and raising them with a metric tensor does not yield a symmetric object. However, to calculate angular momentum, a symmetric energy-momentum complex is required. Landau and Lifshitz (1985) constructed a symmertic energy-momentum pseudotensor which, however, transforms like a vector density. Goldberg (1958) constructed two hierachies of energy-momentum complexes $\theta_{(n)i}^{\ k}$ and $L_{(n)}^{ik}$. They transform like tensor densities of weight $(n+1)$ and $(n+2)$ respectively under linear coordinate transformations and again for $n = 0$ they yield the energy-mometum pseudotensors of Einstein and Landau and Lifshitz (LL) respectively.

He discussed that (a) for the total energy and momentum to form a free vector, the energy and momentum pseudotensor of Einstein $\theta_{(0)i}^{\ k}$ is the desired quantity and similarly the total angualar momentum to be free antisymmetric tensor, the appropriate quantity is $L_{(-1)}^{ik}$ which differs from that of LL and Bergmann and Thomas (1953), and (b) all the mixed quantities $\theta_{(n)i}^{\ k}$ have the same physical content (energy and momentum) whereas the symmetric ones are all different in their physical content. Further he added that amomg all the symmetric quantities, only the LL pseudotensor has the same energy and momentum as that of Einstein.

One knows that the energy-momentum pseudotensors of Einstein, Tolman, and Landau and Lifshitz can yield meaningful result only if the calculations are carried out in quasi-Cartesian coordinates (that in which increasing spatial distance convereges the components of the metric tensor to their values of special relativity).

It is worth discussing here that using LL pseudotensor, Rosen (1956) evaluated the energy and momentum of cylindrical gravitational waves in Cylindrical polar coordinates. He found that the waves in empty space do not appear to carry energy and momentum. This result was again confirmed by Weber and Wheeler (1957). When Rosen (1958) realized that the pseudotensor should be evaluated in quasi-Cartesian coordinates, he evaluated the same in quasi-Cartesian coordinates and found the energy and momentum associated with the gravitational waves to be finite and reasonable.

However, the evaluation of energy-momentum pseudotensors of Einstein, Tolman, or LL in quasi-Cartesian coordinates is usually very lengthy to work out. Moreover, Møller (1958) argued that singling out a particular coordinate system (i.e; quasi-Cartesian) is somehow unsatisfactory from the standpoint of general relativity. However, he agreed that it does not seriously affect the usefulness of the Einstein's expression for the total energy and momentum of closed physical systems. Further he said that for non-closed systems , the pseudotensor of Einstein does not give consistent result for energy. Realizing this problem, he (Møller 1958) constructed a new energy-momentum pseudotensor and claimed that one is not anymore restricted to the use of quasi-Galilean coordinates. Further he discussed that the energy density component in his prescription transforms like a scalar density under purely spatial transformations. Komar (1959) constructed a set of covariant laws in general relativity. Only three years after the new energy-mometum was proposed, Møller realized that unlike the case of the Einstein's pseudotensor, the total energy-momentum vector of a closed physical system in his prescription does not transform like a four-vector under Lorentz transformations. Later Kovacs (1985) showed that Møller was wrong in concluding that. However, Novotny (1987) pointed out the mistake of Kovacs and wrote that Møller was correct to say that the energy-momentum vector in his definition is not a Lorentz four-vector. Weinberg (1972) constructed a new energy and momentum complex and calculated the total energy and angular momnetum in Kerr metric which are M and Ma respectively (M and a stand for the mass and rotation parameter in Kerr metric).

Beig (1978) showed for a stationary, asymptotically flat spacetime that the Komar energy (associated with a time like Killing vector) and the ADM energy are equal if the latter is calculated on a Cauchy surface which is asymptotically at rest relative to the Killing vector. Further he discussed the implication of the result on the problem of positivity of energy in general relativity. Ashtekar and Magnon-Ashtekar (1979) showed that for isolated gravitating systems, the difference between the ADM (Arnowitt, Deser, and Misner 1961) four-momenta and the Bondi four-momenta (Bondi et al 1962) associated with a retarded instant of time is equal to the four-momentum carried away by the gravitational radiation emitted between the infinite past and the given retarded instant. Persides (1979) showed that the LL pseudotensor gives the Bondi four-momentum at null infinity.

Using Komar's definition of energy, Cohen and de Felice (1984), evaluated energy in the Kerr-Newman (K-N) spacetime and found the energy to be shared by the exterior as well as the interior of the K-N object. However, switching off the charge parameter, they found no energy shared by the exterior of the Kerr black hole. The energy in the Reissner-Nordström spacetime is $M - Q^2/R$ whereas that obtained in Tolman's prescription (calculated by Vaidya 1952) is $M - Q^2/2R$. Cohen (1967, 1968) found angular momentum of a Kerr black hole to be $-Ma$.

Tod (1983) calculated the Penrose quasi-local mass for the Reissner-Nordström (R-N) and Friedman-Roberson-Walker metrics. The energy so obtained for the R-N metric is the same as found by Vaidya (1952) in Tolman's prescription. Using the Penrose definition of mass, Tod (1985) showed that a static black hole satisfies the inequality $A \leq 16\pi M^2$ where A is its area. Recently he (Tod 1990) has calculated the Penrose mass for a Cylinder of finite length in a cylindrically symmetric spacetime. He found that for a cylindrically symmetric gravitational pulse, the mass associated with a finite cylinder rises and falls with the passage of the wave.

Based on some physical arguments, Lynden-Bell and Katz (1985) showed that the static spherical systems have a coordinate-independent gravitational field energy-density which shows that the entire field energy for the Schwarzschild object remains outside its horizon. Comparing their result with those of Penrose, Witten, and Horowitz and Strominger, they wrote, "The Penrose mass of a Schwarzschild hole is all within the hole, whereas ours is all outside it, see Tod (1983). Witten (1981) has given a positive definite expression for energy in relativity which has been generalized by Horowitz and Strominger (1983). Both give expressions as integrals of positive quantities which one might be led to interpret as energy densities. Witten's expression does not have our matter term in its classical limit; this can be obtained by taking $n = 2$ in Horowitz and Strominger's expression but evaluating their integrand in Schwarzschild spacetime leads to a different energy density with part of the mass left outside the hole". Further Katz, Lynden-Bell, and Israel (1988) showed that the energy within equipotential surfaces in any spacetime is also well defined. The subject of gravitational energy density is a controversial issue even in simple case of the spherically symmetric objects. To this end, Nahmad-Achar (1987) showed that one can also give physical arguments by which the entire energy in the Schwarzschild object should be within the hole's horizon. Grøn (1986) expressed the covariant definition of gravitational field energy density proposed by Lynden-Bell and Katz in terms of the Israel's theory of surface layers in general relativity. He obtained an expression for the gravitational field energy in a static spherically symmetric spacetime which gives the same result for the Reissner-Nordström spacetime in isotropic coordinates as found using the Einstein's pseudotensor.

