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M. Shillor M. Sofonea J.J. Telega

Models and Analysis of Quasistatic Contact

Variational Methods

Springer

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Preface

Currently the Mathematical Theory of Contact Mechanics is emerging from its infancy, and a point has been reached where a unified presentation of the results, scattered throughout a variety of publications, is needed.

The aim of this monograph is to partially address this need by providing state-of-the-art mathematical modelling and analysis of some of the phenomena that take place when a deformable body comes into quasistatic contact. We present models for the processes, describe the mathematical results, and provide representative proofs. A comprehensive list of recent references supplements this work. Between the time we started writing this monograph and the present, W. Han and M. Sofonea published the book "Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity," which focuses on mathematical and numerical analysis of contact problems for viscoelastic and viscoplastic materials.

Our book, divided into three parts, with 14 chapters, is intended as a unified and readily accessible source for mathematicians, applied mathematicians, mechanicians, engineers and scientists, as well as advanced students. It is organized in three different levels, so that readers who are not fluent in the Theory of Variational Inequalities can easily access the modelling part and the main mathematical results.

Representative proofs, which may be skipped upon first reading, are provided for those who are interested in the mathematical methods. Part I contains models of the processes involved in contact. It is written at the first level for those who have an interest in Contact Mechanics or Tribology, and minimal background in differential equations and initial-boundary value problems. The processes for which we provide various models are friction, heat generation and thermal effects, wear, adhesion and damage. Several sections are devoted to each one of these topics and the relationships among them.

The second level of the book, which focuses on the settings of the models as initial-boundary value problems and their variational formulations, can be found in Part II. It requires some background in modern mathematics, although preliminary material is provided in the first chapter. Each chapter describes a few problems with a common underlying theme. The third level deals with the proofs of the theorems. In each chapter in Part II, the proofs of one or two theorems can be found as examples of the mathematical tools used. This is also the level for those mathematicians interested in the Theory of Variational Inequalities and its applications.

We observe that as a result of the specific problems posed by contact models, the theory had to be extended and some of these generalizations are also provided. Part III presents a short review and many references of recent results for dynamic contact, one-dimensional contact and miscellaneous problems not covered in the book. The concluding chapter is a summary and a discussion of open problems and future directions. The topics of static and evolution geometrically nonlinear contact problems, including structures, are currently in preparation by the authors.

We would like to acknowledge and thank all of our collaborators for their contributions that led to this book, especially to Professors Kevin T. Andrews, Weimin Han and Kenneth L. Kuttler. We would also like to thank Prof. Dr. Wolf Beiglböck, Senior Physics Editor, and his staff for their help in bringing this monograph to your hand.

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Contents

Part II Models and Their Variational Analysis

1 Introduction

Considerable progress has been achieved recently in modeling and mathematical analysis of various processes involved in contact between deformable bodies. Indeed, a general Mathematical Theory of Contact Mechanics is currently emerging.

Extensive technical literature, mainly in engineering but also in geophysics, covers frictional or frictionless contact. In geophysics, the literature focuses on the motion of tectonic plates and, in particular, on earthquakes. The engineering literature deals with many aspects and facets of the functioning and operation of machines, mechanisms and structures. The publications in these disciplines, however, are often concerned with specific settings, geometries or materials. Their aim is usually related to particular applied aspects of the problems.

The emerging Mathematical Theory of Contact Mechanics is concerned with the mathematical structures which underlie general contact problems with different constitutive laws, i.e., materials, varied geometries, and different contact conditions. The aim is to provide a sound, clear and rigorous background to the following:

- construction of models based on thermodynamic principles which are motivated by applications;
- assigning precise meaning to solutions of the models;
- establishing the existence of solutions;
- proving the uniqueness of the solutions, or establishing their nonuniqueness and finding criteria for choosing the appropriate solution;
- determining the generic regularity or smoothness of the solutions;
- investigating the stability of solutions and their asymptotic behavior;
- describing the qualitative behavior of the solutions.

Once existence, uniqueness or nonuniqueness, and stability of solutions have been established, related important questions arise, such as: mathematical analysis of the solutions and how to construct reliable and efficient algorithms for their numerical approximations with guaranteed convergence.

The theory provides an environment or a structure, where questions of optimal control and of system parameter identification can be addressed. These are of considerable theoretical and applied interest. Moreover, optimal

M. Shillor, M. Sofonea, J.J. Telega: *Models and Analysis of Quasistatic Contact*, Lect. Notes Phys. **655**, (2001), 1–6
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shape design, which is one of the main interests of the design engineer can be investigated and reliable results found.

Of the list above, the progress we describe in this monograph is mainly in the first three items; namely, modelling, weak or variational formulation, and existence of solutions. We also present results of uniqueness, a part of the fourth item, but when the uniqueness is not known, the characterization of the nonuniqueness is still missing.

Clearly, much remains to be done, and we mention some of the open problems and future directions in the final chapter. As this monograph shows, however, recent progress is extensive and the field is vibrant and continues to evolve.

The first recognized publication on contact between deformable bodies was that of Hertz [1]. The next one was by Signorini [2], where the problem was posed, in what is now termed a variational form, and was subsequently solved by Fichera [3, 4]. However, the general theory of contact mechanics began with the monograph by Duvaut and Lions [5], who first presented variational formulations of contact problems and proved some basic existence and uniqueness results. Then, Duvaut [6], followed by Nečas et al. [7], Jarusek [8], Cocu [9] and Kato [10], established the existence of a weak solution for the static frictional contact problem involving linearly elastic materials, where in [6,9] the friction condition was regularized. The normal compliance contact condition was introduced by Oden and Martins in [11, 12], and the existence of weak solutions for contact problems with this condition was established in [11, 12] and also in Telega [13] and Klarbring et al. [14, 15]. These papers (except [11,12]) dealt with the static frictional problem, which was considered as a step in a time marching scheme for an evolutionary problem. The static problem with nonlocal friction law was considered in Demkowicz and Oden [16] and in Oden and Pires [17].

Andersson [18] and Klarbring et al. [19] were the first to obtain existence results for the quasistatic frictional contact problem for an elastic material with normal compliance. Then, Rochdi et al. [20] reported the first existence and uniqueness result for the quasistatic frictional contact problem with normal compliance for viscoelastic materials.

This was followed by Amassad et al. [21] who proved the first existence and uniqueness theorems for the problems of viscoelastic bilateral contact with slip rate, or total slip rate dependent friction coefficient. The latter problem takes into account the history of the sliding, and in this manner the rearrangement of the contact surface due to friction. Next, Shillor and Sofonea [22] established the first existence and uniqueness theorem for the viscoelastic bilateral contact problem with friction, and in [23] they proved the unique solvability of the frictional problem for a viscoplastic material with damage.

Recently, Andersson [24] obtained the first existence result for solutions of the quasistatic contact problem with friction and Signorini's condition for an elastic material. Similar results were also proved in Cocu et al. [25], Rocca [26] and Cocu and Rocca [27]. In [25] the contact was modeled with a non-local version of Coulomb's law, in [26] it was modeled with a local version of Coulomb's law, and in [27] the model was assumed to involve friction and adhesion. The variational analysis of the frictionless Signorini problem was provided in Sofonea [28], in the case of rate-type viscoplastic materials, and the numerical analysis of this problem was performed in Chen et al. [29]. These results were extended to the frictionless Signorini problem between two viscoplastic bodies in Rochdi and Sofonea [30] and Han and Sofonea [31], respectively.

Although it is customary in engineering and certain mathematical publications to consider the normal compliance as an approximation of the Signorini nonpenetration condition, the results in this monograph and in the study of dynamic contact problems indicate that considering the Signorini condition as an idealization (or even over-idealization) of normal compliance makes more sense for practical reasons. Such an idealization seems to be useful in some quasistatic problems, but in others it is not. Moreover, in dynamic contact or impact problems it seems not to be very useful, since a perfectly rigid body has to support infinite impulsive stresses upon contact or impact, resulting from the discontinuity of the velocity upon contact. This fact shows up, mathematically, in the form of weak and likely nonunique solutions, in which the contact stress is a measure and not an ordinary function. Models with the Signorini condition are mathematically very complicated, the solutions weak (when existing), and should be of limited practical use. The wide-spread use of the condition is due to the ease of writing it and of implementing it in computer codes for numerical approximations. In most cases the mathematical difficulties associated with it are simply disregarded. However, there seems to be an important exception, and that is in biomechanics, in the contact of tissue with bone or implant. The tissue is often modelled as a viscoelastic material, while the bone or the implant are considered as elastic, but for the purposes of modelling of the contact they are assumed to be essentially rigid. In such a case the Signorini condition is a reasonable choice.

To the best of our knowledge, the first result for dynamic frictional contact with Tresca's friction condition was obtained in [5], and the one for contact with normal compliance can be found in [11,12]. The existence of the unique solution for the dynamic problem with normal compliance and slip rate dependent friction coefficient is given in Kuttler and Shillor [32]. First existence results when the friction coefficient is discontinuous can be found in Kuttler and Shillor [33, 34] and the first existence result for the Signorini problem with nonlocal friction in Cocu [35] and the same problem with slip rate dependent discontinuous friction coefficient in Kuttler and Shillor [36]. A major regularity result for dynamic frictionless contact has been obtained in Kuttler and Shillor [37]. A related regularity result for the problem with adhesion and damage can be found in Kuttler et al. [38].

The first existence result for quasistatic frictionless contact of a thermoelastic body with a rigid foundation was proved in Shi and Shillor [39] and then extended by Xu [40]. The existence of solutions for a thermoviscoelastic problem with frictional contact was first established in Figueredo and Trabucho [41], and can be found in Amassad et al. [42], Andrews et al. [43, 44] and Muñoz Rivera and Racke [45]. In [44], the wear of the contacting surfaces due to friction has been taken into account by using the Archard law, and in [42], the friction coefficient was assumed to depend on the slip rate or on the total slip, i.e., on the process history.

By now the breadth of published results is such that a single survey or monograph cannot do justice to the field. Therefore, we concentrate exclusively on modelling and variational analysis of multidimensional quasistatic contact problems for deformable bodies within the framework of small deformations. Some of the issues related to large deformations can be found in a monograph which is currently in preparation.

We do not describe in any detail publications which deal solely with:

– dynamic contact problems;
– numerical analysis, error bounds or numerical simulations;
– one-dimensional contact problems;
– static or rolling frictional contact;
– problems of crack development;
– earthquakes and geological processes;
– dynamics of contacting rigid bodies;
– impact of rigid bodies.

However, we mention in passing some of those directly related to the main topics of this monograph, especially in the modelling Chaps. 2 and 3. Moreover, in Chap. 13 we provide a very brief survey of results dealing with dynamic, one-dimensional, and miscellaneous contact problems.

Recent papers, reviews, monographs, and books on mathematical and related problems in contact mechanics include [46–66], and we refer the reader there for a wealth of additional information about these and related topics. Rolling frictional contact, a very important topic in transportation, can be found in the monograph by Kalker [67]. Results on contact with lubrication can be found in Bayada et al. [68] and references therein.

Quasistatic contact problems are invariably formulated as variational or quasivariational inequalities. The standard reference for variational inequalities is Kinderlehrer and Stampacchia [69], in addition, useful information can be found in Barbu [70], Hlaváček et al. [71] and Panagiotopoulos [47, 50], among others.

References for the physical and engineering aspects of contact or constitutive relations are [65, 72–79], among many others.

This book synthesizes the mathematical models for the various processes involved in contact, their variational formulations, the assumptions made on the data and the statements of the existence and uniqueness theorems. Some of the new results are described in detail while others are portrayed only briefly.

The monograph is structured as follows.

Part I includes the classical description of the equations, material constitutive relations, boundary conditions and models. It is meant to be self contained. Thermal effects, which often accompany friction are presented. Recent models for wear and adhesion are described fully, and so are models for material damage. The derivation of some of the models from thermodynamic potentials, using thermodynamic laws and subdifferentiation, is provided in Chap. 4. Our aim is to provide a comprehensive overview of the currently employed models for the physical phenomena involved in contact. As can be seen there, the subject is broad, and to have the models reflect the engineering and industrial needs, some of them use sophisticated mathematics to describe rather complicated processes. In Chap. 5 we assemble the equations and relevant conditions into a representative problem and describe it in full detail. This chapter is meant to serve as a bridge between Part I and Part II.

Part II describes in detail the models and for each one we provide the classical formulation, and then the variational or weak formulation, detail the assumptions on the problem data and state the relevant existence and possibly uniqueness results. Elastic, viscoelastic and viscoplastic constitutive laws are used to describe the material behavior. Contact is modelled with the Signorini, normal compliance, or normal damped response conditions. Friction is modelled with general versions of the Coulomb and Tresca laws. Models with slip dependent friction, wear or adhesion are also presented. For each type of problems we provide one or two complete proofs. These indicate the methods employed and the kind of results that can be obtained by using them. We note that for some of the models presented there, the numerical analysis, which includes error estimates, convergence results and numerical simulations, can be found in [51, 56, 80, 81] and the references therein. We do not present these results here, since numerical analysis of contact problems has reached a point where it deserves separate monographs of its own, the first one of which is [51].

Part III is very short, and only lists some of the more recent references on dynamic, one-dimensional, and miscellaneous problems. Although we do comment on some of the papers, the topics have reached a state where they deserve their own comprehensive presentations. Finally, we summarize the topics in Chap. 14. While considerable progress has been achieved, much remains to be done. Therefore, we present some open or unresolved questions and problems that need to be addressed in the near future to continue the expansion and deepening of the subject. We also present our personal views on the future direction of the Mathematical Theory of Contact Mechanics.

The hallmark of contact problems is that the 'action,' or the interesting phenomena take place on the surface or boundary of the body or domain. Mathematically, the processes are described as boundary conditions, and even when the equations of motion, which describe the behaviour of the bulk material, are linear, the initial-boundary value problems with contact conditions are strongly nonlinear and quite nonstandard. This peculiarity may be viewed as an obstacle, introducing considerable difficulties both in the mathematical investigation and in the numerical analysis, and may lead to the non-convergence of computer algorithms. On the other hand, it may be viewed as a challenge, forcing the creation of new mathematical ideas and tools and, thus, leading to the expansion of the theory.

This monograph clearly shows that the close interaction and cross-fertilization between Contact Mechanics and the Theory of Variational Inequalities has been beneficial to both. New contact conditions have led to the expansion of the Theory as new problems were posed, new operators have been introduced and analyzed, and extended existence and uniqueness results have been established. These, in turn, have allowed for a better and more detailed description of the processes, leading to even more sophisticated and challenging theoretical problems. This close interaction is essentially creating and expanding the Mathematical Theory of Contact Mechanics.

2 Evolution Equations, Contact and Friction

Contact processes take place on the surface, and, therefore, are described by boundary conditions. However, these are the boundaries or surfaces of mechanical bodies or structures, and one has to describe the evolution of the mechanical state of the body, as well. The problems, in their classical formulation, consist of the constitutive laws, the equations of motion and the relevant initial and boundary conditions. And contact enters naturally via the boundary conditions.

In this part and the next one the models are constructed within the framework of linearized or small deformations theory. Models for contact within the theory of large deformations will be described in the future.

We begin with Sect. 2.1 which gives a short description of the modelling of contact processes and the general structure of the mathematical problems to provide the reader with an overview of the models and conditions to come. The physical settings of the problems that will be described in this monograph and the quasistatic equations of motion are given in Sect. 2.2. The constitutive conditions for elastic, viscoelastic and elastic-viscoplastic materials are described in Sect. 2.3. Standard boundary conditions can be found in Sect. 2.4. A short note on the dimensionless form of the various variables is provided in Sect. 2.5. Then, we devote Sect. 2.6 to several contact boundary conditions: bilateral contact in which contact is always maintained; normal compliance, which describes contact with a reactive foundation; the Signorini condition, which describes contact with a rigid foundation; and the normal damped response, in which the response of the foundation depends on the speed. Then, we describe the conditions in the tangential directions. One may use the frictionless condition, or various versions of friction conditions, which are described in detail. This leads us, in Sect. 2.7, to discuss the concept of friction coefficient. It turns out to be a very complex issue, especially if one wishes to have it depend on the slip speed, on the temperature and on other surface parameters. Finally, in Sect. 2.8 the transition from the Coulomb-like behavior to that of Tresca's is detailed.

This chapter is a basis for most of what follows in this book, and effort has been made to provide a clear presentation, possibly at the expense of some redundancy.

M. Shillor, M. Sofonea, J.J. Telega: *Models and Analysis of Quasistatic Contact*, Lect. Notes Phys. **655**, (2001), 9–29
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2.1 The Modelling of Contact Processes

In this short section we present a general description of the mathematical approach to the modelling of the processes involved in contact between deformable bodies, which will be found in this part of the monograph.

A mathematical model for a process involving a continuous medium consist of clearly and precisely specifying: the geometrical setting, the variables which determine the state of the system, the material behaviour which is reflected in a constitutive relation or law, the input data, the equations of evolution for the state of the system, the initial and boundary conditions for the system variables, and, finally, clarifying the sense in which the equations and the conditions are to be satisfied by the solutions. In the mathematical literature it is customary to put all the equations and conditions in one place, and call it the model. Usually, the geometry is specified beforehand, and the various assumptions that underlie the model are spelled out, even if it may be seen as somewhat pedantic.

It turns out that when dealing with models for the various processes involved in contact often some of the equations or conditions cannot be satisfied in the usual sense. This is related to the insufficient regularity of the solutions, which often fail to be continuous or have continuous derivatives. Therefore, weak or variational formulations are necessary. Moreover, the first step in the analysis of the models is often carried out using the variational formulations. However, it is instructive, and often necessary, to have a clear and precise classical formulation of the various elements of the model. Indeed, in Chap. 5 we describe in detail how to obtain a variational formulation from a classical one.

This and the following two chapters are dedicated to a thorough presentation and discussion of the various constitutive laws and contact conditions.

We describe elastic, viscoelastic, thermoviscoelastic, and elastic-viscoplastic constitutive laws, as well as the possible development of material damage.

We present contact conditions for a rigid or reactive foundation, with or without friction. The foundation may be stationary or moving. There may be adhesive on the contact surface, or the wear of the surface material may be of importance and has to be included.

A material object the behaviour of which we wish to describe is usually called the 'body.' The 'reference configuration' is the set of points in space that the body occupies when it is free of forces or tractions, and has a uniform temperature. Tractions are just surface forces. Finally, in most of the monograph, unless stated to the contrary, we use variables in dimensionless form, which means that all the physical quantities were rescaled appropriately. The issue is discussed in some detail in Sect. 2.5.

2.2 Physical Setting and Equations of Evolution

The setting we consider consists of a deformable body that may come in contact with another object, the so-called foundation. This term is used when the internal processes inside it are not a part of the problem under consideration. When the internal processes are important, the problem becomes that of contact between two deformable bodies. In this book we deal exclusively with the first case. Indeed, in most situations the problem of contact between two deformable bodies is very similar, both in the structure of the model and in its mathematical study, to that of contact with a foundation, and then the main difficulty becomes in handling correctly cumbersome notation.

The contact with a foundation is also referred to as an *obstacle problem*, since the foundation acts as an obstacle, preventing the body from moving freely.

In this section all the quantities are assumed to have their physical dimensions.

Let Ω be a domain in $\mathbb{R}^d$ (in the applications we have in mind $d = 2, 3$ since one-dimensional problems are excluded) representing the reference configuration of a deformable body which, as a result of forces and tractions acting on it, may come in contact with a rigid or a reactive foundation.

The surface of the body $\Gamma = \partial\Omega$ is assumed to be composed of three parts: Γ_D – over which the body is held fixed; Γ_N – over which known tractions act; Γ_C – over which contact may take place. At each time instant the potential contact surface Γ_C is divided into the part Γ_C^{con} where the body and the foundation are in contact, and the other part Γ_C^{sep} where they are separated. The boundary of the set Γ_C^{con} is a free boundary, dictated by the solution of the problem. The structure of the set Γ_C^{con} is of considerable interest, and we shall remark on this point in Chap. 14.

We denote vectors and tensors by bold-face letters, such as the position vector $\mathbf{x} = (x_1, \ldots, x_d)$, and the (small) strain tensor $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$, for $i, j = 1, \ldots, d$, respectively.

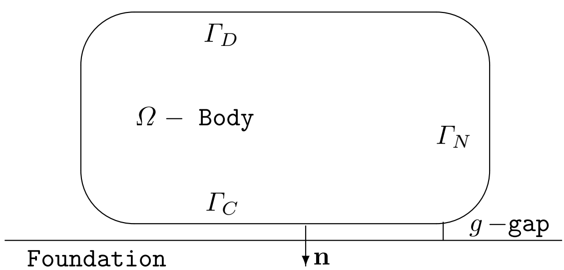

Fig. 1. Schematic physical setting; Γ_C is the potential contact surface

We assume, for the sake of generality, that in the reference configuration there exists a gap, denoted by $g = g(\mathbf{x})$, between Γ_C and the foundation, that is measured along the outer normal $\mathbf{n} = (n_1, \ldots, n_d)$ to Γ_C. A schematic two-dimensional setting is depicted in Fig. 1, however, what follows applies to very general two- or three-dimensional settings.

We denote by $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u_1(\mathbf{x}, t), \ldots, u_d(\mathbf{x}, t))$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, t) = (\sigma_{ij}(\mathbf{x}, t))$, and $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{u})$ the displacement vector, the stress tensor, and the linearized strain tensor, respectively. The mechanical (isothermal) state of the system is completely determined by the pair $(\mathbf{u}, \boldsymbol{\sigma})$. The components of the linearized strain tensor are given by

$$\varepsilon_{ij} = \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\,(u_{i,j} + u_{j,i}),$$

where, here and below, $i, j, k, l = 1, \ldots, d$; a coma separates the components from partial derivatives, i.e., $u_{i,j} = \partial u_i/\partial x_j$, and we employ the summation convention whenever an index appears exactly twice.

The dynamic equations of motion, representing momentum conservation, that govern the evolution of the state of the body are

$$\rho \ddot{u}_i - \sigma_{ij,j} = f_{Bi},$$

where ρ is the material density and $\mathbf{f}_B = (f_{B1}(x, t), \ldots, f_{Bd}(x, t))$ is the density (per unit volume) of applied forces, such as gravity. Here and below, a dot above a variable denotes the derivative with respect to time, and $\ddot{u}_i = \partial^2 u_i/\partial t^2$. These equations are valid for all systems and materials, since they are derived from the fundamental principle of momentum conservation (see, e.g., [82]).

In this book we are interested in situations in which the system configuration and the external forces and tractions vary in time in such a way that the accelerations in the system are rather small, so that the inertial terms $\rho \ddot{u}_i$ can be neglected. Thus, we obtain the quasistatic approximation for the equations of motion,

$$\mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0}, \tag{2.2.1}$$

where 'Div' is the divergence operator, that is $(\mathrm{Div}\,\boldsymbol{\sigma})_i = \sigma_{i1,1} + \ldots + \sigma_{id,d}$. Equations (2.2.1) are the equilibrium equation used in the sequel. In this approximation, at each time instant the system is in equilibrium, and the external forces are balanced by the internal stresses. Thus, the trajectories of the system in the phase space lie on the equilibrium hyper-surfaces.

2.3 Constitutive Relations

The relationship between the stresses in the body and the resulting strains characterizes a specific material the body is made of, and is given by the constitutive law or relation. It describes the deformations of the body resulting from the local action of forces and tractions. In this book, we consider, within the framework of small deformations, linear or nonlinear elastic, viscoelastic and viscoplastic materials. We also consider problems involving constitutive laws for thermoelastic and thermoviscoelastic materials in Sects. 10.4 and 10.5, respectively. In this section the variables have physical dimensions.

A linear elastic constitutive law is given by

$$\boldsymbol{\sigma} = \mathcal{B}_{el}\boldsymbol{\varepsilon}, \tag{2.3.1}$$

where $\mathcal{B}_{el}$ is a fourth-order elasticity tensor. In component form, the constitutive equation (2.3.1) reads

$$\sigma_{ij} = b_{ijkl}\varepsilon_{kl},$$

where the b_{ijkl} are the elasticity coefficients, which may be functions of position in a nonhomogeneous material, and $\varepsilon_{kl} = \varepsilon_{kl}(\mathbf{u})$.

A general viscoelastic constitutive law is given by

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\dot{\boldsymbol{\varepsilon}} + \mathcal{B}_{ve}\boldsymbol{\varepsilon}, \tag{2.3.2}$$

where, $\dot{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}(\dot{\mathbf{u}})$, $\mathcal{B}_{ve}$ is a nonlinear elasticity operator and $\mathcal{A}_{ve}$ is the (local) viscosity operator, both of which may depend on the position.

In linear viscoelasticity $\boldsymbol{\sigma} = (\sigma_{ij})$ is given by the Kelvin-Voigt type of relation

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}(\dot{\mathbf{u}}) + b_{ijkl}\varepsilon_{kl}(\mathbf{u}), \tag{2.3.3}$$

where the b_{ijkl} and a_{ijkl} are the elasticity and viscosity coefficients, respectively, which may be functions of position. For symmetry reasons, when $d = 3$, there are at most 21 different coefficients in each tensor. When the material is isotropic and homogeneous, it is characterized by only four constants: the two Lamé coefficients λ_1 and λ_2 and two viscosity coefficients a_1 and a_2,

$$\sigma_{ij} = \left(\lambda_1\varepsilon_{kk}(\mathbf{u}) + a_1\varepsilon_{kk}(\dot{\mathbf{u}})\right)\delta_{ij} + 2\left(\lambda_2\varepsilon_{ij}(\mathbf{u}) + a_2\varepsilon_{ij}(\dot{\mathbf{u}})\right).$$

Here, δ_{ij} is the Kronecker symbol, i.e. δ_{ij} represent the components of the unit matrix $\mathbf{I}_d$. If we wish to use a model with one viscosity coefficient a, we may use

$$\sigma_{ij} = \lambda_1\varepsilon_{kk}(\mathbf{u})\delta_{ij} + 2\lambda_2\varepsilon_{ij}(\mathbf{u}) + a\varepsilon_{ij}(\dot{\mathbf{u}}).$$

The viscosity terms in either one of the conditions above depend on the velocity, are local or pointwise in time, and represent short term memory. Nonlocal, or long term memory viscoelastic terms can be found in the literature, see, e.g., [5, Ch. 7] or [73, Ch. 3] (see also [83] and references therein), and have the form

$$\int_0^t a_{ijkl}(t-s)\varepsilon_{kl}(\mathbf{u}(s))\,ds,$$

where now the a_{ijkl} depend on time and may be viewed as the components of an integral kernel of the relaxation tensor.

A Maxwell-Norton model of a viscoelastic material was employed in [84] to describe the solidification of aluminium (see also references therein). Since in this model the viscosity rate is assumed to be a function of the stress, the model is given by

$$\mathcal{A}_{er}\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\sigma}} = \lambda_0 \|\boldsymbol{\sigma}^D\|^{q-2}\boldsymbol{\sigma}^D.$$

Here, $\boldsymbol{\sigma}^D$ is the deviatory part of $\boldsymbol{\sigma}$, i.e., $\boldsymbol{\sigma}^D = \boldsymbol{\sigma} - (1/3)Tr(\boldsymbol{\sigma})\mathbf{I}_d$, $(1/3)Tr(\boldsymbol{\sigma})\mathbf{I}_d$ is the hydrostatic pressure, $\mathbf{I}_d$ is the $d \times d$ identity matrix and $\|\cdot\|$ is a matrix norm; $\mathcal{A}_{er}$ is the tensor of elasticity; and λ_0 is a material constant and $q \geq 2$ is a material exponent, both determined experimentally.

Other ways of modelling the viscous response of materials can be found in the references above and in the references therein.

We shall comment on the relationship between problems for viscoelastic and elastic materials below. Formally, one obtains the elastic constitutive relation from the viscoelastic one when the viscosity vanishes, and there are few contact problems for which this limit has been rigorously established. However, as we indicate below, passing to the limit often results in a drop (sometimes quite dramatic) in the regularity of the solutions, and may also result in the loss of uniqueness.

To describe an elastic-viscoplastic material we use a rate-type constitutive relation

$$\dot{\boldsymbol{\sigma}} = \mathcal{A}_{vp}\dot{\boldsymbol{\varepsilon}} + \mathcal{G}_{vp}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}). \tag{2.3.4}$$

Here, $\mathcal{A}_{vp}$ is the elasticity operator, assumed to be linear, and $\mathcal{G}_{vp}$ is a nonlinear viscoplastic operator, that depends on both the stress and the strain tensors, and may depend on the position, as well.

Rate-type viscoplastic relations of this form have been used to describe the behavior of rubber, metals, pastes and rocks, among others (see, e.g., [74,75] and references therein). Perzyna's law, given in (6.4.10) below, is of this type (see, e.g., [5]).

A one-dimensional example of a viscoplastic operator $\mathcal{G}_{vp}$ in (2.3.4) is

$$\mathcal{G}_{vp}(\sigma, \varepsilon) = \begin{cases} -a_1 F_1(\sigma - f^*(\varepsilon)) & \text{if} \quad \sigma > f^*(\varepsilon) \\ 0 & \text{if} \quad f_*(\varepsilon) \leq \sigma \leq f^*(\varepsilon) \\ a_2 F_2(f_*(\varepsilon) - \sigma) & \text{if} \quad \sigma < f_*(\varepsilon). \end{cases}$$

Here, a_1 and a_2 are viscosity coefficients, f_*, f^* are plastic yield limits, and F_1, F_2 are given functions. We refer the reader to [75,85] for the details.

Constitutive relations that include thermal effects will be described in Sect. 3.1, and those with material damage in Sect. 3.4.

Finally, we remark that for historical reasons the elasticity operator, in the constitutive relations above, is denoted by $\mathcal{B}_{el}$, $\mathcal{B}_{ve}$ and $\mathcal{A}_{vp}$, however, we hope that no confusion will arise, since the indices make the distinction among them. We continue to use this notation since this is what the reader will find in the quoted literature.

2.4 Boundary Conditions

We now turn to the boundary conditions. The surface Γ is assumed to be Lipschitz, and thus, at almost every point the outer unit normal vector $\mathbf{n} = (n_1, \ldots, n_d)$ is defined. We also assume that $\mathrm{meas}(\Gamma_D) > 0$, and remark on this assumption below. The following decompositions of vectors and tensors on Γ will be used frequently. If $\mathbf{v}$ is a vector field defined on Γ then v_n denotes the normal component of $\mathbf{v}$ and $\mathbf{v}_\tau$ denotes the projection of $\mathbf{v}$ on the tangent plane of Γ, and they are given by

$$v_n = \mathbf{v} \cdot \mathbf{n}, \qquad \mathbf{v}_\tau = \mathbf{v} - v_n \mathbf{n}.$$

We note that v_n is a scalar, while $\mathbf{v}_\tau$ is a tangent vector to Γ. Similarly, the normal component and the tangential components of a tensor $\boldsymbol{\sigma}$ are denoted by σ_n and $\boldsymbol{\sigma}_\tau$, and are given by

$$\sigma_n = (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{n}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\mathbf{n} - \sigma_n \mathbf{n},$$

where σ_n is a scalar while $\boldsymbol{\sigma}_\tau$ is a tangent vector to Γ. Here and below, '$\cdot$' represents the inner or the scalar product for vectors and tensors; we also use $\|\cdot\|$ to denote the Euclidean norm of vector and tensor quantities. In components,

$$\sigma_n = \sigma_{ij} n_i n_j, \qquad (\sigma_\tau)_i = \sigma_{ij} n_j - \sigma_n n_i.$$

The body is held fixed on Γ_D, so we use the homogeneous Dirichlet condition,

$$\mathbf{u} = \mathbf{0} \qquad \text{on} \;\; \Gamma_D. \tag{2.4.1}$$

Known tractions $\mathbf{f}_N$ act on Γ_N, so we use the Neumann condition

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \qquad \text{on} \;\; \Gamma_N. \tag{2.4.2}$$

We remark that all the results below hold true when $\Gamma_N = \emptyset$. Also, replacing condition (2.4.1) with $\mathbf{u} = \mathbf{u}_D$, the nonhomogeneous Dirichlet condition, introduces no further difficulties, for a given $\mathbf{u}_D$ lying in an appropriate function space.

On the other hand, the assumption that $\mathrm{meas}(\Gamma_D) > 0$ is essential in quasistatic problems. Otherwise, mathematically, the problem becomes noncoercive and many of the results below do not hold. This accurately reflects the physical situation, since when $\mathrm{meas}(\Gamma_D) = 0$ the body is not held in place, but may move freely in space as a rigid body, such as in the punch problem. In such a case the quasistatic approximation is invalid, unless certain restrictions are made on the direction and size of the applied forces and tractions, and the compatibility of the data is guaranteed.

2.5 Dimensionless Variables

At this point we need to address the physical units of the various variables and quantities, and the related dimensionless variables. The problems we present contain many physical and system coefficients and parameters but very often the solutions depend only on certain combinations of these and not necessarily on each one separately.

In this short section we describe the nondimensionalization of the variables. We denote the quantities with physical dimensions, in *cgs*, (*cm*, *gm*, *sec*) units, with tilde and the dimensionless ones without it. For the sake of simplicity we deal with the three-dimensional case, as the two-dimensional case is similar. We note that there are more than one way to set a given problem in a dimensionless form, and below we present only one such choice.

The components of $\widetilde{\mathbf{x}} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$, are measured in $[cm]$, and so are the components $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3$ of the displacements vector $\widetilde{\mathbf{u}}$, and the gap $\widetilde{g}$. The time $\widetilde{t}$ is measured in $[sec]$, and the material density $\widetilde{\rho}$ in $[gm/cm^3]$, which throughout the book is assumed to possibly depend on the position, but not on the time. The components of the body force $\widetilde{\mathbf{f}}_B = (\widetilde{f}_{B1}, \widetilde{f}_{B2}, \widetilde{f}_{B3})$ have the dimensions of $[dyne/cm^3] = [gm/cm^2 \cdot sec^2]$, those of the surface traction $\widetilde{\mathbf{f}}_N$ the dimensions of force per unit area, $[dyne/cm^2] = [gm/cm \cdot sec^2]$, and the components $\widetilde{\sigma}_{ij}$ of the stress tensor $\widetilde{\sigma}$ have the same dimensions.

Let $L\,[cm]$ be a typical length in the system under consideration, $L^*\,[cm]$ a typical displacement, and $T^*\,[sec]$ a typical time unit. We define the dimensionless variables by

$$x_i = \frac{\widetilde{x}_i}{L}, \quad u_i = \frac{\widetilde{u}_i}{L^*}, \quad \text{for } \ i = 1, 2, 3, \qquad t = \frac{\widetilde{t}}{T^*}.$$

Everywhere below we use the symbols $\Omega, \Gamma_D, \Gamma_N$ and Γ_C to denote the reference configuration and the three portions of its surface in dimensionless variables. Moreover, we let the dimensionless gap function be

$$g(\mathbf{x}) = \frac{1}{L^*}\widetilde{g}(\widetilde{\mathbf{x}}),$$

and we scale it with L^* since we will often be dealing with the difference $\widetilde{u}_n - \widetilde{g}$.

It follows from the above that

$$\frac{\partial}{\partial \widetilde{x}_i} = \frac{\partial}{L \partial x_i}, \quad \text{for } \ i = 1, 2, 3, \qquad \frac{\partial}{\partial \widetilde{t}} = \frac{\partial}{T \partial t},$$

and the new (dimensionless) strain tensor is

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{u}) = \frac{L^*}{L}\widetilde{\boldsymbol{\varepsilon}}(\widetilde{\mathbf{u}}).$$

We assume throughout the book, that $\widetilde{\rho}(\widetilde{\mathbf{x}}) = \rho_0 \rho(\mathbf{x})$, where ρ_0 is a (typical scaling) constant with dimensions $[gm/cm^3]$, and $\rho(\mathbf{x})$ is a dimensionless function of the dimensionless variable $\mathbf{x}$.

We define the dimensionless components of the forces by

$$f_{Bi} = \frac{(T^*)^2}{\rho_0 \, L^*} \widetilde{f}_{Bi},$$

and of the linearized stresses

$$a_{ijkl} = \frac{T^*}{\rho_0 \, L^2} \widetilde{a}_{ijkl}, \qquad b_{ijkl} = \frac{(T^*)^2}{\rho_0 \, L^2} \widetilde{b}_{ijkl},$$

where the $\widetilde{a}$'s and $\widetilde{b}$'s are the components of the viscosity tensor $\mathcal{A}_{ve}$ and either $\mathcal{B}_{el}$ or $\mathcal{B}_{ve}$, as the case may be. Then, in the linear case, we have the stress tensor

$$\sigma_{ij} = a_{ijkl} \varepsilon_{kl}(\dot{\mathbf{u}}) + b_{ijkl} \varepsilon_{kl}(\mathbf{u}).$$

When the tensors are nonlinear one has to perform a similar transformation separately in each case. For the sake of simplicity, in the rest of this section we deal only with the linear cases.

An appropriate choice of the scaling factors L^* and T^* often allows to reduce the number of the coefficients in the equations. Indeed, when the material is viscoelastic with one viscosity coefficient, homogeneous, and isotropic, if we choose

$$T^* = \left(\frac{\rho_0 L^2}{2\widetilde{\lambda}_2} \right)^{1/2}, \quad \lambda = \frac{\widetilde{\lambda}_1}{2\widetilde{\lambda}_2}, \quad a = \frac{\widetilde{a}}{\sqrt{2\rho_0 \widetilde{\lambda}_2 L^2}},$$

we obtain that $2\widetilde{\lambda}_2 = 1$, and

$$\sigma_{ij} = \lambda \varepsilon_{kk}(\mathbf{u}) \delta_{ij} + \varepsilon_{ij}(\mathbf{u}) + a \varepsilon_{ij}(\dot{\mathbf{u}}).$$

In other problems, different choices of L^* and T^* will lead to different scaled coefficients.

The dynamic and quasistatic equations of motion retain exactly the same form in the dimensionless variables and coefficients introduced above.

To deal with the surface tractions we note that condition (2.4.2) is written now as $\widetilde{\boldsymbol{\sigma}} \widetilde{\mathbf{n}} = \widetilde{\mathbf{f}}_N$. Let $\mathbf{n}$ be the dimensionless unit outward normal to Γ_N. If the surface traction $\widetilde{\mathbf{f}}_N$ is nondimensionalized as

$$\mathbf{f}_N = \frac{(T^*)^2}{\rho_0 L^* L} \widetilde{\mathbf{f}}_N,$$

then (2.4.2) reads

$$\boldsymbol{\sigma} \mathbf{n} = \mathbf{f}_N,$$

in dimensionless variables.

In the rest of the book, except for Chap. 4, we use dimensionless variables, and in appropriate places we discuss this issue further. We always denote quantities with dimensions by a tilde, except in Chap. 4.

2.6 Contact Conditions

Real surfaces that are being used in engineering and other applications are very far from idealized mathematical surfaces. They have undulations or asperities, and the description of contact processes by boundary conditions need to address surfaces that may be clean, or contaminated with oil or gas; contain layers of adsorbed gas or an oxide layer; are very smooth, smooth, rough or very rough. There may be debris scattered on the surface which may be harder or softer than the surface material, thus causing degradation of the surface or acting as a lubricant, respectively. The material properties of the surfaces are very often very different from those of the parent material. Moreover, the layers adjacent to the contacting surface have a major influence on the contact processes. For more details we refer the reader to [13] and references therein.

Therefore, we expect that different mathematical boundary conditions may be employed to describe the wide variety of the conditions prevailing on contacting surfaces. Below we describe the ones that are currently in use. But, there is a need to extend and widen the range of possible conditions modelling the responses of real surfaces. This is one of the important areas of future research and we comment on it in Chap. 14. However, there seems to exist a mismatch between the way tribologists and engineers, especially those dealing with experiments, approach the issue and the mathematical need to describe the processes as boundary conditions that can be incorporated into the formulations of the models.

We use the dimensionless variables described above, and explain in some places how to obtain the new ones.

We now turn to the various conditions on the contact surface Γ_C, which is where our main interest lies. These are divided, naturally, into the conditions in the normal direction and those in the tangential direction.

The so-called *bilateral contact* describes the situation when contact between the body and the foundation is maintained at all times. It can be found in many machines and in moving parts and components of mechanical equipment. Then, there is no gap, $g = 0$, and

$$u_n = 0. \tag{2.6.1}$$

The term 'bilateral contact' has this meaning throughout this monograph. However, in the engineering literature 'bilateral contact' sometimes means that the system is restricted to move between two obstacles or foundations. As an example, one may consider the vibrations of a beam between two stops.

The so-called *normal compliance* condition describes a reactive foundation. It assigns a reactive normal traction or pressure that depends on the interpenetration of the asperities on the body's surface and those on the foundation. A general expression for the normal reactive traction on Γ_C is

$$-\sigma_n = p_n(u_n - g), \tag{2.6.2}$$

where $p_n(\cdot)$ is a nonnegative prescribed function which vanishes when its argument is nonpositive. Indeed, when $u_n < g$ there is no contact and the normal pressure vanishes; and when contact takes place $u_n - g$ (≥ 0) is a measure of the interpenetration of the surface asperities. In recent literature one can find the following choice

$$-\sigma_n = c_n(u_n - g)_+^{m_n},$$

where c_n is the surface stiffness coefficient, m_n is the normal compliance exponent, and $(f)_+ = \max\{f, 0\}$ is the positive part of f. The condition was first introduced in [11, 12] and since then used in many publications, see, e.g., [13–15, 20, 32, 44, 48, 51] and references therein. In these articles the values of the exponent m_n were restricted because of the use of the Sobolev embedding theorem. In [86] special spaces were introduced which allowed the use of arbitrary values of the exponent m_n.

If we write the condition in dimensional form, we have

$$-\widetilde{\sigma}_n = \widetilde{c}_n(\widetilde{u}_n - \widetilde{g})_+^{m_n},$$

and $\widetilde{c}_n$ has the dimensions $[gm/cm^{m_n+1} \cdot sec^2]$. To set the condition in dimensionless form we let

$$c_n = \frac{(T^*)^2(L^*)^{m_n-1}}{\rho_0 L}\,\widetilde{c}_n,$$

where the meaning of the various constants can be found in Sect. 2.5. The other contact conditions are nondimensionalized similarly.

The contact pressure σ_n is nonpositive. Indeed, during contact it is compressive and thus negative, and otherwise the point is free and it vanishes.

We note that in the normal compliance condition when contact has just been established or about to be lost $u_n = g$ and the normal pressure vanishes. This is not the case in bilateral contact or when the Signorini condition (to be described shortly) is used. Then, the pressure need not be zero and, generally, will not vanish when $u_n = g$.

An idealization of the normal compliance, which is used often in engineering literature, and can also be found in mathematical publications, is the *Signorini contact condition*, in which the foundation is assumed to be perfectly rigid. It is obtained, formally, from the normal compliance condition in the limit when the surface stiffness coefficient becomes infinite, i.e., $c_n \to \infty$ above, and thus interpenetration is not allowed. It can be stated in the following complementarity form

$$u_n - g \leq 0, \qquad \sigma_n \leq 0, \qquad \sigma_n(u_n - g) = 0. \tag{2.6.3}$$

Here, the foundation is rigid, so $u_n \leq g$, the contact pressure is nonpositive, and either $\sigma_n = 0$ when there is no contact, or $u_n = g$ during contact.

The condition is elegant and easy to write, however, the underlying idealization causes severe mathematical difficulties in dynamic problems. Since in this monograph we deal with relatively slow process, there is some merit in using it in models for static or quasistatic contact, and below we describe various results for problems containing it. Moreover, in finite-dimensional approximation or discretization of such problems it leads to the linear complementarity formulation of the problem, which is sometimes used in numerical simulations. But even in the cases when it seems a reasonable approximation, such as in biomechanical applications, or in the description of the contact of a solidifying material with a mold ([84]) and some other quasistatic processes, it leads to substantial mathematical difficulties and the solutions are considerably less regular than those with the normal compliance condition.

Next, when dealing with granular or wet surfaces, the normal reaction may be a function of the surface velocity, thus, we use the *normal damped response* condition,

$$-\sigma_n = p_{ndr}(\dot{u}_n). \tag{2.6.4}$$

Here, p_{ndr} is a nonnegative function which vanishes when its argument is nonpositive. One may use, more specifically,

$$-\sigma_n = c_{ndr}(\dot{u}_n)_+^{m_{dr}}.$$

This condition describes the normal reaction which is active only when the surface element is moving towards the foundation, and vanishes when it is moving away. It has been used in [51, 87–90] as well as in [91–96].

The normal damping coefficient $\widetilde{c}_{ndr}$ has the dimensions $[gm/cm^{m_{dr}+1} \cdot sec^{2-m_{dr}}]$, and then

$$c_{ndr} = \frac{(L^*)^{m_{dr}-1}}{\rho_0 L (T^*)^{m_{dr}-2}} \widetilde{c}_{ndr}.$$

When the damped response is active in both directions, i.e., there is damping when the body approaches the foundation and when it is receding from it, one may modify the condition and use

$$-\sigma_n = p_{ndr}(|\dot{u}_n|),$$

where p_{ndr} has the same behavior as above, namely, a nonnegative function that vanishes at zero.

We now turn to the conditions in the tangential directions. The simplest one is the so-called *frictionless* contact condition,

$$\boldsymbol{\sigma}_\tau = \mathbf{0}. \tag{2.6.5}$$

This is an idealization of the process, since even completely lubricated surfaces generate shear resistance to tangential motion. Actually, when the surface is fully lubricated, the lubricant flow generates a tangential shear stress which is transmitted to the body's surface. More details can be found in Sect. 2.7 below.

However, following current practice, we refer to (2.6.5) as the frictionless condition. We concentrate only on dry contact, for publications dealing with lubrication we refer the reader to [66, 68] and references therein.

Frictional contact is usually modelled with the Coulomb law of dry friction or a version thereof. According to this law, the tangential traction $\boldsymbol{\sigma}_\tau$ can reach a bound H, the so-called *friction bound,* which is the maximal frictional resistance that the surfaces can generate, and once it has been reached, a relative slip motion commences. Thus,

$$\|\boldsymbol{\sigma}_\tau\| \le H, \qquad \boldsymbol{\sigma}_\tau = -H\,\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if} \quad \dot{\mathbf{u}}_\tau \neq \mathbf{0}. \tag{2.6.6}$$

Here, $\dot{\mathbf{u}}_\tau$ is the relative tangential velocity or slip rate, and once slip starts, the frictional resistance has magnitude H and is opposing the motion. The bound H depends on the process variables, and we describe this dependence next.

Often, especially in engineering publications, the friction bound H is chosen as

$$H = H(\sigma_n) = \mu|\sigma_n|, \tag{2.6.7}$$

where μ is the *coefficient of friction*, which is described in more details in Sect. 2.7. The original formulation by Coulomb or Amontons had been proposed for the description of contact between rigid bodies, and using it to describe contact between deformable bodies in the point-wise sense, that is, assuming that the law holds at each point of contact, is more recent.

In mathematical papers, until very recently, μ was assumed to be a constant, and this still holds true in many mathematical and engineering publications, where it is usually used together with the Signorini condition (2.6.3). However, it is very well know experimentally that μ depends on the relative sliding speed, on the temperature and it varies with the process as the surface topography changes due to wear. We describe the dependence of μ on these variables in the next section.

The choice of (2.6.7) in (2.6.6) leads to severe mathematical difficulties (see, e.g., [13]) since, generally, the stress $\boldsymbol{\sigma}$ is only square-integrable, not necessarily continuous, and does not have a well defined trace or value on Γ_C. Therefore, one must give appropriate meaning to σ_n and $\boldsymbol{\sigma}_\tau$ and to the contact conditions in which they are used. When these are used with (2.6.3), the mathematical difficulties are significantly increased. Indeed, in a number of papers the normal stress σ_n has been regularized so as to give meaning to the frictional contact condition (see, e.g., [6, 9, 16, 17, 22, 25, 33]). A discussion of the regularization can be found in Sect. 8.5, however, as we point out there, currently there is no reasonable derivation of the regularization operator from physical principles, and it remains an important unresolved problem.

The choice

$$H = p_\tau(u_n - g), \tag{2.6.8}$$

is compatible with the normal compliance contact condition (2.6.2). Here, p_τ is a nonnegative function which vanishes when its argument is nonpositive, i.e., when there is no contact. In the literature it is called *tangential compliance*. Combining (2.6.7) with (2.6.2) we find

$$H = \mu p_n(u_n - g). \tag{2.6.9}$$

For this reason, keeping in mind (2.6.8), the choice $p_\tau = \mu p_n$ can be found often in the literature.

In dimensional form, $\widetilde{H}$ has the units of pressure, $[gm/cm \cdot sec^2]$ and so does the normal compliance function $\widetilde{p}_n$, and then the friction coefficient $\widetilde{\mu}$ is dimensionless. To set the quantities in dimensionless units we define

$$H = \frac{(T*)^2}{\rho_0 LL^*}\widetilde{H}, \qquad p_n = \frac{(T*)^2}{\rho_0 LL^*}\widetilde{p}_n, \qquad \mu = \widetilde{\mu},$$

using the scaling in Sect. 2.5, which yields (2.6.9).

In certain applications, especially when the loads are light, or the friction is large, or when the load is very large and the real contact area among the asperities is close to the nominal one, i.e., the area where there is no actual contact is small, the function H behaves as a constant, which in the literature is called the *Tresca friction law*, thus,

$$H = const. \tag{2.6.10}$$

This condition simplifies the analysis considerably, and on occasions makes it possible. Sect. 2.8 deals with the relationship between the Coulomb and the Tresca conditions, and points out to a possible transition from the first to the second.

When the wear of the contacting surface is taken into account, a modified version of the Coulomb law is more appropriate. This condition has been derived in [97–99] from thermodynamic considerations, and is given by

$$H = \mu|\sigma_n|(1 - \delta|\sigma_n|)_+, \tag{2.6.11}$$

where δ is a very small positive parameter related to the wear constant of the surface. However, if we use the normal compliance condition (2.6.2) then

$$H = \mu p_n(1 - \delta p_n)_+, \tag{2.6.12}$$

which may be recovered from (2.6.8) with the choice $p_\tau = \mu p_n(1 - \delta p_n)_+$.

On a nonhomogeneous surface H depends on the position $\mathbf{x}$ on the surface, and on the wear of the surface. As is described in the next section, it also depends on the relative slip speed and on the temperature. In some geological applications it has been assumed to depend on the relative slip, instead of relative slip rate, [100, 101].

Finally, we note that condition (2.6.6) describes an isotropic surface. If the surface is anisotropic, say, it has grooves, then an anisotropic friction condition has to be used. To describe it let $\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2\}$ be a system of two orthogonal unit tangent vectors at each point of the surface, one of which is oriented along the grooves and the other orthogonal to them, and let H_1 and H_2 be the corresponding friction bounds. These may be very different, since it is much easier to move along the grooves than in a direction perpendicular to them. Then, we require that

$$|\sigma_{\tau_1}| \leq H_1, \quad |\sigma_{\tau_2}| \leq H_2,$$

$$\sigma_{\tau_1} = -H_1 \frac{\dot{u}_{\tau_1}}{|\dot{u}_{\tau_1}|} \quad \text{if} \quad \dot{u}_{\tau_1} \neq 0,$$

$$\sigma_{\tau_2} = -H_2 \frac{\dot{u}_{\tau_2}}{|\dot{u}_{\tau_2}|} \quad \text{if} \quad \dot{u}_{\tau_2} \neq 0.$$

Here, σ_{τ_1} and σ_{τ_2} are the components of $\boldsymbol{\sigma}_\tau$ in the directions of $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$, respectively, while u_{τ_1} and u_{τ_2} are the components of $\mathbf{u}_\tau$.

This condition can be found in [90]. Other anisotropic conditions may be used when the tangential stresses are coupled, see, e.g., [13, 102–104] and references therein.

The versions of the the friction law, described above, are characterized by the existence of stick-slip zones on the contact boundary. Slip motion occurs only when the tangential shear reaches a critical value, that of the friction bound. In the case when the contact surface is lubricated, slip takes place even when the tangential shears are small. Such a phenomenon may be modelled by a friction law in which the shear stress is proportional to the tangential speed and can be found in Sect. 7.4.

2.7 Friction Coefficient

We observe that the *friction coefficient* μ is not an intrinsic thermodynamic property of a material, a body or its surface, since it depends on the contact process and the operating conditions. It is defined as the ratio between the normal stress and the modulus of the tangential stress on the contact surface when sliding commences, and there is no theoretical reason for this ratio to be a well defined function. It seems that the case of a rigid body resting on an inclined rigid plane, or more generally, static contact between 'rigid' bodies, is the exception. This may explain the difficulties in the experimental measurements of the friction coefficient.

The issue is considerably complicated by the following facts. Engineering surfaces are not mathematically smooth surfaces, but contain asperities and various irregularities. Moreover, very often they contain some or all of the following: moisture, lubrication oils, various debris, wear particles, oxide layers, and chemicals and materials that are different from those of the parent body. Therefore, it is not surprising that the friction coefficient is found to depend on the surface characteristics, on the surface geometry and structure, on the relative velocity between the contacting surfaces, on the surface temperature, on the wear or rearrangement of the surface and, therefore, on its history, and other factors which we skip here. A very thorough description of these issues can be found in [78] (see also the survey [13]). However, and it is somewhat surprising, the concept of a friction coefficient is found to be sufficiently useful to be employed almost universally in frictional contact problems. Indeed, there seems to be no generally accepted current alternative to it.

In tribology the so-called *Stribeck curve*, Fig. 2, is usually used to describe the variation of the friction coefficient with lubrication, see, e.g., [72, 78, 105] and references therein. Three main regimes of lubrication are identified. In the *boundary lubrication* regime ((bl) in Fig. 2) the lubricant layer is very thin and contact stresses are transmitted via physical contact among the surface asperities on the contacting surfaces, and the friction coefficient has a higher value. In the *hydrodynamic lubrication* regime ((hl) in Fig. 2) the lubricant layer is sufficient to prevent physical contact and contact stresses are transmitted via the lubrication layer, and the friction coefficient is rather low. In the *mixed lubrication* regime ((ml) in Fig. 2) there is physical contact between the tips of the asperities and the space between the contacting surfaces is full of lubricant; contact stresses are transmitted both by asperity contact and by the lubricant, and the friction coefficient varies from its 'bl' value to the 'hl' value.

The Stribeck curve depicts the friction coefficient as a function of a dimensionless parameter that is related to the lubricant viscosity, the relative slip speed, the surface roughness and the averaged contact pressure. It is decreasing in the 'bl' regime, slopes down rather steeply in the 'ml' regime and reaches a lower value in the 'hl' regime, where it curves up slightly.

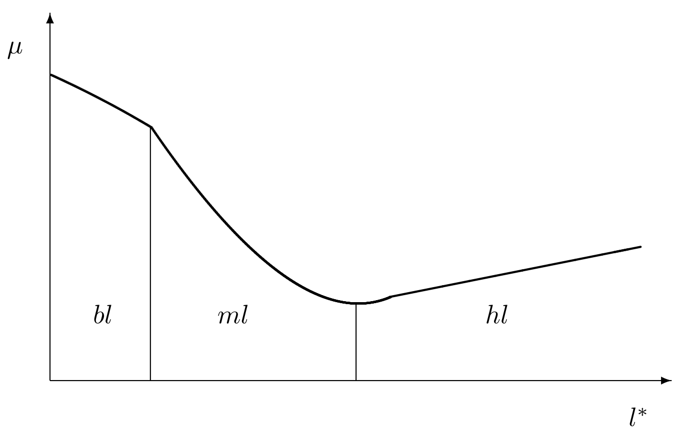

Fig. 2. The Stribeck curve

Typical measured or assumed values of the friction coefficient are as follows ([78]). In lubricated contact (hl) the values of the friction coefficients are in the range $\mu \sim 0.05 - 0.1$. In the 'ml' regime they are in the range is $0.15 - 0.3$, and so are the values in the lower end of dry friction (bl). Large values of the coefficient are $0.5 - 1.0$. When μ exceeds these, it becomes difficult to measure it, since often the system exhibits seizure, and the surfaces become locked. When the tangential shear is increased, a layer of the material near the surface tears out and the whole process changes catastrophically. Currently, there are no mathematical models or results that address such a behavior.

Using the Stribeck curve for the friction coefficient μ is somewhat complicated, when considered from the mathematical point of view, as contrasted with just incorporating it into a numerical code. Indeed, if we choose the dimensionless parameter l^* as

$$l^* = \frac{\eta_l v_{slip}}{pR_S},$$

where η_l is the dynamic viscosity of the lubricant, $v_{slip} = \|\dot{\mathbf{u}}_\tau\|$ is the relative slip speed, p is an averaged contact pressure and R_S is the surface roughness, then, assuming in addition that $\eta_l/R_S = \epsilon_S$ is a constant, we may write

$$\mu = \mu\left(\epsilon_S \frac{\|\dot{\mathbf{u}}_\tau\|}{|\sigma_n|}\right).$$

Therefore, if we assume that the friction bound is given by $H = \mu|\sigma_n|$, then we have

$$H = \mu\left(\epsilon_S \frac{\|\dot{\mathbf{u}}_\tau\|}{|\sigma_n|}\right)|\sigma_n|.$$

This dependence of the friction coefficient on the contact stress makes the formulation of the friction condition more complicated than seems to be necessary. As will be seen below, such friction conditions with slip rate dependent friction coefficient have been studied recently, but without the term $|\sigma_n|$ in the denominator of the argument of μ.

We note that a change of labels in Fig. 2 makes it more useful mathematically. If we multiply both variables by $p = |\sigma_n|$, then, the horizontal axis is labeled with $l^{**} = \eta_l v_{slip}/R_S$ (instead of l^*), and the vertical axis is now labeled with the friction bound $H = \mu p$. Retaining the same curve, the relationship between the friction bound and the dimensionless or scaled slip speed seems to be reasonable. However, the friction coefficient is a more complicated function since it is inversely proportional to the contact pressure. Moreover, as is pointed out in Sect. 2.8, such a condition has to be modified at large contact pressures.

Until very recently, mathematical models for frictional contact used a constant friction coefficient, mainly for mathematical reasons. This is rapidly changing, and the dependence of μ on the process parameters has been incorporated into the models in recent publications. The first result on quasistatic contact with slip rate or total slip rate dependent friction coefficient can be found in [21], where the friction coefficient was assumed to depend on the slip rate or on the total slip rate, thus,

$$\mu = \mu(\|\dot{\mathbf{u}}_\tau\|), \qquad \text{or} \quad \mu = \mu(\int_0^t \|\dot{\mathbf{u}}_\tau\|\, ds). \tag{2.7.1}$$

The dependence on the process history via the total slip rate $\int_0^t ||\dot{\mathbf{u}}_\tau||\, ds$ takes into account the morphological changes undergone by the contacting surfaces as the process goes on.

A friction coefficient which depends on the slip rate has been employed in dynamic cases in [106–108] where the nonuniqueness of the solution and possible solutions with shocks were investigated in a special setting.

In some geological publications on the motion of tectonic plates the friction coefficient is assumed to depend on the slip and not the slip rate, thus,

$$\mu = \mu(\|\mathbf{u}_\tau\|),$$

see [100, 101] and references therein.

The dependence of μ on the temperature may be considerable, as can be seen in [78] and references therein. The quasistatic problem with temperature dependent coefficient, in addition to the slip rate, i.e.,

$$\mu = \mu(\|\dot{\mathbf{u}}_\tau\|, \theta)$$

has been investigated in [42].

The usual assumption when μ is not constant is that it is a Lipschitz function of its arguments. This seems very reasonable in many applications. However, there are cases when the transition from the static to the dynamic value is rather sharp, and a graph may better describe the situation. It has been assumed in some engineering publications, and there is clear evidence for it in simple experimental settings, that the drop in the value of the friction coefficient from a higher *static* value to a lower *dynamic* value may cause instabilities (see [106–108]). Indeed, such a system may exhibit shocks, i.e., discontinuous or sharply changing solutions. It has been established in [106] that in the dynamic case for an elastic body there may be a continuum of solutions, and it was conjectured there that the maximum delay condition would yield the unique solution that the system would realize physically. It may very well be that the solution chosen by the maximum delay condition coincides with the unique viscosity solution, obtained in the limit of vanishing viscosity when the material is assumed to be viscoelastic. These two conjectures are related, and of considerable importance and settling them is a very challenging problem. Dynamic problems with discontinuous friction coefficient, assumed to be a graph with a vertical segment at zero slip rate (connecting the dynamic and static values), have been investigated in [33, 34, 36].

The dependence of the friction coefficient μ on the location $\mathbf{x}$ on the contacting surface, when the surface is not homogeneous, is easy to incorporate into the mathematical models, but is rarely made explicit, except for possibly mentioning it in passing. On the other hand, it is well documented that such dependence may be very pronounced. Indeed, in experiments on axisymmetric stretch forming in [105, 109] the friction coefficient was found to vary steeply from a value close to zero at the center to about 0.3 at the edge, with a very sharp transition region in between which was found to depend on the forming speed (Fig. 3 on page 142 in [105]). We will remark on this dependence in the problems below. It may be of interest to model such sharp transitions using a discontinuous idealization, as in [33, 34]. This topic is of fundamental importance and it is likely to be investigated in the near future.

Frictional contact in metal forming processes was investigated in [105,109, 110]. Since in such processes contact stresses are very large and plastic flow of the surface asperities is considerable, the friction coefficient was assumed to depend on the surface strain rate. By adding an empirical correction for surface roughening they obtained a good correlation with experiments. To our knowledge, the dependence of μ on the strain has not been investigated in the mathematical literature, although the normal compliance condition may turn out to approximate it well.