Dadhich and Chellathurai (1986), using the Lynden-Bell and Katz invariant definition for gravitational field energy, calculated the field energy density for a charged object. They found that the energy is shared by the interior as well as the exterior of the hole and for a maximally symmetric charged black hole ($M = Q$), the energy is shared equally by the interior and the exterior of the charged object. Further for the energy inside and outside of a charged black hole, they established a complementary relationship between the definition of Penrose and Lynden-Bell and Katz.

Nahmad-Achar and Schutz (1987*a*) generalized the method of Persides which allows an asymptotic evaluation of the energy, momentum, and angular momentum in general relativistic systems in any coordinate system. They evaluated energy associated with a Kerr black hole and found the total energy of the system to be shared by its interior as well as exterior. However, switching off the rotation parameter gives the energy of a Schwarzschild black hole which is confined to its interior only. Further they (Nahmad-Achar and Schutz 1987*b*) developed and proved extremum theorems for angular momentum of solutions of Einstein's equations and discussed their usefulness in detail. Cohen and de Felice (1984), using the prescription of Komar, found that there is no energy in the exterior of the Kerr black hole. Kulkarni et al (1988) argued that a modification of the Komar integral was required since that was not sensitive to the repulsive effect arising from the rotation. They gave a definition of the effective gravitational mass of the Kerr black hole that incorporates contribution due to the rotation and that agress with Komar integral asymptotically. Using the above definition, Chellathurai and Dadhich (1990) have recently evaluated the effective mass of a Kerr-Newman black hole and a Kerr black hole embedded in magnetic field. They have found that the rotation on the hole decreases the mass, being least at the horizon and goes to zero for the extremal case ($M^2 = a^2 + Q^2$). However, the magnetic field in the case of the magnetized Kerr black hole increases the effective mass.

Using the Einstein spacetime as a reference space, Horowitz and Katz (1988) defined energy for the Robertson-Walker spacetime and showed it to be equivalent to the dilation operator. They said that their investigation leads to a useful formulation for the dynamics and for a quantum field theory on a classical Robertson-Walker spacetime and it also gives a basis for a microcanonical entropy. Bartnik (1990) proposed a new definition of quasi-local mass function in general relativity which he said to be uniquely defined, manifestly non-negative, and increases with increase in domain. He found the mass function to agree with the Schwarzschild mass for spherically symmetric spacetimes.

Many authors (Schoen and Yau 1979, Nester 1981, Horowitz and Perry

1982, Jezerski 1989) discussed the positivity of gravitational mass in detail. Bonnor and Cooperstock (1989), modelling the electron as a charged sphere obeying Einstein-Maxwell theory, pointed out that it must contain some negative rest mass. The total active gravitational mass within the sphere being negative questions one of the assumptions made in singularities theorems of general relativity.

We (Virbhadra 1990*a*, 1990*b*, 1990*c*) have calculated energy in Kerr-Newman spacetime in the prescriptions of Einstein, Tolman, Landau and Lifshitz(LL), and Møller, and angular momentum in the same spacetime using the LL pseudotensor. We have carried out the calculations upto the third power of the rotation parameter. Recently Cooperstock and Richardson (1990) have calculated energy in Kerr-Newman spacetime upto the seventh power of the rotation parameter. Further they have succeeded in obtaining an exact expression (without neglecting higher orders of rotation parameter) for energy in the same spacetime in Tolman's prescription. These are the subjects of discussion in the following sections of this chapter.

4.2 Energy-momentum pseudotensors

A large number of energy-momentum complexes have been proposed by many authors. However, we have considered pseudotensors of Einstein (Møller 1958), Tolman (Tolman 1930; Vaidya 1952), Landau and Lifshitz (1985), and Møller (1958). These pseudotensors which are given below are well known quantities as these are extensively discussed in the literature.

$$\theta_i{}^k = \frac{1}{16\pi}\left[\frac{g_{in}}{\sqrt{-g}}\left\{-g\left(g^{kn}\,g^{lm} - g^{ln}g^{km}\right)\right\}_{,m}\right]_{,l} \tag{4.2.5}$$

(g stands for the determinant of the metric).

$$t_i^k = \frac{1}{8\pi}\left[\sqrt{-g}\left\{-g^{li}\,V^j_{kl} + \frac{1}{2}\,g^i{}_k\,g^{lm}\,V^j_{lm}\right\}\right]_{,j} \tag{4.2.6}$$

with

$$V^i{}_{jk} = -\Gamma^i{}_{jk} + \frac{1}{2}\,g^i{}_j\,\Gamma^m_{mk} + \frac{1}{2}\,g^i{}_k\,\Gamma^m_{mj} \tag{4.2.7}$$

$$L^{mn} = L^{nm} = \frac{1}{16\pi}\left[-g\left(g^{mn}g^{jk} - g^{mk}g^{jn}\right)\right]_{,jk} \tag{4.2.8}$$

$$\mathcal{T}_i{}^k = \frac{1}{8\pi}\left[\sqrt{-g}\,(g_{in,m} - g_{im,n})\,g^{km}g^{ln}\right]_{,l} \tag{4.2.9}$$

$\theta_i{}^k, t_i{}^k, L^{ik}$, and $\mathcal{T}_i{}^k$ are energy-momentum complexes of Einstein, Tolman, Landau and Lifshitz, and Møller, respectively.

4.3 Kerr-Newman metric

The exterior gravitational field of a completely gravitationally collapsed charged rotating object is described by an axisymmetric, stationary metric which is asymptotically flat. Such an exterior field is given by the well known Kerr-Newman metric which in Kerr-Schild Cartesian coordinates is given by the line-element:

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{2\left[M - Q^2/(2r_0)\right] r_0^3}{r_0^4 + a^2 z^2}\left[dt + \frac{z}{r_0}dz + \frac{r_0}{r_0^2 + a^2}(xdx + ydy) + \frac{a}{r_0^2 + a^2}(xdy - ydx)\right]^2 \quad (4.3.10)$$

where r_0 is defined by

$$\frac{x^2 + y^2}{r_0^2 + a^2} + \frac{z^2}{r_0^2} = 1 \quad (4.3.11)$$

with $x^2 + y^2 + z^2 = r^2$ (r is the spherical radial coordinate). $Q = 0$ gives the Kerr metric whereas $a = 0$ gives the Reissner-Nordström metric.

4.4 Calculations

It is well known that the energy-momentum pseudotensors of Einstein, Tolman, and Landau and Lifshitz yield meaningful results only in quasi-Cartesian coordinates. The energy density component of the energy-momentum complex of Møller transforms like a scalar density under purely spatial transformations and therefore there is no such restriction of quasi-Cartesian coordinates on it. However, in the following we have evaluated all the components of aforesaid pseudotensors in the Kerr-Schild Cartesian coordinates. Such calculations without any approximation are obviously very lengthy. Therefore, for convenience in calculation, we have worked for small values of the rotation parameter a and have neglected the terms beyond its third order. The following are the components of the aforesaid pseudotensors for the Kerr-Newman metric (Virbhadra 1990a, 1990b, 1990c, 1991a, 1991b, 1991c).

$$\theta_0{}^0 = t_0{}^0 = L^{00} = \frac{1}{2}\mathcal{T}_0{}^0 = \frac{Q^2}{8\pi r^8}\left[2a^2\left(2r^2 - 3z^2\right) + r^4\right] \qquad (4.4.12)$$

$$\theta_1{}^0 = -\theta_0{}^1 = t_1{}^0 = -t_0{}^1 = -L^{01} = -L^{10} = -\frac{1}{2}\mathcal{T}_0{}^1$$
$$= \frac{Q^2 ay}{4\pi r^{10}}\left[3a^2\left(2z^2 - r^2\right) - r^4\right] \qquad (4.4.13)$$

$$\theta_2{}^0 = -\theta_0{}^2 = t_2{}^0 = -t_0{}^2 = -L^{02} = -L^{20} = -\frac{1}{2}\mathcal{T}_0{}^2$$

$$= -\frac{Q^2 ax}{4\pi r^{10}}\left[3a^2\left(2z^2 - r^2\right) - r^4\right] \quad (4.4.14)$$

$$\theta_3{}^0 = \theta_0{}^3 = t_3{}^0 = t_0{}^3 = L^{03} = L^{30} = \mathcal{T}_0{}^3 = 0 \quad (4.4.15)$$

$$\theta_1{}^1 = t_1{}^1 = -L^{11} = \frac{Q^2}{8\pi r^{10}}\left[2a^2\left\{-6x^2z^2 + r^2\left(r^2 - 3y^2\right)\right\} + r^4\left(2x^2 - r^2\right)\right] \quad (4.4.16)$$

$$\theta_2{}^2 = t_2{}^2 = -L^{22} = \frac{Q^2}{8\pi r^{10}}\left[2a^2\left\{-6y^2z^2 + r^2\left(r^2 - 3x^2\right)\right\} + r^4\left(2y^2 - r^2\right)\right] \quad (4.4.17)$$

$$\theta_3{}^3 = t_3{}^3 = -L^{33} = \frac{Q^2}{8\pi r^{10}}\left[2a^2\left(-6z^4 + 6z^2r^2 - r^4\right) + r^4\left(2z^2 - r^2\right)\right] \quad (4.4.18)$$

$$\theta_1{}^2 = \theta_2{}^1 = t_1{}^2 = t_2{}^1 = -L^{12} = -L^{21} = \frac{Q^2 xy}{4\pi r^{10}}\left[-3a^2\left(2z^2 - r^2\right) + r^4\right] \quad (4.4.19)$$

$$\theta_2{}^3 = \theta_3{}^2 = t_2{}^3 = t_3{}^2 = -L^{23} = -L^{32} = \frac{Q^2 yz}{4\pi r^{10}}\left[-3a^2\left(2z^2 - r^2\right) + r^4\right] \quad (4.4.20)$$

$$\theta_3{}^1 = \theta_1{}^3 = t_3{}^1 = t_1{}^3 = -L^{31} = -L^{13} = \frac{Q^2 zx}{4\pi r^{10}}\left[-3a^2\left(2z^2 - r^2\right) + r^4\right] \tag{4.4.21}$$

$$\mathcal{T}_1{}^0 = \frac{1}{2\pi r^{10}}\left[3a^3 yQ^2\left(2z^2 - r^2\right) + 2a^2 Mxr^2\left(-3z^2 + r^2\right) - ayQ^2 r^4 + Mxr^6\right] \tag{4.4.22}$$

$$\mathcal{T}_2{}^0 = \frac{1}{2\pi r^{10}}\left[3a^3 xQ^2\left(-2z^2 + r^2\right) + 2a^2 Myr^2\left(-3z^2 + r^2\right) + axQ^2 r^4 + Myr^6\right] \tag{4.4.23}$$

$$\mathcal{T}_3{}^0 = \frac{Mz}{2\pi r^8}\left[2a^2\left(-3z^2 + 2r^2\right) + r^4\right] \tag{4.4.24}$$

$$\begin{aligned}\mathcal{T}_1{}^1 = \frac{1}{8\pi r^{10}}\Big[&12a^3 Mxy\left(-4z^2 + r^2\right) + a^2 Mr\left(35x^2z^2 + 5y^2r^2 - 4r^4\right)\\ &+ 4a^2Q^2\left(r^4 - 3r^2y^2 - 6x^2z^2\right) + 8aMxyr^4\\ &+ 2Mr^5\left(r^2 - 3x^2\right) + 2Q^2r^4\left(2x^2 - r^2\right)\Big]\end{aligned} \tag{4.4.25}$$

$$\begin{aligned}\mathcal{T}_2{}^2 = \frac{1}{8\pi r^{10}}\Big[&12a^3 Mxy\left(4z^2 - r^2\right) + a^2 Mr\left(35y^2z^2 + 5x^2r^2 - 4r^4\right)\\ &+ 4a^2Q^2\left(r^4 - 3r^2x^2 - 6y^2z^2\right) - 8aMxyr^4\\ &+ 2Mr^5\left(r^2 - 3y^2\right) + 2Q^2r^4\left(2y^2 - r^2\right)\Big]\end{aligned} \tag{4.4.26}$$

$$\mathcal{T}_3{}^3 = \frac{1}{8\pi r^{10}}\left[a^2Mr\left(35z^4 - 30z^2r^2 + 3r^4\right) + 4a^2Q^2\left(-6z^4 + 6z^2r^2 - r^4\right)\right.$$

$$\left. + \quad 2Mr^5\left(-3z^2 + r^2\right) + 2Q^2r^4\left(2z^2 - r^2\right)\right] \qquad (4.4.27)$$

$$\mathcal{T}_1{}^2 = \frac{1}{8\pi r^{10}}\left[2a^3M\left(-24y^2z^2 + 6y^2r^2 - 24z^4 + 27z^2r^2 - 5r^4\right)\right.$$

$$+ \quad 5a^2Mxyr\left(7z^2 - r^2\right) + 12a^2xyQ^2\left(-2z^2 + r^2\right)$$

$$\left. + \quad 2aMr^4\left(r^2 - 4x^2\right) - 6Mxyr^5 + 4xyQ^2r^4\right] \quad (4.4.28)$$

$$\mathcal{T}_2{}^3 = \frac{yz}{8\pi r^{10}}\left[5a^2Mr\left(7z^2 - 3r^2\right) + 12a^2Q^2\left(-2z^2 + r^2\right) - 6Mr^5 + 4Q^2r^4\right]$$