Finally, in most geological publications dealing with earthquakes the friction coefficient is assumed to depend on the slip rate, but in some it is assumed to depend on the slip, see, e.g., [100, 101]. Moreover, it may depend on an internal variable or several variables and also on the temperature. Indeed, the so-called Dieterich-Ruina model (see, e.g., [101]) is

$$\mu = \mu_0 - A \ln \left(1 + \frac{\|\dot{\mathbf{u}}_\tau\|}{v_\infty} \right) + B \ln \left(1 + \frac{\eta(t)}{\eta_0} \right),$$

where μ_0 is the static friction coefficient, v_∞ is the maximal slip velocity in the system, and η is an internal variable describing the surface, and whose equation of evolution is given by

$$\frac{d\eta}{dt} = 1 - \frac{\eta \|\dot{\mathbf{u}}_\tau\|}{L^*},$$

where L^*, A, B are adjusted system parameters.

The possible dependence of μ on the temperature, discussed below, has been modelled in [111] as

$$\mu = \mu_0 + A \left(\ln \frac{\|\dot{\mathbf{u}}_\tau\|}{v_\infty} + \frac{Q_A}{R} \left(\frac{1}{\theta} - \frac{1}{\theta_*} \right) \right) + B\eta,$$

where θ_* is a reference temperature, and Q_A and R are additional parameters.

More elaborate expressions can be found in [101], and we refer the reader there and the references therein, and also to [60] and the references therein. However, the well-posedness of models with such friction conditions is, as yet, an unsolved problem.

2.8 On Coulomb and Tresca Conditions

We now discuss shortly the relationship between the Coulomb and the Tresca friction conditions. Detailed explanations can be found in [72,78,105] and the references therein, and in [112] where the real contact area issue is discussed.

We consider contact between a hard (rigid) smooth tool and an elastic-plastic workpiece. According to [105, p. 140], the Coulomb condition is useful when the contact is elastic, within the boundary lubrication regime ('bl' in Fig. 2) and when the *nominal contact pressure* (obtained by dividing the total load by the area of the contact surface) is relatively small, as compared to the hardness of the workpiece material. In such a case contact takes place at the tips of the asperities, and there is a considerable difference between the averaged contact pressure and the maximal pointwise pressure at the tips. If we assume that the tips deform plastically, then the contact pressure at the tips has essentially the plastic yield value. Moreover, the fractional real contact area A is small and may be assumed to be proportional to the averaged pressure. When the contact stresses increase the fractional contact area grows, until there is almost complete plastic flattening of the asperities, and the fractional contact area is close to unity. When this state is reached, the frictional shear stress will reach saturation, and any further increase in the normal pressure will not lead to an increase in the tangential frictional resistance. Therefore, we have a natural transition from the Coulomb to the Tresca laws, depicted in Fig. 3.

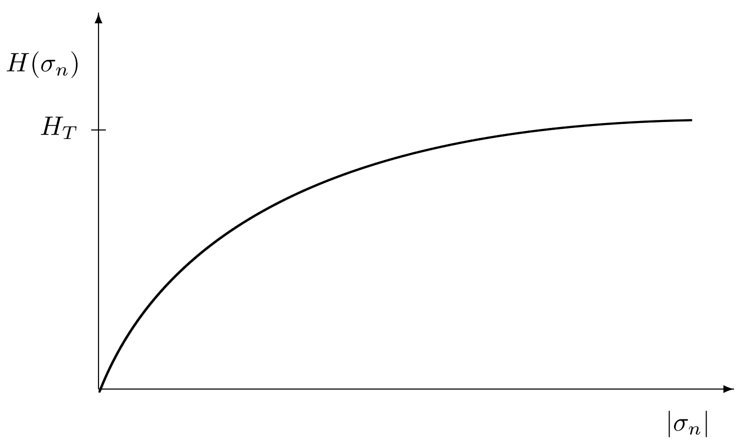

Fig. 3. The friction bound $H(\sigma_n)$ vs. normal stress

We may use the following expression for this relationship between the friction bound H and the normal pressure,

$$H(\sigma_n) = H_T(1 - \exp(-\mu|\mathcal{R}(\sigma_n)|/H_T)),$$

which reduces to the Coulomb law when the contact pressure is small. Here, H_T is the Tresca friction bound which is also a scaling factor and the regularization operator $\mathcal{R}$ has been used either to average the contact pressure or to give it meaning, or both. When using normal compliance, one has to replace the contact pressure $|\mathcal{R}(\sigma_n)|$ with $p_n(u_n - g)$.

The normal compliance condition takes such a behavior into account when the normal compliance function is assumed to be bounded. However, it is rather ad hoc, and the boundedness of the normal compliance function in (2.6.2) is often used for mathematical reasons, to which the discussion above gives some support.

It is of considerable importance to obtain the curve of H vs. $|\sigma_n|$ in Fig. 3 experimentally. Then, such a friction law, with the transition from frictional shear that is proportional to the averaged contact pressure to a constant value when the real fractional contact area is close to unity, may be very useful in applications where contact pressures are likely to vary from very small to very large values. Moreover, this will simplify some of the mathematical manipulations, which typically are more difficult to deal with when the friction bound is unbounded. We note that condition (2.6.11) is not of this form, and is related to the wear of the contacting surface.

Clearly, modelling of contact and friction is an important topic, which is currently under investigation. As better frictional contact conditions for specific applications are obtained, better mathematical models and more accurate and reliable numerical simulations will result.

3 Additional Effects Involved in Contact

More is involved in contact than just friction, although it is the main process. Indeed, during a contact process elastic or plastic deformations of the surface asperities may happen. Also, some or all of the following may take place: squeezing of oil or other fluids, breaking of the asperities' tips and production of debris, motion of the debris, formation or welding of junctions, creeping, fracture, etc. Moreover, frictional contact is associated with heat generation, which, in turn, may influence the process considerably by softening the surface material, causing structural changes, or even melting it.

In this chapter we describe conditions that model some of the other processes involved in contact. We begin, in Sect. 3.1, with ways of including thermal effects into the models. In addition to the heat equation for the body temperature, we discuss frictional heat generation and the heat exchange condition when the heat exchange coefficient depends on the gap or the contact pressure.

In Sect. 3.2 we describe the wear of the surfaces as a result of friction. It is customary in the mathematical literature to model it with a differential or rate version of the Archard condition, (3.2.1), which is a rate condition for wear production. However, it does not allow for the diffusion or motion of the wear debris on the surface. When the wear particles or debris are harder than the surface material additional wear is caused. When the wear particles, such as certain metallic oxides, are softer they may act as solid lubricants.

Some surfaces may include adhesive agents, such as glues. The addition of adhesion into models of contact is recent. The adhesion process may be reversible, as in velcro, or irreversible, as is the case with most common glues. In Sect. 3.3 the topic is presented, and the evolution of the adhesion is also described by (rate equation) (3.3.2) for an irreversible process or the rate equation with right-hand side given in (3.3.4) when it is reversible and with memory or history dependent.

The modelling of material damage, in the form presented in Sect. 3.4 is recent, and, unlike the conditions mentioned above, it enter the models via the constitutive relations. The issue is very important in applications, and is likely to receive increased attention in the near future.

M. Shillor, M. Sofonea, J.J. Telega: *Models and Analysis of Quasistatic Contact*, Lect. Notes Phys. **655**, (2001), 31–47
http://www.springerlink.com/

3.1 Thermal Effects

Contact and friction processes are invariably accompanied by heat generation, which may be considerable. Sudden braking of a car can cause more than 100 HP of power to be dissipated in the form of heat which needs to be transported away from the wheels and the braking system.

Thermal effects in contact processes affect the composition and stiffness of the contacting surfaces, and cause thermal stresses in the contacting bodies. Moreover, the contacting surfaces exchange heat, and energy is lost to the surroundings. The way heat affects the mechanical properties of a contact surface can be, partially, taken into account by assuming that the friction coefficient is temperature dependent [78] (see also [72, 111, 113, 114]).

To take into account thermal effects we need four elements: frictional heat generation condition, a condition describing heat exchange between the body and the foundation, a constitutive relation and the energy equation. We discuss each one of these in turn.

Frictional heat generation is the power dissipated during the process and is assumed to be proportional to the tangential shear stress and to the slip rate. The power generated by the friction traction on the contact surface is given by ([43, 44, 78, 99]),

$$q_f = \|\boldsymbol{\sigma}_\tau\| \, \|\mathbf{v}^* - \dot{\mathbf{u}}_\tau\| = \mu|\sigma_n| \, \|\mathbf{v}^* - \dot{\mathbf{u}}_\tau\|, \tag{3.1.1}$$

where $\mathbf{v}^*$ is the tangential velocity of the foundation, $\dot{\mathbf{u}}_\tau$ is the tangential velocity of the surface and $\mu = \mu(\|\mathbf{v}^* - \dot{\mathbf{u}}_\tau\|, \theta)$ is the slip rate and temperature dependent friction coefficient. To take into account the effects of this energy flow in the system we need to describe the heat exchange between contacting surfaces. But first, in dimensional form we have

$$\widetilde{q}_f = \|\widetilde{\boldsymbol{\sigma}}_\tau\| \, \|\widetilde{\mathbf{v}}^* - \widetilde{\dot{\mathbf{u}}}_\tau\| = \widetilde{\mu}|\widetilde{\sigma}_n| \, \|\widetilde{\mathbf{v}}^* - \widetilde{\dot{\mathbf{u}}}_\tau\|,$$

where $\widetilde{q}_f$ has the dimensions of $[gm/sec^3] = [erg/cm^2 \cdot sec]$, i.e., energy flow per unit area per unit time; then, using the scaling in Sect. 2.5 and $\mu = \widetilde{\mu}$, we obtain (3.1.1) by setting

$$q_f = \frac{(T^*)^3}{\rho_0 L (L^*)^2} \widetilde{q}_f.$$

The second element is the thermal interaction between surfaces in contact, and it has been the subject of [115–122] (see also references therein for further works), where the stability of the steady states were investigated, and of [123]. In [115, 118] the heat exchange between the contacting point and the foundation was assumed to be ideal, i.e., perfect conduction when there is contact and complete insulation when contact is lost. Then, it was shown that such a condition is actually an over-idealization, since mathematically it leads to infinitely fast oscillations. For this reason two more realistic heat exchange conditions were proposed and investigated in [123]. For the sake of completeness a short description of these two conditions follows.

The pointwise heat exchange condition on Γ_C is customarily given by

$$-k_{ij}n_i\frac{\partial\theta}{\partial x_j} = k_e(\theta - \theta_R) - q_f,$$

where k_{ij} are the components of the thermal conductivity tensor $\mathcal{K}$, n_j are the components of $\mathbf{n}$, θ is the pointwise surface temperature, q_f is the power generated by friction in the form of heat, k_e is the heat exchange coefficient and θ_R is the known temperature of the foundation. The usual assumption, both in engineering and mathematical publications dealing with thermal problems, is that the heat exchange coefficient is constant.

However, as has been indicated above, in processes involving contact k_e is not constant but a function of the gap or distance between the surface and the foundation, if there is no contact, and of the contact pressure otherwise (see [124, 125] and also [113]). The dependence of the heat exchange coefficient on the pressure, under large stresses, is well documented (see the references in [124, 125]). We now obtain an elegant dimensionless variable η which represents both the contact pressure, when there is contact, and the gap, when there is separation. In dimensional form we define it as

$$\widetilde{\eta} = (\widetilde{g} - \widetilde{u}_n) + \eta^*\widetilde{\sigma}_n,$$

where η^* is a conversion factor with dimensions $[cm^2 \cdot sec^2/gm]$, so that $\widetilde{\eta}$ has the dimensions of $[cm]$. Now, $\widetilde{g} = L^*g$ and $\widetilde{u}_n = L^*u_n$, and, also $\widetilde{\sigma}_n = (\rho_0 LL^*/(T^*)^2)\sigma_n$, and if we choose $T^* = (\eta^*\rho_0 LL^*)^{-1/2}$ we obtain that the dependence is on the dimensionless, 'natural' and elegant variable $\eta = \eta^*/L^*$, given by

$$\eta = g - u_n + \sigma_n.$$

It was introduced in [123], and it takes into account both the contact pressure and the separating gap: when there is no contact $u_n < g$, $\sigma_n = 0$, therefore $\eta > 0$ and it measures the distance between the point and the foundation; when there is contact $u_n = g$ and $\sigma_n \leq 0$, thus, $\eta \leq 0$ and it measures the contact pressure. When $\eta = 0$ the contact has just been established or lost. In this manner the variable η unifies the description of the heat exchange coefficient, and so $k_e = k_e(\eta)$.

Two forms of dependence of k_e on η were investigated in [123]. In the first case $k_e(\eta)$ was assumed to be a Lipschitz continuous function, in particular, continuous at the point of initiation or loss of contact ($\eta = 0$), Fig.4a. In the second case k_e was chosen as the graph,

$$k_e(\eta) = \begin{cases} k_{con}(\eta) & \text{if} \quad \eta < 0 \\ [\, k_{sep}(0),\, k_{con}(0)] & \text{if} \quad \eta = 0 \\ k_{sep}(\eta) & \text{if} \quad \eta > 0 \end{cases}.$$

Here, when contact takes place $k_e = k_{con}(\eta)$ is a function of the contact

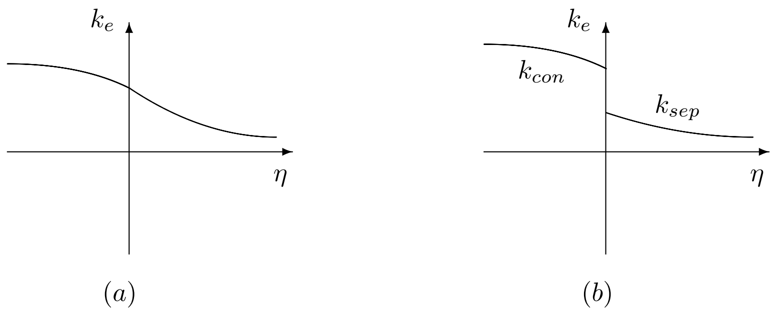

Fig. 4. (a) The function $k_e(\eta)$; (b) the graph k_e;
$\eta < 0$ – contact; $\eta > 0$ – separation.

pressure; when there is no contact $k_e = k_{sep}(\eta)$ is a function of the gap. As contact is established or lost the value of the heat exchange coefficient lies in the interval $[k_{sep}(0),\, k_{con}(0)]$. The graph is depicted in Fig. 4(b).

The graph reflects the experimental fact that it is very complicated to measure the heat exchange coefficient when contact has just been established or lost ([124, 125]). At the moment of initiation of contact certain surface asperities come in contact and deform during the contact period. After contact is lost and reestablished other asperities are likely to come into contact, changing the value of the heat exchange coefficient. Therefore, at least for surfaces with relatively large and possibly soft asperities, the heat exchange coefficient may vary widely at the onset of contact. Thus, the graph (b) in Fig. 4 may represent the process better than the function (a). Indeed, measurements of the electrical conductivity at the onset of contact exhibit a similar behavior (see, e.g., [124, 125] and references therein).

The idealized case discussed in [115, 118] may be described by the graph

$$k_e(\eta) = \begin{cases} \infty & \text{if} \quad \eta < 0 \\ [0,\, \infty) & \text{if} \quad \eta = 0 \\ 0 & \text{if} \quad \eta > 0 \end{cases},$$

that is formally obtained from the one in Fig. 4(b) by setting $k_{con} = \infty$, which means that the contact is perfect, and $k_{sep} = 0$ which represents perfect insulation. The experimental results in [124,125] and in the references therein, clearly indicate that such a choice is an over-idealization that is physically incorrect and mathematically leads to infinitely rapid oscillations.

Contact problems with thermal effects where the coefficient of heat exchange was assumed to be a function of η were investigated in [123] (see also [126] and references therein) in a one-dimensional setting. The problem in which k_e was assumed to be a graph such that $k_{con} = const.$ can be found in [127].

The third element in taking into account thermal effects is related to the constitutive relation of the material. A *thermoviscoelastic relation*, taking into account the thermal expansion of the material is given, generally, in the form

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\dot{\boldsymbol{\varepsilon}} + \mathcal{B}_{ve}\boldsymbol{\varepsilon} - \mathcal{M}\theta,$$

where $\mathcal{B}_{ve}$ is the elasticity operator, $\mathcal{A}_{ve}$ is the viscosity operator and $\mathcal{M}$ is the tensor of thermal expansion. A thermodynamic derivation of this relation can be found in Chap. 4.

We recall that in linear thermoviscoelasticity $\widetilde{\boldsymbol{\sigma}} = (\widetilde{\sigma}_{ij})$ is given by

$$\widetilde{\sigma}_{ij} = \widetilde{a}_{ijkl}\widetilde{\varepsilon}_{kl}(\widetilde{\dot{\mathbf{u}}}) + \widetilde{b}_{ijkl}\widetilde{\varepsilon}_{kl}(\widetilde{\mathbf{u}}) - \widetilde{m}_{ij}\widetilde{\theta},$$

where $\widetilde{\theta}$ is the temperature, measured in degrees centigrade, ${}^0C$, (from some reference temperature Θ_{ref} chosen as zero), the $\widetilde{b}_{ijkl}$, $\widetilde{a}_{ijkl}$, and $\widetilde{m}_{ij}$ are the elasticity, viscosity, and thermal expansion coefficients, respectively. The latter have the dimensions of $[gm/cm \cdot sec^2 \cdot {}^0C]$. To write it in dimensionless form we multiply it by $(T^*)^2/\rho_0 LL^*$, let θ^* be some representative system temperature, and define the new dimensionless temperature and thermal expansion coefficients by

$$\theta = \frac{\widetilde{\theta}}{\theta^*}, \qquad m_{ij} = \frac{(T^*)^2\theta^*}{\rho_0 LL^*}\widetilde{m}_{ij},$$

and we obtain the dimensionless constitutive relation

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}(\dot{\mathbf{u}}) + b_{ijkl}\varepsilon_{kl}(\mathbf{u}) - m_{ij}\theta.$$

The other dimensionless variables can be found in Sect. 2.5.

When the material is isotropic we have

$$\widetilde{\sigma}_{ij} = \widetilde{a}\widetilde{\varepsilon}_{ij}(\widetilde{\dot{\mathbf{u}}}) + \widetilde{\lambda}_1\widetilde{\varepsilon}_{kk}(\widetilde{\mathbf{u}})\delta_{ij} + 2\widetilde{\lambda}_2\widetilde{\varepsilon}_{ij}(\widetilde{\mathbf{u}}) - \widetilde{\alpha}(3\widetilde{\lambda}_1 + 2\widetilde{\lambda}_2)\widetilde{\theta},$$

where $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ are the Lamé coefficients, $\widetilde{a}$ the viscosity coefficient and $\widetilde{\alpha}$ the coefficient of thermal expansion. Thus, the material is characterized by four coefficients.

A *thermoelastic* constitutive relation is obtained by neglecting the viscosity part, thus

$$\boldsymbol{\sigma} = \mathcal{B}_{el}\boldsymbol{\varepsilon} - \mathcal{M}\theta,$$

and the linearized version is

$$\sigma_{ij} = b_{ijkl}\varepsilon_{kl}(\mathbf{u}) - m_{ij}\theta.$$

When, in addition, the material is isotropic the constitutive relation is

$$\sigma_{ij} = \lambda_1\varepsilon_{kk}(\mathbf{u})\delta_{ij} + 2\lambda_2\varepsilon_{ij}(\mathbf{u}) - \alpha\theta\delta_{ij}$$

in dimensionless variables.

The fourth and last element is the equation of energy. An equation that takes into account the viscous heat generation (see, e.g., [96] and references therein and also Chap. 4) is given by

$$\dot{\theta} - (\kappa_{ij}\theta_{,i})_{,j} + m_{ij}(\theta + \Theta_{ref})\dot{u}_{i,j} - a_{ijkl}\varepsilon(\dot{\mathbf{u}})_{ij}\varepsilon(\dot{\mathbf{u}})_{kl} = q_{vol}.$$

Here, θ is the temperature which is measured with respect to Θ_{ref}, where the latter is the (absolute) temperature of the isothermal stress-free reference configuration. The term q_{vol} represents a volume heat source, such as electrical heating. The third term on the left-hand side is nonlinear and is usually linearized by replacing $\theta + \Theta_{ref}$ with the temperature Θ_{ref}. The last term on the left-hand side is the internal viscous heat generation, which is also nonlinear. Often, when the linear heat equation is used, this term is completely neglected.

Quasistatic contact problems with thermal effects can be found in [39, 42, 128–130], where general settings were employed; dynamic problems were investigated in [41, 43–45, 89, 96, 131–134]; while one-dimensional problems can be found in [126, 127, 135–139]. Modelling and numerical simulations can be found in [97–99, 140, 141]. We refer the reader to the references in these publications for further details.

3.2 Wear

As the contact process evolves, the contacting surfaces evolve too, via their wear. Wear in sliding systems is often very slow but it is persisting, continuous and cumulative. There may be increase in the conformity of the surfaces and their smoothness, or increase of the surface roughness, fogging of the surface, generation of scratches and grooves, initiation of cracks and generation of debris which may change the contact characteristics. Friction processes are invariably related to the production of wear particles and debris.

Asperities under large contact stresses may deform plastically or break. In the first case the surface morphology changes and, therefore, both the contact stress and the friction traction are affected. These may be incorporated into a history or memory dependent friction coefficient. In the second case, when asperities break, the surfaces wear out, debris are produced, and again the surface structure changes over time. This must be taken into account if the long time behavior of the system is to be realistically predicted. The broken particles may remain on the surface and act as a lubricant, if they are made of a material that is softer than the surface material, or may cause grooves and damage to the surface, if they are harder. These changes in the surface affect the contact process. There exists a large engineering literature on the subject due to its crucial role in the design of long term proper functioning of mechanical components. However, the processes involved in the wear of contacting surfaces are very complicated and it seems that more sophisticated models will be needed in the near future. Indeed, it is customary to distinguish among the following wear types: adhesive, abrasive, contact fatigue, fretting, oxidation, corrosion and erosion (see, e. g., [46, 54, 78, 97–99, 142–144]). The model for wear that is used very frequently is based on Archard's observations, [145], and will be described shortly. The inclusion of wear in mathematical models is very recent, see, e. g., [20, 34, 44, 54, 135, 136, 146–150] and references therein.

To model the wear of the contacting surfaces the *wear* function $w = w(\mathbf{x}, t)$, for $\mathbf{x}$ in Γ_C, is introduced, measuring the depth, in the normal direction, of the removed material. Therefore, it measures the change in the surface geometry, and represents the cumulative amount of material removed, per unit surface area, in the neighborhood of the point $\mathbf{x}$ up to time t. Since the amounts of material removed are small, as an approximation, one may treat it as a change in the gap, and so g is replaced by $g + w$. Thus, in (2.6.2) we use $p_n(u_n - w - g)$ and in (2.6.8) we use $p_\tau(u_n - w - g)$.

In dimensional form $\widetilde{w}$ has the dimensions of $[cm]$ and we scale it, similarly to $\widetilde{g}$ with L^*, thus $w = \widetilde{w}/L^*$.

It is usually assumed that the rate of wear of the surface is proportional to the contact pressure and to the relative slip speed, that is to the dissipated frictional power. This leads to the rate form of Archard's law of surface wear,

$$\dot{w} = k_w |\sigma_n| \, \|\dot{\mathbf{u}}_\tau\|, \tag{3.2.1}$$

where k_w is the wear coefficient, a very small positive constant in practice. The dimensionless rate equation is obtained from the one with dimensions,

$$\dot{\widetilde{w}} = \widetilde{k}_w |\widetilde{\sigma}_n| \, \|\dot{\widetilde{\mathbf{u}}}_\tau\|,$$

if we use the variables introduces in Sect. 2.5 and define

$$k_w = \frac{\rho_0 L L^*}{(T^*)^2} \widetilde{k}_w.$$

Here, $\widetilde{k}_w$ has the dimensions of $[cm \cdot sec^2/gm]$.

The initial condition is $w(\mathbf{x}, 0) = w_0(\mathbf{x})$ on Γ_C, and when the surface is new or the initial shape is used as the reference configuration $w_0 = 0$. When the stress is given by the normal compliance condition (2.6.2) we obtain

$$\dot{w} = k_w p_n(u_n - w - g) \|\dot{\mathbf{u}}_\tau\|. \tag{3.2.2}$$

Archard's law can be used with the other contact conditions, besides normal compliance, although, to our knowledge, there are very few mathematical results which combine (3.2.1) with the other ones. Indeed, an approximate problem with bilateral contact can be found in [22], and a problem with normal damped response and wear was studied in [90].

In engineering publications Archard's law is usually stated in the form $w = k_w P \overline{L}$, where P is the nominal contact pressure (the total load applied on the system divided by the nominal contact area) and $\overline{L}$ is the total slip. Clearly, this is a surface averaged and time integrated version of condition (3.2.1), namely,

$$\begin{aligned} w(t) &= \frac{1}{|\Gamma_C|} \int_{\Gamma_C} w(\mathbf{x}, t)\, dS \\ &= \frac{1}{|\Gamma_C|} \int_{\Gamma_C} w_0(\mathbf{x}) dS + \frac{k_w}{|\Gamma_C|} \int_{\Gamma_C} \int_0^t |\sigma_n(\mathbf{x}, s)|\, \|\dot{\mathbf{u}}_\tau(\mathbf{x}, s)\|\, ds\, dS, \end{aligned}$$

where $|\Gamma_C|$ denotes the area or measure of Γ_C. Thus, the total volume of material removed over the time period $[0, t]$ is $w(t)|\Gamma_C|$.

When the foundation itself is moving with prescribed velocity $\mathbf{v}^* = \mathbf{v}^*(t)$, we need to replace $\dot{\mathbf{u}}_\tau$ with $\dot{\mathbf{u}}_\tau - \mathbf{v}^*$ in the Coulomb and Archard laws above, and all the results below hold in this case, too.

In the case of contact with normal compliance and wear, condition (3.2.2) can be derived, following [97–99], from the friction and wear pseudo-potential function

$$\mathcal{F}(\boldsymbol{\sigma}_\tau, \sigma_n, w) = \|\boldsymbol{\sigma}_\tau\| - \mu p_n + k_w p_n w.$$

The condition is obtained from the requirement that $(\dot{\mathbf{u}}_\tau, \dot{w})$ lies in $N_F(\boldsymbol{\sigma}_\tau, w)$, the normal cone to the set $F = \{(\boldsymbol{\sigma}_\tau, \sigma_n, w) : \mathcal{F}(\boldsymbol{\sigma}_\tau, \sigma_n, w) \le 0\}$. However, the derivation there depends on the assumption that μ and k_w are constants, and such a derivation needs a more delicate analysis when the friction and wear coefficients depend on the sliding speed and other process variables. See Chap. 4 for a related discussion.

Although condition (3.2.1) is very popular, there is a clear need for more sophisticated description of some of the wear processes. Indeed, condition (3.2.1) does not distinguish between a process that causes deep grooves and a one which produces a smooth and conforming surface. This may be taken into account, in part, by assuming that the friction coefficient depends on the wear, thus,

$$\mu = \mu(\|\dot{\mathbf{u}}_\tau\|, \theta, w, \ldots).$$

Moreover, a basic underlying assumption in (3.2.1) is that the wear particles are instantly removed from the surface and so they are not involved in the process. However, very often the debris remain on the surface, migrate on it and contribute to the wear process. When the wear particles are (relatively) hard they may cause further wear, produce grooves and cracks and cause damage to the surface. When they are (relatively) soft they may act as a lubricant. Therefore, there is a need to take into account the migration of the wear particles and their further contribution to the contact process. The problem is very important in implants and a computational approach can be found in [151]. A modeling step in this direction can be found in [143]. A simple ad hoc mathematical approach to the diffusion of wear particles on the contact surface can be found in [152, 153].

Finally, we note that there is a distinction between *mild wear* and *severe wear*, and the transition from one type to the other depends on the contact pressure, as well as on the chemical composition and the environment (such as humidity, or supply of oxygen). In metals, mild wear describes the process when the oxide layer remains intact and the wear is slow. On the other hand, in severe wear the oxide layer is broken, and there is direct contact between the metallic asperities. The rate of wear in the latter case is usually unacceptable in many applications. The transition from one type of wear to the other seems to be rather abrupt, and may be modelled with a stress dependent wear coefficient. This topic seems to be of practical importance and deserves a mathematical study, which may turn out to be quite involved if one assumes that $k_w = k_w(\sigma_n)$.

3.3 Adhesion

Processes of adhesion are important in many industrial settings where parts, usually nonmetallic, are glued together. Recently, composite materials, made of layers of simple materials, reached prominence since they are very strong and light and, therefore, of considerable importance in aviation, space exploration, and the automotive industry. However, under stress composite materials my undergo delamination in which different layers debond and move relative to each other. This is one of the reasons for the importance of the adhesive process in industrial applications.

The adhesive contact between bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature, because of its industrial interest. Basic modelling can be found in [154–159]. Analysis of models for frictionless adhesive contact can be found in [160–169]. In [163] the problem of a beam in adhesive contact can be found, in [164] the adhesive quasistatic contact of a membrane was investigated and simulated, and in [165] the dynamic problem was shown to have a weak solution. Related models can be found in [148,149] and references therein. Moreover, a new application of the theory is in the medical field of arthoplasty where the bonding between the bone-implant and the tissue is of considerable importance, since debonding may lead to decrease in the persons ability to use the artificial limb or joint (see [148,149] and references therein).

The novelty in these papers (except for [169]) is the introduction of a surface internal variable, the *bonding field* or the *adhesion field* β, which describes the pointwise fractional density of active bonds on the contact surface, and sometimes is referred to as the 'intensity of adhesion.' This variable is dimensionless by its definition. We refer the reader to the extensive bibliography on the subject in [157, 158] and in [148], and a short description of the modelling can be found in Chap. 4. Following Frémond [65,154,155], we introduce the bonding field $\beta = \beta(\mathbf{x}, t)$ defined on the contact surface Γ_C, which has values between 0 and 1. When $\beta = 1$ at a point of Γ_C, the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are severed and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active.

The glue on the contact surface introduces tension that opposes the separation of the surfaces in the normal direction, and opposes the relative motion in the tangential directions. The adhesive tensile traction is assumed proportional to β^2 and the displacements. The evolution of β depends on β and the displacements. We refer the reader to the monograph [65] for full details, and to Chap. 4 for a short derivation.

If we assume, as in [160–162,166], that the compressive part of the normal stress is described by normal compliance (2.6.2) and there is no gap ($g = 0$), then the *normal compliance contact condition with adhesion* is given by

$$-\sigma_n = p_n(u_n) - \gamma_n \beta^2 (-R_{L_b}(u_n))_+,$$

where p_n is the normal compliance function and γ_n is the saturation constant of the surface density of bonding energy. In dimensional form $\widetilde{\gamma}_n$ has the dimensions of $[gm/cm^2 \cdot sec^2]$, and by letting $\gamma_n = ((T^*)^2/\rho_0 L)\widetilde{\gamma}_n$ we obtain the above relation. Moreover, $R_{L_b} : \mathbb{R} \to \mathbb{R}$ is the truncation operator

$$R_{L_b}(s) = \begin{cases} -L_b & \text{if} \quad s < -L_b \\ s & \text{if} \quad -L_b \le s \le L_b \\ L_b & \text{if} \quad s > L_b \end{cases},$$

where $0 < L_b$ is the characteristic length of the bond, beyond which it does not offer any additional traction (see, e.g., [158]). Indeed, the introduction of R_{L_b} is motivated by the observation that if the extension is more than L_b, the glue extends plastically without offering additional tensile traction. However, by choosing L_b sufficiently large, say larger than the size of the system, we recover the case where the traction is linear in the extension. Thus, the contribution of the adhesive to the normal traction is represented by $\gamma_n \beta^2 (-R_{L_b}(u_n))_+$; the adhesive traction is tensile, and is proportional, with proportionality coefficient γ_n, to the square of the adhesion, and to the normal displacement, but as long as it does not exceed the bond length L_b. The maximal tensile traction is $\gamma_n L_b$. More general expressions for this condition can be found in [161,165].

The tangential stiffness generated by the glue is assumed to depend on the adhesion and on the tangential displacement, but again, only up to the bond length L_b, thus,

$$-\boldsymbol{\sigma}_\tau = p_\tau(\beta) R^*_{L_b}(\mathbf{u}_\tau), \tag{3.3.1}$$

where the truncation operator $R^*_{L_b}$ is defined by

$$R^*_{L_b}(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{if} \quad \|\mathbf{v}\| \le L_b \\ L_b \dfrac{\mathbf{v}}{\|\mathbf{v}\|} & \text{if} \quad \|\mathbf{v}\| \ge L_b \end{cases}.$$

Then, $p_\tau(\beta)$ acts as the stiffness or spring constant, increasing with β, and the traction is in direction opposite to the displacement. The maximal modulus of the tangential traction in (3.3.1) is $p_\tau(1)L_b$.

The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted. When it is not negligible one has to add the frictional traction, as has been done in [148, 157, 158] (see Chap. 4 below).

The evolution of the adhesion field, following [154, 155], can be obtained from the principle of virtual work, see Chap. 4. The following adhesion evolution equation has been used recently,

$$\widetilde{k}_\beta \dot{\widetilde{\beta}} = -\widetilde{\gamma}_n \widetilde{\beta}((-R_{L_b}(\widetilde{u}_n))_+)^2.$$

It describes an irreversible process in which once debonding takes place it is permanent, and there is no rebonding. Here, $\widetilde{k}_\beta$ is the bonding rate coefficient with dimensions of $[gm/sec]$, since the dimensions of $\widetilde{\gamma}_n$ are $[gm/cm^2 \cdot sec^2]$.

To set it in dimensionless form let $\beta(\mathbf{x}) = \widetilde{\beta}(\widetilde{\mathbf{x}})$, and define the dimensionless adhesion rate constant by

$$\gamma_\beta = \frac{T^*(L^*)^2}{\widetilde{k}_\beta}\widetilde{\gamma}_n.$$

Then, the dimensionless adhesion rate equation is

$$\dot{\beta} = -\gamma_\beta \beta((-R_{L_b}(u_n))_+)^2. \tag{3.3.2}$$

However, it is possible to consider a more general setting ([160–162, 165]), thus,

$$\dot{\beta} = H_{ad} = H_{ad}(\beta, u_n, \mathbf{u}_\tau, \ldots).$$

Here, H_{ad} is the *adhesion evolution rate function*, and it depends on the bonding and on the, possibly truncated, normal and tangential displacements. In [161] it has been assumed that $H_{ad} = H_{ad}(\beta, R_{L_b}(u_n))$ is a general Lipschitz function, which vanishes when β vanishes. Indeed, when $\beta = 0$ there is no change in β and once $\beta = 0$ debonding is complete and no further evolution of β is allowed, and in particular, no rebonding can take place.

Since H_{ad} may have both positive and negative values, as long as $\beta > 0$ rebonding may take place after debonding. When $H_{ad} \le 0$, as was assumed in [148, 158, 160], the process is irreversible and once the bonds break they cannot be reestablished.

In the description of the adhesion evolutions equations below we shall use standard notation of convex analysis, the explanation of which can be found in Sect. 6.3. In particular, we denote by I_K the indicator function of the set K, (6.3.4), and by ∂I_K its subdifferential, (6.3.5).

A somewhat more general rate equation was used in [165], where the dynamic adhesive contact between a membrane and a rigid obstacle, that had been situated under it, was investigated. The evolution of adhesion was given by

$$\dot{\beta} = H_{ad}(\beta, u - \phi) + \beta_{sub}, \tag{3.3.3}$$

where ϕ described the shape of the obstacle and $u - \phi$ was the distance of the membrane from the obstacle; H_{ad} was assumed to be a general evolution function such that $H_{ad}(\beta, 0) = 0$, and $-\beta_{sub} \in \partial I_{[0,1]}(\beta)$. The term β_{sub} was added to enforce the condition $0 \le \beta \le 1$. The use of subdifferentials to enforce such conditions will be explained in Chap. 4, and a fuller mathematical description can be found in Sect. 6.3. However, for the sake of completeness we describe it shortly here. The indicator function $I_{[0,1]}(\beta)$ is given by

$$I_{[0,1]}(\beta) = \begin{cases} \infty & \text{if} \quad \beta < 0 \\ 0 & \text{if} \quad 0 \le \beta \le 1 \\ \infty & \text{if} \quad \beta > 1 \end{cases},$$

and its subdifferential is the graph (see Fig. 6 below)

$$\partial I_{[0,1]}(\beta) = \begin{cases} (-\infty, 0] & \text{if} \quad \beta = 0 \\ 0 & \text{if} \quad 0 < \beta < 1 \\ [0, \infty) & \text{if} \quad \beta = 1 \\ \emptyset & \text{otherwise} \end{cases}.$$

Mathematically speaking, since $-\beta_{sub} \in \partial I_{[0,1]}(\beta)$, it follows that $\partial I_{[0,1]}(\beta)$ is nonempty, and therefore $0 \le \beta \le 1$.

A more intuitive way to see it is as follows. When $-\beta_{sub} \in \partial I_{[0,1]}(\beta)$ then $\beta_{sub} = 0$ if $0 < \beta < 1$, and as long as the two strong inequalities hold, β_{sub} is inactive in (3.3.3). When $\beta = 0$ then $0 \le \beta_{sub} < \infty$, and its value is chosen automatically by the system to be such as to exactly cancel $H_{ad}(0, u - \phi)$, when the latter is negative. It cannot be larger than $|H_{ad}|$, since in such a case $\dot{\beta} > 0$ and in the next instant we will have $\beta > 0$ and then $\beta_{sub} = 0$. Thus, $\dot{\beta} = 0$, preventing β from becoming negative, and violating the part of the constraint $\beta \ge 0$. When $\beta = 1$, we have $-\infty < \beta_{sub} \le 0$, and its value is such as to exactly cancel $H_{ad}(1, u - \phi)$. Then $\dot{\beta} = 0$, and it prevents β from becoming greater than 1, which violates the constraint $\beta \le 1$. In the case of adhesion the inclusion of the subdifferential $\partial I_{[0,1]}(\beta)$ is done to enforce the interpretation of β as a fraction. In the case of contact forces, say in contact with a rigid obstacle, the subgradient has a physical meaning as it describes exactly the physical force needed to prevent penetration.

The dependence of the adhesion process on its history was taken into account in [162]. The process was assumed reversible, rebonding of broken bonds could take place, however, bonds undergoing rebonding were weaker than the original ones. In such a case $H_{ad} = H_{ad}(\beta, \psi_\beta, R_{L_b}(u_n))$, where

$$\psi_\beta(\mathbf{x}, t) = \int_0^t \beta(\mathbf{x}, s)\, ds$$

is the history of the bonding process at the point $\mathbf{x}$. The assumptions in [162] allowed for cycles of debonding and rebonding, and the rate of the bonding evolution was assumed to be described by

$$H_{ad} = H_{ad}(\beta, \psi_\beta, R_{L_b}(u_n)) = -\gamma_1 \beta \left((-R_{L_b}(u_n))_+\right)^2 + \gamma_2 \frac{\beta_+(1-\beta)_+}{1+d^*\psi_\beta^2}\,. \tag{3.3.4}$$

Here, the normal displacement $R_{L_b}(u_n)$ causes debonding, described by the first term on the right-hand side, while the natural tendency of the adhesive to rebond is represented by the second term on the right-hand side. However, rebonding becomes weaker as the process goes on, which is represented by the factor $1+d^*\psi_\beta^2$ in the denominator, where d^* is the history weight factor, assumed to be positive. Thus, the second term serves a long-time memory term for the process. Also, γ_1 and γ_2 are the debonding and rebonding rate constants, respectively, and are assumed to be positive.

In [160] the following form of H_{ad} has been employed,

$$\dot\beta = -(\gamma_\beta \beta((-R_{L_b}(u_n))_+)^2 - e_{Du})_+, \tag{3.3.5}$$

where e_{Du} is the dimensionless activation energy for debonding, and once debonding occurs bonding cannot be reestablished, since $\dot\beta \le 0$. This is the frictionless version of the debonding condition in [148, 157, 158] which are described in more detail in Chap. 4. The activation energy $\widetilde{e}_{Du}$ has the dimensions of energy per unit area, $[gm/sec^2]$, and $e_{Du} = \widetilde{e}_{Du} \cdot T^*/\widetilde{k}_\beta$. Note that when $e_{Du} \approx 0$ we recover (3.3.2). The activation energy acts as a threshold for debonding, and only when the energy supplied to the adhesive is above it the process of debonding will begin, and it will stop if the energy supplied falls below this threshold.

We note that bonding rate conditions of this form, which are proportional to β, do not allow for the complete debonding in finite time. Indeed, if we assume that the displacements are constant, we find that the bonding field decays as $\beta(t) = A\exp(-at)$. The quasistatic adhesive contact of a rod with an obstacle was investigated recently in [170] where it was shown that when the displacement of the other end if finite debonding will either stop in a finite time or will take infinite time, similarly to the exponential decay above.

To allow for full debonding new specific expressions for H_{ad} need to be derived. A general condition which allows for full debonding ($\beta = 0$) and then rebonding, as well as debonding when fully bonded ($\beta = 1$), together with dependence on the process history, is given by the following extension of (3.3.3),

$$\dot\beta \in H_{ad}(\beta, \psi_\beta, R_{L_b}(u_n)) - \partial I_{[0,1]}(\beta).$$

As above, the addition of the subdifferential to the right-hand side guarantees that $0 \le \beta \le 1$, and the rate function $H_{ad}(\beta, \psi_\beta, R_{L_b}(u_n))$ may be a rather general function of its arguments. When we attempt to model velcro adhesion, the history dependence seems to be unimportant. However, in other applications many cycles of possibly full bonding and debonding take place, and when the adhesive undergoes some degradation in each cycle, the history dependence or the memory has to be included in the model.

The models above deal with the case when an adhesive agent, such as a glue or velcro, is present on the contacting surfaces. The modeling of the intrinsic adhesive component in friction, stemming from chemical bonds of the junctions and their possible welding, needs a different approach, since it affects directly the friction coefficient. This process is important, and may cause the friction coefficient to increase, as the time in which the surfaces are in the stick state increases. It may also explain the often observed fact that the dynamic friction coefficient is smaller than the static one. This surface bonding, especially in metals, may lead to scuffing and damage of the surfaces, processes for which there is no current mathematical description.

Finally, artificial implants of knee and hip prostheses (both cemented and cement-less) clearly show that at the bone-implant interface adhesion plays an important role, see, e.g., [61] and references therein.

3.4 Damage

The constitutive assumptions on the contacting bodies were, up to now, rather standard. However, in many materials, such as concrete, there is an observed decrease in the load bearing capacity over time, caused by the development of internal microcracks. The subject is extremely important in design engineering, since it directly affects the useful life span of the structures or components.

There exists a very large engineering literature on material damage. However, only recently models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. We describe below recent results on contact problems when the damage of the parent material caused by tension or compression is taken into account.

General new models for damage were derived recently in [171, 172] from the virtual power principle. Full details can be found in [65], and a short derivation in Chap. 4. In the engineering literature two types of damage are usually considered, brittle damage and fatigue damage. The models below allow for taking into account both processes; brittle damage is caused by the growth of microscopic cracks in the material and fatigue damage is associated with the accumulation of damage during cycles of loading and unloading.

Mathematical analysis of the evolution of damage in one-dimensional problems can be found in [173, 174], and the three-dimensional case has been investigated in [175, 176]. Here, we describe a variant of one of these models, and we note that a number of other models for damage, based on different considerations, can be found in the engineering literature. Related problems with damage can be found in the publications [81, 129, 137, 177–182] and references therein.

The new idea of [65, 171, 172] involves the introduction of the *damage function* $\widetilde{\zeta} = \widetilde{\zeta}(\widetilde{\mathbf{x}}, \widetilde{t})$, which is the ratio between the elastic modulus of the damaged material and the damage-free one, and so it is dimensionless. In an isotropic and homogeneous elastic material, let E_Y be the Young modulus of the original damage-free material and E_{eff} be the current one, then the damage function is defined by

$$\widetilde{\zeta} = \widetilde{\zeta}(\widetilde{\mathbf{x}}, \widetilde{t}) = \frac{E_{eff}}{E_Y}.$$

To set the problems below in dimensionless form we let

$$\zeta = \zeta(\mathbf{x}, t) = \widetilde{\zeta}(\widetilde{\mathbf{x}}, \widetilde{t}).$$

Clearly, it follows from this definition that the damage function ζ is restricted to values between zero and one. When $\zeta = 1$ the material is damage-free; when $\zeta = 0$ the material is completely damaged; when $0 < \zeta < 1$ there is partial additional damage and the system has a reduced load carrying capacity, relative to the original one.

In anisotropic materials, it is a bit tricky, but one can define the damage field as the factor in the constitutive law that multiplies the strain term, thus providing the relationship between the current stress and the stress that would have been observed in the damage-free material, under the same conditions.

The models for contact processes that take material damage into account contain, in addition, the variable ζ. An evolution equation for the damage field had been derived in [65] from the principle of virtual power. One may consider the case when damage is irreversible, i.e., when cracks open, they can only grow and no self-mending is allowed, or the reversible case when self-mending of the microcracks may take place.

Since ζ is restricted to the values $0 \le \zeta \le 1$, this needs to be enforced in the damage evolution equation. To that end, as in the previous section when describing the bonding field, we use $I_{[0,1]}(\zeta)$ the indicator function of the interval $[0, 1]$, and let $\partial I_{[0,1]}(\zeta)$ be its subdifferential (Fig. 6 on page 58). Then, an evolution equation for the damage field is (see [65] for details)

$$\dot{\zeta} - k_{Dam} \triangle \zeta + \partial I_{[0,1]}(\zeta) \ni \phi(\boldsymbol{\varepsilon}(\mathbf{u}), \zeta), \tag{3.4.1}$$

and the subdifferential term guarantees that ζ remains within the interval $[0, 1]$. Here, k_{Dam} is the microcrack diffusion constant, relating the 'diffusion' or influence of regions with higher density of microcracks on neighbouring regions. The function ϕ is the *damage source function* and is assumed to be a rather general function of the strain and damage itself. Moreover, it is assumed in (3.4.1) that the material may recover from damage and microcracks may close, and therefore there is no restriction on the sign of $\dot{\zeta}$. If one wishes to consider the irreversible process, referred to as 'the unilateral phenomenon' on occasions, when microcracks do not mend or close, one needs, in addition, to impose the restriction $\dot{\zeta} \le 0$. Then, the evolution equation has the form

$$\dot{\zeta} - k_{Dam}\triangle\zeta + \partial I_{[0,1]}(\zeta) + \partial I_{(-\infty,0]}(\dot{\zeta}) \ni \phi(\boldsymbol{\varepsilon}(\mathbf{u}), \zeta). \tag{3.4.2}$$

Here, the subdifferential term $\partial I_{(-\infty,0]}(\dot{\zeta})$ (Fig. 7 on page 58) enforces the condition $\dot{\zeta} \le 0$.

In [171, 172] the damage source was chosen as

$$\phi_{\mathrm{Fr}}(\boldsymbol{\varepsilon}(\mathbf{u}), \zeta) = \lambda_D \left(\frac{1-\zeta}{\zeta}\right) - \frac{1}{2}\lambda_E\, \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{u}) + \lambda_w, \tag{3.4.3}$$

where λ_D, λ_E and λ_w are positive process parameters, to be determined experimentally. Actually, the damage source function there distinguished between damage due to compression and tension by assigning two different rate constants, instead of the rate constant λ_E above. We refer the reader to these papers and to [175, 176] for further details.

We note that this damage source function has a singularity when the damage is complete ($\zeta = 0$), and this does not allow the damage to reach the value zero, i.e., it precludes complete damage.

Other damage source functions can be found in the monograph [65], such as the following one for the irreversible damage process caused only by tension,

$$\phi = -\frac{1}{2}\left(1 - \frac{1-\zeta}{1-m_\zeta\zeta}\right)\left(2\lambda_1(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon})^+ + \lambda_2(tr\,\boldsymbol{\varepsilon})^+)^2\right) + \lambda_w. \tag{3.4.4}$$

Here, λ_1, λ_2 are the Lamé coefficients, $0 < m_\zeta < 1$ is a scaling factor, and since only tension causes damage, one uses the positive part (i.e., the positive eigenvalues) of the strain tensor $\boldsymbol{\varepsilon}$, denoted by $(\cdot)^+$. We note that this is different from the positive part of a scalar function $(f)_+$, as it involves the positive eigenvalues of the strain tensor. For details the reader is referred to [65] and [175, 176]. Unlike the damage source function above, this function allows for complete material damage.

Quasistatic contact problems with damage have been investigated in [23, 173, 178–182]. In [173] the one-dimensional contact problem was reduced to a nonlinear parabolic equation for the damage field, since in this case the part of the model for the displacements decouples from the one for damage. In [178, 179] viscoelastic problems with normal compliance and normal damped response have been considered, respectively. There, (3.4.1) was used for the evolution of the damage field. In [171–174] the evolution equation (3.4.2) for the damage field was used, since the damage was considered irreversible. Finally, in [23, 180–182] viscoplastic problems with damage were considered, where it was assumed that the damage source depended on the stress field too, and the damage was assumed reversible. Therefore, equation (3.4.1) was modified as follows

$$\dot{\zeta} - k_{Dam}\triangle\zeta + \partial I_{[0,1]}(\zeta) \ni \phi(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}), \zeta). \tag{3.4.5}$$

The dynamic contact problem with normal compliance and damage, with a general damage function which may have different rates for compression or tension, has been investigated in [176].

A major topic of interest in models for damage is the behavior of the solutions when complete damage takes place. In the one-dimensional problems complete damage at one point means the collapse of the whole system and, mathematically, is reflected in the quenching of the solutions and blow-up of some of the derivatives. The issue in three-space dimensions is much more complicated and its analysis lies in the future, since quenching of the solution at a point or even over a region may not interfere noticeably with the system's capability to support the applied load.

Clearly, models which include the material damage are considerably more complicated and their mathematical analysis is in its infancy. Moreover, as far as we know, models for the damage of the contacting surfaces, as distinguished from their wear, do not exist, yet.

4 Thermodynamic Derivation

In this chapter we present a short review of thermodynamic principles and potentials and describe their use in derivation of general thermomechanical conditions and equations, as applied to processes involved in contact. Works on Thermodynamics of Continuua abound, and more specialized applications to contact phenomena can be found in [65,97–99,183] and references therein. We use some notions from convex analysis which will be explained in more detail in Sect. 6.3.

The presentation here follows [65, 97, 155] and is directed toward applications to processes involved in contact, in particular in friction, wear or adhesion. In this manner one may obtain some of the equations and conditions introduced in the previous two chapters. Also, this method allows for derivation of other laws and conditions. However, this method has some shortcomings which we describe at the end of the chapter.

In this chapter we use variables with dimensions. To obtain the corresponding dimensionless variables, quantities, and relations one may proceed as in Sect. 2.5. However, we do not use the tilde, for the sake of simplicity, and also we do not describe the units.

In Sect. 4.1 we provide the general formalism, based on the virtual power principle and the entropy inequality, and derive the equations and conditions for a thermoviscoelastic body. In Sect. 4.2 we describe a model for the isothermal friction with adhesion employing the Signorini condition. A model for isothermal friction and adhesion with the normal compliance condition is presented in Sect. 4.3. Finally, in Sect. 4.4 we derive a new model for the evolution of a thermoviscoelastic body with material damage. We conclude with a short summary in Sect. 4.5.

It is seen that the formalism is a very effective method to construct models for complicated phenomena in a rational and thermodynamically consistent way. However, there is a considerable leeway in the choice of the free energy function and the dissipation pseudo-potential. To be useful, in each specific case these must be chosen so that the resulting model is capable of sufficiently accurate prediction of the behaviour of the system under investigation.

M. Shillor, M. Sofonea, J.J. Telega: *Models and Analysis of Quasistatic Contact*, Lect. Notes Phys. **655**, (2001), 49–64
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4.1 The Formalism

We consider a thermodynamically closed system which occupies the domain $\Omega \subset \mathbb{R}^d$ (for $d = 1, 2, 3$ in practice) that is completely described by the vector of extensive variables: the generalized coordinates $\mathbf{z} \in \mathbb{R}^{d_1}$, the absolute temperature $\vartheta \in \mathbb{R}_+$ (or equivalently the entropy S, which is an extensive variable) and a vector of internal variables $\mathbf{y} \in \mathbb{R}^{d_2}$, where d_1 and d_2 are positive integers. The system or the body may come into contact with another body, and then one may consider two deformable bodies in contact over a common surface Γ_C. This leads to somewhat cumbersome notation, since each variable has to have an index denoting its restriction to each one of the bodies (see, e.g., [97, 154, 155, 158] and references therein). To simplify the presentation, and without loss of generality, we describe the process of contact between one deformable body and a foundation which may be rigid or deformable. The term 'foundation' is used when the internal description of the second body is of no interest.

We deal, as in Sect. 2.2, with a body that occupies the domain Ω, it is acted upon by volume forces $\mathbf{f}_B$, is held fixed over the part Γ_D of its surface, is acted upon by tractions $\mathbf{f}_N$ on Γ_N, and which may come into contact with a foundation on Γ_C. The setting is depicted in Fig. 5. Also, since we include thermal effects, we assume that volume heat sources of density r_Ω (per unit volume) are present (such as the Joule heating due to electric current).

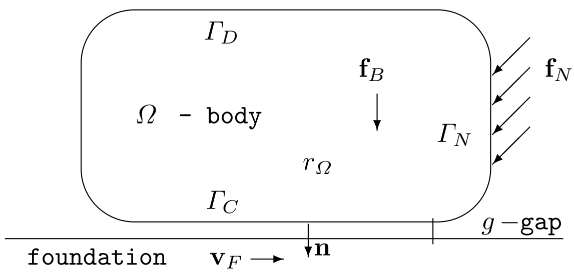

Fig. 5. The setting

The method of virtual power, which is based on the balance of power instead of conservation of energy, and which is equivalent to the latter, has been used in [65, 97], among others, to derive the equations and the relevant contact conditions. The evolution of the state of the system is described by the free or Helmholtz energy potential Ψ_Ω and by a pseudo-potential of dissipation Φ_Ω, combined with the principles of energy conservation, momentum conservation and entropy inequality. This is the usual case when either the Dirichlet or the Neumann conditions are imposed on the boundary. In our case, to describe the evolution of the processes on the contact surface, we introduce, in addition, the surface Helmholtz potential Ψ_C and the dissipation pseudo-potential Φ_C, both defined on Γ_C, and which will be described in detail in the sequel.

First, we describe the processes that take place in Ω. We let $\mathbf{u}$ denote the displacements vector, and recall that

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{u}), \quad \text{with components} \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\,(u_{i,j} + u_{j,i}),$$

represents the small strain tensor, obtained by linearization. Also, the indices i, j, k, l have values in $\{1, \ldots, d\}$; an index denotes the component; an index following a coma indicates a partial derivative with respect to the corresponding spatial variable; the summation convention is employed; a dot above a variable denotes the partial derivative with respect to time; and $\mathbf{v} = \dot{\mathbf{u}}$.

We use the following notation,

$$F_{,i} = \frac{\partial F}{\partial x_i}\,, \qquad F_{i,j} = \frac{\partial F_i}{\partial x_j}\,,$$

for a scalar function F and a vector function $\mathbf{F} = (F_1, \cdots, F_d)$. The gradient operator is given by

$$\nabla F = (F_{,1}, \cdots, F_{,d}).$$

Next, let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d)$ be a vector or a tensor $\boldsymbol{\xi} = (\xi_{ij})$, then by $\boldsymbol{\eta} = \partial F / \partial \boldsymbol{\xi}$ we mean

$$\eta_i = \frac{\partial F}{\partial \xi_i} \quad \text{or} \quad \eta_{ij} = \frac{\partial F}{\partial \xi_{ij}}\,,$$

respectively.

We remark that in this chapter the derivatives are understood in the usual sense when the functions or potentials are differentiable; otherwise, they are obtained by subdifferentiation, and are elements of the subdifferentials.

We assume, following [65, 97], that

$$\Psi_\Omega = \Psi_\Omega(\vartheta_\Omega, \varepsilon(\mathbf{u}), \mathbf{y}^r_\Omega), \qquad \Phi_\Omega = \Phi_\Omega(\nabla\vartheta_\Omega, \varepsilon(\dot{\mathbf{u}}), \mathbf{y}^{ir}_\Omega),$$

where ϑ_Ω is the absolute temperature in the body, $\mathbf{y}^r_\Omega$ are the reversible components of the vector of internal variables, and $\mathbf{y}^{ir}_\Omega$ are its irreversible components.

Next, let ρ be the material density, ($\widetilde{\rho} = \rho_0\rho$ in the terminology of Sect. 2.5), and let e_Ω and s_Ω denote the densities (per unit volume) of the energy and entropy in Ω, respectively. It follows from the Helmholtz relation that

$$\Psi_\Omega = e_\Omega - s_\Omega \vartheta_\Omega. \tag{4.1.1}$$

We recall that

$$s_\Omega = -\frac{\partial \Psi_\Omega}{\partial \vartheta_\Omega}. \tag{4.1.2}$$

Let $\boldsymbol{\sigma} = (\sigma_{ij})$ be the stress tensor. We write it as a sum of a reversible part $\boldsymbol{\sigma}^r$ and an irreversible part $\boldsymbol{\sigma}^{ir}$,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^r + \boldsymbol{\sigma}^{ir},$$

where the two parts are given by

$$\boldsymbol{\sigma}^r = \frac{\partial \Psi_\Omega}{\partial \boldsymbol{\varepsilon}(\mathbf{u})}, \qquad \boldsymbol{\sigma}^{ir} = \vartheta_\Omega \frac{\partial \Phi_\Omega}{\partial \boldsymbol{\varepsilon}(\dot{\mathbf{u}})}.$$

The heat flux vector $\mathbf{q}$ is given by

$$\mathbf{q} = \vartheta_\Omega \mathbf{q}^{ir}, \qquad \mathbf{q}^{ir} = -\frac{\partial \Phi_\Omega}{\partial \nabla \vartheta_\Omega}.$$

The generalized forces related to the internal reversible variables are denoted by $\mathbf{Y}^r_\Omega$ and those associated with the irreversible variables as $\mathbf{Y}^{ir}_\Omega$, and are given, respectively, by

$$\mathbf{Y}^r_\Omega = \frac{\partial \Psi_\Omega}{\partial \mathbf{y}^r_\Omega}, \qquad \mathbf{Y}^{ir}_\Omega = -\vartheta_\Omega \frac{\partial \Phi_\Omega}{\partial \mathbf{y}^{ir}_\Omega}.$$

Let $\mathbf{Y}_\Omega = (\mathbf{Y}^r_\Omega, \mathbf{Y}^{ir}_\Omega)$ be the vector of internal generalized forces, and let $\mathbf{y}_\Omega = (\mathbf{y}^r_\Omega, \mathbf{y}^{ir}_\Omega)$ be the vector of internal variables, then, the power associated with them is $P = \mathbf{Y}_\Omega \cdot \dot{\mathbf{y}}_\Omega$.

If Y is the generalized force associated with the internal variable y, and $\mathbf{Z}$ with ∇y, then the conservation law for y (see [65, p. 5]) is

$$Y - \operatorname{div} \mathbf{Z} = f_y, \tag{4.1.3}$$

where f_y is the density of applied volume 'y-forces.' The equation holds for each such pair $(y, \nabla y)$ with the associated 'forces' $(Y, \mathbf{Z})$.

Finally, the energy and momentum conservation equations, respectively, are

$$\dot{e}_\Omega + \operatorname{div} \mathbf{q} = r_\Omega + \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathbf{Y}_\Omega \cdot \dot{\mathbf{y}}_\Omega, \tag{4.1.4}$$

$$\rho \ddot{\mathbf{u}} - \operatorname{Div} \boldsymbol{\sigma} = \mathbf{f}_B. \tag{4.1.5}$$

Here, 'Div' and 'div' denote the divergence operator for tensor valued and vector valued functions, respectively, and the dot represents the scalar product between vectors or tensors.

These are general relations and they apply to every physical system studied in this book. To describe the evolution of a specific system one has to specify the Helmholtz potential and the dissipation pseudo-potential, together with initial and boundary conditions.

Following [65, 97, 155], we choose the Helmholtz potential as

$$\Psi_\Omega(\vartheta_\Omega, \boldsymbol{\varepsilon}(\mathbf{u})) = -c_p \vartheta_\Omega \log(\vartheta_\Omega) + \frac{1}{2}\, \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}) - \vartheta_\Omega \mathcal{M} \varepsilon(\mathbf{u}), \tag{4.1.6}$$

where c_p is the heat capacity per unit volume, $\mathcal{B} = (b_{ijkl})$ is the tensor of elastic coefficients, and $\mathcal{M} = (m_{ij})$ is the tensor of coefficients of thermal expansion. In isotropic materials $\mathcal{M} = \alpha \mathbf{I}_d$, where α is the scaled coefficient of thermal expansion and $\mathbf{I}_d$ is the identity matrix, sometimes denoted by the Kronecker symbol δ_{ij}.

We choose the dissipation pseudo-potential as

$$\Phi_\Omega(\nabla\vartheta_\Omega, \boldsymbol{\varepsilon}(\dot{\mathbf{u}})) = \frac{k}{2\vartheta_\Omega^2}\,\|\nabla\vartheta_\Omega\|^2 + \frac{1}{2\vartheta_\Omega}\,\varepsilon(\dot{\mathbf{u}})\cdot\mathcal{A}\varepsilon(\dot{\mathbf{u}}). \tag{4.1.7}$$

Here, $k > 0$ is the coefficient of heat conduction, $\mathcal{A} = (a_{ijkl})$ is the viscosity tensor. In homogeneous materials all the coefficients in $\mathcal{A}, \mathcal{B}$, and $\mathcal{M}$ are constants, independent of the position. It is also assumed, usually implicitly, that the coefficients are time independent.

We assumed in (4.1.6) and (4.1.7) that there are no internal variables.