$$(4.4.29)$$

$$\mathcal{T}_3{}^1 = \frac{1}{8\pi r^{10}}\left[24a^3Myz\left(-2z^2 + r^2\right) + 5a^2Mxzr\left(7z^2 - 3r^2\right) + 12a^2xzQ^2\right.$$

$$\left.\left(-2z^2 + r^2\right) + 8aMyzr^4 - 6Mxzr^5 + 4xzQ^2r^4\right]$$

$$(4.4.30)$$

$$\mathcal{T}_2{}^1 = \frac{1}{8\pi r^{10}}\left[\ 2a^3M\left(-24y^2z^2 + 6y^2r^2 + 3z^2r^2 - r^4\right) + 5a^2Mxyr\left(7z^2 - r^2\right)\right.$$

$$+ \quad 12a^2xyQ^2\left(-2z^2 + r^2\right) + 2aMr^4\left(4y^2 - r^2\right)$$

$$\left. - \quad 6Mxyr^5 + 4xyQ^2r^4\right] \qquad (4.4.31)$$

$$\mathcal{T}_1{}^3 = \frac{xz}{8\pi r^{10}}\left[5a^2Mr\left(7z^2-3r^2\right)+12a^2Q^2\left(-2z^2+r^2\right)-6Mr^5+4Q^2r^4\right]$$

(4.4.32)

$$\mathcal{T}_3{}^2 = \frac{1}{8\pi r^{10}}\left[\ 24a^3Mxz\left(2z^2-r^2\right)+5a^2Myzr\left(7z^2-3r^2\right)+12a^2yzQ^2\right.$$
$$\left.\left(-2z^2+r^2\right)-8aMxzr^4-6Myzr^5+4yzQ^2r^4\right]$$

(4.4.33)

Traces of the pseudotensors are

$$\theta_i{}^i = t_i{}^i = L_i{}^i = \mathcal{T}_i{}^i = 0 \qquad (4.4.34)$$

4.5 Energy and angular momentum

It is clear that the pseudotensors of Einstein, Tolman, and Landau and Lifshitz give same energy density for the Kerr-Newman field. Transforming the expression in spherical polar coordinates, the complexes of Einstein, Tolman, and Landau and Lifshitz give

$$E_{exterior} = \frac{Q^2}{8\pi}\int\int\int\frac{1}{r^4}\left[2a^2\left(3sin^2\theta-1\right)+r^2\right]dr\ sin\theta\ d\theta\ d\phi \qquad (4.5.35)$$

whereas that of Møller gives twice the above value. After performing the above integration, one gets

$$\begin{aligned} E_{exterior} &= -Q^2\left[\frac{a^2}{3r^3}+\frac{1}{2r}\right]_R^\infty \\ &= \frac{Q^2}{R}\left[\frac{a^2}{3R^2}+\frac{1}{2}\right] \end{aligned} \tag{4.5.36}$$

As the total gravitational mass is given by the mass parameter M, one concludes that the energy asssociated with a Kerr-Newman black hole is given by

$$E = M - \frac{Q^2}{2R}\left[\frac{2a^2}{3R^2}+1\right] \tag{4.5.37}$$

in prescriptions of Einstein, Tolman, and Landau and Lifshitz (Virbhadra 1990*a*, 1990*c*) whereas

$$E = M - \frac{Q^2}{R}\left[\frac{2a^2}{3R^2}+1\right] \tag{4.5.38}$$

in Møller's prescription (Virbhadra 1990*c*).

The pseudotensors of LL, being symmetric in indices, is capable of giving angular momentum in asymptotically flat spacetimes. The three components of angular momentum are given by

$$J^{\alpha} = \int\int\int \left(x^{\beta} L^{0\gamma} - x^{\gamma} L^{0\beta} \right) d^3x \tag{4.5.39}$$

where α, β, γ take cyclic values 1,2,3. Substituting (4.4.13-15) in (4.5.39), transforming to spherical polar coordinates and then performing the integration, one gets

$$J^x = J^y = 0,$$
$$J^z = 2aQ^2 \left[\frac{a^2}{5r^3} + \frac{1}{3r} \right]_{R_1}^{R_2} \tag{4.5.40}$$

4.6 Recent investigations by Cooperstock and Richardson

Recently Cooperstock and Richardson (1991) have extended our calculations for the Kerr-Newman energy upto the seventh order of the rotation parameter and have got

$$E = M - \frac{Q^2}{2R} \Upsilon \tag{4.6.41}$$

with

$$\Upsilon \equiv 1 + \frac{2}{3}\left(\frac{a}{R}\right)^2 + \frac{3}{5}\left(\frac{a}{R}\right)^4 + \frac{4}{7}\left(\frac{a}{R}\right)^6 \tag{4.6.42}$$

in the prescriptions of Einstein, Tolman, and Landau and Lifshitz, and

$$E = M - \frac{Q^2}{R}\,\Upsilon \qquad (4.6.43)$$

in the prescription of Møller.

Again they have succeeded in obtaining the Kerr-Newman energy in the prescriptions of Tolman without taking any approximation, given as follows:

$$E_{exterior} = \frac{Q^2}{4}\int_R^\infty \int_0^\pi \frac{(r^2 + a^2)\ r^2}{\left[(r^2 - a^2)^2 + 4a^2r^2cos^2\theta\right]^{3/2}}\, dr\ sin\theta\ d\theta \qquad (4.6.44)$$

Performing the integration, they have got

$$E_{exterior} = \frac{Q^2}{8}\left[\frac{2R}{R^2 - a^2} + \frac{1}{a} ln\frac{R + a}{R - a}\right] \quad R > a \qquad (4.6.45)$$

and they have expressed the above result as following by power series expansion :

$$E_{exterior} = \frac{Q^2}{2R}\sum_{n=0}^{\infty}\left(\frac{a}{R}\right)^{2n}\frac{n + 1}{2n + 1} \qquad (4.6.46)$$

which agrees with the previous expressions for $n = 0$ to $n = 3$.

4.7 Discussion

Due to non-tensorial character, the importance of energy-momentum pseudotensors to provide energy and momentum in curved spacetime was taken with a suspicion. Many authors argued (partly because they shared a sceptic viewpoint that different pseudotensors would yield different results for energy and momentum in a finite volume of a general relativistic system) that only the total energy in asymptotically flat spacetimes makes physical sense. Cohen and de Felice (1984), using the Komar's definition, eval. uated energy in Kerr-Newman spacetime in Boyer-Lindquist coordinates which is given by

$$E = M - \frac{Q^2}{2R} - \frac{Q^2\left(R^2 + a^2\right)}{2aR^2} arctan \frac{a}{R} \tag{4.7.47}$$

Generalizing the method of Persides, Nahmad-Achar and Schutz (1987a) gave new definitions for energy, momentum, and angular momentum in curved spacetime and evaluated energy in Kerr field in Boyer-Lindquist (B-L) coordinates. The energy expression for the Kerr metric in B-L coordiantes obtained by them is lengthy. For small value of the rotation parameter they found that to be

$$E = M - \frac{2Ma^2}{3\Delta} + \frac{2M^2a^2}{3R\Delta} + O\left(a^4\right) \tag{4.7.48}$$

where

$$\Delta = R^2 - 2MR + a^2 \tag{4.7.49}$$

We have earlier used the symbol Δ in chapter three. In the present context, it has been redefined by (4.7.49).

Chellathurai and Dadhich (1990) obtained energy in the Kerr-Newman spacetime in B-L coordinates which they further wrote for the small value of the rotation parameter :

$$E = M - \frac{Q^2}{R} - \frac{(12M^2 + Q^2)\,a^2}{3R^3} + \frac{14Ma^2Q^2}{3R^4} + \ldots\ldots \tag{4.7.50}$$

Kulkarni, Chellathurai, and Dadhich (1989) had evaluated the energy in Kerr metric which one can get by putting $Q = 0$ in the energy expression obtained by Chellathurai and Dadhich.

Cohen and de Felice, Nahmad-Achar and Schutz, and Kulkarni et al, they all agree that there is no energy in the exterior of a Schwarzschild object and the entire energy is confined to its interior only. However, their results disagree for the Kerr metric. The result obtained by Cohen and de Felice gives no energy to the exterior of the Kerr black hole whereas that of Nahmad-Achar and Schutz and Kulkarni et al disagree with the results of Cohen and de Felice as the total energy associated with a Kerr black hole in their prescriptions is shared by its interior as well as the exterior. However, it is clear that the energy for the Kerr metric obtained by Nahmad-Achar and Schutz and Kulkarni et al are different quantitatively.

At this point, we (the present author) share a different opinion. Once one starts with a definition of energy which gives the entire energy associated with a Schwarzschild object to its interior only, the same definition of energy should not give energy for the Kerr object such that it is shared by its exterior as well, as one must remember that the Kerr black hole is isolated having no medium surrounding it to transfer its rotational energy to its exterior (rotation parameter a is a constant). Recently Cooperstock and Richardson (1991) have argued that in the absence of gravitational field the R-N energy must reduce to the flat space electrostatic energy, i,e; $Q^2/2R$. Further they have written that switching off the rotation parameter in the energy expression for the Kerr-Newman metric, obtained by Cohen and de Felice in Komar's prescription, gives $M - Q^2/R$ and therefore it fails to give the correct flat space limit.

Now we have learned that the pseudotensors of Einstein, Tolman, and Landau and Lifshitz give same energy for the Kerr-Newman metric (calculations have been carried out upto the seventh order of the rotation parameter) and that of Møller gives twice the value obtained in aforesaid three prescriptions. Switching off the rotation parameter one gets energy for the Reissner-Nordström metric $M - Q^2/2R$ and therefore the definitions of Einstein, Tolman, and LL yield correct flat space limit for the electrostatic energy. However, like in the case of the Komar energy, Møller's complex does not give the correct flat space limit for the electrostatic energy. Tod (1983), using the Penrose's definition, got the same result for the Reissner-Nordström metric as obtained by us in the prescriptions of Einstein, Tolman, and LL. Switching off the charge parameter, one finds that the definitions of Einstein, Tolman, Landau and Lifshitz, Møller, and Komar give no energy to the exterior of a Kerr black hole. These expressions for the Kerr-Newman energy contain three parts: the first is the mass parameter, the second is the electrostatic energy ($Q^2/2R$ as given by complexes of Einstein, Tolman, and LL and Q^2/R as given by those of Møller and Komar), and the rest are due to the magnetic field in the K-N spacetime due to the rotation of the charged object.

There is no energy as well as momentum density in the Kerr field. It is the rotation of the charged object which gives angular momentum (due to the electromagnetic field present there) in the Kerr-Newman spacetime. As the Kerr-Newman solution describes the exterior field due to a charged object rotating about Z-axix, one gets only Z-component of the angular momentum. The angular momentum, obtained in the prescription of LL, contains even power to the charge parameter and odd powers to the rotation parameter, and therefore the direction of the angular momentum vector depends on the direction of the rotation of the charged object and not on the sign of the charge on it.

To see if the aforesaid pseudotensors give same result for all of their components and to see if there is symmetry or lack thereof, we have evaluataed all the components of these pseudotensors for the Kerr-Newman metric (upto to the third power of the rotation parameter). One finds that the pseudotensors of Einstein and Tolman give exactly same result, respectively, for all of their components. The axial symmnetry is clearly reflected from the components of the pseudotensors. We have learned that the entire energy in the exterior of a Kerr-Newman black hole is due to the electromagnetic field and therefore we have calculated the traces of these pseudotensors for the K-N metric. Despite their non-tensor character, we have again got an encouraging result that the pseudotensors are traceless for the K-N metric. In passing, it is worth mentioning that it is desirable to extend the calculations without taking any approximation to see if the conclusions made above are sustained.

Now one would naturally like to ask if the energy expressions obtained in the aforesaid prescriptions will maintain the same relationship and the pseudotensors of Einstein and Tolman will yield exactly same result for all of their components for non-static spacetimes. To answer these questions, we have considered the Vaidya radiating spacetime and this will be the subject of discussion in the next chapter.

Chapter 5

Energy and momentum in Vaidya spacetime

5.1 Introduction

We have now learned that the energy-momentum pseudotensors of Einstein, Tolman, Landau and Lifshitz (LL) give same energy in the Kerr-Newman spacetime. However, that of Møller gives twice the value obtained using these prescriptions. The entire energy in the Kerr-Newman field is only due to the the electromagnetic field present there and therefore one finds these pseudotensors to be traceless for this spacetime. Moreover, the complexes of Einstein and Tolman give the same result for all of their components. Now one would like to pursue these investigations for non-static spacetimes. Lindquist, Schwartz, and Misner (1965) wrote that the problem of non-static solutions was discussed by Vaidya (1951,1953), by Raychaudhuri (1953), and by Israel (1958), and the most convenient form of the solution is that by Vaidya (1953). At present, we consider the well known Vaidya radiating spacetime which is expressed by a non-static spherically symmetric line-element.

Vaidya (1952) started with a general non-static spherically symmetric line-element in Schwarzschild coordinates:

$$d\tau^2 = Bdt^2 - Adr^2 - D\, r^2 \left(d\theta^2 + sin^2\theta\, d\phi^2\right) \tag{5.1.1}$$

with $B = B(r,t),\ A = A(r,t),\ D = D(r,t)$. He transformed the above line-element in quasi-Cartesian coordinates according to

$$\begin{aligned} x &= r\, sin\theta\, cos\phi, \\ y &= r\, sin\theta\, sin\phi, \\ z &= r\, cos\theta \end{aligned} \tag{5.1.2}$$

and got

$$d\tau^2 = Bdt^2 - D(dx^2 + dy^2 + dz^2) - \frac{A-D}{r^2}(xdx + ydy + zdz)^2 \tag{5.1.3}$$

Further he obtained energy density for the above line-element in Tolman's prescription :

$$t_0{}^0 = \frac{1}{8\pi r^2}\frac{\partial}{\partial r}\left[\sqrt{\frac{B}{A}}\, r\,(A - D - rD')\right] \qquad (5.1.4)$$

The prime denotes the partial derivative with respect to the radial coordinate. The above expression gives energy density for the Reissner-Nordström as well as for the Vaidya radiating spacetimes.