With this choice of Ψ_Ω and Φ_Ω we obtain below the system of partial differential equations of the model. But first, we use the above in (4.1.4) to obtain, after some manipulations, an equivalent energy balance

$$\begin{aligned}\vartheta_\Omega\left(\dot{s}_\Omega + \operatorname{div}\mathbf{q}^{ir} - \frac{r}{\vartheta_\Omega}\right) &= \boldsymbol{\sigma}^{ir}\cdot\varepsilon(\dot{\mathbf{u}}) - \mathbf{q}^{ir}\cdot\nabla\vartheta_\Omega \\ &= \partial\Phi_\Omega(\nabla\vartheta_\Omega, \boldsymbol{\varepsilon}(\dot{\mathbf{u}}))\cdot(\nabla\vartheta_\Omega, \boldsymbol{\varepsilon}(\dot{\mathbf{u}})), \end{aligned}\tag{4.1.8}$$

where $\partial\Phi_\Omega$ denotes the subdifferential of Φ_Ω. Actually, since Φ_Ω is convex and differentiable, it is just its gradient. Since $\Phi_\Omega(0,0) = 0$ and the absolute temperature is positive, we conclude that the right-hand side of (4.1.8) is nonnegative and, therefore, the model is thermodynamically consistent, because the expression within the parenthesis on the left-hand side is nonnegative, as required by the Second Law of Thermodynamics.

Next, we note that with this choice the heat flux is

$$\mathbf{q} = -k\nabla\vartheta_\Omega.$$

Also,

$$\boldsymbol{\sigma}^r = \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}) - \vartheta_\Omega\mathcal{M}, \qquad \boldsymbol{\sigma}^{ir} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}),$$

and the constitutive law for the thermoviscoelastic material is

$$\boldsymbol{\sigma} = \mathcal{B}\varepsilon(\mathbf{u}) + \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) - \vartheta_\Omega\mathcal{M}.$$

Now, using all of the above in (4.1.4) and (4.1.5) yields

$$c_p\dot{\vartheta}_\Omega - \operatorname{div}(k\nabla\vartheta_\Omega) = -\vartheta_\Omega\mathcal{M}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \varepsilon(\dot{\mathbf{u}})\cdot\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + r_\Omega, \tag{4.1.9}$$

$$\rho\ddot{\mathbf{u}} - \operatorname{Div}\left(\mathcal{B}\varepsilon(\mathbf{u}) + \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) - \vartheta_\Omega\mathcal{M}\right) = \mathbf{f}_B. \tag{4.1.10}$$

When $\mathcal{M} = \alpha\mathbf{I}_d$ the first term on the right-hand side of (4.1.9) simplifies and reads $-\alpha\vartheta_\Omega\operatorname{div}\dot{\mathbf{u}}$.

We note that most papers on thermoviscoelastic problems employ linearization of the constitutive law, and neglect the dissipation. We recall that ϑ_Ω is the absolute temperature. If we let Θ_{ref} represent a reference temperature, such as the ambient temperature, or the temperature of the reference configuration, usually the relative temperature $\theta \equiv \vartheta_\Omega - \Theta_{ref}$ is used. Then, ϑ_Ω is replaced with θ on the left-hand side of the energy equation (4.1.9), and the linearization consists of replacing ϑ_Ω with Θ_{ref} in the first term on the right-hand side, and omitting the second term. The latter term represents viscous dissipation, is quadratic in the strain rates, and causes substantial mathematical difficulties. However, recently in [184] both terms in (4.1.9) were retained in their original form, and the existence of the unique weak solution established for the homogeneous Dirichlet boundary condition for the displacements, and the Neumann condition for the temperature.

In cases when thermal effects are not important, it is enough to set $\vartheta_\Omega = \Theta_{ref} = const.$ above; to neglect the heat equation (4.1.9), and the thermal expansion term in (4.1.10), and then the constitutive law reduces then to a generalized form of the Kelvin-Voigt law for anisotropic materials, (2.3.2),

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{B}\varepsilon(\mathbf{u}).$$

When viscous effect are neglected, too, one obtains

$$\boldsymbol{\sigma} = \mathcal{B}\varepsilon(\mathbf{u}),$$

which represents an anisotropic elastic material (2.3.1). The modifications to Ψ_Ω and Φ_Ω in such cases are obvious.

To complete the system (4.1.9) and (4.1.10) into an initial-boundary value problem we need to prescribe initial and boundary conditions. We assume that we have, in addition, a pair $(y, \nabla y)$ and the associated generalized forces $(Y, \mathbf{Z})$. We shall use the notation $\mathbf{u}(t)$ and $\vartheta_\Omega(t)$ as a shortcut for the functions

$$\mathbf{u}(t) = \mathbf{u}(\mathbf{x}, t), \quad \vartheta_\Omega(t) = \vartheta_\Omega(\mathbf{x}, t), \qquad \mathbf{x} \in \Omega,$$

respectively. The precise meaning of this statement, in terms of membership in a function space, will be described in Sect. 6.1.

The initial conditions for the system are

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad \vartheta_\Omega(0) = \vartheta_{\Omega 0}, \quad y(0) = y_0,$$

where, $\mathbf{u}_0, \mathbf{v}_0$, $\vartheta_{\Omega 0}$ and y_0 are known functions, and it is assumed that the equation (4.1.3) for y is of the first order in time, otherwise one has to modify the initial condition appropriately.

Now, on Γ_D we prescribe a Dirichlet condition for the displacements, the internal variable and for the temperature, say,

$$\mathbf{u} = \mathbf{u}_D, \quad \vartheta_\Omega = \vartheta_{\Omega D}, \quad y = y_D.$$

On Γ_N we prescribe Neumann conditions,

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N, \quad -k\frac{\partial \vartheta_\Omega}{\partial n} = h(\vartheta_\Omega - \vartheta_{amb}), \quad -\lambda_y \mathbf{Z}\cdot\mathbf{n} = f_{yN},$$

where h is the coefficient of heat exchange, and ϑ_{amb} is the ambient temperature, which may be different from Θ_{ref}. Also, f_{yN} is the surface 'force' related to y.

Other combinations of boundary conditions are possible, but we shall not dwell upon them here, except for mentioning that below ϑ_Ω is prescribed on $\Gamma_D \cup \Gamma_N$.

Our interest lies in the processes of friction and adhesion on the contact surface Γ_C. We assume that the contact surface represents a fictitious thin boundary layer, the interface, which is assigned thermodynamic properties that are different from those of the parent material. This accurately reflects the fact that engineering surface have characteristics that are often markedly different from those of the bulk materials.

We introduce the surface Helmholtz potential Ψ_C, the surface dissipation pseudo-potential Φ_C, and we let e_C and s_C be the surface energy and entropy densities, per unit area, respectively. Moreover, we assign it its own absolute temperature ϑ_C. Then, as in (4.1.1), we have $\Psi_C = e_C - s_C\vartheta_C$, and $s_C = -\partial\Psi_C/\partial\vartheta_C$, as in (4.1.2).

For the sake of simplicity, we omit the subscript C, but, below, we retain the subscript Ω for the traces or values on Γ_C of quantities belonging to the parent material.

We follow the same ideas and steps as above. First, we rewrite $\mathbf{y}^r$, which contains the reversible components of the vector of internal variables, and $\mathbf{y}^{ir}$ the irreversible components. To add adhesion, we introduce, following [65, 97, 158], the surface variable β, the fractional density of active bonds.

The adhesion process was described in Sect. 3.3, and we recall that $\beta \in [0, 1]$ to retain its interpretation as a fraction. When $\beta = 1$ all the bonds are active, when $\beta = 0$ they are all broken, and when $0 < \beta < 1$ it measures the fraction of active bonds. We associate with β the thermodynamic force Y_β. Then $Y_\beta\dot{\beta}$ is the rate of energy or power needed to break the bonds, causing debonding on the surface. It is possible to introduce the reversible and irreversible parts of the adhesion process, as is indicated below.

The wear process was described in Sect. 3.2. As a result of the contact surface debris are formed and the surface topography changes. We associate with wear function w, which measures the debris depth, the thermodynamic force Y_w, and the rate of energy loss due to wear is $Y_w\dot{w}$. We note that wear is an irreversible process. However, as was shown in [97], a derivation of the wear condition is somewhat more complicated, it uses the notion of the dual of the dissipation pseudo-potential, and will not be pursued here any further.

Next, we assume that the normal displacement u_n is reversible, while in the tangential direction the displacement can be decomposed as $\mathbf{u}_\tau =$

$\mathbf{u}_\tau^r + \mathbf{u}_\tau^{ir}$, where $\mathbf{u}_\tau^r$ is the reversible part of the displacement and $\mathbf{u}_\tau^{ir}$ is the irreversible part.

Now, the surface energy potential and dissipation pseudo-potential are

$$\Psi = \Psi(u_n, \mathbf{u}_\tau^r, \beta, \vartheta), \qquad \Phi = \Phi(\mathbf{u}_\tau^{ir}, \dot{\mathbf{u}}_\tau^{ir}, \dot{\beta}, \nabla\vartheta).$$

For the sake of generality we assume that the dissipation includes the gradient of ϑ on the boundary.

As above,

$$\sigma_n = \frac{\partial \Psi}{\partial u_n}, \qquad \boldsymbol{\sigma}_\tau^r = \frac{\partial \Psi}{\partial \mathbf{u}_\tau^r}, \qquad Y_\beta^r = \frac{\partial \Psi}{\partial \beta},$$

are the associated reversible thermodynamic forces. The irreversible or dissipative ones are given by

$$\boldsymbol{\sigma}_\tau^{ir} = \theta \frac{\partial \Phi}{\partial \mathbf{u}_\tau^{ir}}, \qquad Y_\beta^{ir} = \theta \frac{\partial \Phi}{\partial \dot{\beta}},$$

and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}_\tau^r + \boldsymbol{\sigma}_\tau^{ir}$ and $Y_\beta = Y_\beta^r + Y_\beta^{ir}$.

Moreover, we allow for heat conduction on the surface and the heat flux vector $\mathbf{q}$ is given by

$$\mathbf{q} = -\vartheta \frac{\partial \Phi}{\partial \nabla\vartheta}.$$

The virtual power or energy rate equation on the surface Γ_C is

$$\dot{e} + \operatorname{div} \mathbf{q} = \sigma_n \dot{u}_n + \boldsymbol{\sigma}_\tau \cdot \dot{\mathbf{u}}_\tau + q_\Omega, \tag{4.1.11}$$

where q_Ω is the limiting value or trace of the body heat flux on Γ_C, i.e., the heat flux from the body into the (fictitious) surface.

At this stage we do not consider the momentum equation on the boundary. This may be of interest when a rigorous asymptotic derivation, using a thin surface layer and passing to the limit, is attempted. The issue remains an interesting open problem; however, some progress in special cases has been reported in [185].

Since $\Psi = e - s\vartheta$, and $\dot{s} \geq q_\Omega/\vartheta_\Omega$ it follows from (4.1.11) that

$$\dot{\Psi} \leq \sigma_n \dot{u}_n + \boldsymbol{\sigma}_\tau \cdot \dot{\mathbf{u}}_\tau - s\dot{\vartheta} - \mathbf{q} \cdot \frac{\nabla\vartheta}{\vartheta} + \frac{q_\Omega}{\vartheta_\Omega}(\vartheta_\Omega - \vartheta),$$

where the last term on the right-hand side describes the heat exchange between the surface, with temperature ϑ, and the body, with temperature on the surface (the trace) ϑ_Ω. By using the dissipation pseudo-potential we guarantee that the dissipation condition holds true.

To obtain specific models we need to specify Ψ and Φ. Once these are given, the rest follows from the procedure above.

We provide, next, three detailed examples of such choices and the models they yield.

4.2 Isothermal Unilateral Contact with Friction and Adhesion

In the first example we follow [158] and derive a model for the isothermal frictional contact with irreversible adhesion. Here, the absolute temperature ϑ is constant and by scaling the variables we may set it as $\vartheta = 1$, and it will not be mentioned any further in this subsection.

First, we assume that the foundation is completely rigid. Let $I_{(-\infty,0]}$ and $I_{[0,1]}$ be the indicator functions of the sets $(-\infty, 0]$ and $[0, 1]$, respectively. These functions have the value zero when the argument is within the set, and ∞ when it is not. Adding them to the energy potential makes it energetically 'too expensive' for the system to violate the constraints they represent. Since the foundation is rigid we have the constraint $u_n - g \leq 0$, where g is the gap function (see Fig. 5).

The free energy is chosen as

$$\begin{aligned} \Psi(u_n, \mathbf{u}_\tau, \beta) = \frac{1}{2}\lambda_n(u_n - g)^2\beta^2 + \frac{1}{2}\lambda_\tau\|\mathbf{u}_\tau\|^2\beta^2 - e_{Du}\beta \\ +I_{(-\infty,0]}(u_n - g) + I_{[0,1]}(\beta). \end{aligned} \tag{4.2.1}$$

Here, λ_n and λ_τ are the normal and tangential stiffness coefficients, assumed to be positive constants. Mechanically, the normal and tangential stiffnesses are $\lambda_n\beta^2$ and $\lambda_\tau\beta^2$, respectively. The so-called Dupré surface energy e_{Du} measures the amount of energy needed to debond a unit of surface area. Finally, the term $I_{(-\infty,0]}(u_n - g)$ enforces the condition $u_n - g \leq 0$, and $I_{[0,1]}(\beta)$ enforces the constraints $0 \leq \beta \leq 1$. These will be explained in more detail below.

Clearly, $\Psi(0, 0, 0) = 0$, $\Psi \geq 0$, and Ψ is convex in each one of the variables, separately.

The dissipation pseudo-potential is chosen as

$$\Phi(\dot{\mathbf{u}}_\tau, \dot\beta) = \mu|\sigma_n - \lambda_n(u_n - g)\beta^2|\,\|\dot{\mathbf{u}}_\tau\| + \frac{1}{2}\lambda_\beta\dot\beta^2 + I_{(-\infty,0]}(\dot\beta). \tag{4.2.2}$$

Here, μ is the coefficient of friction, assumed to be a positive constant and λ_β is the debonding rate constant (which is independent of β). The term $I_{(-\infty,0]}(\dot\beta)$ enforces the irreversibility of the adhesion process, since it means that $\dot\beta \leq 0$ and rebonding cannot happen.

We shall use the notation

$$\partial_f\Phi(f, \cdots)$$

for the subdifferential of Φ with respect to f, holding all other variables fixed, and similarly for Ψ.

Using Ψ and Φ we obtain the following conditions on the contact surface:

$$\boldsymbol{\sigma}_\tau^r \in \partial_{\mathbf{u}_\tau}\Psi = \lambda_\tau \beta^2 \mathbf{u}_\tau, \tag{4.2.3}$$

$$-\sigma_n^r \in \partial_{u_n}\Psi = \lambda_n (u_n - g)\beta^2 + \partial_{u_n} I_{(-\infty,0]}(u_n - g), \tag{4.2.4}$$

$$-Y_\beta^r \in \partial_\beta \Psi = (\lambda_n (u_n - g)^2 + \lambda_\tau \|\mathbf{u}_\tau\|^2)\beta - e_{Du} + \partial_\beta I_{[0,1]}(\beta), \tag{4.2.5}$$

$$\sigma_n^{ir} \in \partial_{\dot{u}_n}\Phi = 0, \tag{4.2.6}$$

$$\boldsymbol{\sigma}_\tau^{ir} \in \partial_{\dot{\mathbf{u}}_\tau}\Phi = \mu|\sigma_n - \lambda_n (u_n - g)\beta^2|\, \partial_{\dot{\mathbf{u}}_\tau} \|\dot{\mathbf{u}}_\tau\|, \tag{4.2.7}$$

$$Y_\beta^{ir} \in \partial_{\dot\beta}\Phi = \lambda_\beta \dot\beta + \partial_{\dot\beta} I_{(-\infty,0]}(\dot\beta). \tag{4.2.8}$$

Here and below, we use the notion of a subdifferential explained in Sect. 6.3. In particular, the subdifferential of the indicator function I_K, (6.3.4), is given by (6.3.5).

Before we proceed and for the sake of completeness, we present the explicit forms of the subdifferentials in (4.2.3)–(4.2.8). We begin with

$$\partial I_{[0,1]}(s) = \begin{cases} (-\infty, 0] & \text{if} \quad s = 0, \\ 0 & \text{if} \quad 0 < s < 1, \\ [0, \infty) & \text{if} \quad s = 1, \\ \emptyset & \text{otherwise.} \end{cases} \tag{4.2.9}$$

The subdifferential is depicted in Fig. 6 (thick lines).

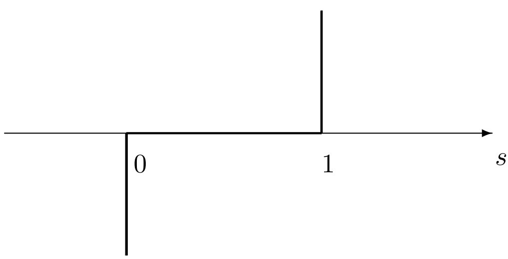

Fig. 6. The subdifferential $\partial I_{[0,1]}(s)$

The way this subdifferential is used in (4.2.5) is as follows (see also Sect. 3.3). Let $-Y_\beta \in \partial I_{[0,1]}(\beta)$. If $0 < \beta < 1$, then it follows from (4.2.9) that $Y_\beta = 0$, and the subdifferential does not contribute to the condition. If $\beta = 0$ then $-Y_\beta \in (-\infty, 0]$, so that $0 \le Y_\beta$, and the generalized force is acting exactly to prevent β from becoming negative. If $\beta = 1$ then $-Y_\beta \in [0, \infty)$, so that $Y_\beta \le 0$, and the generalized force is acting to prevent β from exceeding $\beta = 1$.

Next,

$$\partial I_{(-\infty,0]}(s) = \begin{cases} 0 & \text{if} \quad -\infty < s < 0, \\ [0, \infty) & \text{if} \quad s = 0, \\ \emptyset & \text{otherwise.} \end{cases} \tag{4.2.10}$$

The subdifferential is depicted in Fig. 7 (thick lines).

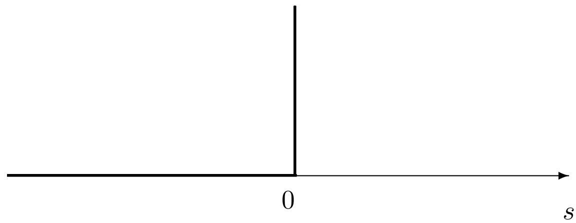

Fig. 7. The subdifferential $\partial I_{(-\infty,0]}(s)$

We use it in (4.2.4) to enforce the nonpenetration condition $u_n - g \leq 0$. Let $-\sigma_n \in \partial I_{(-\infty,0]}(u_n - g)$. It follows from (4.2.10) that $\sigma_n = 0$ when $u_n - g < 0$. But, if $u_n - g = 0$, then $-\sigma_n \in [0, \infty)$, thus $\sigma_n \leq 0$ and it has the exact value that prevents interpenetration, i.e., $g < u_n$.

Also,

$$\partial|s| = \begin{cases} -1 & \text{if} \quad -\infty < s < 0, \\ [-1, 1] & \text{if} \quad s = 0, \\ 1 & \text{if} \quad 0 < s < \infty. \end{cases} \tag{4.2.11}$$

The subdifferential is depicted in Fig. 8 (thick lines).

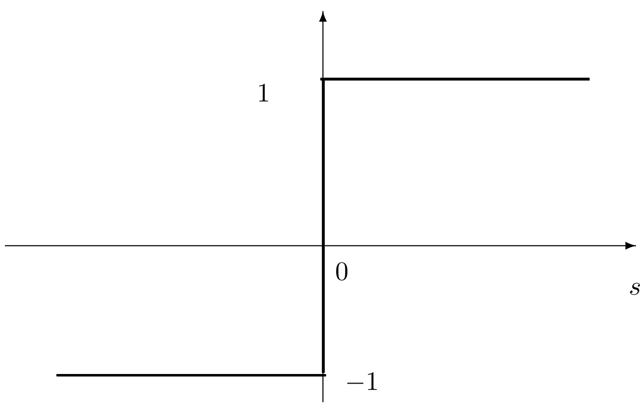

Fig. 8. The subdifferential $\partial|s|$

Finally, let $B_1 = \{\mathbf{s} \in \mathbb{R}^d : \|\mathbf{s}\| \leq 1\}$ be the unit ball in $\mathbb{R}^d$, then for a vector $\mathbf{s}$ we have,

$$\partial\|\mathbf{s}\| = \begin{cases} B_1 & \text{if} \quad \|\mathbf{s}\| = 0, \\ \dfrac{\mathbf{s}}{\|\mathbf{s}\|}, & \text{if} \quad 0 < \|\mathbf{s}\| < \infty. \end{cases} \tag{4.2.12}$$

That is, the subdifferential of the function $f(\mathbf{s}) = \|\mathbf{s}\|$ is the unit vector in the direction of $\mathbf{s}$ when $\|\mathbf{s}\| \neq 0$, and is the whole unit ball B_1 when $\|\mathbf{s}\| = 0$.

We note now that $\sigma_n^r = \sigma_n$, since it is assumed reversible and $\sigma_n^{ir} = 0$; $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}_\tau^r + \boldsymbol{\sigma}_\tau^{ir}$. Following [158] we assume that there are no sources f_β, and since Ψ does not depend on $\nabla\beta$ then $\mathbf{Z}_\beta = 0$, and it follows now from (4.1.3) that $Y_\beta = 0$. Using these facts and the expressions for the subdifferentials we may rewrite ths system (4.2.3)–(4.2.8) in the following form.

The Signorini condition with adhesion in the normal direction is

$$u_n - g \leq 0, \quad -\sigma_n - \lambda_n(u_n - g)\beta^2 \leq 0, \quad (\sigma_n - \lambda_n(u_n - g)\beta^2)(u_n - g) = 0,$$

which is condition (2.6.3) when adhesion is absent, i.e., $\lambda_n = 0$. Note that with adhesion, when there is no contact $(u_n < g)$ the adhesive normal traction is $\sigma_n = \lambda_n(g - u_n)\beta^2$, and the normal stiffness is $\lambda_n\beta^2$.

In the tangential direction we find the friction condition with adhesion,

$$\begin{aligned}
\boldsymbol{\sigma}_\tau^r &= \lambda_\tau \mathbf{u}_\tau (u_n - g)\beta^2, \\
\|\boldsymbol{\sigma}_\tau^{ir}\| &= \|\boldsymbol{\sigma}_\tau - \lambda_\tau \mathbf{u}_\tau (u_n - g)\beta^2\| \leq \mu|\sigma_n - \lambda_n(u_n - g)\beta^2|, \\
\dot{\mathbf{u}}_\tau \neq \mathbf{0} \;\Rightarrow\; \boldsymbol{\sigma}_\tau^{ir} &= -\mu|\sigma_n - \lambda_n(u_n - g)\beta^2| \, \frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \,.
\end{aligned}$$

This condition reduces to the usual Coulomb law, (2.6.6), when there is no adhesion, $\lambda_n = \lambda_\tau = 0$ and $H = \mu|\sigma_n|$.

Next, we obtain the equation of evolution for the adhesion field,

$$\lambda_\beta \dot{\beta} \in e_{Du} - (\lambda_n(u_n - g)^2 + \lambda_\tau\|\mathbf{u}_\tau\|^2)\beta - \partial I_{[0,1]}(\beta) - \partial I_{(-\infty,0]}(\dot{\beta}),$$

which may be written as

$$\begin{aligned}
&\dot{\beta} \leq 0, \\
&\lambda_\beta \dot{\beta} = -\big(e_{Du} - (\lambda_n(u_n - g)^2 + \lambda_\tau\|\mathbf{u}_\tau\|^2)\beta\big)_- \,, \qquad \text{if } \; 0 \leq \beta < 1, \\
&\lambda_\beta \dot{\beta} \geq -\big(e_{Du} - (\lambda_n(u_n - g)^2 + \lambda_\tau\|\mathbf{u}_\tau\|^2)\beta\big)_- \,, \qquad \text{if } \; \beta = 1.
\end{aligned}$$

We note that debonding starts only when the energy term $(\lambda_n(u_n - g)^2 + \lambda_\tau\|\mathbf{u}_\tau\|^2)\beta$ is larger than the Dupré energy e_{Du}, and the process is irreversible, as $\dot{\beta} \leq 0$.

The frictionless condition with adhesion is obtained from the above when one sets $\mu = 0$.

4.3 Isothermal Contact with Normal Compliance, Friction and Adhesion

If we wish to study the frictional contact problem with adhesion between a viscoelastic body and a reactive foundation, modelled with the normal compliance condition (2.6.2), instead of the Signorini condition, we need to modify both Ψ and Φ as follows. Here too, since the absolute temperature ϑ is constant we may scale it as $\vartheta = 1$.

Let $P_n(r)$ be a smooth, nonnegative and nondecreasing function such that $P_n(r) = 0$ when $r \le 0$, and let $p_n = P_n'$ (the prime denotes the derivative), and so $p_n \ge 0$, and $p_n(r) = 0$ when $r \le 0$. Then, we replace the indicator function $I_{(-\infty,0]}(u_n - g)$ in (4.2.1) with $P_n(u_n - g)$. As an example, one may choose $P_n(r) = (1/2)\lambda_{nc}(r_+)^2$, and we note that the Signorini condition is obtained, formally, in the limit $\lambda_{nc} \to \infty$. Moreover, since the adhesive traction acts only when there is separation, and does not contribute when there is interpenetration, we use $(u_n - g)_- = \max\{0, g - u_n\}$. Thus,

$$\begin{aligned}\Psi_{nc}(u_n, \mathbf{u}_\tau, \beta) = \frac{1}{2}\lambda_n((u_n - g)_-)^2\beta^2 + \frac{1}{2}\lambda_\tau\|\mathbf{u}_\tau\|^2\beta^2 - e_{Du}\beta \\ +P_n(u_n - g) + I_{[0,1]}(\beta).\end{aligned}$$

The modified dissipation pseudo-potential, (4.2.2), is

$$\Phi_{nc}(\dot{\mathbf{u}}_\tau, \dot\beta) = \mu p_n(u_n - g)\,\|\dot{\mathbf{u}}_\tau\| + \frac{1}{2}\lambda_\beta\dot\beta^2 + I_{(-\infty,0]}(\dot\beta).$$

Then, following the procedure above we obtain

$$\begin{aligned}-\sigma_n^r \in \partial_{u_n}\Psi_{nc} = \lambda_n((u_n - g)_-)\,\beta^2 + p_n(u_n - g),\\ \boldsymbol{\sigma}_\tau^{ir} \in \partial_{\dot{\mathbf{u}}_\tau}\Phi_{nc} = \mu p_n(u_n - g)\partial_{\dot{\mathbf{u}}_\tau}\|\dot{\mathbf{u}}_\tau\|.\end{aligned}$$

While the other variables remain the same as in (4.2.3)–(4.2.8). Therefore, the normal compliance with adhesion is described by,

$$-\sigma_n = p_n(u_n - g) + \lambda_n((u_n - g)_-)\beta^2,$$

and the friction condition is,

$$\begin{aligned}\boldsymbol{\sigma}_\tau^r &= \lambda_\tau\mathbf{u}_\tau((u_n - g)_-)\beta^2,\\ \|\boldsymbol{\sigma}_\tau^{ir}\| &= \|\boldsymbol{\sigma}_\tau - \lambda_\tau\mathbf{u}_\tau((u_n - g)_-)\beta^2\| \le \mu p_n(u_n - g),\\ \dot{\mathbf{u}}_\tau \ne \mathbf{0} \;\Rightarrow\; \boldsymbol{\sigma}_\tau^{ir} &= -\mu p_n(u_n - g)\,\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|}\,.\end{aligned}$$

We recall that $\sigma_n^r = \sigma_n$, since $\sigma_n^{ir} = 0$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}_\tau^r + \boldsymbol{\sigma}_\tau^{ir}$.

We note that in this model the adhesive contributes only to the reversible part of the tangential traction.

4.4 Thermoviscoelastic Material with Damage

We follow [65] and derive the constitutive relations and the evolution equations for a thermoviscoelastic material with damage. We assume that the damage process is reversible or self-mending, depends on the damage gradient, and the damage field satisfies $0 \le \zeta \le 1$.

We choose the Helmholtz energy as

$$\Psi_\Omega(\vartheta_\Omega, \boldsymbol{\varepsilon}(\mathbf{u}), \zeta, \nabla\zeta) = -c_p\vartheta_\Omega \log(\vartheta_\Omega) + \frac{1}{2}\zeta\boldsymbol{\varepsilon}(\mathbf{u})\cdot\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}) + \bar{w}(1-\zeta) \\ -\zeta\vartheta_\Omega\mathcal{M}\boldsymbol{\varepsilon}(\mathbf{u}) + \frac{1}{2}k_{Dam}(\nabla\zeta)^2 + I_{[0,1]}(\zeta). \quad (4.4.1)$$

Here, $\bar{w}$ is the damage threshold energy, k_{Dam} is the damage diffusion coefficient, both assumed to be positive constants, $\mathcal{M} = (m_{ij})$ is the tensor of coefficients of thermal expansion, and $I_{[0,1]}(\zeta)$ enforces the condition $0 \le \zeta \le 1$. We note that the elastic strain energy and the thermal expansion term are multiplied by the damage function.

The dissipation pseudo-potential is chosen as

$$\Phi_\Omega(\nabla\vartheta_\Omega, \boldsymbol{\varepsilon}(\dot{\mathbf{u}}), \dot{\zeta}) = \frac{k}{2\vartheta_\Omega^2}\|\nabla\vartheta_\Omega\|^2 + \frac{1}{2\vartheta_\Omega}\zeta\,\boldsymbol{\varepsilon}(\dot{\mathbf{u}})\cdot\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \frac{c_\zeta}{2\vartheta_\Omega}(\dot{\zeta})^2. \quad (4.4.2)$$

Here, $k > 0$ is the coefficient of heat conduction; $\mathcal{A} = (a_{ijkl})$ is the viscosity tensor; and c_ζ is the damage rate coefficient.

If one deals with an irreversible damage process, the term $I_{(-\infty,0]}(\dot{\zeta})$ has to be added to the right hand-side of (4.4.2) to enforce the condition $\dot{\zeta} \le 0$.

For the sake of generality, we multiplied the elastic, the strain-rate, and the thermal expansion terms with ζ, thus assuming that all three decrease as the damage grows (i.e., ζ decreases). It is seen that we have a choice and can multiply only one of them, or none. This should be dictated by the particular system one is modelling. The changes to the model for the different choices are easy to obtain from the model that follows.

Following the formalism, we obtain:

$$\boldsymbol{\sigma}^r = \partial_{\varepsilon(\mathbf{u})}\Psi = \zeta\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}) - \zeta\vartheta_\Omega\mathcal{M}, \quad (4.4.3)$$

$$\boldsymbol{\sigma}^{ir} = \vartheta_\Omega\partial_{\boldsymbol{\varepsilon}(\dot{\mathbf{u}})}\Phi = \zeta\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}), \quad (4.4.4)$$

$$Y^r = \partial_\zeta\Psi = \frac{1}{2}\boldsymbol{\varepsilon}(\mathbf{u})\cdot\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}) - \vartheta_\Omega\mathcal{M}\boldsymbol{\varepsilon}(\mathbf{u}) - \bar{w}, \quad (4.4.5)$$

$$\tilde{Y} \in \partial I_{[0,1]}(\zeta), \quad (4.4.6)$$

$$Y^{ir} = \vartheta_\Omega\partial_{\dot{\zeta}}\Phi = c_\zeta\dot{\zeta}, \quad (4.4.7)$$

$$\mathbf{Z}^r = \partial_{\nabla\zeta}\Psi = k_{Dam}\nabla\zeta, \quad (4.4.8)$$

$$\mathbf{Z}^{ir} = \partial_{\nabla\dot{\zeta}}\Phi = 0. \quad (4.4.9)$$

From these we obtain the constitutive relations

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^r + \boldsymbol{\sigma}^{ir} = \zeta\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}) + \zeta\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) - \zeta\vartheta_\Omega\mathcal{M}, \quad (4.4.10)$$

$$Y = Y^r + Y^{ir} + Y^{ir}$$

$$\in \frac{1}{2}\,\boldsymbol{\varepsilon}(\mathbf{u})\cdot\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}) - \vartheta_\Omega\mathcal{M}\boldsymbol{\varepsilon}(\mathbf{u}) - \bar{w} + c_\zeta\dot{\zeta} + \partial I_{[0,1]}(\zeta), \qquad (4.4.11)$$

$$\mathbf{Z} = \mathbf{Z}^r + \mathbf{Z}^{ir} = k_{Dam}\nabla\zeta. \qquad (4.4.12)$$

We see that in addition to the usual thermoviscoelastic constitutive relation (but with ζ), we also have one for the damage 'force' Y and one for the 'damage diffusion' $\mathbf{Z}$.

The fact that ζ multiplies the right-hand side of (4.4.10) and is restricted to the values in the interval $[0,1]$ allows us to call it a damage variable.

If we choose to have ζ multiply only the elastic term in Ψ and not appear in Φ, then we would obtain, instead, the following constitutive relation,

$$\boldsymbol{\sigma} = \zeta\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}) + \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) - \vartheta_\Omega\mathcal{M}.$$

Next, the equations of evolution are as follows. The equations of motion are obtained from (4.1.5),

$$\rho\ddot{\mathbf{u}} - \mathrm{Div}\,(\zeta\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}) + \zeta\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) - \zeta\vartheta_\Omega\mathcal{M}) = \mathbf{f}_B. \qquad (4.4.13)$$

The energy equation follows from (4.1.4),

$$c_p\dot{\vartheta}_\Omega - \mathrm{div}\,(k\nabla\vartheta_\Omega) = r_\Omega + \zeta\,\boldsymbol{\varepsilon}(\dot{\mathbf{u}})\cdot\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) - \zeta\vartheta_\Omega\mathcal{M}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + c_\zeta(\dot{\zeta})^2. \qquad (4.4.14)$$

Finally, (4.1.3) yields the rate equation for the damage field,

$$c_\zeta\dot{\zeta} - \mathrm{div}\,(k_{Dam}\nabla\zeta) \in f_\zeta + \bar{w} - \frac{1}{2}\,\boldsymbol{\varepsilon}(\mathbf{u})\cdot\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}) + \vartheta_\Omega\mathcal{M}\boldsymbol{\varepsilon}(\mathbf{u}) - \partial I_{[0,1]}(\zeta). \qquad (4.4.15)$$

We note that the dependence of Ψ on the damage gradient leads to a parabolic equation for ζ, while omitting it ($k_{Dam} = 0$) leads to an ordinary differential equation.

The model consists of a coupled nonlinear system of equations with a hyperbolic system for the displacements, a parabolic equation for the temperature and a parabolic inclusion for the damage field. To complete the model we need to supply the initial and boundary conditions, as explained above.

The model is new, and there are, yet, no mathematical results on its well-posedness.

If we wish to add contact to the model, we need to introduce the surface Helmholtz potential and a dissipation pseudo-potential, and derive the relevant conditions and equations in a similar manner.

4.5 Short Summary

The formalism is very elegant, and when it can be applied the models are thermodynamically correct or consistent. Moreover, the use of indicator functions in the Helmholtz potential and in the dissipation pseudo-potential makes it easy to incorporate various constraints into the models.

However, it has certain shortcomings, some of which we mention briefly.

(i) A model for the frictional contact with wear was constructed in [97] using the dual of the dissipation pseudo-potential. We did not pursue it here, since the formalism above does not lead to an Archard type of wear condition (3.2.1). We refer the reader to [97] and references therein.

(ii) In the derivation above the friction coefficient μ was assumed to be constant. On the other hand, as has been explained in Sect. 2.7, in many engineering applications μ is a rather complicated function of the surface speed $\|\dot{\mathbf{u}}_\tau\|$, of the temperature ϑ, possibly the wear w, and of other surface characteristics. By using the formalism as presented, but with a nonconstant μ in Φ, derivatives of μ would appear in the expression (4.2.7) for $\boldsymbol{\sigma}_\tau^{ir}$, which currently are not known to be there.

We also note that the choice of the Helmholtz energy function and the dissipation pseudo-potential has to made in each application of the theory.

This formalism has a considerable flexibility in what can be achieved using it, and will be used to derive other contact and boundary conditions. This direction of research seems to have the potential to lead to many interesting models for processes taking place on the surface.

5 A Detailed Representative Problem

In the previous three chapters various constitutive laws for the behaviour of the material and different contact conditions were described in some detail. In this chapter we take the next step and assemble, in full detail, the relevant equations and conditions for one particular and representative problem into a mathematical model. then we analyze the model in detail. The sections in Part II of this monograph will follow the same format.

This chapter is meant for those readers who are not fluent in the Theory of Variational Inequalities, and is intended to serve as a bridge between the models, the variational formulations of the problems, and their analysis. It may be skipped by those acquainted and familiar with such a mathematical approach, since the problem is described in Sect. 8.3 but following the usual way, and without the additional details.

We describe the classical formulation of the problem with detailed explanations in Sect. 5.1. Then we explain the need for a weak or variational formulation and derive such a formulation in Sect. 5.2. Also, we describe the relevant properties of functions in some of the function spaces that are needed in the formulation. Finally, in Sect. 5.3, we present an existence and uniqueness result for the solution of the variational form of the model.

The arguments we detail in this chapter may be used in the study of all the contact problems presented in Part II. The pattern is very similar for each problem: First, the classical formulation of the model is described, which amounts to choosing the constitutive law and the contact conditions. Then, a variational formulation of the problem is derived by performing, formally, integration by parts and using the equations and the conditions. Next, a statement of the existence and uniqueness results, under appropriate assumptions on the problem data, is provided. Finally, for some of the problems a sketch of the proof is described. For representative problems, the full proof is given but in a separate section. In this chapter we provide only the first three steps for the chosen problem.

Throughout this chapter we use dimensionless variables.

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http://www.springerlink.com/

5.1 Problem Statement

The problem we chose to detail describes a viscoelastic body, occupying the domain Ω, which is referred to as the reference configuration, that may come into frictional contact with a reactive stationary foundation over the part Γ_C of its surface. For the sake of simplicity, we assume in this chapter that $d = 3$, so the body is three-dimensional and the contact surface is two-dimensional. The material is assumed viscoelastic (2.3.2), and contact is modelled with the normal compliance condition (2.6.2) and the associated friction law (2.6.6) and (2.6.8).

As a result of applied volume forces and surface tractions the mechanical state of the system evolves over the time interval $[0, T]$, where $T > 0$. To describe this evolution we need to find at least one vector function $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t))$, which describes the displacement at time t of a particle that has in the reference configuration the position $\mathbf{x} = (x_1, x_2, x_3)$; and at least one stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, t) = (\sigma_{ij}(\mathbf{x}, t))$, at time t and position $\mathbf{x}$.

It is customary to use the notation $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^3$, which means that to each pair $(\mathbf{x}, t)$, with $\mathbf{x} \in \Omega$ and $t \in [0, T]$, the function assigns a vector in $\mathbb{R}^3$. Next, let $\mathbb{S}^3$ denote the vector space of 3×3 symmetric tensors or, equivalently, the vector space of 3×3 symmetric matrices; then, we write $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^3$, which means that to each pair $(\mathbf{x}, t)$, with $\mathbf{x} \in \Omega$ and $t \in [0, T]$, the function assigns a symmetric matrix in $\mathbb{S}^3$, the stresses at the point. For simplicity, in what follows we do not indicate explicitly the dependence of various functions on $\mathbf{x}$ and t. We use " $\cdot$ " to denote the scalar product on the spaces $\mathbb{R}^3$ and $\mathbb{S}^3$ and $\|\cdot\|$ will represent the associated norm on these spaces. Also, the subscripts n and τ will represent normal and tangential components of vectors and tensors, respectively, and a dot above a variable denotes its derivative with respect to time.

To describe the material response, the relationship between strains and stresses, a viscoelastic constitutive relation is assumed, given by

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u})$$

in $\Omega_T \equiv \Omega \times (0, T)$. When the viscosity and elasticity functions are linear, this may be written in components as

$$\sigma_{ij} = a_{ijkl}\dot{u}_{k,l} + b_{ijkl}u_{k,l},$$

where $i, j, k, l = 1, 2, 3$, $\mathcal{A}_{ve} = (a_{ijkl})$ and $\mathcal{B}_{ve} = (b_{ijkl})$, and as is the custom, the dependence of the coefficients on the position $\mathbf{x}$, if the material is nonhomogeneous, is suppressed. An index that follows a comma indicates partial derivative with respect to the corresponding spatial variable, and summation over an index repeated twice is implied. When the viscosity or elasticity functions are nonlinear, we have

$$\sigma_{ij} = (\mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}))_{ij} + (\mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}))_{ij}.$$

The quasistatic equation of motion, or more appropriately, the equations of equilibrium, when a force $\mathbf{f}_B = \mathbf{f}_B(\mathbf{x}, t) = (f_{B1}(\mathbf{x}, t), f_{B2}(\mathbf{x}, t), (f_{B3}(\mathbf{x}, t))$ is applied, are

$$\mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0},$$

and, in components, we have three partial differential equations,

$$\sigma_{ij,j}(\mathbf{x}, t) + f_{Bi}(\mathbf{x}, t) = 0,$$

for $i = 1, 2, 3$. These equations are valid in Ω_T; thus, at each time instant $t \in (0, T)$ the system is in a state of equilibrium at each point $\mathbf{x} \in \Omega$.

Next, we need to describe what happens on the boundary. We assume that Γ, the boundary of Ω, is partitioned into three disjoint surfaces Γ_D, Γ_N, and Γ_C. The body is assumed to be held fixed on the part Γ_D of the surface, so

$$\mathbf{u} = \mathbf{0}$$

on $\Gamma_D \times (0, T)$. On the part Γ_N a prescribed surface force or traction

$$\mathbf{f}_N = \mathbf{f}_N((\mathbf{x}, t) = (f_{N1}(\mathbf{x}, t), f_{N2}(\mathbf{x}, t), (f_{N3}(\mathbf{x}, t))$$

is applied, thus, we have the condition

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N$$

on $\Gamma_N \times (0, T)$, which written in components is $\sigma_{ij} n_j = f_{Ni}$. Here, $\mathbf{n} = (n_1, n_2, n_3)$ denotes the outer unit normal to Γ and $\boldsymbol{\sigma}\mathbf{n}$ represents the stress vector, that is the product of the matrix $\boldsymbol{\sigma} = (\sigma_{ij})$ and the vector $\mathbf{n} = (n_i)$.

The potential contact surface is Γ_C where contact may take place. Within a small displacements or small strains theory contact can take place only on a prescribed part of the surface. This is not the case for large deformations or displacements where the body may rotate and contact may be established on different parts of Γ, which are unknown beforehand, and are a part of the solution.

The contact in the normal direction is assumed to satisfy the normal compliance condition ((2.6.2) in Sect. 2.6), so the contact pressure $-\sigma_n$ is related to the interpenetration of surface asperities by

$$-\sigma_n = p_n(u_n - g),$$

on $\Gamma_C \times (0, T)$, where $g = g(\mathbf{x})$ is the gap between the body and the foundation, measured along the normal in the reference configuration. The current gap at the point $\mathbf{x}$ at time t is $g(\mathbf{x}) - u_n(\mathbf{x}, t)$, when it is nonnegative. When it is negative, then $u_n(\mathbf{x}, t) - g(\mathbf{x})$ measures the asperities interpenetration. The function p_n has to vanish when its argument is negative, since then the current gap is positive and there is no contact. The requirements that p_n has to satisfy will be spelled out below, and it will be seen that a very general family of functions with very 'mild' restrictions may be used.

In the tangential direction we use the associated law of dry friction (2.6.6) and (2.6.8), thus

$$\|\boldsymbol{\sigma}_\tau\| \le p_\tau(u_n - g),$$

on $\Gamma_C \times (0,T)$, where p_τ is the friction bound function which the tangential shear cannot exceed. Its properties will be described below. When sliding takes place the shear is opposite to the direction of the velocity, thus

$$\boldsymbol{\sigma}_\tau = -p_\tau(\mathbf{u}_\tau - g)\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if} \quad \dot{\mathbf{u}}_\tau \neq \mathbf{0}.$$

Finally, although the problem deals with an equilibrium equation, we need to prescribe an initial condition because of the velocity terms in the constitutive law and, hence, in the equation, and in the friction condition. Thus, we let

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}),$$

in Ω, where $\mathbf{u}_0$ is the prescribed initial displacement field.

Collecting all the equations and the conditions yields the following formulation of the mechanical problem, which we denote by Problem P_{ve-nc}. Typically, the notation for the various problems in the sequel involves some indication of the type of the conditions used, and here it is 'viscoelastic' (ve) material and 'normal compliance' (nc) contact condition.

Problem P_{ve-nc}. *Find a displacements field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^3$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^3$, *such that*

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{in } \Omega_T, \tag{5.1.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \qquad \text{in } \Omega_T, \tag{5.1.2}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_D \times (0,T), \tag{5.1.3}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \qquad \text{on } \Gamma_N \times (0,T), \tag{5.1.4}$$

$$\left.\begin{aligned} &-\sigma_n = p_n(u_n - g), \\ &\|\boldsymbol{\sigma}_\tau\| \le p_\tau(u_n - g), \\ &\boldsymbol{\sigma}_\tau = -p_\tau(u_n - g)\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if} \quad \dot{\mathbf{u}}_\tau \neq \mathbf{0} \end{aligned}\right\} \text{on } \Gamma_C \times (0,T), \tag{5.1.5}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \text{in } \Omega. \tag{5.1.6}$$

This is the *classical formulation* of the problem, and by this we mean that the unknowns and the data are smooth functions such that all the derivatives and all the conditions are satisfied in the usual sense at each point and at each time instance. However, the friction condition introduces a mathematical difficulty since it is 'nondifferentiable,' and belongs to the conditions dealt with in the part of mechanics called 'nonsmooth mechanics.' Indeed, in general, the problem will not have any 'classical solutions,' i.e., solutions which have all the necessary classical derivatives, and some of the conditions will be satisfied in a weaker sense that has to be made precise. Moreover, the friction condition and the differentiability of the normal compliance function impose a ceiling on the regularity or smoothness of the solutions, even if all the problem data are very smooth. This is in contrast with the usual smooth problems where more regular data lead to more regular solutions.

5.2 Variational Formulation

To allow for the conditions to be satisfied in a 'weaker sense,' we need to reformulate the problem in the so-called 'variational form,' which we proceed to do. But first, we note that the variational formulation is not only a mathematical necessity, but also very useful practically since it leads directly to the finite element approximations for the problem.

We turn to the variational formulation of problem (5.1.1) – (5.1.6).

For the purpose of the derivation, we assume that all the functions to be used are as smooth as is needed for the various mathematical operations to be justified, and so the derivation is formal. We shall return to this point once we obtain the formulation. Assume that $\mathbf{u}$ and $\boldsymbol{\sigma}$ solve the problem, and let $\mathbf{v} = (v_1, v_2, v_3)$ be a *test function*, i.e., an arbitrary function which is smooth and such that $\mathbf{v} = \mathbf{0}$ on Γ_D. The latter is sometimes called an *essential boundary condition*, and it is imposed on the solutions and all test functions.

Let $t \in [0, T]$ be fixed. We multiply equation (5.1.2) with $\mathbf{v} - \dot{\mathbf{u}}$ and integrate over Ω. We use the Gauss divergence theorem, and note that $\sigma_{ij}(v_i - \dot{u}_i)_{,j} = \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}})$ in Ω, and $\sigma_{ij} n_j = \boldsymbol{\sigma}\mathbf{n}$ on Γ, thus,

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}})\, dx = \int_\Gamma \boldsymbol{\sigma}\mathbf{n} \cdot (\mathbf{v} - \dot{\mathbf{u}})\, dS + \int_\Omega \mathbf{f}_B \cdot (\mathbf{v} - \dot{\mathbf{u}})\, dx. \tag{5.2.1}$$

Here, dx denotes a volume element (usually written as $dV = dx_1 dx_2 dx_3$) and dS a surface element.

We now deal with the boundary term on the right-hand side of (5.2.1). We split the boundary integral into integrals over Γ_D, Γ_N and Γ_C and obtain

$$\begin{aligned}\int_\Gamma \boldsymbol{\sigma}\mathbf{n} \cdot (\mathbf{v} - \dot{\mathbf{u}})\, dS = &\int_{\Gamma_D} \boldsymbol{\sigma}\mathbf{n} \cdot (\mathbf{v} - \dot{\mathbf{u}})\, dS + \int_{\Gamma_N} \boldsymbol{\sigma}\mathbf{n} \cdot (\mathbf{v} - \dot{\mathbf{u}})\, dS \\ &+ \int_{\Gamma_C} \boldsymbol{\sigma}\mathbf{n} \cdot (\mathbf{v} - \dot{\mathbf{u}})\, dS.\end{aligned} \tag{5.2.2}$$

Both $\dot{\mathbf{u}}$ and $\mathbf{v}$ vanish on Γ_D, so

$$\int_{\Gamma_D} \boldsymbol{\sigma}\mathbf{n} \cdot (\mathbf{v} - \dot{\mathbf{u}})\, dS = 0. \tag{5.2.3}$$

On Γ_N we have (5.1.4), thus $\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N$, and so

$$\int_{\Gamma_N} \boldsymbol{\sigma}\mathbf{n} \cdot (\mathbf{v} - \dot{\mathbf{u}})\, dS = \int_{\Gamma_N} \mathbf{f}_N \cdot (\mathbf{v} - \dot{\mathbf{u}})\, dS. \tag{5.2.4}$$

Next, we turn to the integral over Γ_C. We decompose the vectors and tensors into their normal and tangential components as follows

$$\boldsymbol{\sigma}\mathbf{n} \cdot (\mathbf{v} - \dot{\mathbf{u}}) = \sigma_n(v_n - \dot{u}_n) + \boldsymbol{\sigma}_\tau \cdot (\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau),$$

and, therefore,

$$\int_{\Gamma_C} \boldsymbol{\sigma}\mathbf{n} \cdot (\mathbf{v} - \dot{\mathbf{u}})\, dS = \int_{\Gamma_C} \sigma_n(v_n - \dot{u}_n)\, dS + \int_{\Gamma_C} \boldsymbol{\sigma}_\tau \cdot (\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau)\, dS. \tag{5.2.5}$$

By using the first part in (5.1.5) we obtain

$$\int_{\Gamma_C} \sigma_n(v_n - \dot{u}_n)\, dS = -\int_{\Gamma_C} p_n(u_n - g)(v_n - \dot{u}_n)\, dS. \tag{5.2.6}$$

Finally, it follows from the second part in (5.1.5) that

$$\begin{aligned} \boldsymbol{\sigma}_\tau \cdot \mathbf{v}_\tau &\geq -\|\boldsymbol{\sigma}_\tau\|\,\|\mathbf{v}_\tau\| \geq -p_\tau(u_n - g)\|\mathbf{v}_\tau\|, \\ -\boldsymbol{\sigma}_\tau \cdot \dot{\mathbf{u}}_\tau &= \|\boldsymbol{\sigma}_\tau\|\,\|\dot{\mathbf{u}}_\tau\| = p_\tau(u_n - g)\|\dot{\mathbf{u}}_\tau\|, \end{aligned}$$

and thus,

$$-\int_{\Gamma_C} \boldsymbol{\sigma}_\tau \cdot (\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau)\, dS \leq \int_{\Gamma_C} p_\tau(u_n - g)(\|\mathbf{v}_\tau\| - \|\dot{\mathbf{u}}_\tau\|)\, dS. \tag{5.2.7}$$

We combine (5.2.1)–(5.2.7), and after a rearrangement find that any solution $(\mathbf{u}, \boldsymbol{\sigma})$ of the original problem has to satisfy the inequality

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}})\, dx &+ \int_{\Gamma_C} p_n(u_n - g)(v_n - \dot{u}_n))\, dS \\ &+ \int_{\Gamma_C} p_\tau(u_n - g)(\|\mathbf{v}_\tau\| - \|\dot{\mathbf{u}}_\tau\|)\, dS \\ &\geq \int_\Omega \mathbf{f}_B \cdot (\mathbf{v} - \dot{\mathbf{u}})\, dx + \int_{\Gamma_N} \mathbf{f}_N \cdot (\mathbf{v} - \dot{\mathbf{u}})\, dS. \end{aligned} \tag{5.2.8}$$

We use now (5.1.1) in (5.2.8) to obtain

$$\begin{aligned} \int_\Omega (\mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u})) \cdot \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}})\, dx &+ \int_{\Gamma_C} p_n(u_n - g)(v_n - \dot{u}_n))\, dS \\ &+ \int_{\Gamma_C} p_\tau(u_n - g)(\|\mathbf{v}_\tau\| - \|\dot{\mathbf{u}}_\tau\|)\, dS \\ &\geq \int_\Omega \mathbf{f}_B \cdot (\mathbf{v} - \dot{\mathbf{u}})\, dx + \int_{\Gamma_N} \mathbf{f}_N \cdot (\mathbf{v} - \dot{\mathbf{u}})\, dS, \end{aligned} \tag{5.2.9}$$

which holds for every test function $\mathbf{v}$ and at each time instant $t \in [0, T]$.

Keeping in mind (5.1.3) and (5.1.6), we obtain the following *variational formulation* of the problem, in terms of the displacements.

Problem P^v_{ve-nc}. *Find a displacement function* $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^3$ *such that* (5.2.9), (5.1.3) *and* (5.1.6) *hold.*

Typically, the notation for a variational formulation of a contact problem presented in this book involves the superscript V. Here, however, we use the superscript v since P^v_{ve-nc} is only an intermediate problem, and will be written in a more elegant way as problem P^V_{ve-nc} below.

The main difficulty at this point is this: it is not clear that a function $\mathbf{u}$ which satisfies this variational formulation has any relationship whatsoever with a solution of problem (5.1.1) – (5.1.6). To address this concern we establish the relationship and show that if $\mathbf{u}$ satisfies the variational formulation and is sufficiently smooth, then is solves problem (5.1.1) – (5.1.6). The catch is in the phrase 'sufficiently smooth,' which the solution, typically, will not be. However, this justifies calling such a function $\mathbf{u}$ a 'weak solution' of problem (5.1.1) – (5.1.6).

To that end, let $t \in [0, T]$ be fixed, assume that $\mathbf{u}$ satisfies the variational formulation and is sufficiently smooth, let $\boldsymbol{\varphi}$ be a smooth function which vanishes on Γ and let $\mathbf{v} = \dot{\mathbf{u}} + \boldsymbol{\varphi}$. It follows that $\mathbf{v}$ is a smooth function which vanishes on Γ_D, and so we may use it as a test function; we denote by $\boldsymbol{\sigma}$ the stress field given by (5.1.1) and we use the fact that the boundary integrals in (5.2.9) vanish, thus,

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\, dx \geq \int_\Omega \mathbf{f}_B \cdot \boldsymbol{\varphi}\, dx.$$

Similarly, if we use $\mathbf{v} = \dot{\mathbf{u}} - \boldsymbol{\varphi}$ as a test function in (5.2.9), we obtain

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\, dx \leq \int_\Omega \mathbf{f}_B \cdot \boldsymbol{\varphi}\, dx.$$

These two inequalities show that $\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\, dx = \int_\Omega \mathbf{f}_B \cdot \boldsymbol{\varphi}\, dx$, and by using the Gauss divergence theorem we obtain

$$\int_\Omega (\mathrm{Div}\, \boldsymbol{\sigma} + \mathbf{f}_B) \cdot \boldsymbol{\varphi}\, dx = 0. \tag{5.2.10}$$

Since $\boldsymbol{\varphi}$ is an arbitrary function which vanishes on Γ, it can be chosen to be nonzero only in an arbitrary small neighborhood of each point in Ω, which implies that the integrand vanishes in Ω. Indeed, assume that there exists $\mathbf{x}_0 \in \Omega$ such that $\mathrm{Div}\, \boldsymbol{\sigma}(\mathbf{x}_0, t) + \mathbf{f}_B(\mathbf{x}_0, t) \neq \mathbf{0}$. Then, because of the assumed continuity of the functions, there exists an open neighborhood $\mathcal{U}_0$ of $\mathbf{x}_0$ in Ω such that

$$\mathrm{Div}\, \boldsymbol{\sigma}(\mathbf{x}, t) + \mathbf{f}_B(\mathbf{x}, t) \neq \mathbf{0} \qquad \text{for } \ \mathbf{x} \in \mathcal{U}_0. \tag{5.2.11}$$

Now, we may choose $\boldsymbol{\varphi}$ such that $\boldsymbol{\varphi}(\mathbf{x}) = \phi(\mathbf{x})(\mathrm{Div}\,\boldsymbol{\sigma}(\mathbf{x},t) + \mathbf{f}_B(\mathbf{x},t))$ where ϕ is a smooth real-valued function such that

$$\phi(\mathbf{x}) > 0 \quad \text{for } \mathbf{x} \in \mathcal{U}_0, \quad \phi(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \Omega - \mathcal{U}_0. \tag{5.2.12}$$

Now, if f is a smooth function and $f > 0$ on $\mathcal{U}_0$ then, necessarily, $\int_{\mathcal{U}_0} f\,dx > 0$, and thus it follows from (5.2.11) and (5.2.12) that

$$\int_\Omega (\mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{f}_B) \cdot \boldsymbol{\varphi}\,dx = \int_{\mathcal{U}_0} \phi\,\|\mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{f}_B\|^2\,dx > 0,$$

which contradicts (5.2.10). We conclude that

$$\mathrm{Div}\,\boldsymbol{\sigma}(\mathbf{x},t) + \mathbf{f}_B(\mathbf{x},t) = \mathbf{0} \quad \text{for } \mathbf{x} \in \Omega,$$

and, therefore, we recover the equilibrium equations (5.1.2).

Next, we choose an arbitrary smooth function $\boldsymbol{\varphi}$ which vanishes on $\Gamma_D \cup \Gamma_C$ and use in (5.2.9) the test functions $\mathbf{v} = \dot{\mathbf{u}} \pm \boldsymbol{\varphi}$. Then, the integral terms on Γ_C vanish and using (5.1.1) we obtain

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\,dx = \int_\Omega \mathbf{f}_B \cdot \boldsymbol{\varphi}\,dx + \int_{\Gamma_N} \mathbf{f}_N \cdot \boldsymbol{\varphi}\,dS.$$

We use again the Gauss divergence theorem and (5.1.2), and after some manipulations find that

$$\int_{\Gamma_N} (\boldsymbol{\sigma}\mathbf{n} - \mathbf{f}_N) \cdot \boldsymbol{\varphi}\,dS = 0. \tag{5.2.13}$$

Now, for each point of Γ_N, the function $\boldsymbol{\varphi}$ can be chosen to be nonzero only in a small neighborhood of the point, and arguing as above we find that the integrand must vanish. Thus, if $\boldsymbol{\sigma}\mathbf{n}$ is sufficiently smooth we recover the boundary condition (5.1.4); and if it is not, then condition (5.2.13) provides the precise sense in which the boundary condition is satisfied in the variational formulation.

By choosing a smooth test function $\boldsymbol{\varphi}$ that vanishes on $\Gamma_D \cup \Gamma_N$ and its tangential component $\boldsymbol{\varphi}_\tau$ vanishes on Γ_C, and using similar arguments as those leading to (5.2.13), we obtain

$$\int_{\Gamma_C} (\sigma_n + p_n(u_n - g))\varphi_n\,dS = 0.$$

Since φ_n is arbitrary, the integrand must vanish on Γ_C and we recover the first part of (5.1.5) when σ_n is continuous, and when it is not, it provides the precise meaning in which it holds.

We are left with the tangential part on Γ_C, which deals with the friction condition. Let $\mathbf{v}$ be a smooth test function that vanishes on Γ_D, and let $\boldsymbol{\sigma}$

denote the stress field given by (5.1.1). We use, again, the Gauss divergence theorem and obtain

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\dot{\mathbf{u}})\,dx+\int_\Omega \operatorname{Div}\boldsymbol{\sigma}\cdot(\mathbf{v}-\dot{\mathbf{u}})\,dx=\int_\Gamma \boldsymbol{\sigma}\mathbf{n}\cdot(\mathbf{v}-\dot{\mathbf{u}})\,dS.$$

Now, using (5.1.2)– (5.1.4) and (5.2.2), yields

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\dot{\mathbf{u}})\,dx=\int_{\Gamma_N}\mathbf{f}_N\cdot(\mathbf{v}-\dot{\mathbf{u}})\,dS+\int_{\Gamma_C}\boldsymbol{\sigma}\mathbf{n}\cdot(\mathbf{v}-\dot{\mathbf{u}})\,dS+\int_\Omega \mathbf{f}_B\cdot(\mathbf{v}-\dot{\mathbf{u}})\,dx.$$

Next, we use (5.2.5) and the first part of (5.1.5), and it follows from this equality that

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\dot{\mathbf{u}})\,dx=\int_{\Gamma_N}\mathbf{f}_N\cdot(\mathbf{v}-\dot{\mathbf{u}})\,dS-\int_{\Gamma_C}p_n(u_n-g)(v_n-\dot{u}_n)\,dS$$
$$+\int_{\Gamma_C}\boldsymbol{\sigma}_\tau\cdot(\mathbf{v}_\tau-\dot{\mathbf{u}}_\tau)\,dS+\int_\Omega \mathbf{f}_B\cdot(\mathbf{v}-\dot{\mathbf{u}})\,dx. \quad (5.2.14)$$

On the other hand, substituting (5.1.1) in (5.2.9) we obtain (5.2.8) and, by using (5.2.8) and (5.2.14), we deduce that

$$\int_{\Gamma_C}\left(\boldsymbol{\sigma}_\tau\cdot(\mathbf{v}_\tau-\dot{\mathbf{u}}_\tau)+p_\tau(u_n-g)(\|\mathbf{v}_\tau\|-\|\dot{\mathbf{u}}_\tau\|)\right)\,dS\geq 0. \quad (5.2.15)$$

Since $\mathbf{u}$ satisfies (5.1.3), it follows that $\dot{\mathbf{u}}=\mathbf{0}$ on Γ_D and, therefore, $\mathbf{v}=2\dot{\mathbf{u}}$ may be used as a test function, and so we substitute it in (5.2.15) and obtain

$$\int_{\Gamma_C}\left(\boldsymbol{\sigma}_\tau\cdot\dot{\mathbf{u}}_\tau+p_\tau(u_n-g)\|\dot{\mathbf{u}}_\tau\|\right)\,dS\geq 0.$$

We also use $\mathbf{v}=\mathbf{0}$ as a test function in (5.2.15) and find

$$\int_{\Gamma_C}\left(\boldsymbol{\sigma}_\tau\cdot\dot{\mathbf{u}}_\tau+p_\tau(u_n-g)\|\dot{\mathbf{u}}_\tau\|\right)\,dS\leq 0.$$

We conclude from the two inequalities that

$$\int_{\Gamma_C}\left(\boldsymbol{\sigma}_\tau\cdot\dot{\mathbf{u}}_\tau+p_\tau(u_n-g)\|\dot{\mathbf{u}}_\tau\|\right)\,dS=0, \quad (5.2.16)$$

and using (5.2.16) in (5.2.15) yields

$$\int_{\Gamma_C}\left(\boldsymbol{\sigma}_\tau\cdot\mathbf{v}_\tau+p_\tau(u_n-g)\|\mathbf{v}_\tau\|\right)\,dS\geq 0, \quad (5.2.17)$$

which holds for all possible test functions $\mathbf{v}$.