Later Lindquist, Schwartz, and Misner (1965) calculated energy and momentum in Vaidya radiating spacetime and got the energy-momentum given by $p^i = (M, 0, 0, 0)$ as expected.

5.2 Vaidya radiating spacetime

The Vaidya radiating spacetime (Vaidya 1951) is a non-static generalization of the Schwarzschild spacetime and it gives the gravitational field due to a spherically symmetric radiating star. The line-element describing this spacetime in the Schwarzschild coordinates is

$$d\tau^2 = Bdt^2 - Adr^2 - r^2\,(d\theta^2 + sin^2\theta d\phi^2) \qquad (5.2.5)$$

with

$$B = \left\{\frac{\dot{M}}{f(M)}\right\}^2 \left(1 - \frac{2M}{r}\right) \qquad (5.2.6)$$

and

$$A = \left(1 - \frac{2M}{r}\right)^{-1} \tag{5.2.7}$$

where $M = M(r,t), \dot{M} = \frac{\partial M}{\partial t}$ and $f(M)$ is an arbitrary function of the mass parameter $M(r,t)$. Vaidya metric in quasi-Cartesian coordinates is given by (5.1.3) with $D = 1$, and B and A as given by (5.2.6) and (5.2.7) respectively.

However, the same spacetime in Kerr-Schild form is given by the line-element (Vaidya 1953; Lindquist et al 1965):

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{2M(u)}{r}\left[dt - \frac{1}{r}\ (xdx + ydy + zdz)\right]^2 \tag{5.2.8}$$

with $u = t - r$ and r as defined by (5.1.2).

5.3 Calculations

As the Vaidya radiating spacetime in the Kerr-Schild form is simpler in structure, we, starting with the line-element (5.2.8), have evaluated all the components of the pseudotensors given in the section two of chapter four. The components of energy-momentum complexes of Einstein, Tolman, LL, and Møller for the Vaidya spacetime are listed below (few symbols like α, β, γ which were used in chapter three are redefined here):

$$\theta_0{}^0 = t_0{}^0 = L^{00} = \mathcal{T}_0{}^0 = -\frac{M'}{4\pi r^2} \tag{5.3.9}$$

$$\theta_1{}^0 = -\theta_0{}^1 = t_1{}^0 = -t_0{}^1 = -L^{01} = -L^{10} = -\mathcal{T}_0{}^1 = \alpha x M' \tag{5.3.10}$$

$$\theta_2{}^0 = -\theta_0{}^2 = t_2{}^0 = -t_0{}^2 = -L^{02} = -L^{20} = -\mathcal{T}_0{}^2 = \alpha y M' \tag{5.3.11}$$

$$\theta_3{}^0 = -\theta_0{}^3 = t_3{}^0 = -t_0{}^3 = -L^{03} = -L^{30} = -\mathcal{T}_0{}^3 = \alpha z M' \tag{5.3.12}$$

$$\theta_1{}^1 = t_1{}^1 = -L^{11} = \beta x^2 M' \tag{5.3.13}$$

$$\theta_2{}^2 = t_2{}^2 = -L^{22} = \beta y^2 M' \tag{5.3.14}$$

$$\theta_3{}^3 = t_3{}^3 = -L^{33} = \beta z^2 M' \tag{5.3.15}$$

$$\theta_1{}^2 = \theta_2{}^1 = t_1{}^2 = t_2{}^1 = -L^{12} = -L^{21} = \beta xy M' \tag{5.3.16}$$

$$\theta_2{}^3 = \theta_3{}^2 = t_2{}^3 = t_3{}^2 = -L^{23} = -L^{32} = \beta yz M' \tag{5.3.17}$$

$$\theta_3{}^1 = \theta_1{}^3 = t_3{}^1 = t_1{}^3 = -L^{31} = -L^{13} = \beta zy M' \tag{5.3.18}$$

$$\mathcal{T}_1{}^0 = x\beta\mu, \quad \mathcal{T}_2{}^0 = y\beta\mu, \quad \mathcal{T}_3{}^0 = z\beta\mu \tag{5.3.19}$$

$$\mathcal{T}_1{}^1 = \gamma\left[M(r^2 - 3x^2) + M'rx^2\right] \tag{5.3.20}$$

$$\mathcal{T}_2{}^2 = \gamma\left[M(r^2 - 3y^2) + M'ry^2\right] \tag{5.3.21}$$

$$\mathcal{T}_3{}^3 = \gamma\left[M(r^2 - 3z^2) + M'rz^2\right] \tag{5.3.22}$$

$$\mathcal{T}_1{}^2 = \mathcal{T}_2{}^1 = xy\nu\gamma, \tag{5.3.23}$$

$$\mathcal{T}_2{}^3 = \mathcal{T}_3{}^2 = yz\nu\gamma, \tag{5.3.24}$$

$$\mathcal{T}_3{}^1 = \mathcal{T}_1{}^3 = zx\nu\gamma, \tag{5.3.25}$$

where

$$\alpha = \frac{1}{4\pi r^3},$$
$$\beta = \frac{\alpha}{r},$$
$$\gamma = \frac{\beta}{r},$$
$$\mu = -2M + rM',$$
$$\nu = \mu - M \tag{5.3.26}$$

The prime denotes the derivative with respect to the coordinate u. Traces of these pseudotensors for the spacetime under investigation are

$$\theta_i{}^i = t_i{}^i = L_i{}^i = \mathcal{T}_i{}^i = 0 \tag{5.3.27}$$

5.4 Discussion

We have already discussed in chapter four that the pseudotensors of Einstein, Tolman, and LL give the same energy density in the Kerr-Newman spacetime whereas that of Møller gives twice the value obtained in these prescriptions. However, one finds that all these prescriptions give the same energy density in the Vaidya spacetime. The total energy-momentum associated with the Vaidya spacetime is given by $p^i = (M, 0, 0, 0)$ in all these four prescriptions. Despite the non-tensorial character of these pseudotensors, like in the case of the Kerr-Newman metric, one finds these energy-momentum complexes to be traceless for the radiating spacetime and $\theta_i{}^k = t_i{}^k$ for all values of indices i and k.