We use this inequality to show that $\|\boldsymbol{\sigma}_\tau\|\leq p_\tau$. Suppose that there exists $\mathbf{x}_0\in\Gamma_C$ such that

$$\|\boldsymbol{\sigma}_\tau(\mathbf{x}_0,t)\| > p_\tau(u_n(\mathbf{x}_0,t) - g(\mathbf{x}_0,t)).$$

Since we deal with smooth functions and p_τ is nonnegative, there exists a neighborhood $\mathcal{U}_0$ of $\mathbf{x}_0$ in Γ_C such that

$$\|\boldsymbol{\sigma}_\tau(\mathbf{x},t)\| > p_\tau(u_n(\mathbf{x},t) - g(\mathbf{x},t)) \geq 0 \quad \text{for } \mathbf{x} \in \mathcal{U}_0. \tag{5.2.18}$$

We can choose the test function as $\mathbf{v}(\mathbf{x}) = -\phi(\mathbf{x})\boldsymbol{\sigma}_\tau(\mathbf{x},t)$ on Γ_C, where ϕ is a smooth function such that

$$\phi(\mathbf{x}) > 0 \quad \text{for } \mathbf{x} \in \mathcal{U}_0; \quad \phi(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \Gamma_C - \mathcal{U}_0. \tag{5.2.19}$$

Now, if f is a smooth function and $f < 0$ on $\mathcal{U}_0$ then $\int_{\mathcal{U}_0} f\,dS < 0$, therefore, it follows from (5.2.18) and (5.2.19) that

$$\int_{\Gamma_C} (\boldsymbol{\sigma}_\tau \cdot \mathbf{v}_\tau + p_\tau(u_n - g)\|\mathbf{v}_\tau\|)\,dS = \int_{\mathcal{U}_0} \left(-\phi\|\boldsymbol{\sigma}_\tau\|^2 + p_\tau(u_n - g)\phi\|\boldsymbol{\sigma}_\tau\|\right)\,dS$$

$$= \int_{\mathcal{U}_0} \phi\|\boldsymbol{\sigma}_\tau\|(p_\tau(u_n - g) - \|\boldsymbol{\sigma}_\tau\|)\,dS < 0,$$

which contradicts (5.2.17). We conclude that

$$\|\boldsymbol{\sigma}_\tau\| \leq p_\tau(u_n - g) \quad \text{on} \quad \Gamma_C. \tag{5.2.20}$$

Next, let α denote the angle between the vectors $\dot{\mathbf{u}}_\tau$ and $\boldsymbol{\sigma}_\tau$. Since $\cos\alpha \geq -1$, we have

$$\int_{\Gamma_C} (\boldsymbol{\sigma}_\tau \cdot \dot{\mathbf{u}}_\tau + p_\tau(u_n - g)\|\dot{\mathbf{u}}_\tau\|)\,dS = \int_{\Gamma_C} \|\dot{\mathbf{u}}_\tau\|(\|\boldsymbol{\sigma}_\tau\|\cos\alpha + p_\tau(u_n - g))\,dS$$

$$\geq \int_{\Gamma_C} \|\dot{\mathbf{u}}_\tau\|(p_\tau(u_n - g) - \|\boldsymbol{\sigma}_\tau\|)\,dS,$$

and by using (5.2.16) we obtain

$$\int_{\Gamma_C} \|\dot{\mathbf{u}}_\tau\|(\|\boldsymbol{\sigma}_\tau\|\cos\alpha + p_\tau(u_n - g))\,dS = 0, \tag{5.2.21}$$

and also

$$\int_{\Gamma_C} \|\dot{\mathbf{u}}_\tau\|(p_\tau(u_n - g) - \|\boldsymbol{\sigma}_\tau\|)\,dS \leq 0. \tag{5.2.22}$$

It follows from (5.2.20) and (5.2.22) that

$$\|\dot{\mathbf{u}}_\tau\|(p_\tau(u_n - g) - \|\boldsymbol{\sigma}_\tau\|) = 0. \tag{5.2.23}$$

Indeed, if f is a smooth function such that $f \geq 0$ on Γ_C and $\int_{\Gamma_C} f\,dS \leq 0$ then, necessarily, $f = 0$ on Γ_C, and we deduce that equality holds in (5.2.23). It follows now that

$$\text{if} \quad \dot{\mathbf{u}}_\tau \neq \mathbf{0} \quad \text{then} \quad \|\boldsymbol{\sigma}_\tau\| = p_\tau(u_n - g). \tag{5.2.24}$$

Using again the inequality $\cos\,\alpha \geq -1$ and (5.2.20) we have

$$\|\dot{\mathbf{u}}_\tau\|(\|\boldsymbol{\sigma}_\tau\|\cos\,\alpha + p_\tau(u_n - g)) \geq \|\dot{\mathbf{u}}_\tau\|\,(p_\tau(u_n - g) - \|\boldsymbol{\sigma}_\tau\|) \geq 0,$$

and by using arguments similar to those used to obtain (5.2.24), condition (5.2.21) implies

$$\|\dot{\mathbf{u}}_\tau\|(\|\boldsymbol{\sigma}_\tau\|\cos\alpha + p_\tau(u_n - g)) = 0. \tag{5.2.25}$$

We conclude from (5.2.25) that if $\dot{\mathbf{u}}_\tau \neq \mathbf{0}$ then $\|\boldsymbol{\sigma}_\tau\|\cos\alpha = -p_\tau(u_n - g)$, and it follows form (5.2.24) that $\cos\alpha = -1$, unless $\|\boldsymbol{\sigma}_\tau\| = p_\tau(u_n - g) = 0$. We conclude that if $\mathbf{x} \in \Gamma_C$ is a point where $\dot{\mathbf{u}}_\tau(\mathbf{x}, t) \neq \mathbf{0}$ and $p_\tau(u_n(\mathbf{x}, t) - g(\mathbf{x})) \neq \mathbf{0}$, then, since $\cos\alpha = -1$, the vectors $\dot{\mathbf{u}}_\tau$ and $\boldsymbol{\sigma}_\tau$ have opposite direction, thus,

$$\boldsymbol{\sigma}_\tau = -p_\tau(u_n - g)\,\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|}. \tag{5.2.26}$$

Let now $\mathbf{x} \in \Gamma_C$ be a point in which $\dot{\mathbf{u}}_\tau(\mathbf{x}, t) \neq \mathbf{0}$ and $p_\tau = \mathbf{0}$, then (5.2.24) implies that $\boldsymbol{\sigma}_\tau = \mathbf{0}$, and equality (5.2.26) still holds true. We conclude that

$$\text{if} \quad \dot{\mathbf{u}}_\tau \neq \mathbf{0} \quad \text{then} \quad \boldsymbol{\sigma}_\tau = -p_\tau(u_n - g)\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{on} \quad \Gamma_C. \tag{5.2.27}$$

The tangential part of the condition is now a consequence of relations (5.2.20) and (5.2.27). If $\boldsymbol{\sigma}_\tau$ is continuous on Γ_C then the conditions hold in the usual sense, otherwise, the precise manner in which they are satisfied is provided by (5.2.15).

We conclude that if the variational problem P^v_{ve-nc} has a sufficiently smooth solution for all of the operations above to be justified, it is also a solution of the original problem in the usual sense. Indeed, equality (5.1.1) is satisfied by the construction of the stress field; the equilibrium equation (5.1.2), the traction boundary condition (5.1.4) and the frictional contact conditions (5.1.5) were derived from the variational inequality (5.2.9), and, finally, the displacement boundary condition (5.1.3) and the initial boundary condition (5.1.6) are satisfied too, since they represent an integral part of the statement of Problem P^v_{ve-nc}. However, now we have a formulation, the so-called *variational formulation*, that may have solutions which do not have the necessary regularity or smoothness, and we still call them solutions of the problem, but now we write that such a function $\mathbf{u}$ is a *weak solution* of the original problem.

This is the heart of the matter, and shows why it is necessary to obtain and study variational formulations, since generally, these problems do not have solutions that satisfy the original problems in the usual sense. This also indicates that once the existence of a weak solution has been established, there is considerable interest to establish its optimal regularity, since if one can find a sufficiently well behaved weak solution, it is a classical or usual solution, as well.

We note in passing that even if a problem possess smooth or classical solutions, the variational formulation is usually the first step in its analysis, since many of the modern mathematical tools are better suited for such a formulation. Moreover, the variational formulation can often be employed directly in the finite element method for the problem's numerical approximations.

In the formulation of problem P^v_{ve-nc} the set, or the linear space in this case, where the test functions lie is not indicated clearly. To correct this a modification of the problem formulation is needed, and we proceed to rewrite problem P^v_{ve-nc} in a more precise and elegant way.

To this end we now describe some mathematical concepts which will also allow us to reformulate the problem in a form that allows for the use of certain abstract mathematical results. Here it is where the terms 'almost everywhere (a.e.),' 'measurable set,' 'measurable function,' 'Lebesgue integral,' 'Sobolev space,' 'weak derivative,' and the rest of the terminology of measure theory and functional analysis come in to make these concepts meaningful and workable.

We begin with the observation that the variational formulation P^v_{nc-ve} contains integrals over Ω and on its boundary Γ; these integrals involve the solution $\mathbf{u}$, its time derivative $\dot{\mathbf{u}}$ as well as the spatial derivatives in the strain tensor $\boldsymbol{\varepsilon}(\mathbf{u})$ and in the strain rate tensor $\boldsymbol{\varepsilon}(\dot{\mathbf{u}})$. These determine, to a large degree, the choice of the appropriate function spaces in which we shall seek the solutions as well as the appropriate assumptions on the problem data, in order for these integrals to makes sense, or to be 'well-defined.' These function spaces are presented in Sects. 6.1 and 6.2. However, below we will describe some of their features that are important here.

First, we employ the space $L^2(\Omega)^3$; where the '3' means that each function in this space has three scalar components, and the '2' means that that each component is square-integrable on Ω. We recall that a product of two square-integrable functions is an integrable function, but not necessarily square-integrable.

We also use the space $Q = L^2(\Omega)^{3\times 3}_s$ where 3×3 indicates that each function $\boldsymbol{\sigma} = (\sigma_{ij})$ in this space has nine scalar components arranged in a matrix form, s means that the matrix is symmetric $(\sigma_{ij} = \sigma_{ji})$, and the '2' means that all the components are square-integrable on Ω. The notation $L^2(\Gamma)^3$ and $L^2(\Gamma_N)^3$ has a similar meaning. We also use the Sobolev space $H^1(\Omega)^3$, where the '3' shows that we deal with vector functions with three scalar components. Each one of these components is square-integrable and has all the partial spatial derivatives (in the sense of the theory of distributions) of the first order, indicated by the '1,' which are square-integrable, too. That is, the components v_i as well as all the spatial derivatives $v_{i,j}$ of a vector function $\mathbf{v} = (v_1, v_2, v_3) \in H^1(\Omega)^3$ are square-integrable functions on Ω.

Next, we use these notions in the assumptions we impose on the problem data. In part, the assumptions are such as to make the formulation meaningful, and in part to allow for the use of abstract existence and uniqueness theorems. Then we shall return to the spaces of test functions. We note that a property holds a.e. in a set if it holds at all the points of the set, except possibly on a very small subset which doesn't really count (a set of zero Lebesgue measure). Also, a function $\mathbf{f}$ is said to be *Lipschitz continuous,* or Lipschitz for short, if there is a positive constant C_f such that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \le C_f \|\mathbf{x} - \mathbf{y}\|,$$

and it is said to be *strongly monotone* if there exists a positive constant m_f such that

$$(\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})) \cdot (\mathbf{x} - \mathbf{y}) \ge m_f \|\mathbf{x} - \mathbf{y}\|^2.$$

We begin with the constitutive law. We shall present the assumptions the way it will be done in Part II, and then explain the main points.

Assume that the viscosity operator $\mathcal{A}_{ve}$ and the elasticity operator $\mathcal{B}_{ve}$ satisfy the following conditions:

$$\left.\begin{array}{l}
\text{(a) } \mathcal{A}_{ve} : \Omega \times \mathbb{S}^3 \to \mathbb{S}^3. \\
\text{(b) There exists } \mathcal{L}_{\mathcal{A}} > 0 \text{ such that} \\
\qquad \|\mathcal{A}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \le \mathcal{L}_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_1 \in \mathbb{S}^3, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\
\qquad (\mathcal{A}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_1 \in \mathbb{S}^3, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(d) For any } \boldsymbol{\varepsilon} \in \mathbb{S}^3, \ x \mapsto \mathcal{A}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega. \\
\text{(e) The mapping } \mathbf{x} \mapsto \mathcal{A}_{ve}(\mathbf{x}, \mathbf{0}) \in Q.
\end{array}\right\} \quad (5.2.28)$$

$$\left.\begin{array}{l}
\text{(a) } \mathcal{B}_{ve} : \Omega \times \mathbb{S}^3 \to \mathbb{S}^3. \\
\text{(b) There exists an } \mathcal{L}_{\mathcal{B}} > 0 \text{ such that} \\
\qquad \|\mathcal{B}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \le \mathcal{L}_{\mathcal{B}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^3, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(c) For any } \boldsymbol{\varepsilon} \in \mathbb{S}^3, \ \mathbf{x} \mapsto \mathcal{B}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega. \\
\text{(d) The mapping } \mathbf{x} \mapsto \mathcal{B}_{ve}(\mathbf{x}, \mathbf{0}) \in Q.
\end{array}\right\} \quad (5.2.29)$$

Part (a) in each condition means that the operator is a symmetric matrix of functions. Part (b) means that each one is Lipschitz with respect to the second variable; this is a substantial assumption, it excludes terms with power greater than one, but is satisfied within linearized viscoelasticity, and is satisfied by truncated operators. It allows for the use of important results from the Theory of Variational Inequalities. Part (c) in (5.2.28) means that the viscosity operator $\mathcal{A}_{ve} = \mathcal{A}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}(\dot{\mathbf{u}}))$ is strongly monotone with respect to the strain rate $\boldsymbol{\varepsilon}(\dot{\mathbf{u}})$. This assumption is quite natural, and nonmonotone operators are more difficult to deal with. Conditions (5.2.28) (d),(e) and (5.2.29) (c), (d) are needed for mathematical reasons. Then, if we assume that the test functions belong to $H^1(\Omega)^3$, the first integral on the left-hand side of (5.2.9) makes sense. If we also require that the volume force $\mathbf{f}_B$ has square-integrable components, then, the first integral on the right-hand side of (5.2.9) makes sense, too.

Next, we need to discus the boundary terms. Now, here things get a bit complicated, since a function that is only square-integrable on Ω does not need to have any continuity properties, and as a matter of fact, if we change its values at a point or on a line or even on a surface, the volume integral does not change, and we are still dealing with the same 'function.' Therefore, a square-integrable function does not have meaningful values at any prescribed point, line or surface. How do we deal with such a situation? In particular, if a square-integrable function does not have definite values on a surface, how do we make sense of the surface integrals in (5.2.9)?

We now give an intuitive answer to these questions, and refer the reader to Chap. 6 for a summary and to any one of the books mentioned there for further details.

Let V be the function space (6.2.3) in which we shall look for solutions of the problem,

$$V = \{\mathbf{v} \in H^1(\Omega)^3 : \mathbf{v} = \mathbf{0} \ \text{ on } \ \Gamma_D\}.$$

It is a linear space and we use it as the space of test functions.

It turns out that although a square-integrable function does not have a meaningful value on the boundary, a function whose spatial derivatives are square-integrable too does have such meaningful values. The precise statement is that there exists an operator γ, called a *trace operator*, usually written as $\gamma : H^1(\Omega)^3 \to L^2(\Gamma)^3$, that assigns to each function $\mathbf{v} \in H^1(\Omega)^3$ a square-integrable function $\gamma\mathbf{v}$ on the boundary Γ. It is constructed in such a way that if $\mathbf{v}$ is a smooth function, its usual values on the boundary coincide with those of $\gamma\mathbf{v}$. Therefore, the statement $\mathbf{v} = \mathbf{0}$ on Γ_D is understood as $\gamma\mathbf{v} = \mathbf{0}$ on Γ_D. It is straightforward to see that any function that is smooth (is continuous and has continuous partial derivatives) on $\overline{\Omega}$, the closure of Ω, belongs to V if it vanishes on Γ_D. For a test function $\mathbf{v} \in V$ by v_n and $\mathbf{v}_\tau$ we denote the normal and tangential components of $\mathbf{v}$, respectively, in the sense of traces, i.e., $v_n = (\gamma v)_n = (\gamma\mathbf{v})_i n_i$, and $\mathbf{v}_\tau = (\gamma\mathbf{v})_\tau$.

We can now describe the assumptions on the boundary data.

Assume that the normal and tangential compliance functions p_e $(e = n, \tau)$ satisfy

$$\left.\begin{array}{l}
\text{(a) } p_e : \Gamma_C \times \mathbb{R} \to \mathbb{R}_+. \\
\text{(b) There exists } \mathcal{L}_e > 0 \text{ such that} \\
\qquad |p_e(\mathbf{x}, r_1) - p_e(\mathbf{x}, r_2)| \le \mathcal{L}_e\, |r_1 - r_2|, \\
\qquad\quad \forall\, r_1, r_2 \in \mathbb{R}, \ \text{a.e. } \mathbf{x} \in \Omega. \\
\text{(c) For each } r \in \mathbb{R},\ \mathbf{x} \mapsto p_e(\mathbf{x}, r) \text{ is measurable on } \Gamma_C. \\
\text{(d) The mapping } \mathbf{x} \mapsto p_e(\mathbf{x}, 0) \in L^2(\Gamma_C).
\end{array}\right\} \tag{5.2.30}$$

Part (a) means that for each $\mathbf{x} \in \Gamma_C$ and each $u_n - g$, the function $p_e(\mathbf{x}, u_n - g)$ is nonnegative. Part (b) means that both functions are Lipschitz with respect to the second variable, the normal displacement. Parts (c) and (d) are mathematical ornaments and mean that the dependence on $\mathbf{x}$ is not too wild.

If we wish to conform to the usual practice, we may write $p_\tau = \mu p_n$ or $p_\tau = \mu p_n(1 - \delta p_n)_+$, where μ is the friction coefficient (see page 22), and we have that if p_n satisfies the condition (5.2.30)(b) then p_τ also satisfies it, with $\mathcal{L}_\tau = \mu\,\mathcal{L}_n$, where $\mu = const$. So, the results below are valid for the boundary value problems associated with these choices of the normal and tangential compliance functions.

The gap function, which in many applications is likely to be a constant, is assumed, for the sake of generality, to satisfy

$$g \in L^2(\Gamma_C), \quad g \geq 0 \text{ a.e. on } \Gamma_C. \tag{5.2.31}$$

So it too has to be square-integrable, and it should be nonnegative to preserve its physical meaning. When dealing only with existence and uniqueness we do not need it smoother, however, if we wish to have better regularity of the solutions, we will have to impose a better smoothness on it.

Under the conditions (5.2.30) and (5.2.31) the integrals on Γ_C on the left-hand side of (5.2.9) make sense, for all $\dot{\mathbf{u}}, \mathbf{u}, \mathbf{v} \in V$.

Finally, we require that the traction $\mathbf{f}_N$ has square-integrable components, and then the last integral on the right-hand side of (5.2.9) makes sense, too.

We now return to the function spaces, and begin with the 'inner product' on V, which is the analogue of the dot product between vectors, and is defined by

$$(\mathbf{v}, \mathbf{w})_V = \frac{1}{4} \int_\Omega (v_{i,j} + v_{j,i})(w_{i,j} + w_{j,i})\,dx,$$

for any $\mathbf{v}, \mathbf{w} \in V$, where the integral is the Lebesgue integral, since the functions are only square-integrable. If both functions have continuous partial derivatives, then this integral is the usual one. The fact that $(\cdot,\cdot)_V$ is an inner product on V is related to the vanishing of the functions on Γ_D, and we skip the explanation, we just mention that it has all the properties one would expect from a dot product. In components,

$$\boldsymbol{\varepsilon}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\mathbf{w}) = \varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{w}) = \frac{1}{4}\,(v_{i,j} + v_{j,i})(w_{i,j} + w_{j,i}),$$

and thus, $(\mathbf{v}, \mathbf{w})_V = \int_\Omega \boldsymbol{\varepsilon}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\mathbf{w})\,dx$, which shows that

$$(\mathbf{v}, \mathbf{w})_V = (\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\varepsilon}(\mathbf{w}))_Q,$$

where $(\cdot,\cdot)_Q$ denotes the inner product on the space Q, and is given by $(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\,dx$. With the inner product $(\cdot,\cdot)_V$ we associate, on V, the norm

$$\|\mathbf{v}\|_V = (\mathbf{v}, \mathbf{v})_V^{1/2}.$$

The space V has the property that if $\{\mathbf{v}_k\}$ is a sequence in V such that $\|\mathbf{v}_k - \mathbf{v}_m\|_V \to 0$ as k, $m \to \infty$, then there exist an element $\mathbf{v} \in V$ such that $\|\mathbf{v}_k - \mathbf{v}\|_V \to 0$ as $k \to \infty$. In this case we say that the sequence $\{\mathbf{v}_k\}$ *converges in norm* to $\mathbf{v}$. A vector space with an inner product which satisfies this property is called a *Hilbert space*, and so V is a Hilbert space. This, in turn, allows us to use a host of results for Hilbert spaces.

We now proceed to rewrite the variational problem in an abstract way. This will allow the use of sophisticated results for variational inequalities. We begin by letting $\mathbf{F}$ denote the element of V which has the following property

$$(\mathbf{F}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_B(t) \cdot \mathbf{v}\, dx + \int_{\Gamma_N} \mathbf{f}_N(t) \cdot \mathbf{v}\, dS,$$

for all $\mathbf{v} \in V$ and $t \in [0, T]$. Thus, $\mathbf{F}(t)$ represents the combined action of the volume forces and surface tractions, in one function. The existence of $\mathbf{F}(t)$ is guaranteed by the Riesz Representation Theorem (see for instance [186, p. 64], since V is a Hilbert space. But there is no formula or a simple way to find $\mathbf{F}(t)$, we only know that it exists and that is sufficient for our purposes. Next, we introduce the *contact functional* $j : V \times V \to \mathbb{R}$,

$$j(\mathbf{v}, \mathbf{w}) = \int_{\Gamma_C} p_n(v_n - g)\, w_n\, dS + \int_{\Gamma_C} p_\tau(v_n - g)\, \|\mathbf{w}_\tau\|\, dS,$$

for all $\mathbf{v}, \mathbf{w} \in V$. It represents the combined normal and tangential compliances, and is related to the work, if $\mathbf{w}$ is a displacement, or the power, if $\mathbf{w}$ is a velocity, of the contact forces.

We use now the Hilbert space structure of V, discussed above, and the notation just introduced, and write problem P^v_{ve-nc} (page 71) in the following way.

Problem P^V_{ve-nc}. *Find a displacement function* $\mathbf{u} : [0, T] \to V$ *such that*

$$\begin{aligned}(\mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + (\mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q \\ + j(\mathbf{u}(t), \mathbf{v}) - j(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \\ \text{for all } \mathbf{v} \in V,\ t \in [0, T], \qquad (5.2.32)\end{aligned}$$

$$\mathbf{u}(0) = \mathbf{u}_0. \qquad (5.2.33)$$

Problem P^V_{ve-nc} represents the *variational formulation* of the mechanical problem P_{ve-nc}. It follows from the discussion above that if $(\mathbf{u}, \boldsymbol{\sigma})$ is a smooth solution of the contact problem P_{ve-nc} then $\mathbf{u}$ is a solution of the variational problem P^V_{ve-nc} and, conversely, if $\mathbf{u}$ is a smooth solution of the variational problem P^V_{ve-nc} and $\boldsymbol{\sigma}$ is defined by (5.1.1), then $(\mathbf{u}, \boldsymbol{\sigma})$ is a solution to the contact problem P_{ve-nc}. However, generally, any solution of Problem P^V_{ve-nc} does not satisfy all the conditions of the problem in the usual sense.

Finally, we remark that the variational formulation P^V_{ve-nc} can be directly discretized by using the finite element method, and, thus, it provides the basis for various numerical schemes for its numerical approximations.

5.3 An Existence and Uniqueness Result

We now present an existence and uniqueness result for problem P^V_{ve-nc}. To this end we have to specify precisely the regularity in time of the forces $\mathbf{f}_B$ and tractions $\mathbf{f}_N$. But first, a short explanation. We denote by $C([0,T];X)$ the linear space of continuous functions from $[0,T]$ to X. Below, we also need $C^1([0,T];X)$, which is the linear space of continuously differentiable functions from $[0,T]$ to X. A function $f:[0,T]\to X$ belongs to the space $C^1([0,T];X)$ if $f\in C([0,T];X)$, f is differentiable at each time instant $t\in[0,T]$ and its time derivative, denoted by $\dot{f}$, belongs to $C([0,T];X)$, as well.

Now, we assume that the force and traction densities satisfy

$$\mathbf{f}_B \in C([0,T];L^2(\Omega)^3), \qquad \mathbf{f}_N \in C([0,T];L^2(\Gamma_N)^3). \tag{5.3.1}$$

Finally, we assume that the initial displacements fulfill

$$\mathbf{u}_0 \in V. \tag{5.3.2}$$

The statement of the existence and uniqueness of the solution for Problem P^V_{ve-nc} is as follows.

Theorem 5.3.1. *Assume that* (5.2.28)–(5.2.31), (5.3.1) *and* (5.3.2) *hold. Then Problem* P^V_{ve-nc} *has a unique solution* $\mathbf{u}\in C^1([0,T];V)$.

A proof of the theorem that is based on results for elliptic variational inequalities and fixed point arguments can be found in [20]. In Sect. 8.4 we provide a different proof, based on an abstract result for evolutionary variational inequalities.

For each time instant $t\in[0,T]$, the solution $\mathbf{u}(t)$ is a function that belongs to the space V, where we were seeking it in the first place, it is continuous in time, it has a derivative in time $\dot{\mathbf{u}}(t)$, and this derivative belongs to V, too. However, being in V is insufficient for all the the conditions of the classical formulation to be satisfied in the usual sense. We say that a pair of functions $(\mathbf{u},\boldsymbol{\sigma})$, which satisfies (5.1.1), (5.2.32) and (5.2.33), is a *weak solution* of problem P_{ve-nc}. We conclude that the mechanical problem P_{ve-nc} has a unique weak solution. The regularity or the smoothness of the solution is an open question at this time.

We end this chapter with the remark that in Part II, for each one of the problems we follow a similar pattern as in this chapter. First, the classical formulation of the problem is presented, then the variational formulation, the assumptions on the problem data are described carefully, and the existence and uniqueness result stated. For some of the problems we also provide a sketch of the proof, indicating its main steps. In each chapter we also provide a detailed proof of one of the theorems, showing the methods of proof. Each of these proofs is different and deals with somewhat different aspects or settings of the problem, and needs different considerations.

6 Mathematical Preliminaries

Part II provides mathematical formulations of the models, the assumptions that underlie them, their weak or variational formulations and the statements of the results. Each chapter also describes representative proofs which show the mathematical methods used, and indicate the types of problems that can be analyzed using them. These may be skipped on first reading.

Preliminary mathematical material and notations used in Part II are given in the first chapter. It is assumed that the reader has some basic knowledge in functional analysis. This is mainly needed for understanding the variational formulation of the problems, and the statements on the intrinsic regularity or smoothness of the solutions. Indeed, the regularity of a function is described by its membership in a function space. Better spaces mean improved regularity and smoother functions.

A partial list of books and monographs, where mathematical notations, definitions and methods have been applied to quasistatic contact problems includes [5, 47, 48, 50, 51, 66, 69, 71, 85, 187–193]. Although long, the list is by no means exhaustive. As can be seen from the list the subject has received considerable attention recently.

In Sect. 6.1 we introduce the notation used throughout this part of the book. The various function spaces and their subspaces or subsets, where the solutions of the problems are to be found, are described in Sect. 6.2. The Projection Theorem, the notion of the subdifferential, fixed-point theorems and solvability results for variational inequalities are recalled in Sect. 6.3. Finally, a description of the operators used to model the constitutive relations can be found in Sect. 6.4, where the assumptions imposed on them are detailed.

6.1 Notation

Throughout the book $\mathbb{R}$ stands for the set of real numbers or the real line, $\mathbb{R}_+$ for the set of non-negative numbers and $\mathbb{R}^d$ for the d-dimensional Euclidean space. In the applications we have in mind $d = 2, 3$, but many of the results hold true or are easy to generalize to any dimension $d \geq 1$.

We use $r_+ = \max\{0, r\}$ and $r_- = \max\{0, -r\}$ to denote the positive and negative parts of $r \in \mathbb{R}$, respectively, thus, $r = r_+ - r_-$. Other symbols frequently used are:

M. Shillor, M. Sofonea, J.J. Telega: *Models and Analysis of Quasistatic Contact*, Lect. Notes Phys. **655**, (2001), 85–99
http://www.springerlink.com/

$\mathbb{N}$ – the set of positive integers;
c – a generic positive constant the value of which may change from place to place;
$n! = 1 \cdot 2 \cdot \ldots \cdot n$;
$\forall$ – for each;
$\Rightarrow$ – implies;
$\overline{A}$ – closure of the set A;
δ_{ij} – the Kronecker delta, the unit matrix $\mathbf{I}_d$;
a.e. – almost everywhere with respect to the Lebesgue measure;
iff – if and only if.

We always assume that $\Omega \subset \mathbb{R}^d$ is open, connected and bounded with a Lipschitz boundary Γ. Since the boundary is Lipschitz, the outward unit normal $\mathbf{n}$ exists a.e. on Γ. If $[0,T]$ represents the time interval of interest, where $T > 0$, we use the notation $\Omega_T = \Omega \times (0,T)$.

We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i, \qquad \|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{1/2} \qquad \forall\, \mathbf{u}, \mathbf{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}, \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2} \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

Here and throughout this book, the indices i, j, k, l have values from 1 to d, the summation convention over repeated indices is used, and the index following a comma indicates a partial derivative with respect to the corresponding spatial variable.

We use standard notation for spaces of real-valued functions associated with Ω and Γ:

$L^p(\Omega)$ – the Lebesgue space of p-integrable functions on Ω, with the usual modification if $p = \infty$;
$L^p(\Gamma_0)$ – the Lebesgue space of p-integrable functions on Γ_0, where Γ_0 is a measurable part of Γ, with the usual modification if $p = \infty$;
$C^m(\overline{\Omega})$ – the space of functions whose derivatives up to and including order m are continuous up to the boundary Γ;
$C_0^\infty(\Omega)$ – the space of infinitely differentiable functions with compact support in Ω;
$C^{m,\beta}(\overline{\Omega})$ – the Hölder space of nonnegative integer m and index $\beta \in (0,1)$;
$W^{k,p}(\Omega)$ – the Sobolev space of functions whose weak derivatives of orders k or less are p-integrable on Ω;
$W^{s,p}(\Omega)$ – the Sobolev space of non-integer order;
$H^s(\Omega) \equiv W^{s,2}(\Omega)$, for positive real s;
$W_0^{k,p}(\Omega)$ – the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$;
$H_0^k(\Omega) \equiv W_0^{k,2}(\Omega)$;
$H^{-1}(\Omega)$ – the dual of $H_0^1(\Omega)$;
$H^{\frac{1}{2}}(\Gamma)$ – the Sobolev space on Γ, defined as the range of the trace operator on $H^1(\Omega)$;
$H^{-\frac{1}{2}}(\Gamma)$ – the dual of $H^{\frac{1}{2}}(\Gamma)$.

All the linear spaces considered in this book, including abstract Banach spaces, Hilbert spaces and various function spaces, are assumed to be real spaces.

Let X and Y be two Hilbert spaces endowed with the inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$, and the associated norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. We denote by X^d the space

$$X^d = \{\ \mathbf{x} = (x_1, \ldots, x_d) : x_i \in X,\ 1 \le i \le d\ \},$$

with the canonical inner product

$$(\mathbf{x}, \mathbf{y})_{X^d} = (x_i, y_i)_X$$

and the associated norm $\|\cdot\|_{X^d}$. We also denote by $X \times Y$ the product space

$$X \times Y = \{\ \mathbf{z} = (x, y) : x \in X,\ y \in Y\ \},$$

with the canonical inner product

$$(\mathbf{z}_1, \mathbf{z}_2)_{X\times Y} = (x_1, x_2)_X + (y_1, y_2)_Y, \qquad \mathbf{z}_k = (x_k, y_k), \in X \times Y,$$

for $k = 1, 2$, and the associated norm by $\|\cdot\|_{X\times Y}$.

Let $(X, \|\cdot\|_X)$ be a real Banach space. We denote by $C([0,T];X)$ and $C^1([0,T];X)$, for $T > 0$, the spaces of continuous and continuously differentiable functions from $[0,T]$ to X, with norms

$$\|u\|_{C([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X,$$

and

$$\|u\|_{C^1([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X + \max_{t\in[0,T]} \|\dot{u}(t)\|_X,$$

respectively. Here and below, a dot above a variable denotes its time derivative.

For $p \in [1,\infty)$, we define $L^p(0,T;X)$ to be the space of all measurable functions $v : [0,T] \to X$ such that $\int_0^T \|v(t)\|_X^p \, dt < \infty$. With the norm

$$\|v\|_{L^p(0,T;X)} = \Big(\int_0^T \|v(t)\|_X^p \, dt \Big)^{1/p},$$

the space $L^p(0,T;X)$ is a Banach space. We define $L^\infty(0,T;X)$ to be the space of all measurable functions $v : [0,T] \to X$ such that $t \mapsto \|v(t)\|_X$ is essentially bounded on $[0,T]$. The space $L^\infty(0,T;X)$ is a Banach space with the norm

$$\|v\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{t\in[0,T]} \|v(t)\|_X.$$

For $p \in [1,\infty]$ we also use the Sobolev space

$$W^{1,p}(0,T;X) = \{\, v \in L^p(0,T;X) : \|\dot v\|_{L^p(0,T;X)} < \infty \,\}.$$

When $p < \infty$, $W^{1,p}(0,T;X)$ is a Banach space with the norm

$$\|u\|_{W^{1,p}(0,T;X)} = \left(\int_0^T (\|u(t)\|_X^p + \|\dot u(t)\|_X^p)\, dt \right)^{1/p},$$

and, when $p = \infty$, $W^{1,p}(0,T;X)$ is a Banach space with the norm

$$\|u\|_{W^{1,\infty}(0,T;X)} = \max\{\|u(t)\|_{L^\infty(0,T;X)},\ \|\dot u(t)\|_{L^\infty(0,T;X)}\}.$$

When $(X, (\cdot,\cdot)_X)$ is a Hilbert space, the spaces $L^2(0,T;X)$ and $W^{1,2}(0,T;X)$ are also Hilbert spaces, with inner products

$$(u,v)_{L^2(0,T;X)} = \int_0^T (u(t), v(t))_X\, dt,$$

and

$$(u,v)_{W^{1,2}(0,T;X)} = \int_0^T (u(t), v(t))_X\, dt + \int_0^T (\dot u(t), \dot v(t))_X\, dt,$$

respectively.

6.2 Function Spaces

In the study of contact problems we use spaces of functions with values in $\mathbb{R}^d$ or $\mathbb{S}^d$. These are the spaces in which the displacements or the stress tensor are sought, and they provide the intrinsic regularity of the solutions.

In the following chapters the spaces that are used frequently are:

$$L^2(\Omega)^d = \{\ \mathbf{v} = (v_1, \ldots, v_d) : v_i \in L^2(\Omega),\ 1 \le i \le d\ \}, \tag{6.2.1}$$

$$Q = \{\ \boldsymbol{\tau} = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega),\ 1 \le i, j \le d\ \}. \tag{6.2.2}$$

These are Hilbert spaces with the respective canonical inner products

$$(\mathbf{u}, \mathbf{v})_{L^2(\Omega)^d} = \int_\Omega u_i(x)\, v_i(x)\, dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \sigma_{ij}(x)\, \tau_{ij}(x)\, dx.$$

The associated norms on these spaces are $\|\cdot\|_{L^2(\Omega)^d}$ and $\|\cdot\|_Q$, respectively.

For the displacement field we use the space

$$H^1(\Omega)^d = \{\ \mathbf{u} = (u_1, \ldots, u_d) : u_i \in H^1(\Omega),\ 1 \le i \le d\ \},$$

or its subspace which describes a homogeneous boundary condition on a part of the boundary. The space $H^1(\Omega)^d$ is a Hilbert space with the canonical inner product

$$(\mathbf{u},\mathbf{v})_{H^1(\Omega)^d} = (\mathbf{u},\mathbf{v})_{L^2(\Omega)^d} + (\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}))_Q,$$

and the corresponding norm is

$$\|\mathbf{v}\|_{H^1(\Omega)^d} = (\mathbf{v},\mathbf{v})^{1/2}_{H^1(\Omega)^d}\,.$$

Here, $\boldsymbol{\varepsilon} : H^1(\Omega)^d \to Q$ is the linearized *deformation* operator defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\,(u_{i,j} + u_{j,i}).$$

This is the so-called *small strain tensor*. Here and below the index following a comma indicates a weak partial derivative, that is a derivative in the sense of distributions on Ω. We denote by $\gamma\mathbf{v} \in L^2(\Gamma)^d$ the trace of a function $\mathbf{v}$ in $H^1(\Omega)^d$ on the boundary Γ. When no ambiguity may occur, we write $\mathbf{v}$ instead of $\gamma\mathbf{v}$. We note that for $\mathbf{v} \in H^1(\Omega)^d$ the trace $\gamma\mathbf{v}$ is a well defined function, but for $\mathbf{v} \in L^2(\Omega)^d$ it does not make any sense.

Now, let Γ_D be a measurable subset of Γ with meas$\,(\Gamma_D) > 0$. In the study of contact problems, we frequently use the subspace of $H^1(\Omega)^d$ given by

$$V = \{\ \mathbf{v} \in H^1(\Omega)^d\ :\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\ \}. \tag{6.2.3}$$

Here, the equality $\mathbf{v} = \mathbf{0}$ on Γ_D is understood in the trace sense, i. e., $\gamma\mathbf{v} = \mathbf{0}$ on Γ_D. Since the trace operator is continuous, V is a closed subspace of $H^1(\Omega)^d$. Moreover, since meas$\,(\Gamma_D) > 0$, Korn's inequality (see, e.g., [192, p. 79]) holds and

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|_Q \geq c_K\,\|\mathbf{v}\|_{H^1(\Omega)^d} \qquad \forall\,\mathbf{v} \in V, \tag{6.2.4}$$

where c_K denotes a positive constant depending only on Ω and Γ_D. Everywhere in the book, unless stated otherwise, we use the inner product $(\cdot\,,\cdot)_V$ on V defined by

$$(\mathbf{u},\mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}))_Q, \tag{6.2.5}$$

which induces the norm

$$\|\mathbf{v}\|_V = \|\boldsymbol{\varepsilon}(\mathbf{v})\|_Q. \tag{6.2.6}$$

It follows from (6.2.4) and (6.2.6) that $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent norms on V, and so $(V, \|\cdot\|_V)$ is a real Hilbert space.

For an element $\mathbf{v} \in H^1(\Omega)^d$ we may consider its *normal* and *tangential* components on the boundary Γ, denoted by v_n and $\mathbf{v}_\tau$ and given, respectively, by

$$v_n = \mathbf{v}\cdot\mathbf{n}, \qquad \mathbf{v}_\tau = \mathbf{v} - v_n\mathbf{n}.$$

We use this notation for vectors with subscripts too, thus, $v_{\eta n}$ and $\mathbf{v}_{\eta\tau}$ represent the normal and tangential components, on the boundary, of the vector $\mathbf{v}_\eta \in H^1(\Omega)^d$.

In analysis of contact problems, the boundary Γ is decomposed into three parts $\overline{\Gamma}_D$, $\overline{\Gamma}_N$ and $\overline{\Gamma}_C$, with Γ_D, Γ_N and Γ_C being relatively open and mutually disjoint. To avoid noncoerciveness in quasistatic problems, we assume that meas$\,(\Gamma_D) > 0$. In addition to the space V, we use the subspace

$$V_1 = \{\ \mathbf{v} \in V\ :\ v_n = 0 \text{ on } \Gamma_C\ \} \tag{6.2.7}$$

and the subset

$$V_2 = \{\ \mathbf{v} \in V\ :\ v_n \le 0 \text{ on } \Gamma_C\ \}. \tag{6.2.8}$$

Over V_1 and V_2 we use the inner product $(\cdot,\cdot)_V$ of V. Then V_1 is itself a Hilbert space, while V_2 is a non-empty, closed and convex set in V. By the Sobolev trace theorem, there is a constant $c_B > 0$, depending only on Ω, Γ_D, and Γ_C, such that,

$$\|\mathbf{v}\|_{L^2(\Gamma_C)^d} \le c_B\, \|\mathbf{v}\|_V \qquad \forall\, \mathbf{v} \in V. \tag{6.2.9}$$

We seek the stress fields in the space

$$Q_1 = \{\ \boldsymbol{\tau} \in Q\ :\ \mathrm{Div}\, \boldsymbol{\tau} \in L^2(\Omega)^d\ \}, \tag{6.2.10}$$

which is a Hilbert space when endowed with the inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{Q_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q + (\mathrm{Div}\, \boldsymbol{\sigma}, \mathrm{Div}\, \boldsymbol{\tau})_{L^2(\Omega)^d},$$

and the associated norm $\|\cdot\|_{Q_1}$. We recall that $\mathrm{Div} : Q_1 \to L^2(\Omega)^d$ is the *divergence* operator,

$$\mathrm{Div}\, \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

If $\boldsymbol{\sigma} = (\sigma_{ij})$ is a symmetric ($\sigma_{ij} = \sigma_{ji}$) and sufficiently regular tensor function, e.g., $\boldsymbol{\sigma} \in C^1(\overline{\Omega})^{d^2}$, then the following Green's formula holds,

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + (\mathrm{Div}\, \boldsymbol{\sigma}, \mathbf{v})_{L^2(\Omega)^d} = \int_\Gamma \boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{v}\, dS \qquad \forall\, \mathbf{v} \in H^1(\Omega)^d. \tag{6.2.11}$$

Here, dS denotes the surface measure on Γ. The *normal* and *tangential* components of $\boldsymbol{\sigma}$ on the boundary, denoted by σ_n and $\boldsymbol{\sigma}_\tau$, respectively, are given by

$$\sigma_n = (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{n}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\mathbf{n} - \sigma_n \mathbf{n}.$$

More generally, if $\boldsymbol{\sigma}$ is an arbitrary element of Q_1, we still denote by σ_n and $\boldsymbol{\sigma}_\tau$ the normal and tangential traces of $\boldsymbol{\sigma}$ on the boundary, see, e.g., [51, Ch. 7] for details.

6.3 Auxiliary Material

In this section we provide various results which will be applied repeatedly in the proofs presented below. These are standard results in convex analysis, and the theories of evolution equations and variational inequalities, the Banach and Schauder fixed-point theorems, and Gronwall–type inequalities. Additional general results needed in our study of the problems will be presented in the following chapters.

We start with the definition of the projection operator in a Hilbert space.

Theorem 6.3.1. (Projection theorem) *Let K be a nonempty, closed and convex subset in a Hilbert space X. Then, for each $f \in X$ there is a unique element $u = P_K f \in K$ such that*

$$\|f - u\|_X = \min_{v \in K} \|f - v\|_X .$$

The operator $P_K : X \to K$ is called the *projection operator* onto K. The element $u = P_K f$ is called the *projection* of f on K and is characterized by the inequality

$$u \in K, \quad (u - f, v - u)_X \geq 0 \quad \forall\, v \in K. \tag{6.3.1}$$

Using inequality (6.3.1) it is easy to verify that the projection operator is *nonexpansive*, that is

$$\|P_K u - P_K v\|_X \leq \|u - v\|_X \quad \forall\, u, v \in X. \tag{6.3.2}$$

We present now some results on convex functions defined on inner product spaces. Let $(X, (\cdot,\cdot)_X)$ be such a space and let φ be a function

$$\varphi : X \to (-\infty, \infty].$$

Below, we adopt the convention that $\infty + \infty = \infty$ while an expression of the form $\infty - \infty$ is undefined. The *effective domain* of φ is the set

$$D(\varphi) = \{\ u \in X \ :\ \varphi(u) < \infty\ \},$$

and we say that the function φ is *proper* if $D(\varphi) \neq \emptyset$, that is there exists $u \in X$ such that $\varphi(u) < \infty$. The function φ is *convex* if

$$\varphi((1 - r)\, u + r\, v) \leq (1 - r)\, \varphi(u) + r\, \varphi(v)$$

for all $u, v \in X$ and $r \in (0, 1)$. The function is is said to be *lower semicontinuous* (l.s.c.) at $u \in X$ if

$$\liminf_{n \to \infty} \varphi(u_n) \geq \varphi(u) \tag{6.3.3}$$

for each sequence $\{u_n\} \subset X$ converging to u in X. A function φ is l.s.c. on a subset Y of X if it is l.s.c. at every point of $u \in Y$. We say that φ is l.s.c. if it is l.s.c. on X. When inequality (6.3.3) holds for each sequence $\{u_n\} \subset X$ that converges weakly to u, the function φ is said to be *weakly lower semicontinuous* at u. The notions of a weakly l.s.c. function on a subset or weakly l.s.c. function on X are defined similarly. If φ is a continuous function then it is also l.s.c. The converse is not true and a lower semicontinuous function can be discontinuous. Since strong convergence in X implies weak convergence, it follows that a lower semicontinuous function is weakly lower semicontinuous. Moreover, it can be shown that a proper convex function $\varphi : X \to (-\infty, \infty]$ is lower semicontinuous if and only if it is weakly lower semicontinuous.

The notion of the subdifferential is useful in describing constraints in many branches of engineering and in mechanics, including those arising in contact problems. Indeed, one can find examples in Sects. 3.3, 3.4 and in Chap. 4. We use the notation

$$\partial\varphi(u) = \{\, f \in X \ :\ \varphi(v) - \varphi(u) \geq (f, v-u)_X \quad \forall\, v \in X\,\},$$

$$D(\partial\varphi) = \{\ u \in X \ :\ \partial\varphi(u) \neq \emptyset\ \}$$

for the subdifferential of φ and for its domain of definition, respectively.

A function φ is said to be *subdifferentiable* at $u \in X$ if $u \in D(\partial\varphi)$, and each element $f \in \partial\varphi(u)$ is called a *subgradient* of φ at u. A function φ is said to be subdifferentiable on a subset Y if it is subdifferentiable at each point $u \in Y$, and it is said to be subdifferentiable, if it is subdifferentiable at each point $u \in X$, i.e., if $D(\partial\varphi) = X$.

We used in the models of adhesion, damage, and in the thermodynamic derivation, in Sects. 3.3, 3.4 and Chap. 4, respectively, the indicator functions of various sets, whose definition is as follows. Let $K \subset X$, then, the *indicator function* of the set K is the function $I_K : X \to (-\infty, \infty]$ such that

$$I_K(v) = \begin{cases} 0 & \text{if} \quad v \in K \\ \infty & \text{if} \quad v \notin K \end{cases}. \tag{6.3.4}$$

It can be shown that the set K is a non-empty closed convex set of X if and only if its indicator function I_K is proper, convex and lower semicontinuous. Consider now the subdifferential of the indicator function $I_K(v)$, denoted ∂I_K. When $u \in K$, then $I_K(u) = 0$ and $f \in \partial I_K(u)$ if and only if

$$I_K(v) \geq (f, v-u)_X \quad \forall\, v \in X,$$

i.e.,

$$(f, v-u)_X \leq 0 \quad \forall\, v \in K.$$

Thus, for $u \in K$ we have the characterization

$$\partial I_K(u) = \{\, f \in X \ :\ (f, v-u)_X \leq 0 \ \ \forall\, v \in K\,\}, \tag{6.3.5}$$

and for $u \notin K$, we just have $\partial I_K(u) = \emptyset$.

Each subgradient $f \in \partial I_K(u)$ is called a *support functional* to K at u. We always have $0_X \in \partial I_K(u)$ for $u \in K$, where 0_X denotes the zero element of X. It is easily seen that if $u \in \text{int}\,(K)$ (the interior of K) then $\partial I_K(u) = \{0_X\}$.

The subdifferentials of the indicator functions $I_{[0,1]}$, $I_{(-\infty,0]}$ and of the functions $|\cdot|$ and $\|\cdot\|$ are given in (4.2.9) – (4.2.12), respectively, and the graphs of the first three are depicted in Figs. 6, 7, and 8.

We turn now to results on elliptic variational inequalities involving a nonlinear operator on a real Hilbert space X. Assume that $A : X \to X$ is a Lipschitz continuous and strongly monotone operator on X, i.e.,

$$\left.\begin{array}{l}\text{(a) there exists } M>0 \text{ such that}\\ \qquad \|Au-Av\|_X \le M\|u-v\|_X \quad \forall\, u,v\in X,\\ \text{(b) there exists } m>0 \text{ such that}\\ \qquad (Au-Av,u-v)_X \ge m\,\|u-v\|_X^2 \quad \forall\, u,v\in X.\end{array}\right\} \tag{6.3.6}$$

The operator A is said to be monotone if it satisfies (6.3.6) (b) with $m = 0$. Then we have the following result.

Theorem 6.3.2. *Let X be a Hilbert space. Assume* (6.3.6) *and that $\varphi : X \to (-\infty, \infty]$ is a proper, convex and lower semicontinuous function. Then, for each $f \in X$, the elliptic variational inequality of the second kind,*

$$u \in X, \quad (Au, v-u)_X + \varphi(v) - \varphi(u) \ge (f, v-u)_X \quad \forall\, v \in X,$$

has a unique solution. Moreover, the solution depends Lipschitz continuously on f.

A proof of Theorem 6.3.2 can be found in [51, p. 60]. We choose now φ to be the indicator function of a non-empty, closed, and convex set $K \subset X$ and obtain the following result, which will be used in Sect. 9.1.

Corollary 6.3.3. *Let X be a Hilbert space and $K \subset X$ be a nonempty, convex and closed subset. Assume that A satisfies* (6.3.6)*. Then, for every $f \in X$ there exists a unique solution of the elliptic variational inequality of the first kind*

$$u \in K, \quad (Au, v-u)_X \ge (f, v-u)_X \quad \forall\, v \in K.$$

Moreover, the solution depends Lipschitz continuously on f.

Now, choosing $\varphi \equiv 0$ in Theorem 6.3.2 we obtain the following result, used in Sect. 11.5.

Corollary 6.3.4. *Let X be a Hilbert space and assume that* (6.3.6) *holds. Then, for each $f \in X$ there exists a unique element $u \in X$ such that $Au = f$. Moreover, the mapping $u \mapsto f$ is Lipschitz continuous from X to X.*

We employ the following general result in Sect. 9.5 in the proof of Theorem 9.5.2.

Theorem 6.3.5. *Let X be a Hilbert space and let $\varphi : X \to (-\infty, \infty]$ be a proper, convex, and lower semicontinuous function. Then, for each $u_0 \in D(\varphi)$ and $f \in L^2(0,T;X)$ there exists a unique solution $u \in W^{1,2}(0,T;X)$ which satisfies,*

$$u(t) \in D(\partial\varphi) \quad \text{a.e. } t \in (0,T), \tag{6.3.7}$$

$$\dot{u}(t) + \partial\varphi(u(t)) \ni f(t) \quad \text{a.e. } t \in (0,T), \tag{6.3.8}$$

$$u(0) = u_0. \tag{6.3.9}$$

Theorem 6.3.5 is a simplified version of a more general result that can be found in [187, p. 72] or [189, p. 189].

The following classical theorem of Cauchy-Lipschitz in the space $W^{1,\infty}$ (see, e.g., [194, p. 60]) is used in the proof of Theorem 11.5.

Theorem 6.3.6. *Assume that* $(X, \|\cdot\|_X)$ *is a real Banach space and* $F(t,\cdot) : X \to X$ *is an operator defined a.e. on* $(0,T)$*, and satisfies:*
(i) There exists $L_F > 0$ *such that*

$$\|F(t,x) - F(t,y)\|_X \le L_F \|x - y\|_X$$

for all $x, y \in X$*, a.e.* $t \in (0,T)$.
(ii) There exists $p \in [1,\infty]$ *such that* $t \mapsto F(t,x) \in L^p(0,T;X)$ *for all* $x \in X$.
Then, for each $x_0 \in X$ *there exists a unique function* $x \in W^{1,p}(0,T;X)$ *such that*

$$\dot{x}(t) = F(t, x(t)) \quad \text{a.e. } t \in (0,T),$$
$$x(0) = x_0.$$

In the study of quasistatic contact problems with damage, in Chap. 12, we use the following general result on parabolic variational inequalities. Let V and H be real Hilbert spaces such that V is dense in H and the injection map is continuous; the space H is identified with its own dual and with a subspace of the dual V' of V. We write

$$V \subset H \subset V',$$

and we say that the inclusions above define a *Gelfand triplet*. We denote by $\|\cdot\|_V$, $\|\cdot\|_H$, and $\|\cdot\|_{V'}$ the norms on the spaces V, H, and V', respectively. We use $\langle\cdot,\cdot\rangle_{V'\times V}$ for the duality pairing between V' and V, and note that if $f \in H$ then it coincides with the inner product on H,

$$\langle f, v\rangle_{V'\times V} = (f, v)_H \quad \forall v \in V.$$

The proof of the following result can be found in [70, p. 124].

Theorem 6.3.7. *Let* $V \subset H \subset V'$ *be a Gelfand triplet and let* K *be a nonempty, closed and convex set of* V*. Assume that* $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ *is a continuous and symmetric bilinear form and there are two real constants* $c_a > 0$ *and* c_0 *such that*

$$a(v,v) + c_0\|v\|_H^2 \ge c_a\, \|v\|_V^2 \qquad \forall\, v \in V.$$

Then, for each $u_0 \in K$ *and each* $f \in L^2(0,T;H)$*, there exists a unique function* $u \in W^{1,2}(0,T;H) \cap L^2(0,T;V)$ *such that*

$$u(t) \in K \qquad \forall t \in [0,T],$$
$$\langle \dot{u}(t), v - u(t)\rangle_{V'\times V} + a(u(t), v - u(t)) \ge (f(t), v - u(t))_H \quad \forall\, v \in K, \quad a.e.\ t \in (0,T),$$
$$u(0) = u_0.$$

The Banach fixed-point theorem that follows, will be used repeatedly in proving the existence of solutions for variational problems.

Theorem 6.3.8. *Let K be a nonempty closed set in a Banach space $(X, \|\cdot\|_X)$. Assume that $\Lambda : K \to K$ is a contraction mapping, i.e., there exists $c_\Lambda \in [0, 1)$, such that*

$$\|\Lambda u - \Lambda v\|_X \le c_\Lambda \|u - v\|_X \qquad \forall\, u, v \in K.$$

Then, there exists a unique $u \in K$ such that $\Lambda u = u$.

We note that an element u such that $\Lambda u = u$ is called a *fixed point* of the operator Λ.

Later in this book we will need a variant of the Banach Theorem which we recall now. But first, given an operator Λ, we define its powers inductively by $\Lambda^n = \Lambda(\Lambda^{n-1})$ for $n \ge 2$.

Theorem 6.3.9. *Assume that K is a nonempty closed set in a Banach space X, and that $\Lambda : K \to K$. Suppose Λ^n is a contraction mapping for some positive integer n. Then Λ has a unique fixed point in K.*

The proofs of Theorems 6.3.8 and 6.3.9 can be found in [51, Ch.1].

We now present the Schauder fixed point theorem which will be used in the Proof of Theorem 9.3.1 in Chap. 9. To this end we need some preliminaries. Consider two Banach spaces X and Y and let $K \subset X$. An operator $\mathcal{T} : K \to Y$ is said to be *compact* if for every bounded set $B \subset K$ the image set $\mathcal{T}(B)$ has compact closure in Y. This is equivalent to saying that for every bounded sequence $\{x_n\} \subset K$, the sequence $\{\mathcal{T}(x_n)\} \subset Y$ has a subsequence that converges to a point in Y. If $\mathcal{T}$ is both compact and continuous we say that it is a *completely continuous* operator. When $\mathcal{T}$ is a linear operator then if $\mathcal{T}$ is compact it implies that $\mathcal{T}$ is bounded and hence continuous; this is not true in general when $\mathcal{T}$ is nonlinear, therefore, the continuity of $\mathcal{T}$ must be assumed separately.

We can now state the following well-known Schauder fixed-point theorem.

Theorem 6.3.10. *Let X be a Banach space and let K be a nonempty, bounded, closed and convex subset of X. Let $\mathcal{T} : K \to K$ be a completely continuous operator. Then $\mathcal{T}$ has at least one fixed point in the set K.*

A proof of Theorem 6.3.10 can be found in [195, p. 90] or [196, p. 482].

We end this section with two Gronwall-type inequalities which will be used often in what follows in obtaining estimates on various functions.

Lemma 6.3.11. *Assume that $f, g : [a, b] \to \mathbb{R}$ are continuous functions which satisfy*

$$f(t) \le g(t) + c \int_a^t f(s)\, ds \qquad \forall t \in [a, b],$$

where $c > 0$ is a constant. Then,

$$f(t) \le g(t) + c \int_a^t g(s)\, e^{c\,(t-s)}\, ds \qquad \forall t \in [a,b].$$

Moreover, if g is nondecreasing, then

$$f(t) \le g(t)\, e^{c\,(t-a)} \qquad \forall t \in [a,b].$$

A proof of Lemma 6.3.11 can be found in [51, p. 162].

6.4 Constitutive Operators

We present the assumptions on the operators involved in elastic, viscoelastic, and viscoplastic constitutive relations used in this book.

Linearly elastic materials have a constitutive relation of the form

$$\boldsymbol{\sigma} = \mathcal{B}_{el}\boldsymbol{\varepsilon}(\mathbf{u}), \tag{6.4.1}$$

in which $\mathcal{B}_{el} = (b_{ijkl})$ is a fourth-order tensor. We allow it to depend on the location, i.e., $\mathcal{B}_{el} = \mathcal{B}_{el}(\mathbf{x}) = (b_{ijkl}(\mathbf{x}))$, which means that the material is nonhomogeneous and, possibly, anisotropic.

In the study of mechanical problems involving elastic materials we assume that the elasticity tensor satisfies the usual properties of ellipticity and symmetry, i.e.

$$\left.\begin{array}{l}
\text{(a) } \mathcal{B}_{el} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d.\\
\text{(b) } b_{ijkl} \in L^\infty(\Omega).\\
\text{(c) } \mathcal{B}_{el}(\mathbf{x})\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \boldsymbol{\sigma}\cdot\mathcal{B}_{el}(\mathbf{x})\boldsymbol{\tau} \ \ \forall\, \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbb{S}^d, \text{a.e. } \mathbf{x} \in \Omega.\\
\text{(d) There exists } m_{el} > 0 \text{ such that}\\
\qquad \mathcal{B}_{el}(\mathbf{x})\boldsymbol{\tau}\cdot\boldsymbol{\tau} \ge m_{el}\,\|\boldsymbol{\tau}\|^2 \quad \forall\, \boldsymbol{\tau} \in \mathbb{S}^d, \ \text{a.e. } \mathbf{x} \in \Omega.
\end{array}\right\} \tag{6.4.2}$$

Generally, due to the symmetries of the elasticity tensor $\mathcal{B}_{el}$, in the three-dimensional case ($d = 3$), there may be up to 21 independent coefficients among the b_{ijkl}. An isotropic material is characterized by only two independent coefficients, usually chosen as the Young modulus and Poisson's ratio, or the two Lamé coefficients. For the sake of generality, we allow for fully anisotropic and nonhomogeneous materials.

We consider viscoelastic materials with constitutive relation of the form

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}). \tag{6.4.3}$$

We allow the nonlinear operators of viscosity $\mathcal{A}_{ve}$ and elasticity $\mathcal{B}_{ve}$ to depend on the location. Thus, $\mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}})$ and $\mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u})$ are short-hand notations for

$\mathcal{A}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}(\dot{\mathbf{u}}))$ and $\mathcal{B}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}(\mathbf{u}))$, respectively. When $\mathcal{A}_{ve}$ and $\mathcal{B}_{ve}$ are linear one recovers the Kelvin-Voigt constitutive law (see (2.3.3) on page 13).

In the study of mechanical problems involving viscoelastic materials, we assume the following.

$$\left.\begin{array}{l}
\text{(a) } \mathcal{A}_{ve} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } \mathcal{L}_{\mathcal{A}} > 0 \text{ such that} \\
\quad \|\mathcal{A}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \le \mathcal{L}_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_1 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\
\quad (\mathcal{A}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\
\quad \forall\, \boldsymbol{\varepsilon}_1 \boldsymbol{\varepsilon}_1 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(d) For any } \boldsymbol{\varepsilon} \in \mathbb{S}^d,\ \mathbf{x} \mapsto \mathcal{A}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega. \\
\text{(e) The mapping } \mathbf{x} \mapsto \mathcal{A}_{ve}(\mathbf{x}, \mathbf{0}) \in Q.
\end{array}\right\} \qquad (6.4.4)$$

$$\left.\begin{array}{l}
\text{(a) } \mathcal{B}_{ve} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists an } \mathcal{L}_{\mathcal{B}} > 0 \text{ such that} \\
\quad \|\mathcal{B}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \le \mathcal{L}_{\mathcal{B}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(c) For any } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \mathbf{x} \mapsto \mathcal{B}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega. \\
\text{(d) The mapping } \mathbf{x} \mapsto \mathcal{B}_{ve}(\mathbf{x}, \mathbf{0}) \in Q.
\end{array}\right\} \qquad (6.4.5)$$

Clearly, assumption (6.4.4) is satisfied in the case of the linear viscoelastic constitutive law (2.3.3) if the components a_{ijkl} belong to $L^\infty(\Omega)$ and satisfy the usual properties of symmetry and ellipticity. Assumption (6.4.5) holds for (2.3.3) if the coefficients b_{ijkl} belong to $L^\infty(\Omega)$ and satisfy the usual symmetry property.

An example of a nonlinear viscoelastic constitutive law is

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \gamma\,(\boldsymbol{\varepsilon}(\mathbf{u}) - P_K\boldsymbol{\varepsilon}(\mathbf{u})). \qquad (6.4.6)$$

Here, $\mathcal{A}_{ve}$ is a fourth-oder tensor which satisfies (6.4.4), $\gamma > 0$, K is a closed convex subset of $\mathbb{S}^d$ such that $\mathbf{0} \in K$, and $P_K : \mathbb{S}^d \to K$ denotes the projection map on K. Since the projection map is non-expansive, (6.3.2), it follows that the elasticity operator $\mathcal{B}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}) = \gamma\,(\boldsymbol{\varepsilon} - P_K\boldsymbol{\varepsilon})$ satisfies condition (6.4.5). We conclude that the results presented below are valid for viscoelastic materials described by (2.3.3) or by (6.4.6), under the above assumptions.

Finally we consider rate-type viscoplastic constitutive law of the form

$$\dot{\boldsymbol{\sigma}} = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{G}_{vp}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u})), \qquad (6.4.7)$$

where $\mathcal{A}_{vp}$ and $\mathcal{G}_{vp}$ are the constitutive functions. The elasticity tensor $\mathcal{A}_{vp}$ is assumed to be linear, while $\mathcal{G}_{vp}$ may be nonlinear. In Chap. 9 we assume that $\mathcal{A}_{vp} = (a^{vp}_{ijkl})$ and $\mathcal{G}_{vp}$ satisfy the following conditions.

$$\left.\begin{array}{l}\text{(a) } \mathcal{A}_{vp} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d.\\ \text{(b) } a^{vp}_{ijkl} \in L^\infty(\Omega),\ 1 \le i,j,k,l \le d.\\ \text{(c) } \mathcal{A}_{vp}(\mathbf{x})\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \boldsymbol{\sigma}\cdot\mathcal{A}_{vp}(\mathbf{x})\boldsymbol{\tau}\ \forall\,\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d,\ \text{a.e. } \mathbf{x}\in\Omega.\\ \text{(d) There exists } m_{vp} > 0 \text{ such that}\\ \qquad \mathcal{A}_{vp}(\mathbf{x})\boldsymbol{\tau}\cdot\boldsymbol{\tau} \ge m_{vp}\|\boldsymbol{\tau}\|^2\ \ \forall\,\boldsymbol{\tau}\in\mathbb{S}^d,\ \text{a.e. } \mathbf{x}\in\Omega.\end{array}\right\}\qquad(6.4.8)$$

$$\left.\begin{array}{l}\text{(a) } \mathcal{G}_{vp} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d.\\ \text{(b) There exists } \mathcal{L}_{vp} > 0 \text{ such that}\\ \qquad \|\mathcal{G}_{vp}(\mathbf{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1) - \mathcal{G}_{vp}(\mathbf{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2)\|\\ \qquad \le \mathcal{L}_{vp}\,(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|)\\ \qquad \forall\,\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \mathbf{x}\in\Omega.\\ \text{(c) For any } \boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathbb{S}^d,\ \mathbf{x}\mapsto\mathcal{G}_{vp}(\mathbf{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega.\\ \text{(d) The mapping } \mathbf{x}\mapsto\mathcal{G}_{vp}(\mathbf{x},\mathbf{0},\mathbf{0})\in Q.\end{array}\right\}\qquad(6.4.9)$$

An example of a rate-type viscoplastic constitutive relation of this type is the *Perzyna law*, in which $\mathcal{G}_{vp}$ does not depend on $\boldsymbol{\varepsilon}$. It is given by

$$\dot{\boldsymbol{\varepsilon}} = \mathcal{A}_{vp}^{-1}\dot{\boldsymbol{\sigma}} + \frac{1}{\lambda}\,(\boldsymbol{\sigma} - P_K\boldsymbol{\sigma}), \qquad (6.4.10)$$

where $\mathcal{A}_{vp}$ is a fourth order elastic tensor satisfying (6.4.8), $\mathcal{A}_{vp}^{-1}$ is its inverse, $\lambda > 0$ is a viscosity constant, $K \subset \mathbb{S}^d$ is a nonempty, closed, and convex set in the space of symmetric tensors, and P_K is the projection mapping on K. Here, the function $\mathcal{G}_{vp}$ is given by

$$\mathcal{G}_{vp}(\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = -\frac{1}{\lambda}\,\mathcal{A}_{vp}(\boldsymbol{\sigma} - P_K\boldsymbol{\sigma}),$$

and satisfies condition (6.4.9). Since $\boldsymbol{\sigma} = P_K\boldsymbol{\sigma}$ iff $\boldsymbol{\sigma}\in K$, equation (6.4.10) implies that viscoplastic deformations occur only when the stress tensor $\boldsymbol{\sigma}$ lies outside of K. In this way K represents the domain where the material behaves elastically, and plastic deformations take place outside of it. It is often defined by

$$K = \{\boldsymbol{\sigma}\in\mathbb{S}^d :\ \mathcal{F}(\boldsymbol{\sigma}) \le 0\ \},$$

where $\mathcal{F} : \mathbb{S}^d \to \mathbb{R}$ is the so-called *yield function*, a continuous and convex function which satisfies $\mathcal{F}(\mathbf{0}) < 0$. The equation $\mathcal{F}(\boldsymbol{\sigma}) = 0$ is called the *yield condition*. A well-known example of the yield function is that of *von Mises*,

$$\mathcal{F}(\boldsymbol{\sigma}) = \frac{1}{2}\,\|\boldsymbol{\sigma}^D\|^2 - k^2,$$

where $\boldsymbol{\sigma}^D$ is the deviator part of $\boldsymbol{\sigma}$ and k is a positive constant, the so-called *yield limit.* Existence results in the study of displacement-traction boundary-value problems for viscoplastic materials, including those of the form (6.4.10) can be found in [5, 85, 194, 197, 198].

In Chap. 12 of this book we consider viscoelastic and viscoplastic constitutive laws with damage. In the viscoelastic models we assume that the damage does not affect the viscosity of the material, only its elastic behaviour; therefore, we replace (6.4.3) with the constitutive relation

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}(\boldsymbol{\varepsilon}(\mathbf{u}), \zeta), \tag{6.4.11}$$

in which ζ denotes the damage field.

In the viscoplastic models we assume that the damage affects only the viscoplastic properties of the material, and the constitutive law is of the form

$$\dot{\boldsymbol{\sigma}} = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{G}_{vp}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}), \zeta). \tag{6.4.12}$$

In the study of contact problems involving the constitutive relations (6.4.11) and (6.4.12) we assume that the viscosity operator $\mathcal{A}_{ve}$ and the elasticity operator $\mathcal{A}_{vp}$ satisfy (6.4.4) and (6.4.8), respectively. The assumptions on the elasticity operator $\mathcal{B}_{ve}$ and the viscoplastic operator $\mathcal{G}_{vp}$ will be specified when needed in Sects. 12.1 and 12.4, respectively.

7 Elastic Contact

Existence results for the problem of quasistatic contact between an elastic material and a reactive foundation were first obtained in [18] and [19]. In both papers the normal compliance contact condition was employed. Additional results were obtained in [25, 199, 200]. Recently, Andersson in [24] succeeded in passing to the normal compliance limit and established the existence of a weak solution for the problem with the idealized Signorini contact condition. In this chapter we deal with contact models for linearly elastic materials of the form (6.4.1), and we assume that conditions (6.4.2) are satisfied.

The contact problem for an elastic material with friction, when contact is modelled with the normal compliance condition, is presented in Sect. 7.1. The existence of a solution when the friction coefficient is sufficiently small is stated, and a bound on the solution in terms of the rates of the forces is provided. In Sect. 7.2 the same problem, but with the Signorini contact condition, is described. The existence of a weak solution is obtained by using the results of the previous section, and by obtaining the necessary a priori estimates and passing to the limit when normal compliance approaches the rigid body limit.

Bilateral frictional contact between an elastic body and a rigid foundation is described in Sect. 7.3, where the well-posedness of the problem is described. A remark on the problem with large friction coefficient can be found at the end of the section. Problems of frictional contact with a general dissipative surface functional are presented in Sect. 7.4, and some concrete examples are given, as well. The complete proofs of of Theorems 7.3.1 and 7.4.1 are provided in Sect. 7.5. They are based on two abstract results for evolutionary variational inequalities which we state as well.

7.1 Frictional Contact with Normal Compliance

The classical formulation of the problem of quasistatic frictional contact between an elastic body and a deformable foundation, where the response of the latter is represented by the normal compliance condition (2.6.2), is as follows.

Problem P_{el-nc}. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *such that*

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http://www.springerlink.com/

$$\boldsymbol{\sigma} = \mathcal{B}_{el}\boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{in } \Omega_T, \tag{7.1.1}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \qquad \text{in } \Omega_T, \tag{7.1.2}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_D \times (0,T), \tag{7.1.3}$$

$$\boldsymbol{\sigma}\mathbf{n} = f_N \qquad \text{on } \Gamma_N \times (0,T), \tag{7.1.4}$$

$$\left.\begin{aligned} &-\sigma_n = p_n(u_n - g), \\ &\|\boldsymbol{\sigma}_\tau\| \le \mu p_n(u_n - g), \\ &\boldsymbol{\sigma}_\tau = -\mu p_n(u_n - g)\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if } \dot{\mathbf{u}}_\tau \ne \mathbf{0} \end{aligned}\right\} \text{ on } \Gamma_C \times (0,T), \tag{7.1.5}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \text{in } \Omega. \tag{7.1.6}$$

Here, p_n is a general normal compliance function, g denotes the gap between Γ_C and the foundation, measured along the normal $\mathbf{n}$, (see Fig. 1) and the choice $p_\tau = \mu p_n$ for the friction bound has been made. We note that although there are no time derivatives in the equations, an initial condition is needed because of the friction law.

In [14, 15, 18, 19, 199–201] as well as other publications, the power law $p_n(r) = c_n r_+^{m_n}$ was used. In [18, 199, 200] the problem was considered with $p_\tau = \mu p_n$ and it was shown that it has a weak solution when μ is sufficiently small. The proofs were based on the backward Euler finite difference time discretization, a priori estimates on the approximate solutions, and passage to the discretization limit.

A version of the problem can be found in [19] where the stress in the contact conditions was regularized with a time mollifier. It was proved in [19] that under smallness assumptions on the data, the forces and the gap function, the mollified problem has a weak solution, which actually is a strong solution.

In these papers the uniqueness of the solutions was left open, and indeed, the question is still unresolved. Moreover, there are restrictions on the size of the data and on the normal compliance exponents. These restrictions are common to models with purely elastic constitutive relations, and arise from the mathematical methods of addressing the problems. Similar results can be found in [25, 202].

We now describe the results obtained in [200] for problem P_{el-nc}. To that end we assume that $d = 3$, $\mathcal{B}_{el}$ satisfies (6.4.2) and the normal and tangential compliance functions satisfy

$$p_n(r) = c_n r_+^{m_n}, \quad p_\tau = \mu p_n, \quad \text{where } c_n \ge 0,\ 1 \le m_n < 3. \tag{7.1.7}$$

For convenience we may let the nonnegative coefficient of friction μ be defined on the whole of the boundary. We assume that μ is a multiplier on $H^{\frac{1}{2}}(\Gamma)$, i.e., that the mapping $\xi \longmapsto \mu\xi : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is bounded with the appropriate norm $\|\mu\|$, so that

$$\|\mu\xi\|_{H^{\frac{1}{2}}(\Gamma)} \le \|\mu\|\,\|\xi\|_{H^{\frac{1}{2}}(\Gamma)} \qquad \forall \xi \in H^{\frac{1}{2}}(\Gamma). \tag{7.1.8}$$

We note that μ is such a multiplier if it is Lipschitz continuous on Γ.

The gap function g is assumed to be the trace of a function g_{ext} such that

$$g_{ext} \in H^1(\Omega), \quad g_{ext} \geq 0 \text{ a.e. on } \Omega, \tag{7.1.9}$$

and then $g = g_{ext}$ on Γ_C.

The force density and the surface tractions satisfy:

$$\mathbf{f}_B \in W^{1,2}(0,T; L^2(\Omega)^3), \quad \mathbf{f}_B(0) = \mathbf{0} \tag{7.1.10}$$

$$\mathbf{f}_N \in W^{1,2}(0,T; H^{-1/2}(\Gamma)^3), \quad \operatorname{supp} \mathbf{f}_N(t) \subset \Gamma_N, \quad \mathbf{f}_N(0) = \mathbf{0}. \tag{7.1.11}$$

Here and below, we denote by $\operatorname{supp} \mathbf{f}_N(t)$ the support of the element $\mathbf{f}_N(t) \in H^{-1/2}(\Gamma)^3$, for all $t \in [0,T]$. Moreover, in what follows $\langle \cdot, \cdot \rangle_{-1/2,1/2}$ denotes the duality pairing between $H^{-1/2}(\Gamma)^m$ and $H^{1/2}(\Gamma)^m$, for an integer m.

To consider the variational formulation of the elastic contact problem with normal compliance, P_{el-nc}, we need the following additional notation. Let V be the space given in (6.2.3), denote by

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega b_{ijkl} u_{i,j} u_{k,l} \, dx,$$

the elasticity bilinear form on $V \times V$, and let the normal compliance functional be given by

$$j_n(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_C} p_n(u_n - g) v_n \, dS.$$

Next, the cut-off function $\psi \in C^\infty(\mathbb{R}^3)$ is introduced, which has the value one in a neighborhood of Γ_C and equals zero on a neighborhood of $\bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $0 \leq \psi \leq 1$. For a choice of the function ψ and further details we refer the reader to [200]. However, we note that the construction of ψ is possible only when the parts $\bar{\Gamma}_C$ and $\bar{\Gamma}_D \cup \bar{\Gamma}_N$ are separated by another portion of the surface that is free of tractions.

A straightforward use of Green's formula (6.2.11) and of (7.1.2) – (7.1.5) yields the following variational formulation of problem P_{el-nc}.

Problem P^V_{el-nc}. *Find a displacement field* $\mathbf{u} : [0,T] \to V$ *such that* $\mathbf{u}(0) = \mathbf{u}_0$ *and*

$$\begin{aligned} a(\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t)) &+ j_n(\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t)) \\ &- \langle \mu \psi \sigma_n(\mathbf{u}(t)), \|\mathbf{v}_\tau\| - \|\dot{\mathbf{u}}_\tau(t)\| \rangle_{-1/2,1/2} \\ &\geq (\mathbf{f}_B, \mathbf{v} - \dot{\mathbf{u}}(t))_{L^2(\Omega)^3} + \langle \mathbf{f}_N, \mathbf{v} - \dot{\mathbf{u}}(t) \rangle_{-1/2,1/2}, \end{aligned} \tag{7.1.12}$$

for all $\mathbf{v} \in V$ and a.e. $t \in (0,T)$.

The following result has been established in [200].

Theorem 7.1.1. *Assume that* (6.4.2) *and* (7.1.7)–(7.1.11) *hold and* $\mathbf{u}_0 = \mathbf{0}$. *Then, there exists a solution* $\mathbf{u} \in W^{1,2}(0,T;V)$ *for problem* P^V_{el-nc}, *provided that* $\|\mu\|$ *is sufficiently small. Moreover,* $\mathbf{u}$ *satisfies*

$$\|\mathbf{u}(t)\|_V \leq C_1 \int_0^t \|\dot{\mathbf{f}}_B(\tau)\|_{L^2(\Omega)^3}\, d\tau + C_2 \int_0^t \|\dot{\mathbf{f}}_N(\tau)\|_{H^{-1/2}(\Gamma)^3}\, d\tau + C_3,$$

for all $t \in [0,T]$.

Here C_1, C_2 and C_3 are three positive constants which depend on the problem data. We note that the growth of $\mathbf{u}$ is controlled by the growth rates of the body force and surface tractions. The solutions were also shown to satisfy certain additional bounds. The proof in [200] has been based on the backward Euler finite difference time discretization, a priori estimates on the approximate solutions, and passage to the discretization limit.

The variational formulation in [19] was obtained from (7.1.12) by replacing the normal compliance function p_n with a time averaged regularization. The existence result in [19] is very similar to Theorem 7.1.1 of [200], however, the smallness of the initial displacements was assumed instead of that of the friction coefficient. The proof was accomplished by using tools from the theory of differential inclusions and duality, together with a priori estimates.

7.2 Frictional Contact with Signorini's Condition

The classical formulation of the limit problem, that is the problem with the Signorini nonpenetration condition, has been addressed recently in [24], and is as follows.

Problem P_{el-S}. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and the stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *such that* (7.1.1)–(7.1.4), (7.1.6), *hold, together with the Signorini and Coulomb conditions,*

$$\sigma_n \leq 0, \quad u_n \leq g, \quad \sigma_n(u_n - g) = 0, \tag{7.2.1}$$

$$\|\boldsymbol{\sigma}_\tau\| \leq -\mu\sigma_n,$$

$$\boldsymbol{\sigma}_\tau = -\mu\sigma_n \frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if} \quad \dot{\mathbf{u}}_\tau \neq \mathbf{0}, \tag{7.2.2}$$

on $\Gamma_C \times (0,T)$.

The problem has a weak solution provided that the friction coefficient is sufficiently small, [24].

To study Problem P_{el-S} we use the space V, (6.2.3), and denote by K_g the convex set of admissible displacements

$$K_g = \{\ \mathbf{w} \in V :\ w_n \leq g \ \text{ on } \Gamma_C\ \}.$$

Next, as above, the cut-off function $\psi \in C_0^\infty(\mathbb{R}^d)$ is introduced, which has the value one in a neighborhood of $\bar{\Gamma}_C$ and equals zero on a neighborhood of $\bar{\Gamma}_D \cup \bar{\Gamma}_{Tr+}$, where Γ_{Tr+} is a part of Γ_N adjacent to Γ_C where the external tractions must vanish. The need to separate the contact surface from the parts of the surface where the Dirichlet condition is prescribed and where active tractions may act is due to the shifting technique used in the proof. Since we do not use the method in this monograph, we refer the reader to the article [24] and the references therein for details.

The variational formulation, in terms of the displacements, is as follows.

Problem P^V_{el-S}. *Find a displacement field* $\mathbf{u} : [0,T] \to V$ *such that* $\mathbf{u}(0) = \mathbf{u}_0$*,* $\mathbf{u}(t) \in K_g$*, for all* $t \in [0,T]$*, and for almost all* $t \in (0,T)$*,*

$$\begin{aligned} a(\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t)) - \langle \psi\sigma_n(\mathbf{u}(t)), v_n - \dot{u}_n(t)\rangle_{-1/2,1/2} \\ -\langle \mu\psi\sigma_n(u(t)), \|\mathbf{v}_\tau\| - \|\dot{\mathbf{u}}_\tau(t)\|\rangle_{-1/2,1/2} \\ \geq (\mathbf{f}_B, \mathbf{v} - \dot{\mathbf{u}}(t))_H + \langle \mathbf{f}_N, \mathbf{v} - \dot{\mathbf{u}}(t)\rangle_{-1/2,1/2} \quad \forall \mathbf{v} \in V, \end{aligned} \tag{7.2.3}$$

and

$$\langle \psi\sigma_n(u(t)), z_n - u_n(t)\rangle_{-1/2,1/2} \geq 0 \quad \forall \mathbf{z} \in K_g. \tag{7.2.4}$$

Here, the initial displacement $\mathbf{u}_0$ is an element of K_g and satisfies some additional compatibility assumptions (see [24]).

Assume that the data satisfy:

$$\mathbf{f}_B \in W^{1,2}(0,T; L^2(\Omega)^d), \tag{7.2.5}$$

$$\mathbf{f}_N \in W^{1,2}(0,T; H^{-1/2}(\Gamma)^d), \qquad \operatorname{supp} \mathbf{f}_N(t) \subset \Gamma_N - \Gamma_{Tr+}, \tag{7.2.6}$$

and the gap g is assumed to be the trace of a function g_{ext} such that

$$g_{ext} \in H^{1+\alpha}(\Omega), \quad g_{ext} \geq 0 \text{ a.e. on } \Omega, \tag{7.2.7}$$

for some $\alpha > 0$. The elasticity tensor $\mathcal{B}_{el} = (b_{ijkl})$ satisfies (6.4.2) and the coefficients b_{ijkl} are locally in $C^{0,\beta}(\overline{\Omega})$, for some $0 < \beta \leq 1$.

The friction coefficient is assumed to satisfy the conditions described in the previous section (page 102), that is it is a positive multiplier on $H^{\frac{1}{2}}(\Gamma)$. Moreover, it is also assumed that $\mu \in L^\infty(\Gamma)$.

Finally, the contact surface Γ_C is of class $C^{1,\beta}$, (see, e.g., [51, Ch. 2]).

Under these assumptions, the following existence theorem was established in [24].

Theorem 7.2.1. *Assume that* (6.4.2), (7.2.5)–(7.2.7) *and the other conditions above hold true. Then, there exists a solution* $\mathbf{u} \in W^{1,2}(0,T;V)$ *for problem* P^V_{el-S}*, provided that* $\|\mu\|$ *and* $\|\mu\|_{L^\infty(\Gamma_C)}$ *are sufficiently small.*

The proof of the theorem was based on a sequence of approximations using the normal compliance. In each approximate problem the Signorini condition in (7.2.1) was replaced with $\sigma_n = -\lambda p_n(u_n - g)$, where $\lambda > 0$ was the regularization parameter and p_n was the normal compliance function. First, the approximate problems with normal compliance were discretized in time and a priori estimates on their solutions obtained. Passing to the time discretization limit yielded a solution for the quasistatic problem with normal compliance. Using now a regularity result, based on the shifting technique, the limit Signorini problem was obtained as $\lambda \to \infty$. Passing to the limit was made possible by this extra regularity.

The uniqueness or nonuniqueness of solutions for problems with elastic materials with normal compliance or the Signorini condition is an unresolved question, and so is the existence of solutions for large μ. Actually, estimating on how small μ has to be for solutions of P_{el-nc} or P_{el-S} to exist are interesting but difficult open problems.

7.3 Bilateral Frictional Contact

We now assume that the contact is bilateral, i.e., there is no loss of contact during the process. We recall that this is the meaning of 'bilateral' in this monograph. This assumption holds for pistons and other relatively moving parts in machinery. In this case the normal displacement u_n vanishes on Γ_C at all times. We model friction with the Tresca law, and assume that the constitutive relation is (6.4.1). Under the previous assumptions, the classical formulation of the mechanical problem is the following.

Problem P_{el-b}. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{B}_{el}\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega_T, \tag{7.3.1}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \quad \text{in } \Omega_T, \tag{7.3.2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \times (0,T), \tag{7.3.3}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \quad \text{on } \Gamma_N \times (0,T), \tag{7.3.4}$$

$$\left.\begin{array}{l} u_n = 0, \\ \|\boldsymbol{\sigma}_\tau\| \le H, \\ \boldsymbol{\sigma}_\tau = -H\dfrac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if } \dot{\mathbf{u}}_\tau \neq \mathbf{0} \end{array}\right\} \quad \text{on } \Gamma_C \times (0,T), \tag{7.3.5}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega. \tag{7.3.6}$$

Here, $\mathbf{u}_0$ is the initial displacements field and H is the friction bound.

We now turn to a variational formulation for the mechanical problem (7.3.1)–(7.3.6). The space of admissible displacements is chosen as V_1, (6.2.7), equipped with the inner product (6.2.5).

We assume that the elasticity operator $\mathcal{B}_{el}$ satisfies conditions (6.4.2), the force and the traction densities satisfy

$$\mathbf{f}_B \in W^{1,\infty}(0,T;L^2(\Omega)^d), \quad \mathbf{f}_N \in W^{1,\infty}(0,T;L^2(\Gamma_N)^d), \tag{7.3.7}$$

and the friction bound satisfies

$$H \in L^\infty(\Gamma_C), \quad H \geq 0 \ \text{ a.e. on } \Gamma_C. \tag{7.3.8}$$

We denote by $(\cdot,\cdot)_Q$ the inner product on the space Q, (6.2.2), and define the bilinear form $a : V_1 \times V_1 \to \mathbb{R}$ by

$$a(\mathbf{u},\mathbf{v}) = (\mathcal{B}_{el}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q, \tag{7.3.9}$$

and the functional $j : V_1 \to \mathbb{R}_+$ by

$$j(\mathbf{v}) = \int_{\Gamma_C} H\, \|\mathbf{v}_\tau\|\, dS. \tag{7.3.10}$$

We let $\mathbf{F}(t)$ be the element of V_1 given by

$$(\mathbf{F}(t),\mathbf{v})_V = \int_\Omega \mathbf{f}_B(t)\cdot\mathbf{v}\,dx + \int_{\Gamma_N} \mathbf{f}_N(t)\cdot\mathbf{v}\,dS, \tag{7.3.11}$$

for all $\mathbf{v} \in V_1$ and $t \in [0,T]$. It follows from (7.3.7) and (7.3.8) that the integrals in (7.3.11) and (7.3.10) are well defined and

$$\mathbf{F} \in W^{1,\infty}(0,T;V_1). \tag{7.3.12}$$

Finally, we assume that the initial data satisfy

$$\mathbf{u}_0 \in V_1, \tag{7.3.13}$$

$$a(\mathbf{u}_0,\mathbf{v}) + j(\mathbf{v}) \geq (\mathbf{F}(0),\mathbf{v})_V \quad \forall\,\mathbf{v} \in V_1. \tag{7.3.14}$$

The last condition is a compatibility condition for the initial displacement field.

A straightforward application of Green's formula (6.2.11) yields the following variational formulation of the contact problem (7.3.1)–(7.3.6).

Problem P^V_{el-b}. *Find a displacement field* $\mathbf{u} : [0,T] \to V_1$ *such that*

$$a(\mathbf{u}(t),\mathbf{v}-\dot{\mathbf{u}}(t)) + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \geq (\mathbf{F}(t),\mathbf{v}-\dot{\mathbf{u}}(t))_V \quad \forall\,\mathbf{v} \in V_1, \text{ a.e. } t \in (0,T), \tag{7.3.15}$$

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{7.3.16}$$

The existence of the unique solution to problem P^V_{el-b} is stated next.

Theorem 7.3.1. *Assume that conditions* (6.4.2), (7.3.7), (7.3.8), (7.3.13) *and* (7.3.14) *hold. Then Problem* P^V_{el-b} *has a unique solution which satisfies* $\mathbf{u} \in W^{1,\infty}(0,T;V_1)$. *Moreover, the mapping* $(\mathbf{F},\mathbf{u}_0) \mapsto \mathbf{u}$ *is Lipschitz continuous from* $W^{1,1}(0,T;V_1) \times V_1$ *to* $L^\infty(0,T;V_1)$.

Theorem 7.3.1 was established in [51] using an abstract result for evolutionary variational inequalities. We note that in addition to the existence of the solutions it guarantees the continuous dependence on the problem data $\mathbf{f}_B, \mathbf{f}_N$, and $\mathbf{u}_0$. Its proof will be presented in Sect. 7.5.

Let $\mathbf{u} \in W^{1,\infty}(0,T;V_1)$ be the solution of Problem P^V_{el-b} and let $\boldsymbol{\sigma}$ be the stress field given by (7.3.1). It can be shown that $\boldsymbol{\sigma} \in W^{1,\infty}(0,T;Q_1)$, where Q_1 is defined in (6.2.10). A *mixed formulation* of the problem is to find a pair of functions $(\mathbf{u},\boldsymbol{\sigma})$ which satisfies (7.3.15), (7.3.16) and (7.3.1). Such a pair is called a *weak solution* of the bilateral contact problem with Tresca's friction law. It follows from Theorem 7.3.1 that problem (7.3.1)–(7.3.6) has a unique weak solution which depends Lipschitz continuously on the data.

We remark here that the two-dimensional frictional sliding contact problem for a neo-Hookean material with constant sliding and Coulomb's friction condition with constant friction coefficient was considered in [203]. It was shown there that the problem is ill-posed (unstable to small perturbations) when the friction coefficient is large. Presently, we are unaware of any related results, except as was mentioned above, restrictions on the friction coefficient in the context of elastic materials are common, and there are no existence results for large μ. Also, in the static case large friction coefficient is known to cause mathematical difficulties, see e.g., [204]. This may indicate that some viscosity is essential in contact models. However, it seems that the restrictions on the friction coefficient are for mathematical reasons, and not because a large friction coefficient may cause mechanical seizure since the latter process is not taken into account in the model. Clearly, the issue deserves further study.

7.4 Contact with Dissipative Friction Potential

In the frictional contact problem considered in this section friction is modelled by a general velocity dependent dissipation functional. Existence and uniqueness of a weak solution for the problem is shown by using arguments from the theory of evolutionary variational inequalities. These results are then applied to a variety of concrete frictional contact problems which we describe in some detail.

The classical formulation of the problem is the following.

Problem P_{el-d}. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{B}_{el}\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega_T, \tag{7.4.1}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \quad \text{in } \Omega_T, \tag{7.4.2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \times (0,T), \tag{7.4.3}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \quad \text{on } \Gamma_N \times (0,T), \tag{7.4.4}$$

$$\mathbf{u} \in U, \quad -\boldsymbol{\sigma}\,\mathbf{n} \cdot (\mathbf{v} - \dot{\mathbf{u}}) \le \varphi(\mathbf{v}) - \varphi(\dot{\mathbf{u}})$$

$$\forall \mathbf{v} \in U \quad \text{on } \Gamma_C \times (0,T), \tag{7.4.5}$$
$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \text{in } \Omega. \tag{7.4.6}$$

Here, frictional contact is modelled with the subdifferential boundary condition (7.4.5) in which U is the set of admissible functions and φ is a given function representing the process. Examples and detailed explanations of inequality problems which lead to boundary conditions of this form will be presented at the end of this section.

We assume that $U \subset H^1(\Omega)^d$ and $\varphi : \Gamma_C \times \mathbb{R}^d \to \mathbb{R}$. To accommodate the homogeneous displacement boundary condition (7.4.3) and the contact condition (7.4.5), we define

$$U_1 = V \cap U. \tag{7.4.7}$$

Let $j : U_1 \to (-\infty, \infty]$ be the functional

$$j(\mathbf{v}) = \begin{cases} \displaystyle\int_{\Gamma_C} \varphi(\mathbf{v})\, dS & \text{if } \varphi(\mathbf{v}) \in L^1(\Gamma_C), \\ \infty & \text{otherwise.} \end{cases} \tag{7.4.8}$$

We assume that the elasticity tensor satisfies condition (6.4.2), and

$$U_1 \text{ is a closed subspace of } H^1(\Omega)^d \text{ such that } C_0^\infty(\Omega)^d \subset U_1; \tag{7.4.9}$$
$$j \text{ is proper, convex and lower semicontinuous on } U_1. \tag{7.4.10}$$

The body forces, the surface tractions, and the initial displacements are assumed to satisfy

$$\mathbf{f}_B \in W^{1,2}(0,T; L^2(\Omega)^d), \quad \mathbf{f}_N \in W^{1,2}(0,T; L^2(\Gamma_2)^d), \tag{7.4.11}$$

$$\mathbf{u}_0 \in U_1. \tag{7.4.12}$$

For $\mathbf{u}, \mathbf{v} \in U_1$ we define

$$(\mathbf{u}, \mathbf{v})_{U_1} = (\mathcal{B}_{el}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q, \qquad \|\mathbf{u}\|_{U_1} = (\mathbf{u}, \mathbf{u})_{U_1}^{1/2}.$$

It follows from assumptions (6.4.2) on $\mathcal{B}_{el}$ and Korn's inequality (6.2.4) that $(\cdot,\cdot)_{U_1}$ is an inner product on U_1, and $\|\cdot\|_{U_1}$ and $\|\cdot\|_{H^1(\Omega)^d}$ are equivalent norms on U_1, thus, $(U_1, \|\cdot\|_{U_1})$ is a real Hilbert space.

The element $\mathbf{F}(t)$ of U_1, for $t \in [0,T]$, is given by

$$(\mathbf{F}(t), \mathbf{v})_{U_1} = \int_\Omega \mathbf{f}_B(t) \cdot \mathbf{v}\, dx + \int_{\Gamma_N} \mathbf{f}_N(t) \cdot \mathbf{v}\, dS \quad \forall \mathbf{v} \in U_1. \tag{7.4.13}$$

It is straightforward to show that if $(\mathbf{u}, \boldsymbol{\sigma})$ is a sufficiently regular pair of functions satisfying (7.4.2)–(7.4.5), then $\mathbf{u}(t) \in U_1$ and

$$\begin{aligned}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \\ \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_{U_1} \qquad \forall \mathbf{v} \in U_1,\end{aligned}$$

for all $t \in [0,T]$. Combining this inequality with the constitutive relation (7.4.1) and the initial condition (7.4.6), we obtain the following variational formulation of problem (7.4.1)–(7.4.6), with the displacements field as the unknown.

Problem P^V_{el-d}. *Find a displacement field* $\mathbf{u} : [0,T] \to U_1$ *such that*

$$(\mathcal{B}_{el}\boldsymbol{\varepsilon}(\mathbf{u}(t)),\, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t))$$
$$\geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_{U_1} \qquad \forall\, \mathbf{v} \in U_1,\ \text{a.e. } t \in (0,T), \tag{7.4.14}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \tag{7.4.15}$$

To study the elastic problem P^V_{el-d} we need the following compatibility assumption on the initial displacements $\mathbf{u}_0$,

$$(\mathbf{u}_0, \mathbf{v})_{U_1} + j(\mathbf{v}) \geq (\mathbf{F}(0), \mathbf{v})_{U_1} \quad \forall\, \mathbf{v} \in U_1, \tag{7.4.16}$$

which means that the initial displacements vector is in an equilibrium state compatible with the force $\mathbf{F}(0)$ acting at $t = 0$.

We note that Problem P^V_{el-d} resembles Problem P^V_{el-b}, however, the functional j is more general.

The well-posedness of problem Problem P^V_{el-d} is stated in the following result and will be proved in the next section.

Theorem 7.4.1. *Assume that conditions* (6.4.2), (7.4.9)–(7.4.12) *and* (7.4.16) *hold. Then Problem* P^V_{el-d} *has a unique solution* $\mathbf{u} \in W^{1,2}(0,T;U_1)$.

Let $\mathbf{u} \in W^{1,2}(0,T;U_1)$ be the solution of Problem P^V_{el-d} and let $\boldsymbol{\sigma}$ be the stress field given by (7.4.1). Using (6.4.2) and (7.4.11) it can be shown that $\boldsymbol{\sigma} \in W^{1,2}(0,T;Q_1)$.

A pair of functions $(\mathbf{u}, \boldsymbol{\sigma})$ which satisfies (7.4.1), (7.4.14) and (7.4.15) is called a *weak solution* of the elastic problem (7.4.1)–(7.4.6). We conclude that the quasistatic elastic problem has a unique weak solution.

Examples of subdifferential conditions with friction. We end this section with six examples of contact and friction laws which lead to the inequality (7.4.5), for which (7.4.9) and (7.4.10) hold. Then, by applying Theorem 7.4.1 we conclude that the initial-boundary value problem in each of the examples has a unique weak solution.

(i) **Bilateral contact with Tresca's friction law**. The contact condition is

$$\left.\begin{aligned} & u_n = 0, \\ & \|\boldsymbol{\sigma}_\tau\| \leq H, \\ & \boldsymbol{\sigma}_\tau = -H\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if } \dot{\mathbf{u}}_\tau \neq \mathbf{0} \end{aligned}\right\} \quad \text{on } \Gamma_C \times (0,T). \tag{7.4.17}$$

Here $H \in L^\infty(\Gamma_C)$ is a positive friction bound and the contact is assumed to be bilateral.

The set of admissible functions U consists of those elements of $H^1(\Omega)^d$ whose normal component vanishes on Γ_C. Thus, by (7.4.7) $U_1 = V_1$, where V_1 is the space given in (6.2.7). It is straightforward to show that if $(\mathbf{u}, \boldsymbol{\sigma})$ is a pair of sufficiently regular functions satisfying (7.4.17) then

$$\boldsymbol{\sigma}\mathbf{n} \cdot (\mathbf{v} - \dot{\mathbf{u}}) \geq H(\|\dot{\mathbf{u}}_\tau\| - \|\mathbf{v}_\tau\|) \qquad \forall\, \mathbf{v} \in U,$$

a.e. on $\Gamma_C \times (0,T)$. So, the contact condition (7.4.5) holds with $\varphi(\mathbf{v}) = H\,\|\mathbf{v}_\tau\|$ and the functional j is given by

$$j(\mathbf{v}) = \int_{\Gamma_C} H\,\|\mathbf{v}_\tau\|\,dS \qquad \forall\, \mathbf{v} \in V_1.$$

In this case conditions (7.4.9) and (7.4.10) hold. We conclude that the elastic problem (7.4.1)–(7.4.4), (7.4.6), (7.4.17) has a unique weak solution. Note that this problem was already presented in a slightly different form in Sect. 7.3.

(ii) **Bilateral contact with power-law friction**. We now consider a problem with the boundary conditions

$$u_n = 0, \quad \boldsymbol{\sigma}_\tau = -\mu\,\|\dot{\mathbf{u}}_\tau\|^{p-1}\dot{\mathbf{u}}_\tau \quad \text{on } \Gamma_C \times (0,T), \tag{7.4.18}$$

where $\mu \geq 0$ is the coefficient of friction and $0 < p \leq 1$. The tangential shear is proportional to the p-th power of the tangential speed. Such a boundary condition arises when the contact surface is lubricated with a thin layer of a non-Newtonian fluid.

It is straightforward to show that if $(\mathbf{u}, \boldsymbol{\sigma})$ is a pair of sufficiently regular functions satisfying (7.4.18) then the contact condition (7.4.5) holds with

$$U = \{\,\mathbf{v} \in H^1(\Omega)^d \;:\; v_n = 0 \text{ on } \Gamma_C\}$$

and

$$\varphi(\mathbf{v}) = \frac{\mu}{p+1}\,\|\mathbf{v}_\tau\|^{p+1}.$$

It follows from (7.4.7) that $U_1 = V_1$ and (7.4.9) holds true. Assuming $\mu \in L^\infty(\Gamma_C)$, we deduce from (7.4.8) that the functional

$$j(\mathbf{v}) = \frac{1}{p+1}\int_{\Gamma_C} \mu\,\|\mathbf{v}_\tau\|^{p+1} dS \qquad \forall\, \mathbf{v} \in V_1$$

satisfies (7.4.10). We conclude that Theorem 7.4.1 applies to the mechanical problem (7.4.1)–(7.4.4), (7.4.6) and (7.4.18).

(iii) **Power-law contact with Tresca's friction law**. We consider a contact problem with the boundary conditions

$$\left.\begin{aligned} &-\sigma_n = \kappa\,|\dot{u}_n|^{q-1}\dot{u}_n,\\ &\|\boldsymbol{\sigma}_\tau\| \leq H,\\ &\boldsymbol{\sigma}_\tau = -H\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if } \dot{\mathbf{u}}_\tau \neq \mathbf{0} \end{aligned}\right\} \quad \text{on } \Gamma_C \times (0,T). \tag{7.4.19}$$

Here $H \in L^\infty(\Gamma_C)$ and $\kappa \in L^\infty(\Gamma_C)$ are positive functions, and $0 < q \le 1$. The normal contact stress depends on a power of the normal speed. Such a condition has been considered in [22, 23]. We have $U = H^1(\Omega)^d$, $U_1 = V$,

$$\varphi(\mathbf{v}) = \frac{\kappa}{q+1} |v_n|^{q+1} + H \, \|\mathbf{v}_\tau\|,$$

and

$$j(\mathbf{v}) = \frac{1}{q+1} \int_{\Gamma_C} (\kappa \, |v_n|^{q+1} + H \, \|\mathbf{v}_\tau\|) \, dS \quad \forall \, \mathbf{v} \in V.$$

Assumptions (7.4.9) and (7.4.10) hold and we conclude that the abstract results apply to the frictional contact problem (7.4.1)–(7.4.4), (7.4.6) and (7.4.19).

(*iv*) **Power-law contact with friction**. In this example the normal stress is proportional to a power of the normal speed, while the tangential shear is proportional to a power of the tangential speed. We choose the following contact boundary conditions

$$-\sigma_n = \kappa \, |\dot{u}_n|^{q-1} \dot{u}_n, \quad \boldsymbol{\sigma}_\tau = -\mu \, \|\dot{\mathbf{u}}_\tau\|^{p-1} \dot{\mathbf{u}}_\tau \quad \text{on } \Gamma_C \times (0, T). \tag{7.4.20}$$

Here $\mu \in L^\infty(\Gamma_C)$ and $\kappa \in L^\infty(\Gamma_C)$ are positive functions, $0 < p, q \le 1$. We choose $U = H^1(\Omega)^d$, $U_1 = V$ and

$$\varphi(\mathbf{v}) = \frac{\kappa}{q+1} |v_n|^{q+1} + \frac{\mu}{p+1} \, \|\mathbf{v}_\tau\|^{p+1}.$$

We may apply Theorem 7.4.1 to problem (7.4.1)–(7.4.4), (7.4.6) and (7.4.20), since assumptions (7.4.9) and (7.4.10) hold.

(*v*) **Normal damped response with Tresca's friction law**. In this problem the contact condition models the normal damped response of a thin lubricant layer. The contact pressure depends on the velocity, but only under compression, and the contact conditions are

$$\left.\begin{aligned} &-\sigma_n = \kappa \, (\dot{u}_n)_+ + p_0, \\ &\|\boldsymbol{\sigma}_\tau\| \le H, \\ &\boldsymbol{\sigma}_\tau = -H \frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if } \dot{\mathbf{u}}_\tau \ne \mathbf{0} \end{aligned}\right\} \quad \text{on } \Gamma_C \times (0, T). \tag{7.4.21}$$

Here, $H \in L^\infty(\Gamma_C)$, $\kappa \in L^\infty(\Gamma_C)$ and $p_0 \in L^\infty(\Gamma_C)$ are positive functions. We choose $U = H^1(\Omega)^d$, $U_1 = V$, and

$$\varphi(\mathbf{v}) = \frac{\kappa}{2} \left((v_n)_+\right)^2 + p_0 v_n + H \, \|\mathbf{v}_\tau\|.$$

The results of Theorem 7.4.1 apply to the viscoelastic problem (7.4.1)–(7.4.4), (7.4.6) and (7.4.21), since (7.4.9) and (7.4.10) are satisfied.

(vi) **Normal damped response with power-law friction**. This is a variant of Examples (iv) and (v) above, and the contact conditions are,

$$-\sigma_n = \kappa\,(\dot{u}_n)_+ + p_0, \quad \boldsymbol{\sigma}_\tau = -\mu\,\|\dot{\mathbf{u}}_\tau\|^{p-1}\dot{u}_\tau \quad \text{on } \Gamma_C \times (0,T). \qquad (7.4.22)$$

Here, $\kappa \in L^\infty(\Gamma_C)$, $\mu \in L^\infty(\Gamma_C)$ and $p_0 \in L^\infty(\Gamma_C)$ are positive functions, $0 < p \le 1$. We choose $U = H^1(\Omega)^d$, $U_1 = V$, and

$$\varphi(\mathbf{v}) = \frac{\kappa}{2}\left((v_n)_+\right)^2 + p_0 v_n + \frac{\mu}{p+1}\,\|\mathbf{v}_\tau\|^{p+1}.$$

Since assumptions (7.4.9) and (7.4.10) hold, our results apply to the problem (7.4.1)–(7.4.4), (7.4.6) and (7.4.22), and we conclude that it has a unique weak solution.

We note that in the examples above the normal pressure and tangential shear stress are related to the normal and tangential velocities, and the set of admissible displacement fields is a linear subspace. These requirements are dictated by the structure of the functional φ (which depends only on the surface velocity) and by condition (7.4.9). Important extensions of the results presented in this section would allow additional dependence of φ on the displacements, such as that in the normal compliance contact condition, or unilateral conditions for the admissible displacement fields, such as those in the Signorini contact condition. However, such extensions are unavailable as of now.

We end this section with the remark that the results presented here in the study of contact with dissipative frictional potential may be extended to viscoelastic materials of the form (6.4.3). Details can be found in [205]. There, the existence of the weak solution of the viscoelastic contact model with dissipative frictional potential was proved and the convergence of the solution to the solution of the problem P^V_{el-d}, as the viscosity tensor $\mathcal{A}_{ve}$ converges to zero, was obtained.

7.5 Proof of Theorems 7.3.1 and 7.4.1

To prove the existence and uniqueness of the solution for Problem P^V_{el-b} we shall use the following abstract result for evolutionary variational inequalities.

Let X be a real Hilbert space with the inner product $(\cdot,\cdot)_X$ and consider the problem of finding $u : [0,T] \to X$ such that

$$a(u(t), v - \dot{u}(t)) + j(v) - j(\dot{u}(t)) \ge (f(t), v - \dot{u}(t))_X \qquad \forall\, v \in X, \quad \text{a.e. } t \in (0,T), \qquad (7.5.1)$$

$$u(0) = u_0. \qquad (7.5.2)$$

In the study of (7.5.1)–(7.5.2) we make the following assumptions:

$$\left.\begin{array}{l} a : X \times X \to \mathbb{R} \text{ is a bilinear symmetric form such that} \\ \text{(a) there exists } M > 0 \text{ such that} \\ \qquad |a(u,v)| \le M\|u\|_X\|v\|_X \quad \forall\, u,v \in X. \\ \text{(b) there exists } m > 0 \text{ such that } a(v,v) \ge m\|v\|_X^2 \quad \forall v \in X. \end{array}\right\} \tag{7.5.3}$$

$$j \text{ is a continuous seminorm on } X. \tag{7.5.4}$$

$$f \in W^{1,p}(0,T;X). \tag{7.5.5}$$

$$u_0 \in X. \tag{7.5.6}$$

$$a(u_0, v) + j(v) \ge (f(0), v)_X \quad \forall\, v \in X. \tag{7.5.7}$$

The well-posedness of problem (7.5.1)–(7.5.2) follows.

Theorem 7.5.1. *Assume that* (7.5.3)–(7.5.7) *hold. Then, there exists a unique solution* $u \in W^{1,p}(0,T;X)$ *of problem* (7.5.1)–7.5.2)*. Moreover, the mapping* $(f, u_0) \longmapsto u$ *is Lipschitz continuous from* $W^{1,1}(0,T;X) \times X$ *to* $L^\infty(0,T;X)$*.*

The proof of Theorem 7.5.1 can be found in [51, p. 69], and it is based on a time-discretization method, and on compactness and lower semicontinuity arguments. A version of this theorem in the case $p = \infty$ can be obtained from Theorem 10.2.1 (on page 168).

We use Theorem 7.5.1 to prove Theorem 7.3.1.

Proof (Theorem 7.3.1). We choose $X = V_1$ and $p = \infty$. Assumption (7.3.8) implies that the functional j, given in (7.3.10), is a continuous seminorm on V_1 and assumption (6.4.2) implies that the form a given by (7.3.9) is a symmetric continuous and coercive bilinear form on V_1. Therefore, it follows from (7.3.12)–(7.3.14) that Theorem 7.3.1 is a consequence of Theorem 7.5.1. □

We turn now to the existence and uniqueness of the solution for Problem P^V_{el-d}. To this end we employ the following abstract result.

Theorem 7.5.2. *Let* $(X, (\cdot,\cdot)_X)$ *be a real Hilbert space and let* $j : X \to (-\infty, \infty]$ *be proper, convex and lower semicontinuous. Assume that* $f \in W^{1,2}(0,T;X)$ *and* $u_0 \in X$ *are such that*

$$\sup_{v \in D(j)} \{(f(0), v)_X - (u_0, v)_X - j(v)\} < \infty,$$

where $D(j)$ *is the effective domain of* j*,*

$$D(j) = \{v \in X : j(v) < +\infty\}.$$

Then, there exists a unique element $u \in W^{1,2}(0,T;X)$ *satisfying*

$$(u(t), v - \dot{u}(t))_X + j(v) - j(\dot{u}(t)) \ge (f(t), v - \dot{u}(t))_X,$$

for all $v \in X$*, a.e.* $t \in (0,T)$*, and* $u(0) = u_0$*.*

Theorem 7.5.2 can be found in [206, p. 117] and was proved by using arguments of evolution equations with maximal monotone operators.

We use Theorem 7.5.2 to prove Theorem 7.4.1.

Proof (Theorem 7.4.1). We choose $X = U_1$, and note that (7.4.11) and (7.4.13) imply $\mathbf{F} \in W^{1,2}(0,T;U_1)$. Therefore, assumptions (7.4.9), (7.4.10), (7.4.12) and (7.4.16) allow for the use of Theorem 7.5.2, implying the existence and uniqueness of an element $\mathbf{u} \in W^{1,2}(0,T;U_1)$ which solves Problem P^V_{el-d}. □

8 Viscoelastic Contact

We describe results for contact problems involving viscoelastic materials. It will be seen that adding viscosity to the models leads to a substantial increase in the regularity or the smoothness of the solutions, and this allows for further analysis. Unlike the case of problems for purely elastic materials, there are numerous results for viscoelastic problems with varying degrees of generality, and with different boundary conditions. Equally important is the observation that for many of these problems the uniqueness of the solutions and their continuous dependence on the problem data are proven.

As can be seen in what follows, the theory for viscoelastic materials is considerably more developed. These results may be used in applications with elastic materials by adding a very small viscosity term to the elastic constitutive law. Moreover, there is a very strong indication that in dynamic contact problems addition of viscosity is essential to the mathematical analysis. This, we believe, reflects the physical fact that there are no perfectly elastic materials when changes in the forces are very rapid.

We consider in this chapter viscoelastic materials with constitutive relation of the form (6.4.3) and we assume, unless stated to the contrary, that conditions (6.4.4) and (6.4.5) are satisfied.

We begin, in Sect. 8.1, with the problem of frictionless contact between a viscoelastic body and a rigid foundation. Contact is modelled with the Signorini condition. The proof of the existence theorem is provided in Sect. 8.2, and we note that in addition to an improved smoothness, the solution is shown to be unique.

In Sect. 8.3 the problem of frictional contact in which normal compliance condition is used is formulated. The existence of its weak solution is proved in Sect. 8.4. This is the representative problem described in Chap. 5 in considerable detail.

Bilateral contact with friction can be found in Sect. 8.5 where the dual formulation is provided, the equivalence of the two problems is shown, and the existence and uniqueness of the solution, when the friction coefficient is sufficiently small, stated.

The contact problem with damped response is described in Sect. 8.6 where the existence of the unique weak solution is presented, under a smallness condition on some of the problem data.

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http://www.springerlink.com/

8.1 Frictionless Contact with Signorini's Condition

We assume frictionless contact with a rigid foundation, which is modelled with the Signorini condition. For the sake of simplicity we assume that $g = 0$, i.e., there is no gap between the reference configuration and the foundation.

The classical formulation of the problem is as follows.

Problem P_{ve-S}. *Find a displacement field* $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{in } \Omega_T, \tag{8.1.1}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \qquad \text{in } \Omega_T, \tag{8.1.2}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_D \times (0, T), \tag{8.1.3}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \qquad \text{on } \Gamma_N \times (0, T), \tag{8.1.4}$$

$$u_n \le 0, \quad \sigma_n \le 0, \quad \sigma_n u_n = 0, \quad \boldsymbol{\sigma}_\tau = \mathbf{0} \qquad \text{on } \Gamma_C \times (0, T), \tag{8.1.5}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \text{in } \Omega. \tag{8.1.6}$$

To study the mechanical problem (8.1.1)–(8.1.6) we use the spaces V, Q, and Q_1 defined in (6.2.3), (6.2.2) and (6.2.10), respectively, and the set V_2 given in (6.2.8).

In this problem we replace (6.4.4) with the stronger assumption that the viscosity operator $\mathcal{A}_{ve}$ is linear, bounded, symmetric, and positive definite, that is,

$$\left.\begin{array}{l}
\text{(a) } \mathcal{A}_{ve} = (a^{ve}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d.\\
\text{(b) } a^{ve}_{ijkl} \in L^\infty(\Omega).\\
\text{(c) } \mathcal{A}_{ve}(\mathbf{x})\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{A}_{ve}(\mathbf{x})\boldsymbol{\tau} \quad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega.\\
\text{(d) There exists } m_{\mathcal{A}} > 0 \text{ such that}\\
\qquad \mathcal{A}_{ve}(\mathbf{x})\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge m_{\mathcal{A}} \|\boldsymbol{\tau}\|^2 \quad \forall\, \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega.
\end{array}\right\} \tag{8.1.7}$$

We also suppose that the body forces and surface tractions satisfy

$$\mathbf{f}_B \in W^{1,1}(0, T; L^2(\Omega)^d), \qquad \mathbf{f}_N \in W^{1,1}(0, T; L^2(\Gamma_N)^d), \tag{8.1.8}$$

and the initial displacements satisfy

$$\mathbf{u}_0 \in V_2. \tag{8.1.9}$$

Keeping in mind (8.1.7) we may use on V the inner product

$$(\mathbf{u}, \mathbf{v})_{\mathcal{A}} = (\mathcal{A}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q$$

and the associated norm $\|\mathbf{u}\|_{\mathcal{A}} = (\mathbf{u}, \mathbf{u})_{\mathcal{A}}^{1/2}$. Then $(V, (\cdot, \cdot)_{\mathcal{A}})$ is a real Hilbert space and $\| \cdot \|_{\mathcal{A}}$ is equivalent to the norm $\| \cdot \|_V$.

Next, we denote by $\mathbf{F}(t)$ the element of V given by

$$(\mathbf{F}(t), \mathbf{v})_{\mathcal{A}} = \int_{\Omega} \mathbf{f}_B(t) \cdot \mathbf{v}\, dx + \int_{\Gamma_N} \mathbf{f}_N(t) \cdot \mathbf{v}\, dS, \tag{8.1.10}$$

for all $\mathbf{v} \in V$ and $t \in [0, T]$, and we note that conditions (8.1.8) imply

$$\mathbf{F} \in W^{1,1}(0, T; V). \tag{8.1.11}$$

It is straightforward to show that if $\mathbf{u}$ and $\boldsymbol{\sigma}$ are two sufficiently regular functions satisfying (8.1.2)–(8.1.5), then $\mathbf{u}(t) \in V_2$, $\boldsymbol{\sigma}(t) \in Q_1$, and for each $t \in [0, T]$, the following variational inequality holds,

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q \geq (\mathbf{F}(t), \mathbf{v} - \mathbf{u}(t))_{\mathcal{A}} \quad \forall\, \mathbf{v} \in V_2. \tag{8.1.12}$$

This inequality, combined with (8.1.1) and (8.1.6), leads to the following variational problem, formulated in terms of displacements.

Problem P^V_{ve-S}. *Find a displacement field* $\mathbf{u} : [0, T] \to V$ *such that* $\mathbf{u}(t) \in V_2$ *for all* $t \in [0, T]$, *and*

$$\begin{aligned}(\mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q + (\mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q \\ \geq (\mathbf{F}(t), \mathbf{v} - \mathbf{u}(t))_{\mathcal{A}} \qquad \forall\, \mathbf{v} \in V_2, \ \text{a.e. } t \in (0, T),\end{aligned} \tag{8.1.13}$$

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{8.1.14}$$

We remark that Problem P^V_{ve-S} is formally equivalent to the mechanical problem (8.1.1)–(8.1.6). Indeed, if $\mathbf{u}$ is a smooth solution of the variational problem P_{ve-S} and $\boldsymbol{\sigma}$ is given by (8.1.1), by using arguments similar to those used in Sect. 5.2 it follows that $(\mathbf{u}, \boldsymbol{\sigma})$ satisfies (8.1.1)–(8.1.6). For this reason we consider Problem P^V_{ve-S} as the variational formulation of problem P_{ve-S}, and a pair of functions $(\mathbf{u}, \boldsymbol{\sigma})$ which satisfies (8.1.1), (8.1.13) and (8.1.14) is called a *weak solution* of the viscoelastic problem P_{ve-S}.

The unique solvability of the variational problem P^V_{ve-S} has been established in [207], and is stated as follows.

Theorem 8.1.1. *Assume* (8.1.7)–(8.1.9) *and* (6.4.5). *Then, there exists a unique solution* $\mathbf{u}$ *of problem* P^V_{ve-S}. *Moreover, the solution satisfies*

$$\mathbf{u} \in W^{1,\infty}(0, T, V). \tag{8.1.15}$$

The proof of the theorem is presented in the next section. It is based on the theory of set-valued maximal monotone operators. We conclude from Theorem 8.1.1 that problem P_{ve-S} has a unique weak solution.

The results presented in this section were recently extended in [208] to include friction. There, a quasistatic frictional process for a viscoelastic material of the form (8.1.1) was considered, under the assumptions (8.1.7) and

(6.4.5) on the operators $\mathcal{A}_{ve}$ and $\mathcal{B}_{ve}$, respectively. Contact was modelled with the Signorini condition, and a regularized Coulomb's law of dry friction was used. The existence of a weak solution for the model was proved under a smallness assumption on the (constant) coefficient of friction. The proof was based on a time-discretization method and fixed point arguments. The question of the uniqueness of the solution was left open.

A model for the contact between a solidifying aluminium body and the pan, in the aluminium casting process, was investigated in [84]. It was set as a frictionless quasistatic contact problem with the Signorini condition for a Maxwell-Norton viscoelastic material (see page 14). They proved the existence of a weak solution for the model by using monotonicity and compensated compactness.

8.2 Proof of Theorem 8.1.1

To prove the theorem, we need the following abstract results. Assume X is a real Hilbert space with the inner product $(\cdot,\cdot)_X$. Let $A : D(A) \subset X \to 2^X$ be a multivalued operator. Here, $D(A)$ denotes the domain of the multivalued operator A, and 2^X represents the set of the subsets of X. We say that the operator A is *monotone* if

$$(u_1 - u_2, w_1 - w_2)_X \geq 0 \quad \forall\, w_1 \in Au_1,\ w_2 \in Au_2,\ \forall\, u_1, u_2 \in D(A).$$

This is an extension of the definition of the monotonicity of a single-valued operator. We say that the multivalued operator A is *maximal monotone* if there exists no monotone multivalued operator $B : D(B) \subset X \to 2^X$ that is a proper extension of A. It can be shown that if $\varphi : X \to (-\infty, \infty]$ is a proper, convex, and l.s.c. function, then its subdifferential $\partial\varphi$ is a maximal monotone operator. It can also be shown that if $A_1 : D(A_1) \subset X \to 2^X$ is a maximal monotone operator and $A_2 : X \to X$ is a single-valued, monotone, and Lipschitz continuous operator, then $A_1 + A_2$ is a maximal monotone operator. Proofs of these results, as well as that of the next one can be found in [70, Ch. 1].

Theorem 8.2.1. *Let X be a real Hilbert space and denote by $\mathcal{I} : X \to X$ the identity operator. If $A : D(A) \subset X \to 2^X$ is a multivalued operator such that the operator $A + \omega\mathcal{I}$ is maximal monotone for some real ω, then, for each $f \in W^{1,1}(0,T;X)$ and $u_0 \in D(A)$, there exists a unique function $u \in W^{1,\infty}(0,T;X)$ which satisfies*

$$\dot{u}(t) + Au(t) \ni f(t) \qquad \text{a.e. } t \in (0,T), \tag{8.2.1}$$

$$u(0) = u_0. \tag{8.2.2}$$

We use this result to prove Theorem 8.1.1.