As the Vaidya spacetime is a non-static generalization of the Schwarzschild spacetime, one finds here all the components of momentum density to be non-zero. However, the total momentum is zero as expected. For the Schwarzschild spacetime (M=constant), all the components of the pseudotensors of Einstein, Tolman, and Landau and Lifshitz vanish , whereas that of Møller, none but only the energy and momentum density components are zero. The symmetry of the spacetime is clearly reflected in the components of the pseudotensors.

Now we come back to the Vaidya's calculation of Tolman energy for a general non-static spherically symmetric metric. The total energy in the Vaidya radiating spacetime obtained by him is the same as obtained by Lindquist et al (used LL pseudotensor) as well as by us (we have used pseudotensors of Einstein, Tolman, and Møller). Moreover, the Vaidya's expression for the Tolman energy for a general non-static spherically symmetric metric yields same energy density for the Reissner-Nordström field as obtained by us (we have taken Reissner-Nordström line-element in Kerr-Schild form) which is desired. Therefore, one concludes that the Tolman energy for the R-N metric is consistent whether one uses for this the line-element (5.1.3) or that expressed in Kerr-Schild form given by (4.3.10) with $a = 0$. Now one would naturally like to learn if the other pseudotensors (those of Einstein, LL, and Møller) yield the same result for the energy in R-N spacetime expressed by the two forms of the line-elements. This is what we will discuss in the next chapter.

Chapter 6

A comment on energy-momentum pseudotensor of Landau and Lifshitz

6.1 Introduction

By previous investigations we have learned that the energy-momentum pseudotensors of Einstein, Tolman, Landau and Lifshitz, and Møller give sensible results for energy in Kerr-Newman as well as the Vaidya radiating spactimes when calculations have been accomplished in Kerr-Schild Cartesian coordinates. The pseudotensor of LL, being symmetric in its indices, have privilege over others to provide angular momentum in, of course, asymptotically flat spacetimes. We have discussed that it yields sensible result for angular momentum in Kerr-Newman spacetime. It is well known that these pseudotensors provide meaningful results in quasi-Cartesian coordinates (Kerr-Schild Cartesian coordinates satisfy this condition). An asymptotically flat spacetime can be expressed in quasi-Cartesian coordinates though it may not be in Kerr-Schild form.

We have already discussed about the Vaidya's expression for Tolman energy for a general non-static spherically symmetric spacetime in the last chapter. However, to maintain the continuity, we will rediscuss this in brief. Vaidya (1952) started with a line-element in Schwarzschild coordinates:

$$d\tau^2 = Bdt^2 - Adr^2 - Dr^2(d\theta^2 + sin^2\theta \, d\phi^2) \tag{6.1.1}$$

where $B = B(r,t)$, $A = A(r,t)$, and $D = D(r,t)$ are such that the line-element is asymptotically flat. Using the usual flat space Cartesian coordinates transformations, he got

$$d\tau^2 = Bdt^2 - D(dx^2 + dy^2 + dz^2) - \frac{A-D}{r^2}(xdx + ydy + zdz)^2 \tag{6.1.2}$$

Further, he found Tolman's energy density for the line-element given above:

$$t_0{}^0 = \frac{1}{8\pi r^2}\frac{\partial}{\partial r}\left[\sqrt{\frac{B}{A}}\, r\,(A - D - rD')\right] \tag{6.1.3}$$

The prime denotes the partial derivative with respect to the radial coordinate. For the Reissner-Nordström spacetime ($B = A^{-1} = 1 - \frac{2M}{R} + \frac{Q^2}{r^2}$, $D = 1$), it yields

$$t_0{}^0 = \frac{Q^2}{8\pi r^4} \tag{6.1.4}$$

which is exactly same as obtained, in chapter four, for the R-N metric in Kerr-Schild Cartesian coordinates ($a = 0$ in the K-N metric gives the R-N metric) and therefore one is happy to find that the energy obtained in the Tolman's prescription is consistent. Now one would like to know if the results are consistent in the prescriptions of Einstein, Tolman, and Møller. This is the subject of investigation in the following.

6.2 Calculations

We have claculated all the components of the energy-momentum complexes of Einstein, Tolman, Landau and Lifshitz, and Møller for the Reissner-Nordström metric expressed by (6.1.2) (with $B = A^{-1} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, D = 1$) which are listed below:

$$\theta_0{}^0 = t_0{}^0 = \frac{1}{2}\mathcal{T}_0{}^0 = \frac{Q^2}{8\pi r^4} \tag{6.2.5}$$

$$L^{00} = \frac{-Q^4 + Q^2 r\,(r + 4M) - 4M^2 r^2}{8\pi r^2 \left[Q^4 + 2Q^2 r\,(r - 2M) + r^2\,(r - 2M)^2\right]} \tag{6.2.6}$$

$$\theta_\alpha{}^0 = \theta_0{}^\alpha = t_\alpha{}^0 = t_0{}^\alpha = L^{0\alpha} = L^{\alpha 0} = \mathcal{T}_\alpha{}^0 = \mathcal{T}_0{}^\alpha = 0 \tag{6.2.7}$$

where index α runs from 1 to 3.

$$\theta_1{}^1 = t_1{}^1 = -L^{11} = \frac{Q^2}{8\pi r^6}\left(2x^2 - r^2\right) \tag{6.2.8}$$

$$\theta_2{}^2 = t_2{}^2 = -L^{22} = \frac{Q^2}{8\pi r^6}\left(2y^2 - r^2\right) \tag{6.2.9}$$

$$\theta_3{}^3 = t_3{}^3 = -L^{33} = \frac{Q^2}{8\pi r^6}\left(2z^2 - r^2\right) \tag{6.2.10}$$

$$\theta_1{}^2 = \theta_2{}^1 = t_1{}^2 = t_2{}^1 = -L^{12} = -L^{21} = \frac{Q^2 xy}{4\pi r^6} \tag{6.2.11}$$

$$\theta_2{}^3 = \theta_3{}^2 = t_2{}^3 = t_3{}^2 = -L^{23} = -L^{32} = \frac{Q^2 yz}{4\pi r^6} \tag{6.2.12}$$

$$\theta_3{}^1 = \theta_1{}^3 = t_3{}^1 = t_1{}^3 = -L^{31} = -L^{13} = \frac{Q^2 zx}{4\pi r^6} \tag{6.2.13}$$

$$\mathcal{T}_1{}^1 = \frac{1}{4\pi r^6}\left[Mr\left(r^2 - 3x^2\right) + Q^2\left(2x^2 - r^2\right)\right] \tag{6.2.14}$$

$$\mathcal{T}_2{}^2 = \frac{1}{4\pi r^6}\left[Mr\left(r^2 - 3y^2\right) + Q^2\left(2y^2 - r^2\right)\right] \tag{6.2.15}$$

$$\mathcal{T}_3{}^3 = \frac{1}{4\pi r^6}\left[Mr\left(r^2 - 3z^2\right) + Q^2\left(2z^2 - r^2\right)\right] \tag{6.2.16}$$

$$\mathcal{T}_1{}^2 = \mathcal{T}_2{}^1 = \frac{xy}{4\pi r^6}\left(2Q^2 - 3Mr\right) \tag{6.2.17}$$

$$\mathcal{T}_2{}^3 = \mathcal{T}_3{}^2 = \frac{yz}{4\pi r^6}\left(2Q^2 - 3Mr\right) \tag{6.2.18}$$

$$\mathcal{T}_3{}^1 = \mathcal{T}_1{}^3 = \frac{zx}{4\pi r^6}\left(2Q^2 - 3Mr\right) \tag{6.2.19}$$