Proof (Theorem 8.1.1). By the Riesz Representation Theorem we can define the operator $B : V \to V$ by

$$(B\mathbf{u}, \mathbf{v})_{\mathcal{A}} = (\mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q \qquad \forall\, \mathbf{u}, \mathbf{v} \in V.$$

It follows from the assumptions (6.4.5) on $\mathcal{B}_{ve}$ and (8.1.7) on $\mathcal{A}_{ve}$ that

$$\|B\mathbf{u}_1 - B\mathbf{u}_2\|_{\mathcal{A}} \le \frac{L_{\mathcal{B}}}{m_{\mathcal{A}}}\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathcal{A}} \qquad \forall\, \mathbf{u}_1,\, \mathbf{u}_2 \in V,$$

i.e., B is a Lipschitz continuous operator. Moreover, the operator

$$B + \frac{L_{\mathcal{B}}}{m_{\mathcal{A}}}\mathcal{I} :\; V \to V$$

is monotone and Lipschitz continuous, where $\mathcal{I}$ denotes the identity operator on V. Let

$$I_{V_2} : V \to (-\infty, \infty]$$

denote the indicator function of the set V_2 and let ∂I_{V_2} be its subdifferential. Since V_2 is a nonempty, convex, and closed subset of V, it follows that ∂I_{V_2} is a maximal monotone operator on V and $D(\partial I_{V_2}) = V_2$. Moreover, the sum

$$\partial I_{V_2} + B + \frac{L_{\mathcal{B}}}{m_{\mathcal{A}}}\mathcal{I} :\; V_2 \subset V \to 2^V$$

is a maximal monotone operator. Thus, conditions (8.1.11) and (8.1.9) allow us to apply Theorem 8.2.1 with the choice $X = V$ endowed with the inner product $(\cdot,\cdot)_{\mathcal{A}}$,

$$A = \partial I_{V_2} + B, \qquad D(A) = V_2 \subset V,$$

and $\omega = \mathcal{L}_{\mathcal{B}}/m_{\mathcal{A}}$. We deduce that there exists a unique element $\mathbf{u} \in W^{1,\infty}(0,T;V)$ such that

$$\dot{\mathbf{u}}(t) + \partial I_{V_2}(\mathbf{u}(t)) + B\mathbf{u}(t) \ni \mathbf{F}(t) \quad \text{a.e. } t \in (0,T), \tag{8.2.3}$$

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{8.2.4}$$

We observe, next, that for each $\mathbf{u}$, $\mathbf{h} \in V$ the following are equivalent,

$$\mathbf{h} \in \partial I_{V_2}(\mathbf{u}) \quad \Longleftrightarrow \quad \mathbf{u} \in V_2, \quad (\mathbf{h}, \mathbf{v} - \mathbf{u})_{\mathcal{A}} \le 0 \quad \forall\, \mathbf{v} \in V_2.$$

Thus, the differential inclusion (8.2.3) is equivalent to the following variational inequality: $\mathbf{u}(t) \in V_2$ and

$$(\dot{\mathbf{u}}(t), \mathbf{v} - \mathbf{u}(t))_{\mathcal{A}} + (B\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t))_{\mathcal{A}} \ge (F(t), \mathbf{v} - \mathbf{u}(t))_{\mathcal{A}} \quad \forall\, \mathbf{v} \in V_2,$$

for a.e. $t \in (0,T)$. It follows that $\mathbf{u}$ satisfies $\mathbf{u}(t) \in V_2$ and the inequality

$$(\mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q + (\mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q$$
$$\ge (\mathbf{F}(t)(t), \mathbf{v} - \mathbf{u}(t))_{\mathcal{A}} \quad \forall \mathbf{v} \in V_2, \tag{8.2.5}$$

for a.e. $t \in (0,T)$. Therefore, we conclude from (8.2.4) and (8.2.5) the existence part in Theorem 8.1.1. The uniqueness of the solution follows from the uniqueness of the element $\mathbf{u} \in W^{1,\infty}(0,T;V)$, which satisfies (8.2.3) and (8.2.4), that is by guaranteed by Theorem 8.2.1. □

Let $\mathbf{u}$ be the solution of Problem P^V_{ve-S} and define $\boldsymbol{\sigma}$ by (8.1.1). Since $\mathbf{u} \in W^{1,\infty}(0,T;V)$ and assumptions (8.1.7) on $\mathcal{A}_{ve}$ and (6.4.5) on $\mathcal{B}_{ve}$ hold, we have $\boldsymbol{\sigma} \in L^\infty(0,T;Q)$. Taking $\mathbf{v} = \mathbf{u}(t) \pm \boldsymbol{\varphi}$ in (8.1.13), where $\boldsymbol{\varphi} \in C_0^\infty(\Omega)^d$ is arbitrary, and using the definition (8.1.10) for $\mathbf{F}(t)$ we find

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \mathbf{f}_B(t) = \mathbf{0} \quad \text{a.e. } t \in (0,T). \tag{8.2.6}$$

The regularity assumption (8.1.8) on $\mathbf{f}_B$ implies $\operatorname{Div} \boldsymbol{\sigma} \in L^\infty(0,T;L^2(\Omega)^d)$, and thus $\boldsymbol{\sigma} \in L^\infty(0,T;Q_1)$.

A pair of functions $(\mathbf{u},\boldsymbol{\sigma})$ which satisfies (8.1.1), (8.1.13) and (8.1.14) is called a *weak solution* of the problem P_{ve-S}. We conclude from Theorem 8.1.1 that problem P_{ve-S} has a unique weak solution.

8.3 Frictional Contact with Normal Compliance

We now describe the problem with normal compliance and friction. The first existence and uniqueness result for quasistatic frictional contact with normal compliance between a reactive foundation and a viscoelastic body has been established in [20]. Related and additional results for these problems can be found in [209–211]. This is the representative problem described in detail in Chap. 5. Here the presentation is condensed, as in the rest of Part II.

The classical formulation of the mechanical problem is as follows.

Problem P_{ve-nc}. *Find a displacements field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega_T, \tag{8.3.1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \quad \text{in } \Omega_T, \tag{8.3.2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \times (0,T), \tag{8.3.3}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \quad \text{on } \Gamma_N \times (0,T), \tag{8.3.4}$$

$$\left.\begin{aligned} &-\sigma_n = p_n(u_n - g),\\ &\|\boldsymbol{\sigma}_\tau\| \le p_\tau(u_n - g),\\ &\boldsymbol{\sigma}_\tau = -p_\tau(u_n - g)\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if } \dot{\mathbf{u}}_\tau \neq \mathbf{0} \end{aligned}\right\} \text{ on } \Gamma_C \times (0,T), \tag{8.3.5}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega. \tag{8.3.6}$$

An example of the normal compliance function p_n is

$$p_n(r) = c_n r_+, \tag{8.3.7}$$

or, more generally, $p_n(r) = c_n(r_+)^m$, where c_n is a positive constant and $m \geq 1$. Formally, Signorini's nonpenetration condition is obtained in the limit $c_n \to \infty$. This leads to the idea of regarding the contact with a rigid support as a limiting case of contact with a deformable support whose stiffness, or resistance to compression grows. We may also consider the normal compliance function

$$p_n(r) = \begin{cases} c_n(r_+)^m & \text{if} \quad r \leq r^* \\ c_n r^{**} & \text{if} \quad r > r^*, \end{cases} \tag{8.3.8}$$

where r^* is a positive cut-off limit, related to the hardness of the surface, m is a positive integer and $r^{**} = (r^*)^m$. This allows us to consider the normal compliance condition with higher differentiability at the onset of contact, i.e., at zero, while the function is Lipschitz on $\mathbb{R}$. In this case the contact condition means that when the asperity interpenetration is large, i.e., when it exceeds the depth r^*, the obstacle offers no additional resistance to the penetration. Practically, this cut-off does not pose any limitation on the use of such a condition, since its purpose is mathematical, and one can always choose r^* to be larger than the dimensions of the system under consideration.

We turn to the variational formulation of problem (8.3.1) – (8.3.6). First, we assume that the viscosity operator $\mathcal{A}_{ve}$ and the elasticity operator $\mathcal{B}_{ve}$ satisfy conditions (6.4.4) and (6.4.5), respectively. The normal and tangential compliance functions p_e $(e = n, \tau)$ satisfy

$$\left.\begin{array}{l} \text{(a) } p_e : \Gamma_C \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } \mathcal{L}_e > 0 \text{ such that} \\ \qquad |p_e(\mathbf{x}, u_1) - p_e(\mathbf{x}, u_2)| \leq \mathcal{L}_e \, |u_1 - u_2|, \\ \qquad \forall\, u_1, u_2 \in \mathbb{R}, \ \text{a.e. } \mathbf{x} \in \Omega. \\ \text{(c) For any } u \in \mathbb{R},\ x \mapsto p_e(\mathbf{x}, u) \text{ is measurable on } \Gamma_C. \\ \text{(d) The mapping } \mathbf{x} \mapsto p_e(\mathbf{x}, 0) \in L^2(\Gamma_C). \end{array}\right\} \tag{8.3.9}$$

We observe that the assumptions on the functions p_n and p_τ are quite general, with the exception of (8.3.9)(b) which requires the functions to grow asymptotically at most linearly. From the practical point of view this is an insignificant restriction, since the interpenetration is likely to be very small. It is easily seen that the functions defined in (8.3.7) and (8.3.8) satisfy the condition (8.3.9)(b). Also, to conform to the usual practice, we may write $p_\tau = \mu p_n$ or $p_\tau = \mu p_n(1 - \delta p_n)_+$, (see page 22 for details), and we notice that if p_n satisfies the condition (8.3.9)(b), then p_τ also satisfies the condition (8.3.9)(b) with $\mathcal{L}_\tau = \mu \, \mathcal{L}_n$. So the results below are valid for the boundary value problems associated with these choices of the normal compliance functions.

We assume that the force and traction densities satisfy

$$\mathbf{f}_B \in C([0,T]; L^2(\Omega)^d), \qquad \mathbf{f}_N \in C([0,T]; L^2(\Gamma_N)^d), \tag{8.3.10}$$

and the gap function satisfies

$$g \in L^2(\Gamma_C), \quad g \geq 0 \text{ a.e. on } \Gamma_C. \tag{8.3.11}$$

Finally, the initial displacements fulfill,

$$\mathbf{u}_0 \in V. \tag{8.3.12}$$

We use the space V, (6.2.3), with the inner product (6.2.5), and denote by $\mathbf{F}(t)$ the element of V given by

$$(\mathbf{F}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_B(t) \cdot \mathbf{v}\, dx + \int_{\Gamma_N} \mathbf{f}_N(t) \cdot \mathbf{v}\, dS, \tag{8.3.13}$$

for all $\mathbf{v} \in V$ and $t \in [0, T]$.

Let $j : V \times V \to \mathbb{R}$ be the contact functional

$$j(\mathbf{v}, \mathbf{w}) = \int_{\Gamma_C} p_n(v_n - g)\, w_n\, dS + \int_{\Gamma_C} p_\tau(v_n - g)\, \|\mathbf{w}_\tau\|\, dS \tag{8.3.14}$$

for all $\mathbf{v}, \mathbf{w} \in V$. From assumption (8.3.9) it follows that the integrals in (8.3.14) are well defined.

If $(\mathbf{u}, \boldsymbol{\sigma})$ are smooth functions satisfying (8.3.2)–(8.3.5), then $\mathbf{u}(t) \in V$ and

$$\begin{aligned}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\mathbf{u}(t), \mathbf{v}) - j(\mathbf{u}(t), \dot{\mathbf{u}}(t))\\ \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \qquad \forall\, \mathbf{v} \in V,\end{aligned} \tag{8.3.15}$$

for all $t \in [0, T]$. A detailed derivation of this inequality can be found in Chap. 5.

The following is a variational formulation of the problem P_{ve-nc}, given in terms of the displacements.

Problem P^V_{ve-nc}. *Find a displacement function* $\mathbf{u} : [0, T] \to V$ *such that*

$$\begin{aligned}(\mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + (\mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\mathbf{u}(t), \mathbf{v})\\ -j(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \quad \forall\, \mathbf{v} \in V,\ t \in [0, T],\end{aligned} \tag{8.3.16}$$

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{8.3.17}$$

The existence of the unique solution of Problem P^V_{ve-nc} is guaranteed in the followinf result.

Theorem 8.3.1. *Assume that* (6.4.4), (6.4.5) *and* (8.3.9)–(8.3.12) *hold. Then Problem* P^V_{ve-nc} *has a unique solution* $\mathbf{u} \in C^1([0, T]; V)$.

A proof of the theorem that was based on results for elliptic variational inequalities and fixed point arguments can be found in [20]. A new and different proof, based on an abstract existence and uniqueness result, is presented in the next section.

We say that a pair of functions $(\mathbf{u}, \boldsymbol{\sigma})$, which satisfies (8.3.1), (8.3.16) and (8.3.17), is a *weak solution* of problem P_{ve-nc}. Let now $\mathbf{u} \in C^1([0, T]; V)$ be the solution of Problem P^V_{ve-nc} and let $\boldsymbol{\sigma}$ be the stress field defined by (8.3.1), then we conclude from the theorem that problem P_{ve-nc} has a unique weak solution, and moreover, $\boldsymbol{\sigma} \in C([0, T]; Q_1)$.

8.4 Proof of Theorem 8.3.1

We provide a proof of Theorem 8.3.1, which is different from the one in [20]. To that end we need the following abstract result for evolutionary variational inequalities.

Let X be a real Hilbert space with inner product $(\cdot,\cdot)_X$ and the associated norm $||\cdot||_X$, and consider the problem of finding $u:\ [0,T]\to X$ such that

$$\begin{aligned}(A\dot{u}(t), v-\dot{u}(t))_X + (Bu(t), v-\dot{u}(t))_X + j(u(t), v)\\ -j(u(t),\dot{u}(t)) \ge (f(t), v-\dot{u}(t))_X \quad \forall\, v\in X,\ t\in[0,T],\end{aligned} \tag{8.4.1}$$

$$u(0) = u_0. \tag{8.4.2}$$

Here, A and B are nonlinear operators on X, j is a functional defined on $X\times X$, while f and u_0 are given data.

In the study of problem (8.4.1)–(8.4.2) we need the following assumptions. The operator $A: X\to X$ is Lipschitz continuous and strongly monotone, i.e.,

$$\left.\begin{array}{l}\text{(a) There exists } M_A>0 \text{ such that}\\ \quad \|Au_1-Au_2\|_X \le M_A\,\|u_1-u_2\|_X \quad \forall\, u_1,u_2\in X.\\ \text{(a) There exists } m_A>0 \text{ such that}\\ \quad (Au_1-Au_2, u_1-u_2)_X \ge m_A\|u_1-u_2\|_X^2 \quad \forall\, u_1,u_2\in X.\end{array}\right\} \tag{8.4.3}$$

The nonlinear operator $B: X\to X$ is Lipschitz continuous, i.e. there exists $M_B>0$ such that

$$\|Bu_1-Bu_2\|_X \le M_B\,\|u_1-u_2\|_X \quad \forall\, u_1,u_2\in X. \tag{8.4.4}$$

The functional $j: X\times X\to\mathbb{R}$ satisfies

$$\left.\begin{array}{l}\text{(a) } j(u,\cdot) \text{ is convex and l.s.c. on } X, \text{ for all } u\in X,\\ \text{(b) There exists } \alpha>0 \text{ such that}\\ \quad j(u_1,v_2)-j(u_1,v_1)+j(u_2,v_1)-j(u_2,v_2)\\ \qquad \le \alpha\,\|u_1-u_2\|_X\,\|v_1-v_2\|_X \quad \forall\, u_1,u_2,v_1,v_2\in X.\end{array}\right\} \tag{8.4.5}$$

Finally, we assume that

$$f\in C([0,T];X) \tag{8.4.6}$$

and

$$u_0\in X. \tag{8.4.7}$$

The following existence and uniqueness result was proved in [211] and may be also found in [51, p. 230].

Theorem 8.4.1. *Let* (8.4.3)–(8.4.7) *hold. Then, there exists a unique solution* $u\in C^1([0,T];X)$ *of problem* (8.4.1)–(8.4.2).

We use Theorem 8.4.1 to prove Theorem 8.3.1.

Proof (Theorem 8.3.1). We choose the space $X = V$ equipped with the inner product $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V$. We use the Riesz Representation Theorem to define the operators $A : V \to V$ and $B : V \to V$ by

$$(A\mathbf{u},\mathbf{v})_V = (\mathcal{A}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}))_Q, \quad (B\mathbf{u},\mathbf{v})_V = (\mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}))_Q,$$

for all $\mathbf{u}$, $\mathbf{v} \in V$. It follows from assumptions (6.4.4) on $\mathcal{A}_{ve}$ and (6.4.5) on $\mathcal{B}_{ve}$ that the operators A and B satisfy conditions (8.4.3) and (8.4.4), respectively. Next, combining the assumptions (8.3.9) on p_n and p_τ with the inequality (6.2.9) it follows that the functional j, defined by (8.3.14), satisfies (8.4.5), since

$$\begin{aligned} &j(\mathbf{u}_1,\mathbf{v}_2) - j(\mathbf{u}_2,\mathbf{v}_1) + j(\mathbf{u}_2,\mathbf{v}_1) - j(\mathbf{u}_2,\mathbf{v}_2)\\ &\quad \le c_B^2(\mathcal{L}_n + \mathcal{L}_\tau)\|\mathbf{u}_1 - \mathbf{u}_2\|_V\|\mathbf{v}_1 - \mathbf{v}_2\|_V, \end{aligned}$$

for all $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}_1, \mathbf{v}_2 \in V$. Also, (8.3.10) implies that the function $\mathbf{F}$, defined by (8.3.13), satisfies $\mathbf{F} \in C([0,T],V)$ and, finally, (8.3.12) shows that condition (8.4.7) is satisfied, too. By applying now Theorem 8.4.1 we obtain that there exists a unique function $\mathbf{u} \in C^1([0,T];V)$ such that (8.3.16) and (8.3.17) holds, which concludes the proof. □

8.5 Bilateral Frictional Contact

We consider, following [212], the mechanical problem describing bilateral frictional contact between a viscoelastic body and a rigid foundation. We recall that by 'bilateral' we mean that the body is permanently in contact with the foundation and there is no gap, so that $g = 0$. As was noted above, such settings can be found in many engineering applications where the relative motion of machine parts is such that contact must be maintained at all times.

The friction condition is assumed to be nonlocal in order to make the contact pressure meaningful, indeed, its value at a point has to be regularized by averaging it over a small nominal contact area. We comment on this requirement below. Such nonlocal friction laws were used in [9,17,22,48,213] in the static and quasistatic cases, and the interested reader will find further details there.

The model for the process is as follows.

Problem P_{ve-b}. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega_T, \tag{8.5.1}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \quad \text{in } \Omega_T, \tag{8.5.2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \times (0,T), \tag{8.5.3}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \quad \text{on } \Gamma_N \times (0,T), \tag{8.5.4}$$

$$\left.\begin{aligned} & u_n = 0, \\ & \|\boldsymbol{\sigma}_\tau\| \le \mu\, p(|\mathcal{R}\sigma_n|), \\ & \boldsymbol{\sigma}_\tau = -\mu p(|\mathcal{R}\sigma_n|)\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if } \dot{\mathbf{u}}_\tau \ne \mathbf{0} \end{aligned}\right\} \text{ on } \Gamma_C \times (0,T), \tag{8.5.5}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \text{in } \Omega. \tag{8.5.6}$$

Conditions (8.5.5) describe a frictional bilateral contact process. Here, μ is the friction coefficient and $\mathcal{R} : H^{-\frac{1}{2}}(\Gamma) \to L^2(\Gamma)$ is a continuous regularizing operator representing the averaging of the normal stress over a small neighborhood of the contact point. We address this issue shortly.

In the case where p is a known function which is independent of σ_n, i.e., $p(r) = h$, the friction law involved in (8.5.5) becomes the Tresca friction law, and $H = \mu\, h$ is the friction bound. By choosing $p(r) = r$ in (8.5.5), we recover the usual regularized Coulomb friction law used in the literature. The choice $p(r) = r_+\,(1-\delta\, r)_+$, where δ is a small positive coefficient related to the wear and hardness of the surface, (2.6.11), was employed in [97].

Problem P_{ve-b} has been investigated in [22], in the case $p(r) = r$, and the existence of the unique weak solution to the model was established. There, only the primal formulation of the problem, in terms of displacements, has been considered.

The inclusion of the regularizing operator $\mathcal{R}$ can be traced to [17,213]. As explained in [17], there seems to be some physical justification in considering the normal stress in the friction condition (8.5.5) as averaged over a small surface area which contains many asperities. However, the main motivation for such a choice is mathematical, to avoid otherwise insurmountable difficulties. Indeed, in the weak formulation the stress $\boldsymbol{\sigma}$ is only square-integrable over Ω and, therefore, its values or trace on the contact surface are not well defined mathematical functions. To overcome this difficulty the operator $\mathcal{R}$ has been introduced in [213]. As an example of such an operator one may use the convolution of $\boldsymbol{\sigma}$ with an infinitely differentiable function with support in a small ball.

We assume that $\mathcal{R} : H^{-\frac{1}{2}}(\Gamma) \to L^2(\Gamma)$ is a continuous operator. Using the continuity of $\mathcal{R}$ and of the normal trace mapping, we deduce the existence of a constant $c_{\mathcal{R}} > 0$, depending only on Ω, Γ_D, Γ_C and $\mathcal{R}$ such that

$$\|\mathcal{R}\xi_n\|_{L^2(\Gamma_C)} \le c_{\mathcal{R}}\, \|\boldsymbol{\xi}\|_{Q_1} \qquad \forall\, \boldsymbol{\xi} \in Q_1. \tag{8.5.7}$$

In the study of the mechanical problem (8.5.1)–(8.5.6) we assume that the viscosity operator $\mathcal{A}_{ve}$ and the elasticity operator $\mathcal{B}_{ve}$ satisfy conditions (6.4.4) and (6.4.5), respectively. The assumptions on the friction function p are:

$$\left.\begin{array}{l}\text{(a) } p : \Gamma_C \times \mathbb{R} \to \mathbb{R}_+.\\ \text{(b) There exists } \mathcal{L}_p > 0 \text{ such that}\\ \qquad |p(\mathbf{x}, u_1) - p(\mathbf{x}, u_2)| \le \mathcal{L}_p\, |u_1 - u_2|\\ \qquad \forall\, u_1, u_2 \in \mathbb{R},\ \text{a.e. } \mathbf{x} \in \Gamma_C.\\ \text{(c) For each } u \in \mathbb{R},\ \mathbf{x} \mapsto p(\mathbf{x}, u) \text{ is measurable on } \Gamma_C.\\ \text{(d) The mapping } \mathbf{x} \mapsto p(\mathbf{x}, 0) \in L^2(\Gamma_C).\end{array}\right\} \qquad (8.5.8)$$

We observe that the assumptions (8.5.8) on p are quite general. Clearly, the functions $p(r) = h$, $p(r) = r$ and $p(r) = r_+\,(1 - \delta\, r)_+$ satisfy these conditions, when h and δ are given positive constants. So, the results presented below hold true for the boundary value problems with each one of these tangential functions.

We assume that the force and traction densities belong to the spaces

$$\mathbf{f}_B \in C([0,T]; L^2(\Omega)^d), \qquad \mathbf{f}_N \in C([0,T]; L^2(\Gamma_N)^d), \qquad (8.5.9)$$

and the coefficient of friction μ satisfies

$$\mu \in L^\infty(\Omega), \quad \mu \ge 0 \ \text{ a.e. on } \Gamma_C. \qquad (8.5.10)$$

Finally, we assume

$$\mathbf{u}_0 \in V_1. \qquad (8.5.11)$$

Recall that the space V_1, defined in (6.2.7), is a real Hilbert space when equipped with the inner product (6.2.5).

Let $\mathbf{F}(t)$ denote the element of V_1 given by (7.3.11), and let $j : Q_1 \times V_1 \to \mathbb{R}$ be the friction functional

$$j(\boldsymbol{\tau}, \mathbf{v}) = \int_{\Gamma_C} \mu\, p(|\mathcal{R}\boldsymbol{\tau}_n|)\, \|\mathbf{v}_\tau\|\, dS \qquad \forall \boldsymbol{\tau} \in Q_1,\ \mathbf{v} \in V_1. \qquad (8.5.12)$$

Since $\mathcal{R}\sigma_n$ lies in $L^2(\Gamma)$, it follows from assumptions (8.5.8) and (8.5.10) that the integral in (8.5.12) is well defined on $Q_1 \times V_1$. Next, for $t \in [0,T]$ and $\boldsymbol{\tau} \in Q_1$, we introduce the set

$$\Sigma(t, \boldsymbol{\tau}) = \{\, \boldsymbol{\xi} \in Q :\ (\boldsymbol{\xi}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + j(\boldsymbol{\tau}, \mathbf{v}) \ge (\mathbf{F}(t), \mathbf{v})_V \ \forall\, \mathbf{v} \in V_1 \,\}.$$

It is straightforward to show that if $\mathbf{u}$ and $\boldsymbol{\sigma}$ are sufficiently regular functions satisfying (8.5.2)–(8.5.5), then for all $t \in [0,T]$,

$$\mathbf{u}(t) \in V_1, \quad \boldsymbol{\sigma}(t) \in \Sigma(t, \boldsymbol{\sigma}(t)), \qquad (8.5.13)$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\boldsymbol{\sigma}(t), \mathbf{v}) - j(\boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t)) \ge (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \quad \forall\, \mathbf{v} \in V_1, \qquad (8.5.14)$$

$$(\boldsymbol{\tau} - \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q \ge 0 \qquad \forall\, \boldsymbol{\tau} \in \Sigma(t, \boldsymbol{\sigma}(t)). \qquad (8.5.15)$$

From (8.5.1), (8.5.6), (8.5.13)–(8.5.15) we obtain the following two variational formulations of problem (8.5.1)–(8.5.6).

Problem P^V_{ve-b1} *Find a displacement field* $\mathbf{u} : [0,T] \to V_1$ *and a stress field* $\boldsymbol{\sigma} : [0,T] \to Q_1$ *such that*

$$\mathbf{u}(0) = \mathbf{u}_0 \tag{8.5.16}$$

and for every $t \in [0,T]$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \tag{8.5.17}$$

$$\begin{aligned}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\boldsymbol{\sigma}(t), \mathbf{v}) - j(\boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t))& \\ \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \quad \forall\, \mathbf{v} \in V_1.&\end{aligned} \tag{8.5.18}$$

Problem P^V_{ve-b2}. *Find a displacement field* $\mathbf{u} : [0,T] \to V_1$ *and a stress field* $\boldsymbol{\sigma} : [0,T] \to Q_1$ *such that*

$$\mathbf{u}(0) = \mathbf{u}_0 \tag{8.5.19}$$

and for every $t \in [0,T]$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \tag{8.5.20}$$

$$\boldsymbol{\sigma}(t) \in \Sigma(t, \boldsymbol{\sigma}(t)),\ (\boldsymbol{\tau} - \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q \geq 0 \quad \forall\, \boldsymbol{\tau} \in \Sigma(t, \boldsymbol{\sigma}(t)). \tag{8.5.21}$$

Whereas in problem P^V_{ve-b1} the main role is played by the displacements field, in problem P^V_{ve-b2}, the so-called *dual formulation*, the main role is played by the stress field. The importance of dual problems in mechanics lies in the fact that in most applications finding the stresses in the system, especially the contact stresses, is more important than obtaining the displacements.

We note in passing that (8.5.20) and (8.5.21) is a *quasi-variational inequality* since the set of admissible test functions Σ depends on the solution.

We have the following result for Problem P^V_{ve-b1}.

Theorem 8.5.1. *Assume* (6.4.4), (6.4.5) *and* (8.5.8)–(8.5.11). *Then, there exists* $\mu_0 > 0$, *which depends only on* $\Omega, \Gamma_D, \Gamma_C, \mathcal{A}_{ve}, p$, *and* $\mathcal{R}$, *such that Problem* P^V_{ve-b1} *has a unique solution* $(\mathbf{u}, \boldsymbol{\sigma})$ *when* $\|\mu\|_{L^\infty(\Gamma_C)} < \mu_0$. *Moreover, the solution satisfies*

$$\mathbf{u} \in C^1([0,T]; V_1), \qquad \boldsymbol{\sigma} \in C([0,T]; Q_1). \tag{8.5.22}$$

We note that in this case there is a restriction on the size of the friction coefficient, which is usually found in results for static and quasistatic friction problems involving Coulomb's law and elastic materials.

Proof. The proof of the theorem is carried out in four steps which we outline below.

(*i*) Let $\boldsymbol{\eta} \in C([0,T];Q)$ be a given function, which is the elastic stress, and let $\boldsymbol{\xi} \in C([0,T];Q_1)$ be a given contact stress. Then, it follows from Theorem 6.3.2 that there exists a unique solution

$$\mathbf{v}_{\eta\xi} \in C([0,T];V_1), \quad \boldsymbol{\sigma}_{\eta\xi} \in C([0,T];Q_1),$$

of the following variational problem,

$$\boldsymbol{\sigma}_{\eta\xi}(t) = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\mathbf{v}_{\eta\xi}(t)) + \boldsymbol{\eta}(t),$$

$$(\boldsymbol{\sigma}_{\eta\xi}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{v}_{\eta\xi}(t)))_Q + j(\boldsymbol{\xi}(t), v) - j(\boldsymbol{\xi}(t), \mathbf{v}_{\eta\xi}(t)) \\ \geq (\mathbf{F}(t), \mathbf{v} - \mathbf{v}_{\eta\xi}(t))_V \quad \forall \mathbf{v} \in V_1,$$

for all $t \in [0,T]$.

Thus, if we input the elastic stress and the contact stress, then we obtain the unique velocity and stress fields.

(*ii*) Next, we construct an operator that associates with a given contact stress $\boldsymbol{\xi}$ the body stress $\boldsymbol{\sigma}_{\eta\xi}$ which is provided by the previous step, when $\boldsymbol{\eta}$ is given. Let

$$\mu_0 = \frac{m_{\mathcal{A}}}{\mathcal{L}_{\mathcal{A}}\mathcal{L}_p c_{\mathcal{R}} c_B},$$

where $m_{\mathcal{A}}$, $\mathcal{L}_{\mathcal{A}}$, $\mathcal{L}_p$, $c_{\mathcal{R}}$, and c_B are the constants found in the assumptions (6.4.4), (8.5.7), (8.5.8), and (6.2.9). Then the operator $\Lambda_\eta : C([0,T];Q_1) \to C([0,T];Q_1)$ defined by

$$\Lambda_\eta \boldsymbol{\xi} = \boldsymbol{\sigma}_{\eta\xi} \quad \forall \xi \in C([0,T];Q_1),$$

has a unique fixed point $\boldsymbol{\xi}_\eta$, when $\|\mu\|_{L^\infty(\Gamma_C)} < \mu_0$.

(*iii*) We construct another operator which relates to each given $\boldsymbol{\eta}$ the corresponding strain $\boldsymbol{\varepsilon}(\mathbf{u}_\eta)$.

When $\|\mu\|_{L^\infty(\Gamma_C)} < \mu_0$ the operator $\Lambda : C([0,T];Q) \to C([0,T];Q)$, defined by

$$\Lambda\boldsymbol{\eta}(t) = \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \quad \mathbf{u}_\eta(t) = \int_0^t \mathbf{v}_{\eta\xi_\eta}(s)\,ds + \mathbf{u}_0,$$

for all $\boldsymbol{\eta} \in C([0,T];Q)$ and $t \in [0,T]$, has a unique fixed point $\boldsymbol{\eta}^* \in C([0,T];Q)$.

(*iv*) Finally, the fixed point thus obtained provides the unique solution of the problem. Let $\|\mu\|_{L^\infty(\Gamma_C)} < \mu_0$, then $(\mathbf{u}, \boldsymbol{\sigma})$, with $\mathbf{u} = \mathbf{u}_{\eta^*}$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\eta^*\xi_{\eta^*}}$ is the unique solution of Problem P^V_{nc-b1} satisfying (8.5.22). □

The next result deals with the relationship between the two formulations and shows that Problems P^V_{ve-b1} and P^V_{ve-b2} are equivalent.

Theorem 8.5.2. *Let conditions* (6.4.4), (6.4.5), *and* (8.5.8)–(8.5.11) *hold, and assume that* $(\mathbf{u}, \boldsymbol{\sigma})$ *satisfies* (8.5.22). *Then* $(\mathbf{u}, \boldsymbol{\sigma})$ *is a solution of Problem* P^V_{ve-b1} *if and only if it is a solution of Problem* P^V_{ve-b2}.

The following result is a consequence of Theorems 8.5.1 and 8.5.2.

Theorem 8.5.3. *Let conditions* (6.4.4)*,* (6.4.5)*, and* (8.5.8)–(8.5.11) *hold, and let* μ_0 *be defined as in Theorem* 8.5.1*. Then, if* $\|\mu\|_{L^\infty(\Gamma_C)} < \mu_0$*, Problem* P^V_{ve-b2} *has a unique solution* $(\mathbf{u}, \boldsymbol{\sigma})$ *which satisfies* (8.5.22)*.*

Since, by Theorem 8.5.2, the problems are equivalent Theorems 8.5.1 and 8.5.3 provide the existence and uniqueness of the (same) solution for Problems P^V_{ve-b1} and P^V_{ve-b2}.

We conclude from Theorems 8.5.1 and 8.5.3 that if the coefficient of friction is sufficiently small the mechanical problem has a unique weak solution which satisfies both Problems P^V_{ve-b1} and P^V_{ve-b2}.

More detailed study of problem (8.5.1)–(8.5.6), including the continuous dependence of the weak solution on the data and on the coefficient of friction, may be found in [212].

8.6 Frictional Contact with Normal Damped Response

We turn to viscoelastic frictional contact when the foundation's reaction depends on the normal velocity. We model it with a general normal damped response condition and the associated version of dry friction law.

The classical formulation of the mechanical problem is as follows.

Problem P_{ve-d}. *Find a displacement* $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega_T, \tag{8.6.1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \quad \text{in } \Omega_T, \tag{8.6.2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \times (0, T), \tag{8.6.3}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \quad \text{on } \Gamma_N \times (0, T), \tag{8.6.4}$$

$$\left.\begin{aligned} &-\sigma_n = p_n(\dot{u}_n), \\ &\|\boldsymbol{\sigma}_\tau\| \le p_\tau(\dot{u}_n), \\ &\boldsymbol{\sigma}_\tau = -p_\tau(\dot{u}_n)\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if } \dot{\mathbf{u}}_\tau \ne \mathbf{0} \end{aligned}\right\} \quad \text{on } \Gamma_C \times (0, T), \tag{8.6.5}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega. \tag{8.6.6}$$

We comment on the frictional contact conditions (8.6.5). In [90] the following form of the function p_n has been employed

$$p_n(r) = \gamma_d\, r_+ + p_0, \tag{8.6.7}$$

modelling a foundation that is covered with a thin lubricant layer, say oil. Here, γ_d is the damping resistance coefficient, assumed positive, and p_0 is the oil pressure, which is given and nonnegative. In this case the lubricant layer presents resistance, or damping, only when the surface moves towards the foundation, but does nothing when it recedes.

Another choice of p_n is a power law,

$$p_n(r) = \kappa\,|r|^{q-1} r. \tag{8.6.8}$$

Here, $\kappa \geq 0$, $0 < q \leq 1$ and the normal contact pressure depends on a power of the normal velocity, which is similar to the behavior of a nonlinear viscous dashpot.

Finally, we may choose

$$p_n(r) = S_C, \tag{8.6.9}$$

where S_C is a given positive function. In this case the normal stress is prescribed on the contact surface.

Given p_n, we may choose the friction bound function p_τ in a number of ways, such as

$$p_\tau = \mu p_n \tag{8.6.10}$$

or

$$p_\tau = \mu p_n (1 - \delta p_n)_+, \tag{8.6.11}$$

where μ and δ are positive coefficients (see page 22 for details).

In the study of the mechanical problem (8.6.1)–(8.6.6) we use assumptions (6.4.4) and (6.4.5). We also assume that the contact functions p_e $(e = n, \tau)$ satisfy

$$\left.\begin{array}{l}
\text{(a) } p_e : \Gamma_C \times \mathbb{R} \to \mathbb{R}_+. \\
\text{(b) There exists } \mathcal{L}_e > 0 \text{ such that} \\
\qquad |p_e(\mathbf{x}, r_1) - p_e(\mathbf{x}, r_2)| \leq \mathcal{L}_e\,|r_1 - r_2| \\
\qquad \forall\, r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_C. \\
\text{(c) For any } r \in \mathbb{R},\ \mathbf{x} \mapsto p_e(\mathbf{x}, r) \text{ is measurable on } \Gamma_C. \\
\text{(d) The mapping } \mathbf{x} \mapsto p_e(\mathbf{x}, 0) \in L^2(\Gamma_C).
\end{array}\right\} \tag{8.6.12}$$

We note that assumptions (8.6.12) on the functions p_n and p_τ are quite general, with the exception assumption (b). The functions defined in (8.6.7) and (8.6.9) satisfy the condition (8.6.12)(b) and the function defined in (8.6.8) satisfies this condition if $q = 1$. We also observe that when the functions p_n and p_τ are related by (8.6.10) or (8.6.11) and p_n satisfies the condition (8.6.12)(b), then p_τ also satisfies condition (8.6.12)(b) with $\mathcal{L}_\tau = \mu\,\mathcal{L}_n$. We conclude that the results below are valid for the boundary value problems with these choices of contact functions.

We assume that the forces and tractions satisfy

$$\mathbf{f}_B \in C([0,T]; L^2(\Omega)^d), \qquad \mathbf{f}_N \in C([0,T]; L^2(\Gamma_2)^d), \tag{8.6.13}$$

and the initial displacements satisfy

$$\mathbf{u}_0 \in V, \tag{8.6.14}$$

where the space V is defined in (6.2.3). We denote by $\mathbf{F}(t)$ the element of V given by (8.3.13), and let $j : V \times V \to \mathbb{R}$ be the contact functional defined by

$$j(\mathbf{v}, \mathbf{w}) = \int_{\Gamma_C} p_n(v_n)\, w_n\, dS + \int_{\Gamma_C} p_\tau(v_n)\, \|\mathbf{w}_\tau\|\, dS \qquad \forall\, \mathbf{v},\, \mathbf{w} \in V. \tag{8.6.15}$$

It follows now from the Green formula (6.2.11) that if $(\mathbf{u}, \boldsymbol{\sigma})$ are smooth functions satisfying (8.6.2)–(8.6.5), then $\mathbf{u}(t) \in V$ and for all $t \in [0, T]$,

$$\begin{aligned}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\dot{\mathbf{u}}(t), \mathbf{v}) - j(\dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t))&\\ \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \quad \forall\, \mathbf{v} \in V.&\end{aligned} \tag{8.6.16}$$

Thus, we obtain from (8.6.1), (8.6.6) and (8.6.16) the following variational formulation of problem (8.6.1)–(8.6.6) in terms of the displacements.

Problem P^V_{ve-d}. Find a displacement field $\mathbf{u} : [0, T] \to V$ such that

$$\begin{aligned}(\mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + (\mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\dot{\mathbf{u}}(t), \mathbf{v})&\\ - j(\dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t)) \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \qquad \forall\, \mathbf{v} \in V,\ t \in [0, T],&\end{aligned} \tag{8.6.17}$$

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{8.6.18}$$

The existence and uniqueness of the solution to Problem P^V_{ve-d} is described in the following result of [179], and it holds when a smallness assumption on a part of the data is satisfied.

Theorem 8.6.1. *Assume* (6.4.4), (6.4.5), *and* (8.6.12)–(8.6.14). *There exists* $\mathcal{L}_0 > 0$, *which depends only on* Ω, Γ_D, Γ_C *and* $\mathcal{A}_{ve}$, *such that if* $\mathcal{L}_n + \mathcal{L}_\tau \leq \mathcal{L}_0$, *then Problem* P^V_{ve-d} *has a unique solution* $\mathbf{u} \in C^1([0, T]; V)$.

Let now $\mathbf{u} \in C^1([0, T]; V)$ be the solution of Problem P^V_{ve-d} and let $\boldsymbol{\sigma}$ be the stress field given by (8.6.1). It follows from (6.4.4) and (6.4.5) that $\boldsymbol{\sigma} \in C([0, T]; Q)$. Moreover, using (8.6.17) and (8.6.13) it can be shown that $\mathrm{Div}\, \boldsymbol{\sigma} \in C([0, T]; L^2(\Omega)^d)$ and then $\boldsymbol{\sigma} \in C([0, T]; Q_1)$.

A pair of functions $(\mathbf{u}, \boldsymbol{\sigma})$ which satisfies (8.6.1), (8.6.17) and (8.6.18) is called a *weak solution* of problem (8.6.1)–(8.6.6). We conclude that problem (8.6.1)–(8.6.6) has a unique weak solution provided $\mathcal{L}_n + \mathcal{L}_\tau$ is sufficiently small. The critical value $\mathcal{L}_0$ depends only on the viscosity operator and on the geometry of the problem, and is independent of the elasticity operator, the external forces, or the initial displacements.

We now discuss possible mechanical interpretations of the restriction $\mathcal{L}_n + \mathcal{L}_\tau < \mathcal{L}_0$, which guarantees the unique solvability of the problem. The verification of this condition as well as its interpretation depend on the specific mechanical problem under consideration. For example, consider problem

(8.6.1)–(8.6.6), in which the function p_n is given by (8.6.7) and the function p_τ is given by (8.6.10) or by (8.6.11). Then, it follows that assumption (8.6.12)(b) is satisfied with $\mathcal{L}_n = \gamma_d$ and $\mathcal{L}_\tau = \mu\,\gamma_d$ and, therefore, the condition $\mathcal{L}_n + \mathcal{L}_\tau \le \mathcal{L}_0$ holds if $\gamma_d \le \mathcal{L}_0/(\mu + 1)$. Thus, the condition imposes a size restriction on the damping resistance coefficient. We conclude that the corresponding mechanical problem has a unique weak solution if the damping resistance coefficient of the oil layer is sufficiently small. Next, consider the mechanical problem (8.6.1)–(8.6.6) when the function p_n is given by (8.6.9) with $S_C \in L^\infty(\Gamma_C)$ and the function p_τ is given by (8.6.10) or by (8.6.11). Assumption (8.6.12)(b) is satisfied with $\mathcal{L}_n = \mathcal{L}_\tau = 0$ and, therefore, the condition $\mathcal{L}_n + \mathcal{L}_\tau \le \mathcal{L}_0$ holds trivially. We conclude that the corresponding mechanical problem has a unique weak solution without any additional restriction on the coefficients μ or δ.

We note that the frictional contact problem with damped response for elastic materials, i.e., problem (7.1.1), (8.6.2)–(8.6.6), seems to be an open problem. Indeed, removing the viscosity operator $\mathcal{A}_{ve}$ in the variational inequality (8.6.17) leads to severe mathematical difficulties and, as far as we know, there are no existence results for such models. A similar comment applies to the frictional contact problem with damped response for viscoplastic materials, i.e., problem (9.1.1), (9.1.6), (8.6.2)–(8.6.5).

9 Viscoplastic Contact

We describe contact problems for viscoplastic materials in which once the stress reaches the so-called yield limit the deformation becomes irreversible. Metals behave as viscoplastic materials, for instance, during metal-forming, and so do some polymers and other materials when hot or under large loads. Such problems are common in industrial processes and deserve considerable attention, both analytical and computational. We use rate-type constitutive laws of the form (6.4.7), and note that unlike the elastic or viscoelastic cases, the variational formulations of viscoplastic contact problems involve both the displacements and the stress fields. Indeed, this feature arises from the fact that, essentially, the stress field can not be eliminated from the constitutive law and, therefore, derivation of a variational formulation in terms of the displacements only leads to considerable mathematical difficulties.

Throughout this chapter and we assume that conditions (6.4.8) and (6.4.9) hold.

In Sect. 9.1 the frictionless contact problem with the Signorini condition is presented. Two mixed variational formulations are given and the existence of the unique weak solution to each one is stated. The main steps in the proofs are provided, as an example of the general method. A detailed proof of the equivalence between the two variational formulations can be found in Sect. 9.2.

Section 9.3 deals with recent results for the viscoplastic contact problem with normal compliance and friction. The existence of a weak solution is stated, under the condition that some of the problem data is small, in an appropriate sense. The proof is provided in Sect. 9.4.

The bilateral problem, when contact is maintained at all times, is described in Sect. 9.5. Two different mixed variational formulations are provided, the existence of the unique weak solution stated, and the main steps in the proofs described.

The chapter concludes with Sect. 9.6 where existence and uniqueness results are provided for a general dissipative contact functional which depends on the velocities.

We would like to emphasize here that the steps in the proofs, which use fixed-point arguments, provide a way to develop reliable numerical algorithms for such problems.

M. Shillor, M. Sofonea, J.J. Telega: *Models and Analysis of Quasistatic Contact*, Lect. Notes Phys. **655**, (2001), 135–162
http://www.springerlink.com/

9.1 Frictionless Contact with Signorini's Condition

We assume that contact is frictionless and model it with the Signorini condition, and zero gap function ($g = 0$). The viscoplastic constitutive law of the material is chosen as (6.4.7), and we follow [28] in the presentation below. Under these assumptions the classical formulation of the mechanical problem is the following.

Problem P_{vp-S}. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\sigma : \Omega \times [0,T] \to \mathbb{S}^d$ *such that*

$$\dot{\boldsymbol{\sigma}} = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{G}_{vp}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u})) \qquad \text{in } \Omega_T, \tag{9.1.1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \qquad \text{in } \Omega_T, \tag{9.1.2}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_D \times (0,T), \tag{9.1.3}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \qquad \text{on } \Gamma_N \times (0,T), \tag{9.1.4}$$

$$u_n \le 0,\ \ \sigma_n \le 0,\ \ \sigma_n u_n = 0,\ \ \boldsymbol{\sigma}_\tau = \mathbf{0} \text{ on } \Gamma_C \times (0,T), \tag{9.1.5}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \qquad \text{in } \Omega. \tag{9.1.6}$$

Here, $\mathbf{u}_0$ and $\boldsymbol{\sigma}_0$ are the initial displacement and stress fields, respectively. These have to be provided because of the time derivatives of u and $\boldsymbol{\sigma}$ in (9.1.1). Physically, these are needed since in viscoplastic materials specifying only the displacements does not determine uniquely the stresses, and therefore, the state of the system.

We assume that the force and traction densities satisfy

$$\mathbf{f}_B \in W^{1,\infty}(0,T; L^2(\Omega)^d), \quad \mathbf{f}_N \in W^{1,\infty}(0,T; L^2(\Gamma_N)^d) \tag{9.1.7}$$

and we denote by $\mathbf{F}(t)$ the element of V given by

$$(\mathbf{F}(\mathbf{t}), \mathbf{v})_V = \int_\Omega \mathbf{f}_B(t) \cdot \mathbf{v}\, dx + \int_{\Gamma_N} \mathbf{f}_N \cdot \mathbf{v}\, dS, \tag{9.1.8}$$

for all $\mathbf{v} \in V$ and $t \in [0,T]$. Conditions (9.1.7) imply

$$\mathbf{F} \in W^{1,\infty}(0,T; V). \tag{9.1.9}$$

Next, we choose the set of admissible displacement fields to be V_2, (6.2.8), and the time dependent set of admissible stress fields $\Sigma(t)$ is given by

$$\Sigma(t) = \{\boldsymbol{\tau} \in Q : (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q \ge (\mathbf{F}(t), \mathbf{v})_V\ \forall\, \mathbf{v} \in V_2\}, \quad t \in [0,T]. \tag{9.1.10}$$

This is the set of all stresses that are compatible with the forces $\mathbf{F}(t)$ over $[0,T]$. Finally, we assume that the initial data satisfy

$$\mathbf{u}_0 \in V_2, \quad \boldsymbol{\sigma}_0 \in \Sigma(0), \quad (\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\mathbf{u}_0))_Q = (\mathbf{F}(0), \mathbf{u}_0)_V. \tag{9.1.11}$$

The last condition is a compatibility condition on the initial data that is necessary in many quasistatic problems. Physically, it is needed so as to guarantee that initially the state is in equilibrium, since otherwise the inertial terms cannot be neglected and the problem becomes dynamic.

It can be shown that if $\mathbf{u}$ and $\boldsymbol{\sigma}$ are smooth functions which satisfy (9.1.2)–(9.1.5), then

$$\mathbf{u}(t) \in V_2,\ (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q \geq (\mathbf{F}(t), \mathbf{v} - \mathbf{u}(t))_V\ \forall\, \mathbf{v} \in V_2, \quad (9.1.12)$$

$$\boldsymbol{\sigma}(t) \in \Sigma(t),\ (\boldsymbol{\tau} - \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q \geq 0 \quad \forall\, \boldsymbol{\tau} \in \Sigma(t), \quad (9.1.13)$$

for all $t \in [0, T]$.

These inequalities lead to the following two mixed weak formulations of problem (9.1.1)–(9.1.6).

Problem P^V_{vp-S1}. *Find a displacement field* $\mathbf{u} : [0, T] \to V$ *and a stress field* $\boldsymbol{\sigma} : [0, T] \to Q_1$ *such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t))) \quad \text{a.e. } t \in (0, T), \quad (9.1.14)$$

$$\mathbf{u}(t) \in V_2, \quad (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q \geq (\mathbf{F}(t), v - \mathbf{u}(t))_V \qquad \forall\, \mathbf{v} \in V_2,\ t \in [0, T], \quad (9.1.15)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0. \quad (9.1.16)$$

Problem P^V_{vp-S2}. *Find a displacement field* $\mathbf{u} : [0, T] \to V$ *and a stress field* $\boldsymbol{\sigma} : [0, T] \to Q_1$ *such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t))) \quad \text{a.e. } t \in (0, T), \quad (9.1.17)$$

$$\boldsymbol{\sigma}(t) \in \Sigma(t),\ (\boldsymbol{\tau} - \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q \geq 0 \quad \forall\, \boldsymbol{\tau} \in \Sigma(t),\ t \in [0, T], \quad (9.1.18)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0. \quad (9.1.19)$$

We note that Problems P^V_{vp-S1} and P^V_{vp-S2} are, formally, equivalent to the mechanical problem (9.1.1)–(9.1.6). Indeed, if $(\mathbf{u}, \boldsymbol{\sigma})$ represents a smooth solution of either one of the variational problems P^V_{vp-S1} or P^V_{vp-S2}, then it follows by using arguments as in [5] (see also Chap. 5), that the pair $(\mathbf{u}, \boldsymbol{\sigma})$ satisfies (9.1.1)–(9.1.6).

We turn to the existence of solutions for Problems P^V_{vp-S1} and P^V_{vp-S2}. We start with Problem P^V_{vp-S1} which was studied in [28].

Theorem 9.1.1. *Assume* (6.4.8), (6.4.9), (9.1.7), *and* (9.1.11). *Then there exists a unique solution* $(\mathbf{u}, \boldsymbol{\sigma})$ *of Problem* P^V_{vp-S1}, *and it satisfies*

$$\mathbf{u} \in W^{1,\infty}(0, T; V), \quad \boldsymbol{\sigma} \in W^{1,\infty}(0, T; Q_1). \quad (9.1.20)$$

Proof. The proof of Theorem 9.1.1 is based on fixed point arguments, and carried out in several steps which we describe below. The fixed point is related to the plastic part of the problem.

(i) Let $\boldsymbol{\eta} \in L^\infty(0,T;Q)$ and define the function $\mathbf{z}_\eta \in W^{1,\infty}(0,T;Q)$ by

$$\mathbf{z}_\eta(t) = \int_0^t \boldsymbol{\eta}(s)\,ds + \boldsymbol{\sigma}_0 - \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\mathbf{u}_0). \tag{9.1.21}$$

We prove, by using Corollary 6.3.3, that there exists a unique pair $(\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta)$ such that $\mathbf{u}_\eta \in W^{1,\infty}(0,T;V)$, $\boldsymbol{\sigma}_\eta \in W^{1,2}(0,T;Q_1)$ and

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)) + \mathbf{z}_\eta(t), \tag{9.1.22}$$

$$\mathbf{u}_\eta(t) \in V_2, \quad (\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)))_Q \geq (\mathbf{F}(t), \mathbf{v} - \mathbf{u}_\eta(t))_V, \quad \forall\, \mathbf{v} \in V_2,\ t \in [0,T]. \tag{9.1.23}$$

Moreover,

$$\mathbf{u}_\eta(0) = \mathbf{u}_0, \quad \boldsymbol{\sigma}_\eta(0) = \boldsymbol{\sigma}_0. \tag{9.1.24}$$

(ii) We consider next the operator $\Lambda : L^\infty(0,T;Q) \to L^\infty(0,T;Q)$ defined by

$$\Lambda\boldsymbol{\eta} = \mathcal{G}_{vp}(\boldsymbol{\sigma}_\eta, \boldsymbol{\varepsilon}(\mathbf{u}_\eta)) \quad \forall\, \boldsymbol{\eta} \in L^\infty(0,T;Q), \tag{9.1.25}$$

where $(\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta)$ is the solution of problem (9.1.22)–(9.1.23). Using Theorem 6.3.9 it follows that the operator Λ has a unique fixed point $\boldsymbol{\eta}^* \in L^\infty(0,T;Q)$.

(iii) *Existence.* Let $\boldsymbol{\eta}^* \in L^\infty(0,T;Q)$ be the fixed point of Λ, and let $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*})$ be the solution obtained in step (i) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$. Using (9.1.22) and (9.1.21) we have

$$\dot{\boldsymbol{\sigma}}_{\eta^*}(t) = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_{\eta^*}(t)) + \boldsymbol{\eta}^* \quad \text{a.e. } t \in (0,T).$$

Since

$$\boldsymbol{\eta}^*(t) = \Lambda\boldsymbol{\eta}^*(t) = \mathcal{G}_{vp}(\boldsymbol{\sigma}_{\eta^*}(t), \boldsymbol{\varepsilon}(\mathbf{u}_{\eta^*}(t))) \quad \forall\, t \in [0,T],$$

it follows that $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*})$ satisfies (9.1.14). Using now (9.1.23) and (9.1.24) we conclude that $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*})$ is a solution of Problem P^V_{vp-S1} which satisfies (9.1.20).

(iv) Uniqueness. Let $(\mathbf{u}, \boldsymbol{\sigma})$ be a solution of Problem P^V_{vp-S1} which satisfies (9.1.20) and let $\boldsymbol{\eta} \in L^\infty(0,T;Q)$ be the function defined by

$$\boldsymbol{\eta}(t) = \mathcal{G}_{vp}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t))) \quad \forall\, t \in [0,T]. \tag{9.1.26}$$

Let $\mathbf{z}_\eta$ be the function given in (9.1.21), then $(\mathbf{u}, \boldsymbol{\sigma})$ is a solution of problem (9.1.22)–(9.1.23). By step (i) it follows that

$$\mathbf{u} = \mathbf{u}_\eta, \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}_\eta. \tag{9.1.27}$$

Using now (9.1.25)–(9.1.27) we deduce that $\Lambda\boldsymbol{\eta} = \boldsymbol{\eta}$, and by the uniqueness of the fixed point of Λ, we obtain

$$\boldsymbol{\eta} = \boldsymbol{\eta}^*. \tag{9.1.28}$$

The uniqueness part in Theorem 9.1.1 is now a consequence of (9.1.27) and (9.1.28). □

We turn to an existence and uniqueness result for Problem P^V_{vp-S2}.

Theorem 9.1.2. *Assume* (6.4.8), (6.4.9), (9.1.7), *and* (9.1.11). *Then there exists a unique solution* $(\mathbf{u}, \boldsymbol{\sigma})$ *of Problem* P^V_{vp-S2}, *and it satisfies* (9.1.20).

Proof. We observe first that the variational inequality (9.1.18) is defined over the time-dependent convex set $\Sigma(t)$. To convert it into a variational inequality associated with a fixed convex set we change the variables. To that end we introduce the notation

$$\Sigma_0 = \{\boldsymbol{\tau} \in Q : (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q \geq 0 \quad \forall\, \mathbf{v} \in V_2\}, \tag{9.1.29}$$

$$\widetilde{\boldsymbol{\sigma}} = \boldsymbol{\varepsilon}(\mathbf{F}), \tag{9.1.30}$$

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \quad \bar{\boldsymbol{\sigma}}_0 = \boldsymbol{\sigma}_0 - \widetilde{\boldsymbol{\sigma}}(0), \tag{9.1.31}$$

and consider the problem

$$\begin{aligned} \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) = \mathcal{A}_{vp}^{-1}\dot{\bar{\boldsymbol{\sigma}}}(t) - \mathcal{A}_{vp}^{-1}\mathcal{G}_{vp}(\bar{\boldsymbol{\sigma}}(t) + \widetilde{\boldsymbol{\sigma}}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)))& \\ + \mathcal{A}_{vp}^{-1}\dot{\widetilde{\boldsymbol{\sigma}}}(t) \quad \text{a.e. } t \in (0,T),& \end{aligned} \tag{9.1.32}$$

$$\bar{\boldsymbol{\sigma}}(t) \in \Sigma_0, \quad (\boldsymbol{\tau} - \bar{\boldsymbol{\sigma}}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q \geq 0 \quad \forall\, \boldsymbol{\tau} \in \Sigma_0,\ \forall\, t \in [0,T], \tag{9.1.33}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \bar{\boldsymbol{\sigma}}(0) = \bar{\boldsymbol{\sigma}}_0, \tag{9.1.34}$$

with the unknowns $\mathbf{u} : [0,T] \to V$ and $\bar{\boldsymbol{\sigma}} : [0,T] \to Q_1$. From (9.1.10) it follows that

$$\Sigma(t) = \Sigma_0 + \{\widetilde{\boldsymbol{\sigma}}(t)\}, \tag{9.1.35}$$

for all $t \in [0,T]$, and using (9.1.7)–(9.1.9), we have

$$\widetilde{\boldsymbol{\sigma}}(t) \in W^{1,\infty}(0,T;Q_1). \tag{9.1.36}$$

Using (9.1.31)–(9.1.36) it is straightforward to show that the pair $(\mathbf{u}, \boldsymbol{\sigma})$ is a solution of Problem P^V_{vp-S2}, and also $\mathbf{u} \in W^{1,\infty}(0,T;V)$ and $\boldsymbol{\sigma} \in W^{1,\infty}(0,T;Q_1)$ if and only if $(\mathbf{u}, \bar{\boldsymbol{\sigma}})$ is a solution of problem (9.1.32)–(9.1.34) and satisfies $\mathbf{u} \in W^{1,\infty}(0,T;V)$ and $\bar{\boldsymbol{\sigma}} \in W^{1,\infty}(0,T;Q_1)$.

We turn now to problem (9.1.32)–(9.1.34) and solve it by using, again, a fixed point method. The proof proceeds in four steps, as follows.

(i) Let $\boldsymbol{\eta} \in L^\infty(0,T;Q)$ and define $\mathbf{z}_\eta \in W^{1,\infty}(0,T;Q)$ by

$$\mathbf{z}_\eta(t) = \int_0^t \boldsymbol{\eta}(s)\,ds + \boldsymbol{\varepsilon}(\mathbf{u}_0) - \mathcal{A}_{vp}^{-1}\boldsymbol{\sigma}_0, \quad t \in [0,T]. \tag{9.1.37}$$

Using Corollary 6.3.3 it follows that there exists a unique pair of functions $(\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta)$ such that $\mathbf{u}_\eta \in W^{1,\infty}(0,T;V)$, $\boldsymbol{\sigma}_\eta \in W^{1,\infty}(0,T;Q_1)$, and

$$\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)) = \mathcal{A}_{vp}^{-1}\boldsymbol{\sigma}_\eta(t) + \mathbf{z}_\eta(t) + \mathcal{A}_{vp}^{-1}\widetilde{\boldsymbol{\sigma}}(t), \tag{9.1.38}$$

$$\boldsymbol{\sigma}_\eta(t) \in \Sigma_0, \quad (\boldsymbol{\tau} - \boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)))_Q \geq 0 \quad \forall\, \boldsymbol{\tau} \in \Sigma_0, \tag{9.1.39}$$

for all $t \in [0, T]$. Moreover,

$$\mathbf{u}_\eta(0) = \mathbf{u}_0, \quad \boldsymbol{\sigma}_\eta(0) = \bar{\boldsymbol{\sigma}}_0. \tag{9.1.40}$$

(ii) We consider next the operator $\Lambda : L^\infty(0, T; Q) \to L^\infty(0, T; Q)$ defined by

$$\Lambda \boldsymbol{\eta} = -\mathcal{A}_{vp}^{-1} \mathcal{G}(\boldsymbol{\sigma}_\eta + \widetilde{\boldsymbol{\sigma}}, \boldsymbol{\varepsilon}(\mathbf{u}_\eta)), \tag{9.1.41}$$

where, for $\boldsymbol{\eta} \in L^\infty(0, T; Q)$, $(\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta)$ is the solution of the variational problem (9.1.38) and (9.1.39). Arguments similar to those used in the proof of Theorem 9.1.1 show that the operator Λ has a unique fixed point $\boldsymbol{\eta}^* \in L^\infty(0, T; Q)$.

(iii) Existence. Let $\boldsymbol{\eta}^* \in L^\infty(0, T; Q)$ be the fixed point of Λ and let $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*})$ be the solution obtained in step *(i)* for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$. From (9.1.38)–(9.1.41) it follows that $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*})$ represents a solution of problem (9.1.32)–(9.1.34) and, moreover, $\mathbf{u}_{\eta^*} \in W^{1,\infty}(0, T; V)$ and $\boldsymbol{\sigma}_{\eta^*} \in W^{1,\infty}(0, T; Q_1)$. It follows now that the pair $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*} + \widetilde{\boldsymbol{\sigma}})$ is a solution of Problem P^V_{vp-S2} and it satisfies (9.1.20).

(iv) Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ. It can also be deduced directly from (9.1.17)–(9.1.19) using (6.4.8), (6.4.9), and the Gronwall inequality. □

We next study the link between Problems P^V_{vp-S1} and P^V_{vp-S2}, which represent two different variational formulations of the mechanical problem (9.1.1)–(9.1.6). We have the following equivalence result.

Theorem 9.1.3. *Assume that conditions* (6.4.8), (6.4.9), (9.1.7), *and* (9.1.11) *hold. Let* $\mathbf{u} \in W^{1,\infty}(0, T; V)$ *and* $\boldsymbol{\sigma} \in W^{1,\infty}(0, T; Q_1)$. *Then the pair* $(\mathbf{u}, \boldsymbol{\sigma})$ *is the solution of* P^V_{vp-S1} *if and only if it solves* P^V_{vp-S2}.

The proof of the Theorem 9.1.3 can be found in the next section. It is based on the properties of projection operators.

Theorems 9.1.1 and 9.1.2 guarantee the unique solvability of Problems P^V_{vp-S1} and P^V_{vp-S1}, respectively, while Theorem 9.1.3 expresses the equivalence of these two problems. From these theorems we conclude that the mechanical problem (9.1.1)–(9.1.6) has a unique *weak solution* which solves both Problems P^V_{vp-S1} and P^V_{vp-S1}.

A fixed point method, similar to the one used in the proof of Theorem 9.1.1, as well as in other results described in this monograph, was employed in [214] in the study of a displacement-traction problem for viscoplastic materials with hardening, but without contact.

Error analysis for Problem P^V_{vp-S1} was performed in [29, 215]. A nonconforming finite element method was used in ([216, 217]) to solve the frictionless contact problems with viscoplastic materials of the form (6.4.7). An extension of the results presented here to the study of viscoplastic materials with internal state variable can be found in [218].

Variational analysis of the Signorini frictionless contact problem between two viscoplastic bodies of the form (6.4.7) can be found in [30], where the existence of the unique weak solution was proved. Its numerical analysis, including error estimates for discrete schemes, has been conducted in [31,219]. The problem of frictional contact between a viscoplastic body and a rigid foundation, with the Signorini condition, was studied in [220], where the existence of a weak solution was obtained, under a smallness assumption on the coefficient of friction.

9.2 Proof of Theorem 9.1.3

We begin the proof by assuming that $(\mathbf{u}, \boldsymbol{\sigma})$ is a solution of the variational Problem P^V_{vp-S1}. Let $t \in [0,T]$. Choosing first $\mathbf{v} = 2\mathbf{u}(t) \in V_2$ and then $\mathbf{v} = \mathbf{0} \in V_2$ in (9.1.15) yields

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q = (\mathbf{F}(t), \mathbf{u}(t))_V. \tag{9.2.1}$$

Using (9.1.15) and (9.2.1) we see that $\boldsymbol{\sigma} \in \Sigma(t)$. The inequality in (9.1.18) follows now from (9.1.10) and (9.2.1). We conclude that $(\mathbf{u}, \boldsymbol{\sigma})$ is a solution of Problem P^V_{vp-S2}.

Conversely, suppose that $(\mathbf{u}, \boldsymbol{\sigma})$ is a solution of Problem P^V_{vp-S2}. We first show that $\mathbf{u}(t) \in V_2$ for all $t \in [0,T]$. Arguing by contradiction, assume that there exists $t \in [0,T]$ such that $\mathbf{u}(t) \notin V_2$. Denote by $P\mathbf{u}(t)$ the projection of $\mathbf{u}(t)$ on the nonempty, closed, and convex set $V_2 \subset V$. Using (6.3.1) we have

$$\begin{aligned}(P\mathbf{u}(t) - \mathbf{u}(t), \mathbf{v})_V &\geq (P\mathbf{u}(t) - \mathbf{u}(t), P\mathbf{u}(t))_V \\ &> (P\mathbf{u}(t) - \mathbf{u}(t), \mathbf{u}(t))_V \quad \forall\, \mathbf{v} \in V_2.\end{aligned}$$

Therefore, there exists $\alpha \in \mathbb{R}$ such that

$$(P\mathbf{u}(t) - \mathbf{u}(t), \mathbf{v})_V > \alpha > (P\mathbf{u}(t) - \mathbf{u}(t), P\mathbf{u}(t))_V \quad \forall\, \mathbf{v} \in V_2. \tag{9.2.2}$$

We denote by $\widetilde{\boldsymbol{\tau}}(t)$ the element of Q given by $\widetilde{\boldsymbol{\tau}}(t) = \boldsymbol{\varepsilon}(P\mathbf{u}(t) - \mathbf{u}(t))$. Using this and (9.2.2) yields

$$(\widetilde{\boldsymbol{\tau}}(t), \boldsymbol{\varepsilon}(\mathbf{v}))_Q > \alpha > (\widetilde{\boldsymbol{\tau}}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q \quad \forall\, \mathbf{v} \in V_2. \tag{9.2.3}$$

Choosing $\mathbf{v} = \mathbf{0}$ in (9.2.3) we deduce that

$$\alpha < 0. \tag{9.2.4}$$

Assume now that there exists $\mathbf{w} \in V_2$ such that

$$(\widetilde{\boldsymbol{\tau}}(t), \boldsymbol{\varepsilon}(\mathbf{w}))_Q < 0. \tag{9.2.5}$$

Since $\lambda\, \mathbf{w} \in V_2$ for $\lambda \geq 0$, it follows from (9.2.3) that

$$\lambda\, (\widetilde{\boldsymbol{\tau}}(t), \boldsymbol{\varepsilon}(\mathbf{w}))_Q > \alpha \quad \forall\, \lambda \geq 0.$$

Passing to the limit $\lambda \to \infty$ and using (9.2.5), we obtain that $\alpha \le -\infty$ which contradicts the assumption $\alpha \in \mathbb{R}$. We conclude that

$$(\widetilde{\boldsymbol{\tau}}(t), \boldsymbol{\varepsilon}(\mathbf{w}))_Q \ge 0 \quad \forall\, \mathbf{w} \in V_2,$$

which implies that $\widetilde{\boldsymbol{\tau}}(t) \in \Sigma_0$ (see (9.1.29)). Using now (9.1.35) we deduce that $\widetilde{\boldsymbol{\tau}}(t) + \widetilde{\boldsymbol{\sigma}}(t) \in \Sigma(t)$ and from (9.2.4), (9.2.3), and (9.1.18) we find

$$0 > (\widetilde{\boldsymbol{\tau}}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q \ge (\boldsymbol{\sigma}(t) - \widetilde{\boldsymbol{\sigma}}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q,$$

which yields

$$(\boldsymbol{\sigma}(t) - \widetilde{\boldsymbol{\sigma}}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q < 0. \tag{9.2.6}$$

On the other hand, it follows from the proof of Theorem 9.1.2 that (9.1.33) holds, where $\bar{\boldsymbol{\sigma}}$ is given by (9.1.31). It is easy to check that $2(\boldsymbol{\sigma}(t) - \widetilde{\boldsymbol{\sigma}}(t)) \in \Sigma_0$ and therefore, by taking $\boldsymbol{\tau} = 2(\boldsymbol{\sigma}(t) - \widetilde{\boldsymbol{\sigma}}(t))$ in (9.1.33) and using (9.1.31) we have

$$(\boldsymbol{\sigma}(t) - \widetilde{\boldsymbol{\sigma}}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q \ge 0. \tag{9.2.7}$$

We note that inequalities (9.2.6) and (9.2.7) contradict each other, thus $\mathbf{u}(t) \in V_2$. Using (9.1.30) it follows that $\widetilde{\boldsymbol{\sigma}}(t) \in \Sigma(t)$ by the definition (9.1.10). By setting $\boldsymbol{\tau} = \widetilde{\boldsymbol{\sigma}}(t)$ in (9.1.18) we find

$$(\mathbf{F}(t), \mathbf{u}(t))_V \ge (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q. \tag{9.2.8}$$

Moreover, since $\boldsymbol{\sigma}(t) \in \Sigma(t)$ and $\mathbf{u}(t) \in V_2$, (9.1.10) implies

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q \ge (\mathbf{F}(t), \mathbf{u}(t))_V. \tag{9.2.9}$$

Combining now inequalities (9.2.8) and (9.2.9) we obtain

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q = (\mathbf{F}(t), \mathbf{u}(t))_V. \tag{9.2.10}$$

The inequality in (9.1.15) follows now from (9.1.10) and (9.2.10). We conclude that the pair $(\mathbf{u}, \boldsymbol{\sigma})$ is the solution of Problem P^V_{vp-S1}. $\square$

9.3 Frictional Contact with Normal Compliance

We describe a recent existence result for the problem of viscoplastic material with normal compliance and friction, obtained in [221]. Related and additional results for viscoplastic frictionless contact problems with normal compliance can be found in [51, 222–224].

The mechanical problem is the following.

Problem P_{vp-nc}. *Find a displacement field* $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$ *such that*

$$\dot{\boldsymbol{\sigma}} = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{G}_{vp}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u})) \qquad \text{in } \Omega_T, \tag{9.3.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \qquad \text{in } \Omega_T, \tag{9.3.2}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_D \times (0,T), \tag{9.3.3}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \qquad \text{on } \Gamma_N \times (0,T), \tag{9.3.4}$$

$$\left.\begin{aligned} & -\sigma_n = p_n(u_n - g), \\ & \|\boldsymbol{\sigma}_\tau\| \le p_\tau(u_n - g), \\ & \boldsymbol{\sigma}_\tau = -p_\tau(u_n - g)\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if } \dot{\mathbf{u}}_\tau \neq \mathbf{0} \end{aligned}\right\} \text{ on } \Gamma_C \times (0,T), \tag{9.3.5}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \qquad \text{in } \Omega. \tag{9.3.6}$$

We turn to the variational formulation of problem (9.3.1) – (9.3.6). We assume that the force and traction densities satisfy (9.1.7) and, as usual, we denote by $\mathbf{F}(t)$ the element of V given by (9.1.8). The normal and tangential compliance functions p_e ($e = n, \tau$) satisfy assumption (8.3.9) and the gap function satisfies (8.3.11). We use in this section the contact functional j defined in (8.3.14) and assume that the initial data satisfy

$$\mathbf{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q_1, \tag{9.3.7}$$

together with the compatibility condition

$$(\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + j(\mathbf{u}_0, \mathbf{v}) \ge (\mathbf{F}(0), \mathbf{v})_V \qquad \forall \mathbf{v} \in V. \tag{9.3.8}$$

The need for compatibility of the initial data was mentioned above, and without it the quasistatic approximation may be invalid.

Using standard arguments we have the following variational formulation of problem P_{vp-nc}.

Problem P^V_{vp-nc}. *Find a displacement field* $\mathbf{u} : [0,T] \to V$ *and a stress field* $\boldsymbol{\sigma} : [0,T] \to Q_1$ *such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{G}_{vp}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)) \quad \text{a.e. } t \in (0,T), \tag{9.3.9}$$

$$\begin{aligned} & (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\mathbf{u}(t), \mathbf{v}) - j(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \\ & \qquad \ge (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \quad \forall \mathbf{v} \in V, \text{ a.e. } t \in (0,T), \end{aligned} \tag{9.3.10}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0. \tag{9.3.11}$$

Existence of a solution to Problem P^V_{vp-nc}, under a smallness assumption on some of the problem data, is given in the following, and has been obtained in [221].

Theorem 9.3.1. *Assume that conditions* (6.4.8), (6.4.9), (8.3.9), (8.3.11), (9.1.7), (9.3.7), *and* (9.3.8) *hold. Then, there exists* $\mathcal{L}_0 > 0$, *depending only on* Ω, Γ_D, Γ_C, $\mathcal{A}_{vp}$, $\mathcal{G}_{vp}$ *and* T, *such that problem* P^V_{vp-nc} *has at least one solution if* $\mathcal{L}_n + \mathcal{L}_\tau < \mathcal{L}_0$. *Moreover, the solution satisfies* (9.1.20).

The proof of the theorem can be found in the next section.

We end this section with few comments on Theorem 9.3.1. First, we conclude from the theorem that problem (9.3.1)–(9.3.6) has at least one *weak solution* provided that $\mathcal{L}_n + \mathcal{L}_\tau$ is sufficiently small. Next, the critical value $\mathcal{L}_0$ depends on the constitutive functions, on the geometry of the problem, and on the duration of the process, but does not depend on the external forces, nor on the initial data. The verification of the condition $\mathcal{L}_n + \mathcal{L}_\tau < \mathcal{L}_0$, which guarantees the solvability of problem P^V_{vp-nc} as well as its physical interpretation, depends on the specific mechanical problem one has in mind. For example, consider the mechanical problem P_{vp-nc} in which the function p_n is given by (8.3.7) or (8.3.8) with $m = 1$ and the function p_τ is given by (8.6.10) or (8.6.11). It follows that assumption (8.3.9)(b) is satisfied with $\mathcal{L}_n = c_n$ and $\mathcal{L}_\tau = \mu c_n$ and, therefore, the condition $\mathcal{L}_n + \mathcal{L}_\tau < \mathcal{L}_0$ holds if $c_n(1 + \mu) < \mathcal{L}_0$. This may be interpreted as a smallness assumption on the coefficients c_n and μ.

The smallness assumption on the contact data seems to be related to the fact that we deal with a viscoplastic material. Indeed, we recall that in the corresponding frictional contact problem with a viscoelastic material, treated in Sect. 8.3, we proved both the existence and the uniqueness of the solution without any smallness assumption on the normal compliance functions.

Finally, the important question of uniqueness of the solution of problem P^V_{vp-nc} remains open. This is the case also for the local elastic problem with normal compliance, treated in [18], when the coefficient of friction and the loads are assumed to be sufficiently small, as well as for the global elastic problem with normal compliance and friction studied in [199].

9.4 Proof of Theorem 9.3.1

The proof is carried out in several steps and is based on the study of two intermediate problems, followed by an application of the Schauder fixed-point theorem. As usual, c denotes a positive generic constant which may depend on $\Omega, \Gamma_D, \Gamma_C, \mathcal{A}_{vp}, \mathcal{G}_{vp}$ and T, and whose value may change from place to place.

The main steps of the proof follow. The full details can be found in [221].

Intermediate Elastic Problem. We start by solving the contact problem when the viscoplastic part of the stress tensor $\boldsymbol{\eta}$, the normal contact stress, and the friction bound $\boldsymbol{\varphi}$ are given. Thus, we consider the functions $\boldsymbol{\eta}$ and $\boldsymbol{\varphi}$ which satisfy

$$\boldsymbol{\eta} \in L^\infty(0,T;Q), \tag{9.4.1}$$

$$\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in W^{1,\infty}(0,T;L^2(\Gamma_C)^2), \tag{9.4.2}$$

$$\boldsymbol{\varphi}(0) = \boldsymbol{\varphi}_0, \tag{9.4.3}$$

where $\boldsymbol{\varphi}_0$ is the element of $L^2(\Gamma_C)^2$ given by

$$\boldsymbol{\varphi}_0 = (p_n(u_{0n} - g), p_\tau(u_{0n} - g)). \tag{9.4.4}$$

We let $\mathbf{z}_\eta \in W^{1,\infty}(0,T;Q)$ and $l(\boldsymbol{\varphi}(t), \cdot) : V \to \mathbb{R}$ be defined by

$$\mathbf{z}_\eta(t) = \int_0^t \boldsymbol{\eta}(s)ds + \boldsymbol{\sigma}_0 - \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\mathbf{u}_0) \quad \forall t \in [0,T], \tag{9.4.5}$$

$$l(\boldsymbol{\varphi}(t), \mathbf{v}) = \int_{\Gamma_C} |\varphi_1(t)|\, v_n\, dS + \int_{\Gamma_C} |\varphi_2(t)|\, \|\mathbf{v}_\tau\|\, dS \quad \forall \mathbf{v} \in V, \tag{9.4.6}$$

and we consider the following intermediate variational problem.

Problem $P_V^{\varphi\eta}$. *Find a displacement field $\mathbf{u}_{\varphi\eta} : [0,T] \to V$ and a stress field $\boldsymbol{\sigma}_{\varphi\eta} : [0,T] \to Q_1$ such that*

$$\boldsymbol{\sigma}_{\varphi\eta}(t) = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\mathbf{u}_{\varphi\eta}(t)) + \mathbf{z}_\eta(t) \qquad \forall t \in [0,T], \tag{9.4.7}$$

$$(\boldsymbol{\sigma}_{\varphi\eta}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_{\varphi\eta}(t)))_Q + l(\boldsymbol{\varphi}(t), \mathbf{v}) - l(\boldsymbol{\varphi}(t), \dot{\mathbf{u}}_{\varphi\eta}(t)) \tag{9.4.8}$$
$$\geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}_{\varphi\eta}(t))_V \qquad \forall \mathbf{v} \in V, \text{ a.e. } t \in (0,T),$$

$$\mathbf{u}_{\varphi\eta}(0) = \mathbf{u}_0. \tag{9.4.9}$$

We note that (9.4.7) represent an elastic-like constitutive law. For this reason we refer to problem $P_V^{\varphi\eta}$ as an intermediate elastic problem for which we have the following result.

Lemma 9.4.1. *Problem $P_V^{\varphi\eta}$ has a unique solution which satisfies*

$$\mathbf{u}_{\varphi\eta} \in W^{1,\infty}(0,T;V), \quad \boldsymbol{\sigma}_{\varphi\eta} \in W^{1,\infty}(0,T;Q_1). \tag{9.4.10}$$

Moreover, there exists a positive constant c such that

$$\begin{aligned} &\|\mathbf{u}_{\varphi\eta}\|_{W^{1,\infty}(0,T;V)} + \|\boldsymbol{\sigma}_{\varphi\eta}\|_{W^{1,\infty}(0,T;Q_1)} \\ &\quad \leq c\,(\|\boldsymbol{\varphi}\|_{W^{1,\infty}(0,T;L^2(\Gamma_C)^2)} + \|\mathbf{F}\|_{W^{1,\infty}(0,T;V)} \\ &\qquad + \|\mathbf{z}_\eta\|_{W^{1,\infty}(0,T;Q)} + \|\mathbf{u}_0\|_V). \end{aligned} \tag{9.4.11}$$

Proof. The proof is carried out in several steps, using arguments similar to those in [220, 225]. Since the modifications are straightforward we omit the details. We employ the bilinear form $a : V \times V \to \mathbb{R}$ which is given by

$$a(\mathbf{u}, \mathbf{v}) = (\mathcal{A}_{vp}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q. \tag{9.4.12}$$

The steps of the proof are as follows.

i) Incremental time-discretized problems. Let $0 = t_0 < t_1 < \cdots < t_M = T$ be a uniform partition of the time interval $[0,T]$ such that $t_m = mh$, for $m = 1, \ldots, M$, where $h = T/M$. For a continuous function $\mathbf{w}(t)$ we use the notation $\mathbf{w}_m = \mathbf{w}(t_m)$. For a sequence $\{\mathbf{w}_m\}_{m=0}^M$, we denote the differences by $\Delta\mathbf{w}_m = \mathbf{w}_m - \mathbf{w}_{m-1}$ and by $\delta\mathbf{w}_m = \Delta\mathbf{w}_m/h$ the corresponding divided differences. No summation is implied over the repeated index m.

Using standard arguments for elliptic variational inequalities yields the existence of the unique sequence $\{\mathbf{u}_m^{\varphi\eta}\}_{m=0}^M \subset V$ such that $\mathbf{u}_0^{\varphi\eta} = \mathbf{u}_0$ and, for $m = 1, \ldots, M$,

$$a(\mathbf{u}_m^{\varphi\eta}, \mathbf{v} - \delta\mathbf{u}_m^{\varphi\eta}) + (\mathbf{z}_{\eta m}, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\delta\mathbf{u}_m^{\varphi\eta}))_Q + l(\boldsymbol{\varphi}_m, \mathbf{v}) - l(\boldsymbol{\varphi}_m, \delta\mathbf{u}_m^{\varphi\eta}) \geq (\mathbf{F}_m, \mathbf{v} - \delta\mathbf{u}_m^{\varphi\eta})_V \quad \forall\, \mathbf{v} \in V. \tag{9.4.13}$$

Moreover, the following two estimates hold:

$$\|\mathbf{u}_m^{\varphi\eta}\|_V \leq c\,(\|\boldsymbol{\varphi}_m\|_{L^2(\Gamma_C)^2} + \|\mathbf{F}_m\|_V + \|\mathbf{z}_{\eta m}\|_Q), \tag{9.4.14}$$

for $0 \leq m \leq M$, and

$$\|\Delta\mathbf{u}_m^{\varphi\eta}\|_V \leq c\,(\|\Delta\boldsymbol{\varphi}_m\|_{L^2(\Gamma_C)^2} + \|\Delta\mathbf{F}_m\|_V + \|\Delta\mathbf{z}_{\eta m}\|_Q), \tag{9.4.15}$$

for $1 \leq m \leq M$. To prove the estimate (9.4.14) for $m = 0$ we have to use the compatibility assumption on the initial data

$$a(\mathbf{u}_0, \mathbf{v}) + (\mathbf{z}_0, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + l(\boldsymbol{\varphi}(0), \mathbf{v}) \geq (\mathbf{F}(0), \mathbf{v})_V,$$

which follows from (9.3.8), (9.4.2)–(9.4.4) and the definition of the functionals j and l.

ii) Weak convergence.* We define a piecewise linear function $\mathbf{u}_M^{\varphi\eta}$ using linear interpolation of the sequence $\{\mathbf{u}_m^{\varphi\eta}\}_{m=0}^M \subset V$ as follows,

$$\mathbf{u}_M^{\varphi\eta}(t) = \mathbf{u}_{m-1}^{\varphi\eta} + \frac{t - t_{m-1}}{h}\,\Delta\mathbf{u}_m^{\varphi\eta}, \tag{9.4.16}$$

for $t \in [t_{m-1}, t_m]$ and $1 \leq m \leq M$.

Each one of the functions $\mathbf{u}_M^{\varphi\eta}$ belongs to the space $W^{1,\infty}(0,T;V)$ and the assertions (9.4.14) and (9.4.15) show that the sequence $\{\mathbf{u}_M^{\varphi\eta}\}_M$ is bounded there. Therefore, there exists a function $\mathbf{u}_{\varphi\eta} \in W^{1,\infty}(0,T;V)$ and a subsequence of $\{\mathbf{u}_M^{\varphi\eta}\}_M$, still denoted by $\{\mathbf{u}_M^{\varphi\eta}\}_M$, such that

$$\mathbf{u}_M^{\varphi\eta} \to \mathbf{u}_{\varphi\eta} \quad \text{weak}\,* \text{ in } \ W^{1,\infty}(0,T;V) \ \text{ as } \ M \to \infty. \tag{9.4.17}$$

Actually, in step v) below we will show that the whole sequence $\{\mathbf{u}_M^{\varphi\eta}\}_M$ converges to $\mathbf{u}_{\varphi\eta}$. Also, there exists a positive constant c such that

$$\|\mathbf{u}_{\varphi\eta}\|_{W^{1,\infty}(0,T;V)} \leq c\,(\|\boldsymbol{\varphi}\|_{W^{1,\infty}(0,T;L^2(\Gamma_C)^2)} + \|\mathbf{F}\|_{W^{1,\infty}(0,T;V)} + \|\mathbf{z}_\eta\|_{W^{1,\infty}(0,T;Q)} + \|\mathbf{u}_0\|_V). \tag{9.4.18}$$

The proof of (9.4.18) is based on inequalities (9.4.14) and (9.4.15), which provide estimates for the functions $\mathbf{u}_M^{\varphi\eta}$ in the norm of $W^{1,\infty}(0,T;V)$.

iii) Convergence and semicontinuity. Let $\mathbf{u}_{\varphi\eta}$ denote an element of $W^{1,\infty}(0,T;V)$ that was obtained in step *ii)* as the weak* limit of a subsequence of $\{\mathbf{u}_M^{\varphi\eta}\}_M$. We introduce the piecewise constant functions $\widetilde{\mathbf{u}}_M^{\varphi\eta} : [0,T] \to V$, $\widetilde{\mathbf{z}}_M^{\eta} : [0,T] \to Q$, $\widetilde{\boldsymbol{\varphi}}_M : [0,T] \to L^2(\Gamma_C)^2$, and $\widetilde{\mathbf{F}}_M : [0,T] \to V$ by

$$\widetilde{\mathbf{u}}_M^{\varphi\eta}(t) = \mathbf{u}_m^{\varphi\eta}, \quad \widetilde{\mathbf{z}}_M^{\eta}(t) = \mathbf{z}_{\eta m}, \quad \widetilde{\boldsymbol{\varphi}}_M(t) = \boldsymbol{\varphi}_m, \quad \widetilde{\mathbf{F}}_M(t) = \mathbf{F}_m, \tag{9.4.19}$$

for $t \in (t_{m-1}, t_m]$, and $m = 1, \dots, M$. For almost every $t \in (0,T)$ we have

$$\|\widetilde{\mathbf{u}}_M^{\varphi\eta}(t) - \mathbf{u}_M^{\varphi\eta}(t)\|_V \le \frac{T}{M}\, \|\dot{\mathbf{u}}_M^{\varphi\eta}(t)\|_V, \tag{9.4.20}$$

and, since $\{\dot{\mathbf{u}}_M^{\varphi\eta}\}_M$ is bounded in $L^\infty(0,T;V)$, we deduce that

$$\widetilde{\mathbf{u}}_M^{\varphi\eta} \to \mathbf{u}_{\varphi\eta} \quad \text{weak}^* \text{ in } L^\infty(0,T;V) \text{ as } M \to \infty. \tag{9.4.21}$$

Furthermore, since $\mathbf{z}_\eta \in W^{1,\infty}(0,T;Q)$, $\boldsymbol{\varphi} \in W^{1,\infty}(0,T;L^2(\Gamma_C)^2)$, and $\mathbf{F} \in W^{1,\infty}(0,T;V)$, we obtain

$$\widetilde{\mathbf{z}}_M^{\eta} \to \mathbf{z}_\eta \quad \text{strongly in} \quad L^2(0,T;Q), \tag{9.4.22}$$

$$\widetilde{\boldsymbol{\varphi}}_M \to \boldsymbol{\varphi} \quad \text{strongly in} \quad L^2(0,T;L^2(\Gamma_C)^2), \tag{9.4.23}$$

$$\widetilde{\mathbf{F}}_M \to \mathbf{F} \quad \text{strongly in} \quad L^2(0,T;V), \tag{9.4.24}$$

as $M \to \infty$. Using (9.4.13) we find that for $\mathbf{v} \in L^2(0,T;V)$ we have

$$\begin{aligned}\int_0^T a(\widetilde{\mathbf{u}}_M^{\varphi\eta}(t), \mathbf{v}(t))dt &+ \int_0^T (\widetilde{\mathbf{z}}_M^{\eta}(t), \boldsymbol{\varepsilon}(\mathbf{v}(t)) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_M^{\varphi\eta}(t)))_Q dt \\ &+ \int_0^T l(\widetilde{\boldsymbol{\varphi}}_M(t), \mathbf{v}(t))dt \ge \int_0^T (\widetilde{\mathbf{F}}_M(t), \mathbf{v}(t) - \dot{\mathbf{u}}_M^{\varphi\eta}(t))_V dt \\ &+ \int_0^T a(\widetilde{\mathbf{u}}_M^{\varphi\eta}(t), \dot{\mathbf{u}}_M^{\varphi\eta}(t))dt + \int_0^T l(\widetilde{\boldsymbol{\varphi}}_M(t), \dot{\mathbf{u}}_M^{\varphi\eta}(t))dt.\end{aligned} \tag{9.4.25}$$

To pass to the lower limit in (9.4.25), we need the following convergence results. First, (9.4.21)–(9.4.24) and the weak* convergence of $\{\dot{\mathbf{u}}_M^{\varphi\eta}\}_M$ to $\dot{\mathbf{u}}_{\varphi\eta}$ yield

$$\int_0^T a(\widetilde{\mathbf{u}}_M^{\varphi\eta}(t), \mathbf{v}(t))\, dt \to \int_0^T a(\mathbf{u}_{\varphi\eta}(t), \mathbf{v}(t))\, dt, \tag{9.4.26}$$

$$\begin{aligned}\int_0^T (\widetilde{\mathbf{z}}_M^{\eta}(t), \boldsymbol{\varepsilon}(\mathbf{v}(t)) \;-\; \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_M^{\varphi\eta}(t)))_Q\, dt \\ \to \int_0^T (\mathbf{z}_\eta(t), \boldsymbol{\varepsilon}(\mathbf{v}(t)) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_{\varphi\eta}(t)))_Q\, dt,\end{aligned} \tag{9.4.27}$$

$$\int_0^T (\widetilde{\mathbf{F}}_M(t), \mathbf{v}(t) - \dot{\mathbf{u}}_M^{\varphi\eta}(t))_V\, dt \to \int_0^T (\mathbf{F}(t), \mathbf{v}(t) - \dot{\mathbf{u}}_{\varphi\eta}(t))_V\, dt, \tag{9.4.28}$$

$$\int_0^T l(\widetilde{\boldsymbol{\varphi}}_M(t), \mathbf{v}(t))dt \to \int_0^T l(\boldsymbol{\varphi}(t), \mathbf{v}(t))dt, \tag{9.4.29}$$

for all $\mathbf{v} \in L^2(0,T;V)$, as $M \to \infty$. Next, using standard semicontinuity arguments and some straightforward manipulations we find

$$\liminf_{M\to\infty} \int_0^T l(\widetilde{\boldsymbol{\varphi}}_M(t), \dot{\mathbf{u}}_M^{\varphi\eta}(t))\, dt \geq \int_0^T l(\boldsymbol{\varphi}(t), \dot{\mathbf{u}}_{\varphi\eta}(t))\, dt, \tag{9.4.30}$$

$$\begin{aligned}\liminf_{M\to\infty} \int_0^T a(\widetilde{\mathbf{u}}_M^{\varphi\eta}(t), \dot{\mathbf{u}}_M^{\varphi\eta}(t))\, dt &\geq \frac{1}{2}\, a(\mathbf{u}_{\varphi\eta}(T), \mathbf{u}_{\varphi\eta}(T)) - \frac{1}{2}\, a(\mathbf{u}_0, \mathbf{u}_0) \\ &= \int_0^T a(\mathbf{u}_{\varphi\eta}(t), \dot{\mathbf{u}}_{\varphi\eta}(t))\, dt. \end{aligned}\tag{9.4.31}$$

iv) Existence. It follows from (9.4.25)–(9.4.31) that

$$\begin{aligned}&a(\mathbf{u}_{\varphi\eta}(t), \mathbf{v}(t) - \dot{\mathbf{u}}_{\varphi\eta}(t)) + (\mathbf{z}_\eta(t), \boldsymbol{\varepsilon}(\mathbf{v}(t)) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_{\varphi\eta}(t)))_Q \\ &\quad + l(\boldsymbol{\varphi}(t), \mathbf{v}) - l(\boldsymbol{\varphi}(t), \dot{\mathbf{u}}_{\varphi\eta}(t)) \ \geq (\mathbf{F}(t), \mathbf{v}(t) - \dot{\mathbf{u}}_{\varphi\eta}(t))_V, \end{aligned}\tag{9.4.32}$$

for all $\mathbf{v} \in V$, a.e. $t \in (0,T)$. Moreover, $\mathbf{u}_{\varphi\eta}(0) = \mathbf{u}_0$ and so (9.4.9) holds. Let $\boldsymbol{\sigma}_{\varphi\eta} \in W^{1,\infty}(0,T;Q)$ be the element given by (9.4.7). We use (9.4.32) and (9.4.12) to obtain (9.4.8). Also, (9.4.8) implies that

$$\mathrm{Div}\, \boldsymbol{\sigma}_{\varphi\eta}(t) + \mathbf{f}_0(t) = \mathbf{0} \quad \forall t \in [0,T], \tag{9.4.33}$$

and then it follows from (9.1.7) that $\boldsymbol{\sigma}_{\varphi\eta} \in W^{1,\infty}(0,T;Q_1)$. We conclude that $(\mathbf{u}_{\varphi\eta}, \boldsymbol{\sigma}_{\varphi\eta})$ is a solution of Problem $P_V^{\varphi\eta}$ and it satisfies (9.4.10).

v) Uniqueness and boundness. The uniqueness of the solution follows from the unique solvability of the Cauchy problem (9.4.32) and (9.4.9). Moreover, this shows that the whole sequence $\{\mathbf{u}_M^{\varphi\eta}\}_M$ converges weak* in $W^{1,\infty}(0,T;V)$ to $\mathbf{u}_{\varphi\eta}$. Finally, estimate (9.4.11) follows from (9.4.18), (9.4.7), (9.4.33), and (6.4.8). □

Intermediate Viscoplastic Problem. We solve the contact problem for the fully elastic-viscoplastic constitutive relation when the normal stress and the friction bound on the contact surface are prescribed. To that end we shall use the Banach fixed point theorem. Let $\boldsymbol{\varphi}$ be a given function which satisfies (9.4.2), (9.4.3) and consider the following intermediate problem.

Problem P_V^φ. *Find a displacement field $\mathbf{u}_\varphi : [0,T] \to V$ and a stress field $\boldsymbol{\sigma}_\varphi : [0,T] \to Q_1$ such that*

$$\dot{\boldsymbol{\sigma}}_\varphi(t) = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\varphi(t)) + \mathcal{G}_{vp}(\boldsymbol{\sigma}_\varphi(t), \boldsymbol{\varepsilon}(\mathbf{u}_\varphi(t))) \quad \text{a.e. } t \in (0,T), \tag{9.4.34}$$

$$\begin{aligned}&(\boldsymbol{\sigma}_\varphi(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\varphi(t)))_Q + l(\boldsymbol{\varphi}(t), \mathbf{v}) - l(\boldsymbol{\varphi}(t), \dot{\mathbf{u}}_\varphi(t)) \\ &\quad \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}_\varphi(t))_V \quad \forall \mathbf{v} \in V, \ \text{a.e. } t \in (0,T), \end{aligned}\tag{9.4.35}$$

$$\mathbf{u}_\varphi(0) = \mathbf{u}_0, \quad \boldsymbol{\sigma}_\varphi(0) = \boldsymbol{\sigma}_0. \tag{9.4.36}$$

We have following the existence and uniqueness result.

Lemma 9.4.2. *Problem P_V^{φ} has a unique solution which satisfies*

$$\mathbf{u}_\varphi \in W^{1,\infty}(0,T;V), \quad \boldsymbol{\sigma}_\varphi \in W^{1,\infty}(0,T;Q_1). \tag{9.4.37}$$

Moreover, there exists a positive constant c such that

$$\begin{aligned}&\|\mathbf{u}_\varphi\|_{W^{1,\infty}(0,T;V)} + \|\boldsymbol{\sigma}_\varphi\|_{W^{1,\infty}(0,T;Q_1)}\\ &\quad \le c\,(\|\boldsymbol{\varphi}\|_{W^{1,\infty}(0,T;L^2(\Gamma_C)^2)} + \|\mathbf{F}\|_{W^{1,\infty}(0,T;V)}\\ &\qquad + \|\mathbf{u}_0\|_V + \|\boldsymbol{\sigma}_0\|_{Q_1} + \|\mathcal{G}(\mathbf{0},\mathbf{0})\|_Q).\end{aligned} \tag{9.4.38}$$

Proof. The proof is carried out in three steps.

i) Using the Banach fixed point theorem. We consider the operator $\Lambda_\varphi : L^\infty(0,T;Q) \to L^\infty(0,T;Q)$ defined by

$$\Lambda_\varphi \boldsymbol{\eta} = \mathcal{G}_{vp}(\boldsymbol{\sigma}_{\varphi\eta}, \boldsymbol{\varepsilon}(\mathbf{u}_{\varphi\eta})) \quad \forall \boldsymbol{\eta} \in L^\infty(0,T;Q), \tag{9.4.39}$$

where $(\mathbf{u}_{\varphi\eta}, \boldsymbol{\sigma}_{\varphi\eta})$ is the solution of the intermediate elastic problem $P_V^{\varphi\eta}$ provided by Lemma 9.4.1. We prove that the operator Λ_φ has a unique fixed point $\boldsymbol{\eta}_\varphi^* \in L^\infty(0,T;Q)$. To this end, let $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in L^\infty(0,T;Q)$ and denote $\mathbf{u}_i = \mathbf{u}_{\varphi\eta_i}$, $\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{\varphi\eta_i}$, and $\mathbf{z}_i = \mathbf{z}_{\eta_i}$ for $i = 1, 2$. We rewrite (9.4.32) as

$$a(\mathbf{u}_1, \mathbf{v} - \dot{\mathbf{u}}_1) + (\mathbf{z}_1, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1))_Q + l(\boldsymbol{\varphi}, \mathbf{v}) - l(\boldsymbol{\varphi}, \dot{\mathbf{u}}_1) \ge (\mathbf{F}, \mathbf{v} - \dot{\mathbf{u}}_1)_V,$$

$$a(\mathbf{u}_2, \mathbf{v} - \dot{\mathbf{u}}_2) + (\mathbf{z}_2, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2))_Q + l(\boldsymbol{\varphi}, \mathbf{v}) - l(\boldsymbol{\varphi}, \dot{\mathbf{u}}_2) \ge (\mathbf{F}, \mathbf{v} - \dot{\mathbf{u}}_2)_V$$

for all $\mathbf{v} \in V$, a.e. on $(0,T)$. Let $t \in [0,T]$. We choose $\mathbf{v} = \dot{\mathbf{u}}_2$ in the first inequality, $\mathbf{v} = \dot{\mathbf{u}}_1$ in the second inequality, add the two inequalities and integrate the result on $[0,t]$, thus

$$\begin{aligned}c\,\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 &\le \|\mathbf{z}_1(t) - \mathbf{z}_2(t)\|_Q \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V\\ &\quad + \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V ds.\end{aligned}$$

Now,

$$\|\mathbf{z}_1(t) - \mathbf{z}_2(t)\|_Q \le \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q ds,$$

and, therefore, we find

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 \le c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q^2 ds + c \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 ds.$$

Applying the Gronwall inequality we obtain

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 \le c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q^2 ds. \tag{9.4.40}$$

From the definition (9.4.7) of $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ we have

$$\begin{aligned}\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t) &= \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\mathbf{u}_1(t) - \mathbf{u}_2(t)) + \mathbf{z}_1(t) - \mathbf{z}_2(t) \\ &= \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\mathbf{u}_1(t) - \mathbf{u}_2(t)) + \int_0^t (\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s))\, ds,\end{aligned}$$

and by using (9.4.40) we deduce

$$\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_Q^2 \le c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q^2 ds. \tag{9.4.41}$$

Finally,

$$\Lambda_\varphi \boldsymbol{\eta}_1(t) - \Lambda_\varphi \boldsymbol{\eta}_2(t) = \mathcal{G}_{vp}(\boldsymbol{\sigma}_1(t), \boldsymbol{\varepsilon}(\mathbf{u}_1(t))) - \mathcal{G}_{vp}(\boldsymbol{\sigma}_2(t), \boldsymbol{\varepsilon}(\mathbf{u}_2(t))).$$

Using the assumptions (6.4.9) and the bounds (9.4.40) and (9.4.41) yields

$$\|\Lambda_\varphi \boldsymbol{\eta}_1(t) - \Lambda_\varphi \boldsymbol{\eta}_2(t)\|_Q^2 \le c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q^2 ds. \tag{9.4.42}$$

We deduce now from (9.4.42) and an application of Banach's fixed point theorem to a suitable iteration power of the map Λ_φ that the operator Λ_φ has a unique fixed point $\boldsymbol{\eta}_\varphi^* \in L^\infty(0, T; Q)$.

ii) Existence. Let $(\mathbf{u}_\varphi, \boldsymbol{\sigma}_\varphi)$ be the solution of problem $P_V^{\varphi\eta}$ for $\boldsymbol{\eta} = \boldsymbol{\eta}_\varphi$, that is, $\mathbf{u}_\varphi = \mathbf{u}_{\varphi\eta_\varphi}$ and $\boldsymbol{\sigma}_\varphi = \boldsymbol{\sigma}_{\varphi\eta_\varphi}$. Using (9.4.7) and (9.4.5) we have

$$\dot{\boldsymbol{\sigma}}_\varphi(t) = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\varphi(t)) + \boldsymbol{\eta}_\varphi(t) \quad \text{a.e. } t \in (0, T)$$

and using (9.4.39) yields

$$\boldsymbol{\eta}_\varphi(t) = \Lambda_\varphi \boldsymbol{\eta}_\varphi(t) = \mathcal{G}_{vp}(\boldsymbol{\sigma}_\varphi(t), \boldsymbol{\varepsilon}(\mathbf{u}_\varphi(t))) \quad \text{a.e. } t \in (0, T).$$

Combining the previous two equalities we find that $(\mathbf{u}_\varphi, \boldsymbol{\sigma}_\varphi)$ satisfies (9.4.34). Moreover, from (9.4.5), (9.4.7), and (9.4.9) it follows that (9.4.36) holds and, finally, (9.4.35) is a consequence of (9.4.8). We conclude that $(\mathbf{u}_\varphi, \boldsymbol{\sigma}_\varphi)$ is a solution of problem P_V^φ and it satisfies (9.4.37).

iii) Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ_φ. Note that a similar argument was already used (page 138) in the proof of the uniqueness part in Theorem 9.1.1.

iv) Boundedness. It follows from (9.4.34)–(9.4.36) and the arguments used in the proof of Lemma 3.1 in [220], after straightforward but tedious manipulations, that

$$\begin{aligned}&\|\mathbf{u}_\varphi(t)\|_V + \|\boldsymbol{\sigma}_\varphi(t)\|_{Q_1} \\ &\quad \le c(\|\boldsymbol{\varphi}\|_{W^{1,\infty}(0,T;L^2(\Gamma_C)^2)} + \|\mathbf{F}\|_{W^{1,\infty}(0,T;V)} \\ &\qquad + \|\mathbf{u}_0\|_V + \|\boldsymbol{\sigma}_0\|_{Q_1} + \|\mathcal{G}_{vp}(\mathbf{0}, \mathbf{0})\|_Q),\end{aligned} \tag{9.4.43}$$

for all $t \in [0,T]$. We let $\mathbf{z}_\varphi = \mathbf{z}_{\eta_\varphi}$, and by using (9.4.5) and (9.4.39) again, we deduce that

$$\mathbf{z}_\varphi(t) = \int_0^t \mathcal{G}_{vp}(\boldsymbol{\sigma}_\varphi(s), \boldsymbol{\varepsilon}(\mathbf{u}_\varphi(s)))ds + \boldsymbol{\sigma}_0 - \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\mathbf{u}_0) \tag{9.4.44}$$

for all $t \in [0,T]$. Writing now

$$\mathcal{G}_{vp}(\boldsymbol{\sigma}_\varphi(s), \boldsymbol{\varepsilon}(\mathbf{u}_\varphi(s))) = \Big[\mathcal{G}_{vp}(\boldsymbol{\sigma}_\varphi(s), \boldsymbol{\varepsilon}(\mathbf{u}_\varphi(s))) - \mathcal{G}_{vp}(\mathbf{0},\mathbf{0})\Big] + \mathcal{G}_{vp}(\mathbf{0},\mathbf{0})$$

and using condition (6.4.9), it follows from (9.4.44) that

$$\begin{aligned}\|\mathbf{z}_\varphi(t)\|_Q \leq c\Big(&\|\mathbf{u}_0\|_V + \|\boldsymbol{\sigma}_0\|_{Q_1} + \|\mathcal{G}_{vp}(\mathbf{0},\mathbf{0})\|_Q \\ &+ \int_0^t (\|\mathbf{u}_\varphi(s)\|_V + \|\boldsymbol{\sigma}_\varphi(s)\|_Q)ds\Big)\end{aligned} \tag{9.4.45}$$

a.e. $t \in (0,T)$, and

$$\|\dot{\mathbf{z}}_\varphi(t)\|_Q \leq c\,(\|\mathbf{u}_\varphi(t)\|_V + \|\boldsymbol{\sigma}_\varphi(t)\|_Q + \|\mathcal{G}_{vp}(\mathbf{0},\mathbf{0})\|_Q) \tag{9.4.46}$$

for all $t \in [0,T]$.

Now, keeping in mind (9.4.43), (9.4.45), and (9.4.46) we obtain

$$\begin{aligned}\|\mathbf{z}_\varphi\|_{W^{1,\infty}(0,T;Q)} \leq c\Big(&\|\mathbf{u}_0\|_V + \|\boldsymbol{\sigma}_0\|_{Q_1} + \|\mathcal{G}_{vp}(\mathbf{0},\mathbf{0})\|_Q \\ &+\|\boldsymbol{\varphi}\|_{W^{1,\infty}(0,T;L^2(\Gamma_C)^2)} \\ &+\|\mathbf{F}\|_{W^{1,\infty}(0,T;V)}\Big) \quad \forall t \in [0,T].\end{aligned} \tag{9.4.47}$$

We use (9.4.11) with $\boldsymbol{\eta} = \boldsymbol{\eta}_\varphi$ and (9.4.47) to obtain (9.4.38). □

The contact boundary operator. Let $k > 0$ and denote by Lip_k^0 the set

$$Lip_k^0 = \{\, \boldsymbol{\varphi} \in W^{1,\infty}(0,T;L^2(\Gamma_C)^2) : \|\dot{\boldsymbol{\varphi}}\|_{L^\infty(0,T;L^2(\Gamma_C)^2)} \leq k,\ \boldsymbol{\varphi}(0) = \boldsymbol{\varphi}_0 \,\},$$

where $\boldsymbol{\varphi}_0$ is the element of $L^2(\Gamma_C)^2$ given by (9.4.3). We note that

$$\|\boldsymbol{\varphi}\|_{W^{1,\infty}(0,T;L^2(\Gamma_C)^2)} \leq k(T+1) + \|\boldsymbol{\varphi}_0\|_{L^2(\Gamma_C)^2} \quad \forall \boldsymbol{\varphi} \in Lip_k^0, \tag{9.4.48}$$

and for every $\boldsymbol{\varphi} \in Lip_k^0$ we denote by $(\mathbf{u}_\varphi, \boldsymbol{\sigma}_\varphi)$ the solution of the intermediate viscoplastic problem P_V^φ provided by Lemma 9.4.2.

We begin with the following estimate.

Lemma 9.4.3. *There exists a positive constant c_k such that*

$$\begin{aligned}&\|\mathbf{u}_{\varphi_1} - \mathbf{u}_{\varphi_2}\|_{C([0,T];V)} + \|\boldsymbol{\sigma}_{\varphi_1} - \boldsymbol{\sigma}_{\varphi_2}\|_{C([0,T];Q_1)} \\ &\qquad \leq c_k\big(\|\mathbf{u}_{\varphi_1} - \mathbf{u}_{\varphi_2}\|_{C([0,T];L^2(\Gamma_C)^d)}\big)^{\frac{1}{2}}\end{aligned} \tag{9.4.49}$$

for all $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2 \in Lip_k^0$.

Proof. Let $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \in Lip_k^0$ and let $t \in [0,T]$. For $i = 1,2$ we use the notation $\mathbf{u}_{\varphi_i} = \mathbf{u}_i$, $\boldsymbol{\sigma}_{\varphi_i} = \boldsymbol{\sigma}_i$ and then it follows from $P_V^{\varphi_i}$ that each $\mathbf{u}_i$ satisfies

$$a(\mathbf{u}_i(t), \mathbf{v}) + l(\boldsymbol{\varphi}_i(t), \mathbf{v}) \geq (\mathbf{F}(t), \mathbf{v})_V - (\boldsymbol{\sigma}_0 - \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\mathbf{u}_0) + \int_0^t \mathcal{G}_{vp}(\boldsymbol{\sigma}_i(s), \boldsymbol{\varepsilon}(\mathbf{u}_i(s)))ds, \boldsymbol{\varepsilon}(\mathbf{v}))_Q \quad \forall \mathbf{v} \in V. \tag{9.4.50}$$

Using standard arguments it follows from (9.4.50) that

$$\begin{aligned} a(\mathbf{u}_2(t) - \mathbf{u}_1(t), \mathbf{u}_2(t) - \mathbf{u}_1(t)) \\ \leq l(\boldsymbol{\varphi}_1(t), \mathbf{u}_2(t) - \mathbf{u}_1(t)) + l(\boldsymbol{\varphi}_2(t), \mathbf{u}_1(t) - \mathbf{u}_2(t)) \end{aligned}$$
$$+(\int_0^t [\mathcal{G}_{vp}(\boldsymbol{\sigma}_2(s), \boldsymbol{\varepsilon}(\mathbf{u}_2(s))) - \mathcal{G}_{vp}(\boldsymbol{\sigma}_1(s), \boldsymbol{\varepsilon}(\mathbf{u}_1(s)))]ds, \boldsymbol{\varepsilon}(\mathbf{u}_1(t)) - \boldsymbol{\varepsilon}(\mathbf{u}_2(t)))_Q.$$

By using (6.4.8) and (6.4.9) we find

$$\begin{aligned} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 \leq c\big(&\|\boldsymbol{\varphi}_1(t)\|_{L^2(\Gamma_C)^2} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{L^2(\Gamma_C)^d} \\ &+\|\boldsymbol{\varphi}_2(t)\|_{L^2(\Gamma_C)^2} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{L^2(\Gamma_C)^d} \\ &+ \int_0^t (\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_Q^2)ds\big). \end{aligned} \tag{9.4.51}$$

Then (9.4.51) and (9.4.48) lead to

$$\begin{aligned} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 \leq c\,\big(&M_k \|\mathbf{u}_1 - \mathbf{u}_2\|_{C([0,T];L^2(\Gamma_C)^d)} \\ &+ \int_0^t (\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_Q^2)ds\big), \end{aligned} \tag{9.4.52}$$

where $M_k = 2\big(k(T+1) + \|\boldsymbol{\varphi}_0\|_{L^2(\Gamma_C)^2}\big)$. On the other hand,

$$\begin{aligned} \boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t) = &\mathcal{A}_{vp}\boldsymbol{\varepsilon}(\mathbf{u}_1 - \mathbf{u}_2)(t) \\ &+ \int_0^t (\mathcal{G}_{vp}(\boldsymbol{\sigma}_1(s), \boldsymbol{\varepsilon}(\mathbf{u}_1(s))) - \mathcal{G}_{vp}(\boldsymbol{\sigma}_2(s), \boldsymbol{\varepsilon}(\mathbf{u}_2(s))))ds, \end{aligned}$$

and since $\mathrm{Div}(\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)) = \mathbf{0}$, we obtain

$$\begin{aligned} \|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{Q_1} \leq c\Big(&\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \\ &+ \int_0^t (\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V + \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_Q)ds\Big), \end{aligned}$$

which implies

$$\begin{aligned} \|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{Q_1}^2 \leq c\Big(&\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 \\ &+ \int_0^t (\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_Q^2)ds\Big). \end{aligned}$$

This inequality and (9.4.52) imply

$$\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{Q_1}^2 \leq c\Big(M_k \|\mathbf{u}_1 - \mathbf{u}_2\|_{C([0,T];L^2(\Gamma_C)^d)} + \int_0^t (\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_Q^2) ds\Big). \quad (9.4.53)$$

Adding (9.4.52) and (9.4.53) and using Gronwall's lemma, we deduce (9.4.49) with $c_k = \mathcal{O}(k) + 1$. □

Next, assumptions (8.3.9) on the normal compliance functions mean that $p_e(u_{\varphi n} - g) \in C([0,T]; L^2(\Gamma_C)^2)$ for $e = n, \tau$ and $\boldsymbol{\varphi} \in Lip_k^0$. This allows us to consider the contact operator

$$\mathcal{T} : Lip_k^0 \subset C([0,T]; L^2(\Gamma_C)^2) \to C([0,T]; L^2(\Gamma_C)^2),$$

defined by

$$\mathcal{T}\boldsymbol{\varphi} = (p_n(u_{\varphi n} - g), p_\tau(u_{\varphi n} - g)). \quad (9.4.54)$$

We turn to investigate the properties of $\mathcal{T}$.

Lemma 9.4.4. *The operator $\mathcal{T}$ is compact.*

Proof. Let $\{\boldsymbol{\varphi}_m\}$ be a sequence of elements of Lip_k^0 and for $m \in \mathbb{N}$ we denote by $(\mathbf{u}_m, \boldsymbol{\sigma}_m)$ the solution of Problem $P_V^{\varphi_m}$ provided by Lemma 9.4.2. Using (9.4.48) it follows that $\{\boldsymbol{\varphi}_m\}$ is a bounded sequence in $W^{1,\infty}(0,T; L^2(\Gamma_C)^2)$ and estimate (9.4.38) shows that $\{\mathbf{u}_m\}$ is a bounded sequence in $W^{1,\infty}(0,T;V)$. By the continuity of the trace map we find that the sequence of traces on Γ_C of the displacements $\{\mathbf{u}_m\}$ is a bounded sequence in $W^{1,\infty}(0,T; L^2(\Gamma_C)^d)$. It follows now from the Arzela-Ascoli theorem that we can extract a subsequence of $\{\mathbf{u}_{m_p}\}$ such that the sequence of its traces on Γ_C converge strongly in $C([0,T]; L^2(\Gamma_C)^d)$, and so is a Cauchy sequence. Using now Lemma 9.4.3 shows that the corresponding subsequences of $\{\mathbf{u}_{m_p}\}$ and $\{\boldsymbol{\sigma}_{m_p}\}$ are Cauchy sequences in $C([0,T];V)$ and $C([0,T];Q_1)$, respectively, and therefore they converge strongly there. Let $\mathbf{u} \in C([0,T];V)$ be the limit in $C([0,T];V)$ of the subsequence $\{\mathbf{u}_{m_p}\}$ and denote by $u_{m_p n}$ the normal component of $\mathbf{u}_{m_p} \in V$. We use now (8.3.14) and (6.2.9) and obtain that

$$p_n(u_{m_p n} - g) \to p_n(u_n - g), \quad p_\tau(u_{m_p n} - g) \to p_\tau(u_n - g)$$

in $C([0,T]; L^2(\Gamma_C))$ as $m_p \to \infty$, which shows that the sequence $\{\mathcal{T}\boldsymbol{\varphi}_{m_p}\}$ converges in $C([0,T]; L^2(\Gamma_C)^2)$. This proves the lemma. □

Lemma 9.4.5. *The operator $\mathcal{T}$ is continuous.*

Proof. Let $\boldsymbol{\varphi} \in Lip_k^0$ and let $\{\boldsymbol{\varphi}_m\}$ be a sequence of elements of Lip_k^0 such that $\boldsymbol{\varphi}_m \to \boldsymbol{\varphi}$ in $C([0,T]; L^2(\Gamma_C)^2)$. For $m \in \mathbb{N}$ we denote by $(\mathbf{u}_m, \boldsymbol{\sigma}_m)$ the solution of Problem $P_V^{\varphi_m}$ provided by Lemma 9.4.2. Using (9.4.48) and

(9.4.38) it follows that $\{(\mathbf{u}_m, \boldsymbol{\sigma}_m)\}$ is a bounded sequence in $W^{1,\infty}(0,T;V \times Q_1)$. Therefore, by using arguments similar to those in the proof of Lemma 6.2 we find that there exists $(\mathbf{u}, \boldsymbol{\sigma}) \in W^{1,\infty}(0,T;V \times Q_1)$ such that, for a subsequence $\{(\mathbf{u}_{m_p}, \boldsymbol{\sigma}_{m_p})\}$, we have

$$(\mathbf{u}_{m_p}, \boldsymbol{\sigma}_{m_p}) \to (\mathbf{u}, \boldsymbol{\sigma}) \text{ weak}^* \text{ in } W^{1,\infty}(0,T;V \times Q_1), \tag{9.4.55}$$

$$(\mathbf{u}_{m_p}, \boldsymbol{\sigma}_{m_p}) \to (\mathbf{u}, \boldsymbol{\sigma}) \text{ in } C([0,T];V \times Q_1), \tag{9.4.56}$$

as $m_p \to \infty$. Using now (9.4.34)–(9.4.36) we obtain

$$\boldsymbol{\sigma}_{m_p}(t) = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\mathbf{u}_{m_p}(t)) + \mathbf{z}_{m_p}(t) \quad \forall t \in [0,T], \tag{9.4.57}$$

$$\begin{aligned} &a(\mathbf{u}_{m_p}(t), \mathbf{v} - \dot{\mathbf{u}}_{m_p}(t)) + (\mathbf{z}_{m_p}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_{m_p}(t)))_Q \\ &\quad + l(\boldsymbol{\varphi}_{m_p}(t), \mathbf{v}) - l(\boldsymbol{\varphi}(t), \dot{\mathbf{u}}_{m_p}(t)) \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}_{m_p}(t))_V, \end{aligned} \tag{9.4.58}$$

for all $\mathbf{v} \in V$, a.e. $t \in (0,T)$, where, for all $t \in [0,T]$,

$$\mathbf{z}_{m_p}(t) = \int_0^t \mathcal{G}_{vp}(\boldsymbol{\sigma}_{m_p}(s), \boldsymbol{\varepsilon}(\mathbf{u}_{m_p}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\mathbf{u}_0). \tag{9.4.59}$$

Moreover,

$$\mathbf{u}_{m_p}(0) = \mathbf{u}_0. \tag{9.4.60}$$

Passing to the limit in (9.4.57)–(9.4.60) as $m_p \to \infty$ and using (9.4.55) and (9.4.56) we obtain

$$\boldsymbol{\sigma}(t) = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \mathbf{z}(t) \quad \forall t \in [0,T], \tag{9.4.61}$$

$$\begin{aligned} a(\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t)) + (\mathbf{z}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + l(\boldsymbol{\varphi}(t), \mathbf{v}) - l(\boldsymbol{\varphi}(t), \dot{\mathbf{u}}(t)) \\ \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V, \end{aligned} \tag{9.4.62}$$

for all $\mathbf{v} \in V$, a.e. $t \in (0,T)$, and

$$\mathbf{u}(0) = \mathbf{u}_0, \tag{9.4.63}$$

where

$$\mathbf{z}(t) = \int_0^t \mathcal{G}_{vp}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\mathbf{u}(s))\, ds + \boldsymbol{\sigma}_0 - \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\mathbf{u}_0) \quad \forall t \in [0,T]. \tag{9.4.64}$$

Indeed, it follows from (6.4.9) and (9.4.56) that $\mathbf{z}_{m_p} \to \mathbf{z}$ in $C([0,T];Q)$ and, therefore, (9.4.57) and (6.4.8) imply (9.4.61). To prove (9.4.62) we integrate (9.4.58) on $[0,T]$, employ arguments similar to those in the proof of Lemma 9.4.1 (see (9.4.26)–(9.4.31)) and then perform a localization argument, based on a classical application of Lebesgue point for L^1 functions.

It follows from (9.4.61)–(9.4.64) that $(\mathbf{u}, \boldsymbol{\sigma})$ is a solution of problem $P_V^{\boldsymbol{\varphi}}$. We use (9.4.56) and (6.2.9) to show that $u_{m_p n} \to u_n$ in $C([0,T];L^2(\Gamma_C))$ and, keeping in mind (8.3.9) and (9.4.54), we find that $\mathcal{T}\boldsymbol{\varphi}_{m_p} \to \mathcal{T}\boldsymbol{\varphi}$ in

$C([0,T];L^2(\Gamma_C)^2)$ as $m_p \to \infty$. These arguments show that $\mathcal{T}\boldsymbol{\varphi}$ is the limit of all the convergent subsequences $\{\mathcal{T}\boldsymbol{\varphi}_{m_p}\} \subset \{\mathcal{T}\boldsymbol{\varphi}_m\}$. Lemma 9.4.4 now implies that $\mathcal{T}$ is a compact operator and we deduce that the whole sequence is convergent, i.e., $\mathcal{T}\boldsymbol{\varphi}_m \to \mathcal{T}\boldsymbol{\varphi}$ in $C([0,T];L^2(\Gamma_C)^2)$ as $m \to \infty$, which concludes the proof of the lemma. □

In the next step we prove that, under a suitable assumption, we can find $k > 0$ such that the set Lip_k^0 is invariant under the operator $\mathcal{T}$.

Lemma 9.4.6. *There exists a positive constant $\mathcal{L}_0$, which depends only on Ω, Γ_D, $\Gamma_C, \mathcal{A}_{vp}, \mathcal{G}_{vp}$, and T, such that whenever $\mathcal{L}_n + \mathcal{L}_\tau < \mathcal{L}_0$ then there exists $k > 0$ such that*

$$\boldsymbol{\varphi} \in Lip_k^0 \quad \Longrightarrow \quad \mathcal{T}\boldsymbol{\varphi} \in Lip_k^0. \tag{9.4.65}$$

Proof. Let $k > 0$, $\boldsymbol{\varphi} \in Lip_k^0$, and let t_1, $t_2 \in [0,T]$. Using (9.4.54), (11.4.12), and (6.2.9) we have

$$\begin{aligned}\|\mathcal{T}\boldsymbol{\varphi}(t_1) - \mathcal{T}\boldsymbol{\varphi}(t_2)\|_{L^2(\Gamma_C)^2} &\le c(\mathcal{L}_n + \mathcal{L}_\tau)\|\mathbf{u}(t_1) - \mathbf{u}(t_2)\|_V \\ &\le c(\mathcal{L}_n + \mathcal{L}_\tau)\|\dot{\mathbf{u}}_\varphi\|_{L^\infty(0,T;V)}|t_1 - t_2|,\end{aligned}$$

and, keeping in mind (9.4.38), we find

$$\begin{aligned}&\|\mathcal{T}\boldsymbol{\varphi}(t_1) - \mathcal{T}\boldsymbol{\varphi}(t_2)\|_{L^2(\Gamma_C)^2} \\ &\qquad \le c(\mathcal{L}_n + \mathcal{L}_\tau)(\|\boldsymbol{\varphi}\|_{W^{1,\infty}(0,T;L^2(\Gamma_C)^2)} + \overline{\theta})|t_1 - t_2|,\end{aligned} \tag{9.4.66}$$

where $\overline{\theta}$ is a positive number which does not depend on k. We now combine (9.4.66) and (9.4.48) and obtain

$$\|\mathcal{T}\boldsymbol{\varphi}(t_1) - \mathcal{T}\boldsymbol{\varphi}(t_2)\|_{L^2(\Gamma_C)^2} \le c(\mathcal{L}_n + \mathcal{L}_\tau)(k(T+1) + \|\boldsymbol{\varphi}_0\|_{L^2(\Gamma_C)^2} + \overline{\theta})|t_1 - t_2|.$$

We conclude that $\mathcal{T}\boldsymbol{\varphi} \in W^{1,\infty}(0,T;L^2(\Gamma_C)^2)$ and, moreover,

$$\Big\|\frac{d}{dt}\mathcal{T}\boldsymbol{\varphi}\Big\|_{L^\infty(0,T;L^2(\Gamma_C)^2)} \le c(\mathcal{L}_n + \mathcal{L}_\tau)(k(T+1) + \|\boldsymbol{\varphi}_0\|_{L^2(\Gamma_C)^2} + \overline{\theta}).$$

We choose $\mathcal{L}_0 = \frac{1}{c(T+1)}$ and it is straightforward to show that when $\mathcal{L}_n + \mathcal{L}_\tau < \mathcal{L}_0$ then we can find $k > 0$ such that

$$c(\mathcal{L}_n + \mathcal{L}_\tau)(k(T+1) + \|\boldsymbol{\varphi}_0\|_{L^2(\Gamma_C)^2} + \overline{\theta}) \le k,$$

and, therefore,

$$\Big\|\frac{d}{dt}\mathcal{T}\boldsymbol{\varphi}\Big\|_{L^\infty(0,T;L^2(\Gamma_C)^2)} \le k.$$

Since by (9.4.36) and (9.4.4) we have $\mathcal{T}\boldsymbol{\varphi}(0) = \boldsymbol{\varphi}_0$, we conclude that $\mathcal{T}\boldsymbol{\varphi} \in Lip_k^0$, which proves the lemma. □

We have now all the ingredients to prove the theorem.