Traces are given by

$$\theta_i{}^i = t_i{}^i = \mathcal{T}_i{}^i = 0 \tag{6.2.20}$$

$$L_i{}^i \;=\; \frac{4Mr\left(2Q^2 - Mr\right) - 3Q^4}{8\pi r^4\left[r^2 - 2Mr + Q^2\right]} \tag{6.2.21}$$

The Reissner-Nordström metric in Kerr-Schild Cartesian coordinates (Carter 1968):

$$d\tau^2 = dT^2 - dx^2 - dy^2 - dz^2 - \left(\frac{2M}{r} - \frac{Q^2}{r^2}\right)(dT + dr)^2 \tag{6.2.22}$$

with $r^2 = x^2 + y^2 + z^2$

The time coordinate is here denoted by T instead of t as used in (4.3.10). The time coordinates in (6.1.2) [with $B = A^{-1} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$] and (6.2.22) are related through

$$T = \pm t - r + \int dr \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} \tag{6.2.23}$$

The chapter four contains all the components of the aforesaid pseudotensors for the Kerr-Newman metric in Kerr-Schild Cartesiãn coordinates. Switching off the rotation parameter, one gets for the R-N metric:

$$\theta_0{}^0 = t_0{}^0 = L^{00} = \frac{1}{2}\mathcal{T}_0{}^0 = \frac{Q^2}{8\pi r^4} \tag{6.2.24}$$

$$\theta_\alpha{}^0 = \theta_0{}^\alpha = t_\alpha{}^0 = t_0{}^\alpha = L^{0\alpha} = L^{\alpha 0} = \mathcal{T}_0{}^\alpha = 0 \tag{6.2.25}$$

(index α runs from 1 to 3).

$$\theta_1{}^1 = t_1{}^1 = -L^{11} = \frac{Q^2}{8\pi r^6}\left(2x^2 - r^2\right) \tag{6.2.26}$$

$$\theta_2{}^2 = t_2{}^2 = -L^{22} = \frac{Q^2}{8\pi r^6}\left(2y^2 - r^2\right) \tag{6.2.27}$$

$$\theta_3{}^3 = t_3{}^3 = -L^{33} = \frac{Q^2}{8\pi r^6}\left(2z^2 - r^2\right) \tag{6.2.28}$$

$$\theta_1{}^2 = \theta_2{}^1 = t_1{}^2 = t_2{}^1 = -L^{12} = -L^{21} = \frac{Q^2 xy}{4\pi r^6} \tag{6.2.29}$$

$$\theta_2{}^3 = \theta_3{}^2 = t_2{}^3 = t_3{}^2 = -L^{23} = -L^{32} = \frac{Q^2 yz}{4\pi r^6} \tag{6.2.30}$$

$$\theta_3{}^1 = \theta_1{}^3 = t_3{}^1 = t_1{}^3 = -L^{31} = -L^{13} = \frac{Q^2 zx}{4\pi r^6} \tag{6.2.31}$$

$$\mathcal{T}_1{}^0 = \frac{Mx}{2\pi r^4} \tag{6.2.32}$$

$$\mathcal{T}_2{}^0 = \frac{My}{2\pi r^4} \tag{6.2.33}$$

$$\mathcal{T}_3{}^0 = \frac{Mz}{2\pi r^4} \tag{6.2.34}$$

$$\mathcal{T}_1{}^1 = \frac{1}{4\pi r^6}\left[Mr\left(r^2 - 3x^2\right) + Q^2\left(2x^2 - r^2\right)\right] \tag{6.2.35}$$

$$\mathcal{T}_2{}^2 = \frac{1}{4\pi r^6}\left[Mr\left(r^2 - 3y^2\right) + Q^2\left(2y^2 - r^2\right)\right] \tag{6.2.36}$$

$$\mathcal{T}_3{}^3 = \frac{1}{4\pi r^6}\left[Mr\left(r^2 - 3z^2\right) + Q^2\left(2z^2 - r^2\right)\right] \tag{6.2.37}$$

$$\mathcal{T}_1{}^2 = \mathcal{T}_2{}^1 = \frac{xy}{4\pi r^6}\left(2Q^2 - 3Mr\right) \tag{6.2.38}$$

$$\mathcal{T}_2{}^3 = \mathcal{T}_3{}^2 = \frac{yz}{4\pi r^6}\left(2Q^2 - 3Mr\right) \qquad (6.2.39)$$

$$\mathcal{T}_3{}^1 = \mathcal{T}_1{}^3 = \frac{zx}{4\pi r^6}\left(2Q^2 - 3Mr\right) \qquad (6.2.40)$$

Traces of the pseudotensors are

$$\theta_i{}^i = t_i{}^i = L_i{}^i = \mathcal{T}_i{}^i = 0 \qquad (6.2.41)$$

6.3 Discussion

A comparision of (6.2.5-19) and (6.2.24-40) makes it clear that the values for the components of the pseudotensors of Einstein, Tolman, LL, and Møller for the Reissner-Nodström spacetime expressed by the line-elements (6.1.2) and (6.2.22) are consistent except the components $L^{00}, \mathcal{T}_\alpha{}^0$ (α takes values 1 to 3). The energy-momentum complexes of Einstein, Tolman, and Møller give respectively same energy density for the Reissner-Nordström spacetime expressed by the line-elements (6.1.2) as well as (6.2.22). However, that of LL does not give a consistent result. Moreover, switching off the charge parameter in (6.2.6) yields negative energy density in the Scharzschild field for all values of the radial distance and the mass parameter. One can see that $\theta_i{}^k = t_i{}^k$ for all values of indices i and k. Despite the non-tensorial character, the energy-momentum complexes of Einstein, Tolman, and Møller have been found to be traceless for the Reissner-Nordström spacetime expressed by both forms of the line-elements. As opposed to above, the pseudotensor of LL is however traceless for the R-N field expressed by the line-element (6.2.22), we find this result to be not consistent for the same spacetime given by (6.1.2), which is an undesired outcome of the LL pseudotensor.

Though one has a predilection for the LL energy-momentum pseudotensor over others, as, being symmetric, it can be used to evaluate angular momentum in asymptotically flat spacetimes, one concludes that it suffers from a relative drawback as well.

However, we believe that an adequate prescription for energy- momentum localization in a general relativistic system should be possible to be obtained which requires further serious effort.

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