Proof (Theorem 9.3.1*).* We choose $\mathcal{L}_0$ as in Lemma 9.4.6 and assume that $\mathcal{L}_n + \mathcal{L}_\tau < \mathcal{L}_0$. We also choose k such that (9.4.65) holds. It is straightforward to show that $Lip\,^0_k$ is a nonempty, closed, bounded, and convex set in the Banach space $C([0,T];L^2(\Gamma_C)^2)$ and our choice of k guarantees that $\mathcal{T}(Lip\,^0_k) \subset Lip\,^0_k$. Moreover, it follows from Lemmas 9.4.4 and 9.4.5 that $\mathcal{T}$ is a completely continuous operator. Therefore, by using the Schauder fixed point theorem (Theorem 6.3.10 on page 95) we deduce that $\mathcal{T}$ has a fixed point, i.e., there exists a an element $\boldsymbol{\varphi}^* \in Lip\,^0_k$ such that $\mathcal{T}\boldsymbol{\varphi}^* = \boldsymbol{\varphi}^*$. Let $(\mathbf{u}^*, \boldsymbol{\sigma}^*)$ denote the solution of Problem $P_V^{\boldsymbol{\varphi}}$ with the choice $\boldsymbol{\varphi} = \boldsymbol{\varphi}^*$. It follows now from (9.4.54), since p_n and p_τ are nonnegative functions, that

$$|\varphi_n^*(t)| = p_n(u_n^*(t) - g), \quad |\varphi_\tau^*(t)| = p_\tau(u_n^*(t) - g) \quad \forall t \in [0,T],$$

and by using (9.4.6) and (8.3.14)) we obtain

$$l(\boldsymbol{\varphi}^*(t), \mathbf{v}) = j(\mathbf{u}^*(t), \mathbf{v}) \qquad \forall \mathbf{v} \in V,\ t \in [0,T].$$

We conclude from (9.4.34)–(9.4.37) that $(\mathbf{u}^*, \boldsymbol{\sigma}^*)$ is a solution of problem P^V_{vp-nc} and it satisfies (9.1.20), and this concludes the proof. □

9.5 Bilateral Frictional Contact

We follow [225] and assume that there is no loss of contact during the process, and so the normal displacement u_n vanishes on Γ_C. We model friction with the Tresca friction law (2.6.10). The viscoplastic constitutive law of the material is assumed to be (6.4.7), with the properties (6.4.8) and (6.4.9).

The classical formulation of the mechanical problem is as follows.

Problem P_{vp-b}. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *such that*

$$\dot{\boldsymbol{\sigma}} = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{G}_{vp}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u})) \quad \text{in } \Omega_T, \tag{9.5.1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \quad \text{in } \Omega_T, \tag{9.5.2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \times (0,T), \tag{9.5.3}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \quad \text{on } \Gamma_N \times (0,T), \tag{9.5.4}$$

$$\left.\begin{array}{l} u_n = 0, \\ \|\boldsymbol{\sigma}_\tau\| \le H, \\ \boldsymbol{\sigma}_\tau = -H\dfrac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if } \dot{\mathbf{u}}_\tau \neq \mathbf{0} \end{array}\right\} \text{on } \Gamma_C \times (0,T), \tag{9.5.5}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in } \Omega. \tag{9.5.6}$$

Here, $\mathbf{u}_0$ and $\boldsymbol{\sigma}_0$ are the initial displacement and stress fields, respectively.

We use the subspace V_1, defined in (6.2.7), for the displacements and the space Q_1, (6.2.10), for the stresses. Over the subspace V_1 we use the inner product of V, and then V_1 is itself a Hilbert space.

We assume that the force and the traction densities satisfy (7.3.7) and the friction bound satisfies (7.3.8). We denote by $\mathbf{F}(t)$ the element of V_1 given by (7.3.11) for all $\mathbf{v} \in V_1$, $t \in [0,T]$, and let $j : V_1 \to \mathbb{R}_+$ be the friction functional

(7.3.10). We also denote by $\Sigma(t)$, for $t \in [0,T]$, the set of admissible stress fields

$$\Sigma(t) = \{\boldsymbol{\tau} \in Q : (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + j(\mathbf{v}) \geq (\mathbf{F}(t), \mathbf{v})_V \ \forall\, \mathbf{v} \in V_1\} \tag{9.5.7}$$

and we assume that the initial data satisfy

$$\mathbf{u}_0 \in V_1, \quad \boldsymbol{\sigma}_0 \in \Sigma(0). \tag{9.5.8}$$

It is straightforward to show that if $\mathbf{u}$ and $\boldsymbol{\sigma}$ are smooth functions satisfying (9.5.2)–(9.5.5), then

$$\begin{aligned} \mathbf{u}(t) \in V_1,\ & (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \\ & \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \quad \forall\, \mathbf{v} \in V_1, \end{aligned} \tag{9.5.9}$$

$$\boldsymbol{\sigma}(t) \in \Sigma(t),\ (\boldsymbol{\tau} - \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q \geq 0 \quad \forall\, \boldsymbol{\tau} \in \Sigma(t), \tag{9.5.10}$$

for all $t \in [0,T]$. These inequalities lead to the following two weak formulations of the frictional problem (9.5.1)–(9.5.6).

Problem P^V_{vp-b1}. *Find a displacement field* $\mathbf{u} : [0,T] \to V_1$ *and a stress field* $\boldsymbol{\sigma} : [0,T] \to Q_1$ *such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t))) \quad \text{a.e. } t \in (0,T), \tag{9.5.11}$$

$$\begin{aligned} (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \\ \forall\, \mathbf{v} \in V_1,\ \text{a.e. } t \in (0,T), \end{aligned} \tag{9.5.12}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0. \tag{9.5.13}$$

Problem P^V_{vp-b2}. *Find a displacement field* $\mathbf{u} : [0,T] \to V_1$ *and a stress field* $\boldsymbol{\sigma} : [0,T] \to Q_1$ *such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t))) \quad \text{a.e. } t \in (0,T), \tag{9.5.14}$$

$$\begin{aligned} \boldsymbol{\sigma}(t) \in \Sigma(t),\ & (\boldsymbol{\tau} - \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q \geq 0 \\ & \forall\, \boldsymbol{\tau} \in \Sigma(t),\ \text{a.e. } t \in (0,T), \end{aligned} \tag{9.5.15}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0. \tag{9.5.16}$$

Note that whereas in problem P^V_{vp-b1} the main role is played by the displacements field, in problem P^V_{vp-b2}, which is the so-called *dual formulation*, the main role in played by the stress field.

The first existence and uniqueness result is the following ([225]).

Theorem 9.5.1. *Assume* (6.4.8), (6.4.9), (7.3.7), (7.3.8), *and* (9.5.8). *Then there exists a unique solution* $(\mathbf{u}, \boldsymbol{\sigma})$ *to Problem* P^V_{vp-b1}. *Moreover, the solution satisfies* (9.1.20).

Proof. The proof of Theorem 9.5.1 is carried out in several steps, using results of evolutionary variational inequalities and the Banach fixed-point theorem. An outline of the steps follows.

(*i*) Let $\boldsymbol{\eta} \in L^\infty(0, T; Q)$ and define the function $\mathbf{z}_\eta \in W^{1,\infty}(0, T; Q)$ by (9.1.21). Then, there exists a unique solution $(\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta)$, which satisfies (9.1.20), of the variational problem

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)) + \mathbf{z}_\eta(t) \quad \forall\, t \in [0, T], \tag{9.5.17}$$

$$\begin{aligned}(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)))_Q + j(\mathbf{v}) - j(\dot{\mathbf{u}}_\eta(t))& \\ \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_V \quad \forall\, \mathbf{v} \in V_1,\ \text{a.e. } t \in (0, T),&\end{aligned} \tag{9.5.18}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0, \quad \boldsymbol{\sigma}_\eta(0) = \boldsymbol{\sigma}_0. \tag{9.5.19}$$

(ii) Consider the operator defined by (9.1.25), where $(\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta)$ is the solution of the variational problem (9.5.17)–(9.5.19). Then, the operator Λ has a unique fixed point $\boldsymbol{\eta}^* \in L^\infty(0, T; Q)$.

(iii) Existence. Let $\boldsymbol{\eta}^* \in L^\infty(0, T; Q)$ be the fixed point of Λ and let $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*})$ be the solution of problem (9.5.17)–(9.5.19) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$. It follows that $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*})$ is a solution of Problem P^V_{vp-b1} satisfying (9.1.20).

(iv) Uniqueness. The uniqueness part follows from the uniqueness of the fixed point of the operator Λ. The details are very similar to those in the proof of Theorem 9.1.1, and so we skip them. □

The second existence and uniqueness result follows ([225]).

Theorem 9.5.2. *Assume* (6.4.8), (6.4.9), (7.3.7), (7.3.8) *and* (9.5.8). *Then there exists a unique solution* $(\mathbf{u}, \boldsymbol{\sigma})$ *of Problem* P^V_{vp-b2} *and it satisfies* (9.1.20).

Proof. We first remark that the variational inequality (9.5.15) is posed over a time-dependent convex set $\Sigma(t)$. To avoid this time dependence of the set we use a change of variable, similar to the one used in the proof of Theorem 9.1.2. To that end, let

$$\Sigma_0 = \{\, \boldsymbol{\tau} \in Q : (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + j(\mathbf{v}) \geq 0 \ \ \forall\, \mathbf{v} \in V_1 \,\}, \tag{9.5.20}$$

$$\widetilde{\boldsymbol{\sigma}} = \boldsymbol{\varepsilon}(\mathbf{F}), \tag{9.5.21}$$

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \quad \bar{\boldsymbol{\sigma}}_0 = \boldsymbol{\sigma}_0 - \widetilde{\boldsymbol{\sigma}}(0). \tag{9.5.22}$$

Consider now the variational problem of finding $\mathbf{u} : [0, T] \to V_1$ and $\bar{\boldsymbol{\sigma}} : [0, T] \to Q_1$ such that

$$\varepsilon(\dot{\mathbf{u}}(t)) = \mathcal{A}_{vp}^{-1}\dot{\bar{\boldsymbol{\sigma}}}(t) - \mathcal{A}_{vp}^{-1}\mathcal{G}_{vp}(\bar{\boldsymbol{\sigma}}(t) + \widetilde{\boldsymbol{\sigma}}(t), \varepsilon(\mathbf{u}(t)))$$
$$+\mathcal{A}_{vp}^{-1}\dot{\widetilde{\boldsymbol{\sigma}}}(t) \quad \text{a.e. } t \in (0,T), \tag{9.5.23}$$
$$\bar{\boldsymbol{\sigma}}(t) \in \Sigma_0,\ (\boldsymbol{\tau} - \bar{\boldsymbol{\sigma}}(t), \varepsilon(\dot{\mathbf{u}}(t)))_Q \geq 0$$
$$\forall \boldsymbol{\tau} \in \Sigma_0,\ \text{a.e.}\, t \in (0,T), \tag{9.5.24}$$
$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \bar{\boldsymbol{\sigma}}(0) = \bar{\boldsymbol{\sigma}}_0. \tag{9.5.25}$$

From (9.5.7) it follows that

$$\Sigma(t) = \Sigma_0 + \{\widetilde{\boldsymbol{\sigma}}(t)\} \quad \forall t \in [0,T]. \tag{9.5.26}$$

Using (7.3.7), (7.3.11) and (9.5.21) we have

$$\widetilde{\boldsymbol{\sigma}}(t) \in W^{1,\infty}(0,T;Q_1). \tag{9.5.27}$$

Using now (9.5.22)–(9.5.27) it is straightforward to verify that the pair $(\mathbf{u}, \boldsymbol{\sigma})$ is a solution of Problem P_{vp-b2}^V and such that $\mathbf{u} \in W^{1,\infty}(0,T;V_1)$, $\boldsymbol{\sigma} \in W^{1,\infty}(0,T;Q_1)$ if and only if $(\mathbf{u}, \bar{\boldsymbol{\sigma}})$ is a solution for problem (9.5.23)–(9.5.25), and $\mathbf{u} \in W^{1,\infty}(0,T;V_1)$, $\bar{\boldsymbol{\sigma}} \in W^{1,\infty}(0,T;Q_1)$.

We turn to problem (9.5.23)–(9.5.25), which we solve, again, by using the fixed point method. The proof proceeds in four steps which we now describe.

(*i*) Let $\boldsymbol{\eta} \in L^\infty(0,T;Q)$ and consider the problem of finding $\mathbf{u}_\eta : [0,T] \to V_1$ and $\boldsymbol{\sigma}_\eta : [0,T] \to Q_1$ such that

$$\varepsilon(\dot{\mathbf{u}}_\eta(t)) = \mathcal{A}_{vp}^{-1}\dot{\boldsymbol{\sigma}}_\eta(t) + \boldsymbol{\eta}(t) \quad \text{a.e. } t \in (0,T), \tag{9.5.28}$$
$$\boldsymbol{\sigma}_\eta(t) \in \Sigma_0, \quad (\boldsymbol{\tau} - \boldsymbol{\sigma}_\eta(t), \varepsilon(\dot{\mathbf{u}}_\eta(t)))_Q \geq 0$$
$$\forall \boldsymbol{\tau} \in \Sigma_0, \quad \text{a.e. } t \in (0,T), \tag{9.5.29}$$
$$\mathbf{u}_\eta(0) = \mathbf{u}_0, \qquad \boldsymbol{\sigma}_\eta(0) = \bar{\boldsymbol{\sigma}}_0. \tag{9.5.30}$$

The unique solvability of problem (9.5.28)–(9.5.30) is based on Theorem 6.3.5. Indeed, Theorem 6.3.5 combined with a regularity result may be used to show that problem (9.5.28)–(9.5.30) has a unique solution $(\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta)$, such that $\mathbf{u}_\eta \in W^{1,\infty}(0,T;V_1)$ and $\boldsymbol{\sigma}_\eta \in W^{1,\infty}(0,T;Q_1)$.

(*ii*) We consider now the operator $\Lambda : L^\infty(0,T;Q) \to L^\infty(0,T;Q)$ defined by

$$\Lambda\boldsymbol{\eta} = \mathcal{A}_{vp}^{-1}\dot{\widetilde{\boldsymbol{\sigma}}} - \mathcal{A}_{vp}^{-1}\mathcal{G}(\boldsymbol{\sigma}_\eta + \widetilde{\boldsymbol{\sigma}}, \varepsilon(\mathbf{u}_\eta)), \tag{9.5.31}$$

and prove that it has a unique fixed point $\boldsymbol{\eta}^* \in L^\infty(0,T;Q)$.

(iii) Existence. Let $\boldsymbol{\eta}^* \in L^\infty(0,T;Q)$ be the fixed point of Λ and let $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*}) \in W^{1,\infty}(0,T;V_1 \times Q_1)$ be the functions obtained at step (*i*) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$. It follows that $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*})$ is a solution of problem (9.5.23)–(9.5.25) and, thus, we obtain the existence part in Theorem 9.5.2.

(iv) Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ. Alternatively, it can be shown to follow directly from (9.5.14)–(9.5.16), by using (6.4.8), (6.4.9) and Lemma 6.3.11. □

P^V_{vp-b1} and P^V_{vp-b2} are two different, but equivalent, variational formulations of the mechanical problem (9.5.1)–(9.5.6), as we establish next.

Theorem 9.5.3. *Assume that* (6.4.8), (6.4.9), (7.3.7), (7.3.8), *and* (9.5.8) *hold. Let* $\mathbf{u} \in W^{1,\infty}(0,T;V_1)$ *and* $\boldsymbol{\sigma} \in W^{1,\infty}(0,T;Q_1)$. *Then, the pair* $(\mathbf{u}, \boldsymbol{\sigma})$ *is the solution of the variational problem* P^V_{vp-b1} *if and only if it is the solution of problem* P^V_{vp-b2}.

Theorems 9.5.1 and 9.5.2 furnish the unique solvability of problems P^V_{vp-b2} and P^V_{vp-b2}, respectively, while Theorem 9.5.3 expresses their equivalence. We conclude that the mechanical problem (9.5.1)–(9.5.6) has a unique weak solution

which solves both Problems P^V_{vp-b1} and P^V_{vp-b2}.

The variational analysis of the mechanical problem P^V_{vp-b} including Theorems 9.5.1–9.5.3 can be found in [225]. The existence of a solution to the bilateral viscoplastic contact problem with regularized Coulomb friction was recently obtained in [220]. There, the friction functional j was assumed to depend on the regularized normal stress too, which introduced a severe mathematical difficulty in the analysis of the model; this difficulty was overcome by using the Schauder fixed-point theorem and arguments similar as those presented in the proof of Theorem 9.3.1.

9.6 Contact with Dissipative Friction Potential

In this section we extend some of the results in Sect. 9.5 to the case when friction is modelled by a general velocity dependent dissipation functional, as in Sect. 7.4. We note that here the functional j is not a continuous seminorm on the space of admissible velocity fields and, thus, the arguments used in the proof of Theorem 9.5.1 do not apply. Moreover, since $j(2\mathbf{v}) \neq 2j(\mathbf{v})$, it is not possible to derive a dual variational formulation similar to that in Sect. 9.5. Therefore, we consider only the primal variational formulation of the mechanical problem, and we use a result of [206] to prove the existence of a unique weak solution to the model. The results of this section were obtained in [23], where the full details can be found.

The classical formulation of the problem is the following.

Problem P_{vp-d}. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *such that*

$$\dot{\boldsymbol{\sigma}} = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{G}_{vp}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u})) \qquad \text{in } \Omega_T, \tag{9.6.1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \qquad \text{in } \Omega_T, \tag{9.6.2}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_D \times (0,T), \tag{9.6.3}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \qquad \text{on } \Gamma_N \times (0,T), \tag{9.6.4}$$

$$\mathbf{u} \in U, \quad -\boldsymbol{\sigma}\,\mathbf{n} \cdot (\mathbf{v} - \dot{\mathbf{u}}) \le \varphi(\mathbf{v}) - \varphi(\dot{\mathbf{u}})$$

$$\forall \mathbf{v} \in U \quad \text{on } \Gamma_C \times (0,T), \tag{9.6.5}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \qquad \text{in } \Omega. \tag{9.6.6}$$

Here, frictional contact is modelled with the subdifferential boundary conditions (9.6.5) in which U represents the set of admissible test functions and φ is the given potential function. Examples of inequality problems in contact mechanics which lead to boundary conditions of this form can be found in Sect. 7.4.

We suppose that $U \subset H^1(\Omega)^d$, $\varphi : \Gamma_C \times \mathbb{R}^d \to (-\infty, \infty]$ and use (7.4.7) and (7.4.8). We denote by $D(j)$ the effective domain of j, i.e., $D(j) = \{v \in U_1 : j(v) < \infty\}$, and use on U_1 the inner product $(\cdot,\cdot)_V$, defined in (6.2.5). We assume in the sequel that (7.4.9)–(7.4.11) hold and the functions $\mathcal{A}_{vp}$ and $\mathcal{G}_{vp}$ satisfy conditions (6.4.8) and (6.4.9), respectively.

Next, for each instant $t \in [0,T]$, let

$$F(t,\mathbf{v}) = \int_\Omega \mathbf{f}_B(t) \cdot \mathbf{v}\, dx + \int_{\Gamma_N} \mathbf{f}_N(t) \cdot \mathbf{v}\, dS \quad \forall \mathbf{v} \in U_1,$$

and

$$\Sigma(t) = \big\{ \boldsymbol{\tau} \in Q : (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + j(\mathbf{v}) \geq F(t,\mathbf{v}) \quad \forall \mathbf{v} \in D(j) \big\}.$$

Finally, we assume that the initial data satisfy

$$\mathbf{u}_0 \in U_1, \qquad \boldsymbol{\sigma}_0 \in \Sigma(0). \tag{9.6.7}$$

It is straightforward to show that when $(\mathbf{u}, \boldsymbol{\sigma})$ are smooth functions satisfying (9.6.2)–(9.6.5), then, for all $t \in [0,T]$, $\mathbf{u}(t) \in U_1$ and

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \geq F(t, \mathbf{v} - \dot{\mathbf{u}}(t)), \tag{9.6.8}$$

for all $\mathbf{v} \in U_1$. Using (9.6.1), (9.6.8) and (9.6.6) we obtain the following variational formulation of the mechanical problem (9.6.1)–(9.6.6).

Problem P^V_{vp-d}. *Find a displacement field* $\mathbf{u} : [0,T] \to U_1$ *and a stress field* $\boldsymbol{\sigma} : [0,T] \to Q_1$ *such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{G}_{vp}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t))), \text{ a.e. } t \in (0,T), \tag{9.6.9}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \geq \mathbf{F}(t, \mathbf{v} - \dot{\mathbf{u}}(t)) \quad \forall \mathbf{v} \in U_1, \text{ a.e. } t \in (0,T), \tag{9.6.10}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0. \tag{9.6.11}$$

The following existence and uniqueness result was proved in ([23]).

Theorem 9.6.1. *Assume* (6.4.8), (6.4.9), (7.4.9)–(7.4.11), *and* (9.6.7). *Then, there exists a unique solution for Problem* P^V_{vp-d} *and it satisfies*

$$\mathbf{u} \in W^{1,2}(0,T;U_1), \qquad \boldsymbol{\sigma} \in W^{1,2}(0,T;Q_1).$$

Proof. The proof was carried out in several steps and was based on Theorem 7.5.2 and a fixed-point argument similar to the one used in Sects. 9.1 and 9.5. □

We end this section by recalling that examples of contact and friction laws which lead to an inequality of the form (9.6.5), such that (7.4.9) and (7.4.10) hold, were presented in Sect. 7.4. We conclude by Theorem 9.6.1 that the relevant boundary value problems in the study of the viscoplastic materials for each of the examples have unique weak solutions.

10 Slip or Temperature Dependent Frictional Contact

The frictional contact problems which have been described up to now contained a constant friction coefficient, although, as was described in Sect. 2.7, in many applications it depends on the slip speed, on the temperature, and on other factors. In this chapter we address this issue and present models of frictional contact when the coefficient of friction depends on the slip, slip rate, or on the process history via the total slip rate.

Quasistatic problems with slip dependent friction can be found in [21,226, 227]. A model for an elastic material with prescribed normal pressure and slip dependent friction coefficient is described in Sect. 10.1. The proof of the the existence of a solution is provided in Sect. 10.2, and it is based on recent results for abstract evolutionary variational inequalities. Models for viscoelastic materials with friction coefficient that depends on the slip rate or on the accumulated slip, i.e., the process history, are described in Sect. 10.3. The existence of the unique weak solution is shown under a smallness assumption on the friction coefficient.

We also describe contact problems which include thermal effects, that usually accompany frictional contact, and consider thermoelastic or thermoviscoelastic material constitutive laws, and include the heat equation for the temperature. Indeed, frictional contact is very often associated with frictional heat generation, which may be considerable, such as the heat generated when one applied the brakes of a car. It also plays an important role in orthopaedic biomechanics [228,229]. The frictionless problem for a thermoelastic material and the Signorini contact condition is presented in Sect. 10.4. The existence of a weak solution is obtained under the assumption that the thermal expansion coefficients are sufficiently small. Thermoviscoelastic bilateral frictional contact problem is presented in Sect. 10.5, and the existence of the unique solution is stated when the friction coefficient is sufficiently small.

The only mathematical publication that we are aware of, where the frictional contact problem for a thermoviscoelastic material with temperature dependent friction coefficient was investigated, is [42].

We use dimensionless variables in this chapter.

M. Shillor, M. Sofonea, J.J. Telega: *Models and Analysis of Quasistatic Contact*, Lect. Notes Phys. **655**, (2001), 163–182
http://www.springerlink.com/

10.1 Elastic Contact with Slip Dependent Friction

In this section the body is assumed linearly elastic. The contact is with friction and is modelled with a condition in which the normal stress on the contact surface is prescribed, and the coefficient of friction depends on the slip. Although, usually, the friction coefficient is assumed to depend on the slip rate, as has been mentioned in Sect. 2.7, in some publications in geology the friction coefficient is assumed to depend on the slip. This is the case when the slip rate is constant, and then the slip is just a multiple of the slip rate.

The model for the process is as follows.

Problem $P_{el-slip}$. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{B}_{el}\boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{in } \Omega_T, \tag{10.1.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \qquad \text{in } \Omega_T, \tag{10.1.2}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_D \times (0,T), \tag{10.1.3}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \qquad \text{on } \Gamma_N \times (0,T), \tag{10.1.4}$$

$$\left.\begin{array}{l} \sigma_n = -S_C, \\ \|\boldsymbol{\sigma}_\tau\| \le \mu(\|\mathbf{u}_\tau\|)S_C, \\ \boldsymbol{\sigma}_\tau = -\mu(\|\mathbf{u}_\tau\|)S_C\dfrac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if } \dot{\mathbf{u}}_\tau \ne \mathbf{0} \end{array}\right\} \quad \text{on } \Gamma_C \times (0,T), \tag{10.1.5}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \text{in } \Omega. \tag{10.1.6}$$

Some comments on the contact conditions (10.1.5) are in order. The first condition in (10.1.5) states that the normal stress σ_n is prescribed on the contact surface, and given by $S_C \ge 0$. Such a condition was used in [5, 47], among others. It makes sense in a situation where the real contact area is close to the nominal one, and the surfaces are conforming. Then S_C is the contact pressure and is given by the ratio of the total applied force to the total nominal contact area. It also seems to be a good approximation when the load is very light and the contact force is transmitted by the asperity tips only.

The coefficient of friction μ in the law of dry friction in (10.1.5) is assumed to depend on the slip $\|\mathbf{u}_\tau\|$, see, e.g., [106]. Such dependence was employed in [78] in order to take into account the changes in the structure of the contact surface that result from sliding. It was used afterwards in various papers (see, e.g. [76, 230] and the references therein) in models for earthquakes. In these applications, usually, the slip velocity has a single direction and a single sense during the slip and, therefore, there is a direct relation between the slip, the slip rate and the total slip rate, (see for instance [11] and [76]).

The static version of the model (10.1.1)–(10.1.6) was considered in [230]. There, the existence of a weak solution for the problem was proved by using a theorem of the Weierstrass type, based on lower semicontinuity arguments.

The uniqueness of the solution was derived under the additional assumption of inequality between two scalar parameters: one related to the geometry and the elastic coefficients of the material, and the other one was a measure of the slip weakening and the normal stress.

In the study of the mechanical problem (10.1.1)–(10.1.6) we assume that the elasticity tensor $\mathcal{B}_{el}$ satisfies condition (6.4.2), the force and the traction densities satisfy (7.3.7), and the given normal stress is such that

$$S_C \in L^\infty(\Gamma_C), \quad S_C \geq 0 \ \text{ a.e. on } \Gamma_C. \tag{10.1.7}$$

The coefficient of friction is nonnegative and Lipschitz, and satisfies:

$$\left.\begin{array}{l}\text{(a) } \mu : \Gamma_C \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+. \\ \text{(b) There exists } \mathcal{L}_\mu > 0 \text{ such that} \\ \qquad |\mu(\mathbf{x}, r_1) - \mu(\mathbf{x}, r_2)| \leq \mathcal{L}_\mu |r_1 - r_2|, \\ \qquad \forall\, r_1,\, r_2 \in \mathbb{R}_+, \ \text{a.e. } \mathbf{x} \in \Gamma_C. \\ \text{(c) For any } r \in \mathbb{R}_+,\ \mathbf{x} \mapsto \mu(\mathbf{x}, r) \text{ is measurable on } \Gamma_C. \\ \text{(d) The mapping } \mathbf{x} \mapsto \mu(\mathbf{x}, 0) \in L^2(\Gamma_C). \end{array}\right\} \tag{10.1.8}$$

We note that assumptions (10.1.8) on the coefficient of friction are fairly general, but are different from those considered in [230], where μ was assumed to be bounded and continuously differentiable with respect to the second argument. However, to obtain the existence result below, we need to impose an additional smallness assumption on μ, which was not needed in the static case treated in [230].

To prove the uniqueness of the solution we need to replace (10.1.8) with the following stronger condition, in which we assume that μ does not depend on the slip $\|\mathbf{u}_\tau\|$. We assume that μ is a given function that satisfies

$$\mu \in L^2(\Gamma_C), \quad \mu \geq 0 \ \text{ a.e. on } \Gamma_C. \tag{10.1.9}$$

Next, we consider the bilinear form $a : V \times V \to \mathbb{R}$ given in (7.3.9) and the friction functional $j : V \times V \to \mathbb{R}$ given by

$$j(\boldsymbol{\eta}, \mathbf{v}) = \int_{\Gamma_C} \mu(\|\boldsymbol{\eta}_\tau\|)\, S_C\, \|\mathbf{v}_\tau\|\, dS. \tag{10.1.10}$$

Recall that V is the space of displacements (6.2.3), a real Hilbert space with the inner product (6.2.5). Let $\mathbf{F} : [0, T] \to V$ be given by

$$(\mathbf{F}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_B(t) \cdot \mathbf{v}\, dx + \int_{\Gamma_N} \mathbf{f}_N(t) \cdot \mathbf{v}\, dS - \int_{\Gamma_C} S_C v_n\, dS, \tag{10.1.11}$$

for all $\mathbf{v} \in V$ and $t \in [0, T]$, and clearly $\mathbf{F} \in W^{1,\infty}(0, T; V)$.

We assume that the initial data satisfies

$$\mathbf{u}_0 \in V, \tag{10.1.12}$$

$$a(\mathbf{u}_0, \mathbf{v}) + j(\mathbf{u}_0, \mathbf{v}) \geq (\mathbf{F}(0), \mathbf{v})_V \qquad \forall \mathbf{v} \in V. \tag{10.1.13}$$

The second condition guarantees the compatibility of the initial displacements and the forces $\mathbf{F}$ at $t = 0$, which is needed in quasistatic problems.

A straightforward application of the Green formula (6.2.11) yields the following variational formulation of the contact problem (10.1.1)–(10.1.6).

Problem $P^V_{el-slip}$. *Find a displacement field* $\mathbf{u} : [0, T] \to V$ *such that*

$$\begin{aligned} a(\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t)) + j(\mathbf{u}(t), \mathbf{v}) - j(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \\ \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \quad \forall\, \mathbf{v} \in V, \ a.e.\ t \in (0, T), \end{aligned} \tag{10.1.14}$$

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{10.1.15}$$

The existence and the possible uniqueness of solutions for problem $P^V_{el-slip}$, using the two different assumptions on the friction coefficient, is described in the following theorem due to [227].

Theorem 10.1.1. *Assume that conditions* (6.4.2), (7.3.7), (10.1.7), (10.1.12) *and* (10.1.13) *hold. Then,*

1) *Under the assumption* (10.1.8)*, there exists* $\mathcal{L}_0 > 0$*, depending only on* Ω*,* Γ_D*,* Γ_C *and* $\mathcal{B}_{el}$*, such that if* $\mathcal{L}_\mu \|S_C\|_{L^\infty(\Gamma_C)} < \mathcal{L}_0$ *then problem* $P^V_{el-slip}$ *has at least one solution* $\mathbf{u}$*, such that* $\mathbf{u} \in W^{1,\infty}(0, T; V)$*.*

2) *Under assumption* (10.1.9)*, there exists a unique solution* $\mathbf{u}$ *of problem* $P^V_{el-slip}$*, and* $\mathbf{u} \in W^{1,\infty}(0, T; V)$*. Moreover, the mapping* $(\mathbf{F}, \mathbf{u}_0) \longmapsto \mathbf{u}$ *is Lipschitz continuous from* $W^{1,\infty}(0, T; V) \times V$ *to* $L^\infty(0, T; V)$*.*

The proof of Theorem 10.1.1 is given in Sect. 10.2. Further details and additional results in the study of Problem $P_{el-slip}$ can be found in [227].

We note that Theorem 10.1.1 guarantees the solvability of the mechanical problem $P_{el-slip}$ under the smallness assumption $\mathcal{L}_\mu \|S_C\|_{L^\infty(\Gamma_C)} < \mathcal{L}_0$. Here, $\mathcal{L}_0$ represents a scalar parameter which depends only on the elasticity operator and on the geometry of the problem, but does not depend on the external forces, nor on the initial displacements. From the mathematical point of view the inequality $\mathcal{L}_\mu \|S_C\|_{L^\infty(\Gamma_C)} < \mathcal{L}_0$ represents a sufficient condition for the solvability of the variational problem $P^V_{el-slip}$, and it is needed for the application of the abstract result on evolutionary variational inequalities provided by Theorem 10.2.1 below. From the mechanical point of view this inequality shows that problem $P_{el-slip}$ has a solution if either the slip weakening or the given normal stress on the contact surface are sufficiently small. A similar condition was used in [230] in order to derive the uniqueness of the solution in the static model. We also note that if the coefficient of friction does not depend on the slip (i.e., assumption (10.1.8) is replaced by the stronger condition (10.1.9)), then Theorem 10.1.1 guarantees the unique solvability of problem $P_{el-slip}$.

10.2 Proof of Theorem 10.1.1

To prove Theorem 10.1.1 we need the following abstract result on evolutionary variational inequalities.

Let X be a real Hilbert space endowed with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$, and we denote by 0_X the zero element of X. We consider the abstract evolutionary inequality

$$\begin{aligned} & a(u(t), v - \dot{u}(t)) + j(u(t), v) - j(u(t), \dot{u}(t)) \\ & \quad \geq (f(t), v - \dot{u}(t))_X \quad \forall\, v \in X, \text{ a.e. } t \in (0,T), \end{aligned} \tag{10.2.1}$$

$$u(0) = u_0, \tag{10.2.2}$$

in which the unknown is the function $u : [0,T] \to X$.

In the study of (10.2.1)–(10.2.2) we consider the following assumptions:

The bilinear symmetric form $a : X \times X \to \mathbb{R}$ satisfies

$$\left.\begin{aligned} & (a) \text{ there exists } M > 0 \text{ such that} \\ & \qquad |a(u,v)| \leq M\|u\|_X\|v\|_X \quad \forall\, u, v \in X. \\ & (b) \text{ there exists } m > 0 \text{ such that } a(v,v) \geq m\|v\|_X^2 \quad \forall v \in X. \end{aligned}\right\} \tag{10.2.3}$$

The functional $j : X \times X \to \mathbb{R}$ is positively homogeneous and subadditive for every $\eta \in X$, $j(\eta,\cdot) : X \to \mathbb{R}$ i.e.,

$$\left.\begin{aligned} & (a)\ j(\eta, \lambda u) = \lambda j(\eta, u) \quad \forall u \in X,\ \lambda \in \mathbb{R}_+. \\ & (b)\ j(\eta, u + v) \leq j(\eta, u) + j(\eta, v) \quad \forall u, v \in X. \end{aligned}\right\} \tag{10.2.4}$$

$$f \in W^{1,\infty}(0,T;X). \tag{10.2.5}$$

$$u_0 \in X. \tag{10.2.6}$$

$$a(u_0, v) + j(u_0, v) \geq (f(0), v)_X \qquad \forall v \in X. \tag{10.2.7}$$

It follows from (10.2.4) that the functional $j(\eta,\cdot) : X \to \mathbb{R}$ is convex for each $\eta \in X$. Therefore, the directional derivative j_2', given by

$$j_2'(\eta, u; v) = \lim_{\lambda \to 0^+} \frac{1}{\lambda}\left(j(\eta, u + \lambda v) - j(\eta, u)\right) \qquad \forall \eta,\ u,\ v \in X, \tag{10.2.8}$$

exists.

We consider now the following additional assumptions on j:

(j_1) For every sequence $\{u_n\} \subset X$ with $\|u_n\|_X \to \infty$, every sequence $\{t_n\} \subset [0,1]$, and each $\overline{u} \in X$, one has

$$\liminf_{n \to \infty} \left(\frac{1}{\|u_n\|_X^2} j_2'(t_n u_n, u_n - \overline{u}; -u_n) \right) < m.$$

(j_2) For every sequence $\{u_n\} \subset X$ with $\|u_n\|_X \to \infty$, every bounded sequence $\{\eta_n\} \subset X$, and each $\overline{u} \in X$, it holds

$$\liminf_{n\to\infty} \left(\frac{1}{\|u_n\|_X^2} j_2'(\eta_n, u_n - \overline{u}; -u_n) \right) < m.$$

(j_3) For every sequence $\{u_n\} \subset X$ and $\{\eta_n\} \subset X$ such that $u_n \to u$ weakly in X, $\eta_n \to \eta$ weakly in X, and for every $v \in X$, one has

$$\limsup_{n\to\infty} \left(j(\eta_n, v) - j(\eta_n, u_n) \right) \le j(\eta, v) - j(\eta, u).$$

(j_4) There exists $c_0 \in (0, m)$ such that

$$j(u, v-u) - j(v, v-u) \le c_0 \|u - v\|_X^2 \quad \forall u, v \in X.$$

(j_5) There exist two functions $e_1 : X \to \mathbb{R}$ and $e_2 : X \to \mathbb{R}$ which map bounded sets in X into bounded sets in $\mathbb{R}$ such that

$$|j(\eta, u)| \le e_1(\eta)\|u\|_X^2 + e_2(\eta) \quad \forall \eta, u \in X, \quad e_1(0_X) < m - c_0.$$

(j_6) For every sequence $\{\eta_n\} \subset X$ with $\eta_n \to \eta$ weakly in X, and every bounded sequence $\{u_n\} \subset X$, one has

$$\lim_{n\to\infty} \left(j(\eta_n, u_n) - j(\eta, u_n) \right) = 0.$$

(j_7) For every $s \in (0, T]$ and every pair of functions $u, v \in W^{1,\infty}(0, T; X)$ such that $u(0) = v(0)$, and $u(s) \ne v(s)$,

$$\int_0^s \left(j(u(t), \dot{v}(t)) - j(u(t), \dot{u}(t)) + j(v(t), \dot{u}(t)) - j(v(t), \dot{v}(t)) \right) dt < \frac{m}{2} \|u(s) - v(s)\|_X^2.$$

(j_8) There exists $q \in (0, m/2)$ such that for every $s \in (0, T]$ and for every pair of functions $u, v \in W^{1,\infty}(0, T; X)$ such that $u(s) \ne v(s)$,

$$\int_0^s \left(j(u(t), \dot{v}(t)) - j(u(t), \dot{u}(t)) + j(v(t), \dot{u}(t)) - j(v(t), \dot{v}(t)) \right) dt < q \|u(s) - v(s)\|_X^2.$$

The following result has been obtained in [202].

Theorem 10.2.1. *Let* (10.2.3)–(10.2.7) *hold. Then,*

(i) Under the assumptions (j_1)–(j_6) *there exists at least one solution* $u \in W^{1,\infty}(0, T; X)$ *to problem* (10.2.1)–(10.2.2).

(ii) Under the assumptions (j_1)–(j_7) *there exists a unique solution* $u \in W^{1,\infty}(0, T; X)$ *to problem* (10.2.1)–(10.2.2).

(iii) Under the assumptions (j_1)–(j_6) *and* (j_8) *there exists a unique solution* $u = u(f, u_0) \in W^{1,\infty}(0, T; X)$ *to problem* (10.2.1)–(10.2.2), *and the mapping* $(f, u_0) \longmapsto u$ *is Lipschitz continuous from* $W^{1,\infty}(0, T; X) \times X$ *to* $L^\infty(0, T; X)$.

We use Theorem 10.2.1 to prove Theorem 10.1.1. The proof will be carried out in several steps. We assume that (6.4.2), (7.3.7), (10.1.7), (10.1.12), and (10.1.13) hold and we use m to denote the coercivity constant of the form a, see (10.2.17) below. We recall (see page 89) that $\mathbf{u}_{n\tau}$ and $\boldsymbol{\eta}_{n\tau}$ denote the tangential components of the elements $\mathbf{u}_n,\ \boldsymbol{\eta}_n \in V$.

We begin by investigating the properties of the friction functional j, given by (10.1.10), and which satisfies (10.2.4). We have the following results.

Lemma 10.2.2. *The functional j satisfies assumptions (j_1) and (j_2).*

Proof. Let $\boldsymbol{\eta}$, $\mathbf{u}$, $\overline{\mathbf{u}} \in V$ and let $\lambda \in (0,1]$. Using (10.1.10), it follows that

$$j(\boldsymbol{\eta}, \mathbf{u} - \overline{\mathbf{u}} - \lambda \mathbf{u}) - j(\boldsymbol{\eta}, \mathbf{u} - \overline{\mathbf{u}}) \le \lambda \int_{\Gamma_C} \mu(\|\boldsymbol{\eta}_\tau\|) S_C \, \|\overline{\mathbf{u}}_\tau\| \, dS.$$

Therefore, we obtain from (10.2.8)

$$j_2'(\boldsymbol{\eta}, \mathbf{u} - \overline{\mathbf{u}}; -\mathbf{u}) \le \int_{\Gamma_C} \mu(\|\boldsymbol{\eta}_\tau\|) S_C \, \|\overline{\mathbf{u}}_\tau\| \; dS \quad \forall \boldsymbol{\eta}, \mathbf{u}, \overline{\mathbf{u}} \in V. \tag{10.2.9}$$

We now consider the sequences $\{\mathbf{u}_n\} \subset V$, $\{t_n\} \subset (0,1]$, and let $\overline{\mathbf{u}} \in V$. Using (6.2.9), (10.1.7), (10.1.8) and (10.2.9), we find

$$\begin{aligned} j_2'(t_n \mathbf{u}_n, \mathbf{u}_n - \overline{\mathbf{u}}; -\mathbf{u}_n) &\le \int_{\Gamma_C} (\mathcal{L}_\mu \|\mathbf{u}_{n\tau}\| + |\mu(0)|) S_C \, \|\overline{\mathbf{u}}_\tau\| \; dS \\ &\le c_B \|S_C\|_{L^\infty(\Gamma_C)} (c_B \mathcal{L}_\mu \|\mathbf{u}_n\|_V + \|\mu(0)\|_{L^2(\Gamma_C)}) \|\overline{\mathbf{u}}\|_V . \end{aligned}$$

It follows that if $\|\mathbf{u}_n\|_V \to \infty$ then

$$\liminf_{n\to\infty} \left(\frac{1}{\|\mathbf{u}_n\|_V^2} j_2'(t_n \mathbf{u}_n, \mathbf{u}_n - \overline{\mathbf{u}}; -\mathbf{u}_n) \right) \le 0,$$

and we conclude that j satisfies assumption (j_1).

Consider now the sequences $\{\mathbf{u}_n\} \subset V$ and $\{\boldsymbol{\eta}_n\} \subset V$ such that

$$\|\mathbf{u}_n\|_V \to \infty, \tag{10.2.10}$$

$$\|\boldsymbol{\eta}_n\|_V \le c \quad \forall n \in \mathbb{N}, \tag{10.2.11}$$

where $c > 0$. Let $\overline{\mathbf{u}} \in V$. Using (6.2.9) again, (10.1.8) and (10.2.9), we obtain

$$\begin{aligned} & j_2'(\boldsymbol{\eta}_n, \mathbf{u}_n - \overline{\mathbf{u}}; -\mathbf{u}_n) \\ & \quad \le c_B \|S_C\|_{L^\infty(\Gamma_C)} \Big(c_B \mathcal{L}_\mu \|\boldsymbol{\eta}_n\|_V + \|\mu(0)\|_{L^2(\Gamma_C)} \Big) \|\overline{\mathbf{u}}\|_V \quad \forall n \in \mathbb{N}. \end{aligned} \tag{10.2.12}$$

We conclude from (10.2.10)–(10.2.12) that j satisfies assumption (j_2). □

Lemma 10.2.3. *The functional j satisfies assumptions (j_3) and (j_6).*

Proof. Let $\{\mathbf{u}_n\} \subset V$ and $\{\boldsymbol{\eta}_n\} \subset V$ be two sequences such that $\mathbf{u}_n \to \mathbf{u}$ weakly in V and $\boldsymbol{\eta}_n \to \boldsymbol{\eta}$ weakly in V. Using the compactness of the trace map and assumption (10.1.8), we obtain

$$\mu(\|\boldsymbol{\eta}_{n\tau}\|) \to \mu(\|\boldsymbol{\eta}_\tau\|) \quad \text{in } L^2(\Gamma_C), \tag{10.2.13}$$

$$\mathbf{u}_n \to \mathbf{u} \quad \text{in } L^2(\Gamma_C)^d. \tag{10.2.14}$$

Therefore, it follows from (10.2.13) and (10.2.14) that $j(\boldsymbol{\eta}_n, \mathbf{v}) \to j(\boldsymbol{\eta}, \mathbf{v})$ for all $\mathbf{v} \in V$ and $j(\boldsymbol{\eta}_n, \mathbf{u}_n) \to j(\boldsymbol{\eta}, \mathbf{u})$, and these show that the functional j satisfies (j_3).

Now, let $\{\mathbf{u}_n\}$ be a bounded sequence of V, i.e.,

$$\|\mathbf{u}_n\|_V \le c \quad \forall n \in \mathbb{N}, \tag{10.2.15}$$

where $c > 0$. We have

$$|j(\boldsymbol{\eta}_n, \mathbf{u}_n) - j(\boldsymbol{\eta}, \mathbf{u}_n)| \le \int_{\Gamma_C} S_C \Big(\mu(\|\boldsymbol{\eta}_{n\tau}\|) - \mu(\|\boldsymbol{\eta}_\tau\|)\|\mathbf{u}_{n\tau}\|\Big)\, dS$$

and, using (6.2.9) and (10.1.7), we deduce

$$|j(\boldsymbol{\eta}_n, \mathbf{u}_n) - j(\boldsymbol{\eta}, \mathbf{u}_n)| \le \\ c_B \|S_C\|_{L^\infty(\Gamma_C)} \big\|\mu(\|\boldsymbol{\eta}_{n\tau}\|) - \mu(\|\boldsymbol{\eta}_\tau\|)\big\|_{L^2(\Gamma_C)} \|\mathbf{u}_n\|_V.$$

It follows now from (10.2.13) and (10.2.15) that j satisfies condition (j_6). □

Lemma 10.2.4. *The functional j satisfies assumption (j_5), for all $c_0 \in (0, m]$. Moreover,*

$$j(\mathbf{u}, \mathbf{v}-\mathbf{u}) - j(\mathbf{v}, \mathbf{v}-\mathbf{u}) \le \mathcal{L}_\mu c_B^2 \|S_C\|_{L^\infty(\Gamma_C)} \|\mathbf{u}-\mathbf{v}\|_V^2 \quad \forall \mathbf{u}, \mathbf{v} \in V. \tag{10.2.16}$$

Proof. Let $\boldsymbol{\eta}, \mathbf{u} \in V$. Using (10.1.8) and (10.1.7) it follows that

$$|j(\boldsymbol{\eta}, \mathbf{u})| \le \int_{\Gamma_C} S_C \mu(\|\boldsymbol{\eta}_\tau\|)\|\mathbf{u}_\tau\|\, dS$$
$$\le \|S_C\|_{L^\infty(\Gamma_C)} \Big(\mathcal{L}_\mu \|\boldsymbol{\eta}_\tau\|_{L^2(\Gamma_C)^d} + \|\mu(0)\|_{L^2(\Gamma_C)}\Big) \|\mathbf{u}_\tau\|_{L^2(\Gamma_C)^d},$$

and, keeping in mind (6.2.9), we find

$$|j(\boldsymbol{\eta}, \mathbf{u})| \le c_B \|S_C\|_{L^\infty(\Gamma_C)} \left(\mathcal{L}_\mu c_B \|\boldsymbol{\eta}\|_V + \|\mu(0)\|_{L^2(\Gamma_C)}\right) \|\mathbf{u}\|_V.$$

We use now the Cauchy inequality with ϵ, $ab \le \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$ for $\epsilon > 0$, and find that condition (j_5) holds for all $c_0 \in (0, m]$.

Let $\mathbf{u}, \mathbf{v} \in V$, and by using (10.1.8) and (10.1.7), again, we find that

$$j(\mathbf{u}, \mathbf{v}-\mathbf{u}) - j(\mathbf{v}, \mathbf{v}-\mathbf{u}) = \int_{\Gamma_C} S_C \Big(\mu(\|\mathbf{u}_\tau\|) - \mu(\|\mathbf{v}_\tau\|)\Big) \|\mathbf{u}_\tau - \mathbf{v}_\tau\|\, dS$$
$$\le \mathcal{L}_\mu \|S_C\|_{L^\infty(\Gamma_C)} \int_{\Gamma_C} \|\mathbf{u}-\mathbf{v}\|^2\, dS.$$

This inequality and (6.2.9) imply (10.2.16). □

We have now all the ingredients needed to prove Theorem 10.1.1.

Proof (Theorem 10.1.1). (*i*) It follows from (6.4.2) that the bilinear form a, defined in (7.3.9), is symmetric and coercive with the constant $m = m_{el}$, i.e.,

$$a(\mathbf{v}, \mathbf{v}) \geq m_{el} \|\mathbf{v}\|_V^2 \qquad \forall \mathbf{v} \in V. \tag{10.2.17}$$

Let $\mathcal{L}_0 = m_{el}/c_B^2$. Clearly, $\mathcal{L}_0$ depends only on Ω, Γ_D, Γ_C and $\mathcal{A}_{el}$. Assume now that $\mathcal{L}_\mu \|S_C\|_{L^\infty(\Gamma_C)} < \mathcal{L}_0$. Then, there exits $c_0 \in \mathbb{R}$ such that

$$\mathcal{L}_\mu c_B^2 \|S_C\|_{L^\infty(\Gamma_C)} < c_0 < m_{el}.$$

Using (10.2.16), we obtain

$$j(\mathbf{u}, \mathbf{v} - \mathbf{u}) - j(\mathbf{v}, \mathbf{v} - \mathbf{u}) \leq c_0 \|\mathbf{u} - \mathbf{v}\|_V^2 \qquad \forall \mathbf{u}, \mathbf{v} \in V.$$

We conclude that the functional j satisfies condition (j_4). Using now Lemmas 10.2.2–10.2.4, conditions (10.1.12)–(10.1.13), the fact that $\mathbf{F} \in W^{1,\infty}(0, T; V)$ and part (*i*) of Theorem 10.2.1, we deduce that Problem $P^V_{el-slip}$ has at least one solution $u \in W^{1,\infty}(0, T; V)$.

(*ii*) Let (10.1.9) hold, and so the functional j does not depend on the the first argument and, therefore, assumptions (j_7) and (j_8) hold. The conclusion follows now from parts (*ii*) and (*iii*) of Theorem 10.2.1. □

10.3 Viscoelastic Contact with Total Slip Rate Dependent Friction

We follow [21] and describe a problem of bilateral frictional contact between a viscoelastic material and a foundation, when the friction coefficient depends on the slip rate or on the total slip rate, i.e., when it depends on the history of the contact process. Here, we describe only the problem with the latter condition, since the analysis of the problem with slip rate dependence is somewhat simpler. We recall that 'bilateral' means that there is no separation of the body from the foundation at any point of Γ_C.

We begin with the definition of the total slip rate operator. For $\mathbf{v} \in C([0, T]; H^1(\Omega)^d)$ we denote by $\delta_t(\mathbf{v})$ the element of $L^2(\Gamma_C)$ given by

$$\delta_t(\mathbf{v}) = \int_0^t \|\mathbf{v}_\tau(s)\|\, ds \quad \text{a.e. on } \Gamma_C,$$

for $t \in [0, T]$. We note that when $\mathbf{v}$ is the surface velocity, $\delta_t(\mathbf{v})$ represents the total or accumulated slip rate, while $\delta_t(\mathbf{u})$, where $\mathbf{u}$ is the surface displacement, represents the total or accumulated slip. Since the wear of the contacting surfaces depends on the friction traction, which in turn depends on the total slip rate, we use $\delta_t(\mathbf{v})$ as an argument of μ. This takes into account the accumulated wear or other morphological changes of the surface. If one wishes to investigate the problem with accumulated slip $\delta_t(\mathbf{u})$ should be used. All the results below hold true for the latter case, and the proofs are slightly simpler.

We assume that the material is viscoelastic, its constitutive law is given by (6.4.3),

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}).$$

As above, both the viscosity operator $\mathcal{A}_{ve}$ and the elasticity operator $\mathcal{B}_{ve}$ depend on the location, although we do not show this explicitly, and satisfy conditions (6.4.4) and (6.4.5), respectively.

We assume that the friction coefficient μ depends on the total slip rate as well as on the position $\mathbf{x} \in \Gamma_C$, thus,

$$\mu = \mu(\mathbf{x}, \delta_t(\dot{\mathbf{u}})),$$

on $\Gamma_C \times (0, T)$, and we do not depict the dependence on $\mathbf{x}$ explicitly. Since we deal with bilateral contact we need to regularize the contact stress, using the regularization operator $\mathcal{R}$ (page 127) and we use the friction bound (2.6.11), i.e.,

$$H = H(\delta_t(\dot{\mathbf{u}}), \sigma_n) = \mu(\delta_t(\dot{\mathbf{u}}))|\mathcal{R}\sigma_n|(1 - \delta|\mathcal{R}\sigma_n|)_+.$$

The classical formulation of the frictional bilateral contact problem with total slip rate dependent friction coefficient is the following.

Problem $P_{ve-slip}$. *Find a displacement field* $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}(\boldsymbol{\varepsilon}(\dot{\mathbf{u}})) + \mathcal{B}_{ve}(\boldsymbol{\varepsilon}(\mathbf{u})) \quad \text{in } \Omega_T, \qquad (10.3.1)$$

$$\operatorname{Div} \boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \quad \text{in } \Omega_T, \qquad (10.3.2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \times (0, T), \qquad (10.3.3)$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \quad \text{on } \Gamma_N \times (0, T), \qquad (10.3.4)$$

$$\left.\begin{aligned} & u_n = 0, \\ & \|\boldsymbol{\sigma}_\tau\| \le H(\delta_t(\dot{\mathbf{u}}), \sigma_n), \\ & \boldsymbol{\sigma}_\tau = -H(\delta_t(\dot{\mathbf{u}}), \sigma_n)\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if } \dot{\mathbf{u}}_\tau \ne \mathbf{0} \end{aligned}\right\} \text{on } \Gamma_C \times (0, T), \qquad (10.3.5)$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega. \qquad (10.3.6)$$

We proceed to the variational formulation of the problem. To this end we use the space (6.2.7), endowed with the inner product (6.2.5).

In the study of the contact problem (10.3.1)–(10.3.6) we assume that the friction coefficient satisfies

$$\left.\begin{array}{l}\text{(a) } \mu : \Gamma_C \times \mathbb{R}_+ \to \mathbb{R}_+.\\[4pt] \text{(b) There exists } \mathcal{L}_\mu > 0 \text{ such that}\\ \quad |\mu(\mathbf{x}, r_1) - \mu(\mathbf{x}, r_2)| \le \mathcal{L}_\mu |r_1 - r_2|\\ \quad \forall\, r_1, r_2 \in \mathbb{R}_+, \text{ a.e. } \mathbf{x} \in \Gamma_C.\\[4pt] \text{(c) For all } r \in \mathbb{R}_+,\ \mathbf{x} \mapsto \mu(\mathbf{x}, r) \text{ is measurable on } \Gamma_C.\\[4pt] \text{(d) There exists } \mu^\star > 0 \text{ such that } \mu(\mathbf{x}, r) \le \mu^\star\\ \quad \forall\, r \in \mathbb{R}_+, \text{ a.e. } \mathbf{x} \in \Gamma_C.\end{array}\right\} \qquad (10.3.7)$$

The forces and the tractions satisfy (8.5.9), the initial displacement satisfies (8.5.11) and, finally, we denote by $\mathbf{F}(t)$ the element of V_1 given by (7.3.11) for all $v \in V_1$ and $t \in [0, T]$.

Let $j : [0, T] \times C([0, T]; V_1) \times Q_1 \times V_1 \to \mathbb{R}$ be the *friction functional*

$$j(t, \mathbf{u}, \boldsymbol{\sigma}, \mathbf{w}) = \int_{\Gamma_C} \mu(\delta_t(\mathbf{u})) |\mathcal{R}\sigma_n| (1 - \delta |\mathcal{R}\sigma_n|)_+ \|\mathbf{w}_\tau\| \, dS.$$

We note that if $(\mathbf{u}, \boldsymbol{\sigma})$ is a sufficiently regular solution of problem $P_{ve-slip}$ then

$$\begin{aligned}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{w}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(t, \dot{\mathbf{u}}, \boldsymbol{\sigma}(t), \mathbf{w}) - j(t, \dot{\mathbf{u}}, \boldsymbol{\sigma}(t), \dot{\mathbf{u}}(t))&\\ \ge (\mathbf{F}(t), \mathbf{w} - \dot{\mathbf{u}}(t))_V \qquad \forall \mathbf{w} \in V_1,\ t \in [0, T].&\quad (10.3.8)\end{aligned}$$

We obtain from (10.3.1), (10.3.6), and (10.3.8) the following variational formulation of problem (10.3.1)–(10.3.6).

Problem $P^V_{ve-slip}$. *Find a displacement field* $\mathbf{u} : [0, T] \to V_1$ *and a stress field* $\boldsymbol{\sigma} : [0, T] \to Q_1$ *such that* $\mathbf{u}(0) = \mathbf{u}_0$*, and for all* $t \in [0, T]$

$$\boldsymbol{\sigma}(t) = \mathcal{A}_{ve}(\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t))) + \mathcal{B}_{ve}(\boldsymbol{\varepsilon}(\mathbf{u}(t))),$$

and (10.3.8) hold.

The main result established in [21], based on the Banach fixed-point theorem, is the following.

Theorem 10.3.1. *Assume that* (6.4.4), (6.4.5), (8.5.9), (8.5.11)*, and* (10.3.7) *hold. Then, there exists* $\mu_0 > 0$*, which depends only on* Ω*,* Γ_D*,* Γ_C *and* $\mathcal{A}_{ve}$*, such that if* $\mu^\star \le \mu_0$*, then there exists a unique solution* $(\mathbf{u}, \boldsymbol{\sigma})$ *of problem* $P^V_{ve-slip}$*. Moreover, the solution satisfies* (8.5.22)*.*

We conclude that problem (10.3.1)–(10.3.6) has a unique weak solution $(u, \boldsymbol{\sigma})$ provided that μ is sufficiently small. We note that in this case we needed to regularize the normal stress and the result holds for small friction coefficient. The question whether the size restriction is due to the mathematical method or is an intrinsic feature of the problem is an open question. However, as has been mentioned on several occasions, it is well known that some friction problems do exhibit considerable difficulties, both mathematical and experimental, when the friction coefficient is not small.

10.4 Thermoelastic Contact with Signorini's Condition

We first present the results for the n-dimensional thermoelastic frictionless contact problem, following [39]. The setting is the same as in Fig. 1, and although all the results below hold for $d = 1, 2$, for the sake of simplicity we consider here the three-dimensional case. Therefore, in this section the indices i, j, k, l have values in the set $\{1, 2, 3\}$.

Let θ and $\mathbf{u}$ denote the temperature and displacement fields of the body, respectively. As a result of the applied forces $\mathbf{f}_B$, surface tractions $\mathbf{f}_N$, and heat sources q_{th} and the resulting thermal expansion, the body's thermomechanical state evolves in time. We assume that the body is thermoelastic, with linear constitutive relation

$$\boldsymbol{\sigma} = \mathcal{B}_{el}\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{M}\theta,$$

in components,

$$\sigma_{ij} = b_{ijkl}\varepsilon(\mathbf{u})_{kl} - m_{ij}\theta.$$

Here, the b_{ijkl} are the components of the elasticity tensor $\mathcal{B}_{el}$ and m_{ij} are the components of the second-order thermal expansion tensor $\mathcal{M}$. We allow for a nonhomogeneous and anisotropic material, therefore, the coefficients are functions of position, and there may be up to 21 independent components in $\mathcal{B}_{el}$ and six in $\mathcal{M}$, taking the necessary symmetries into account. In particular, $\mathcal{M}$ is a symmetric tensor, and in the isotropic case it is represented by only one number, the coefficient of thermal expansion α, i.e., $\mathcal{M} = \alpha I_3$, where I_3 is the 3×3 identity matrix.

We assume that the process is slow, the accelerations are negligible and the quasistatic approximation applies. The energy equation is given by

$$\dot{\theta} - \left(k_{ij}\,\theta_{,i}\right)_{,j} = -m_{ij}\Theta_{ref}\,\dot{u}_{i,j} + q_{th} \quad \text{in } \Omega_T. \tag{10.4.1}$$

Here, the k_{ij} are the components of the thermal conductivity tensor $\mathcal{K}$, which is symmetric. The first term on the right-hand side of (10.4.1) represents the (linearized) internal heat generated by the work of elastic deformations, and Θ_{ref} is a reference temperature, which is a positive constant and for convenience set to be equal to one.

For the sake of simplicity we assume that the temperature on the whole of the boundary Γ is held constant, at the ambient temperature (which is scaled to be zero). In particular, the foundation has the constant temperature zero, too. There is no loss of generality in this assumption, since any given boundary temperature θ_{bd} can be incorporated by a simple change of variable. Since we consider the frictionless case, there is no heat generated at the contact surface.

The foundation is assumed to be rigid and, thus, the contact condition is Signorini's. The classical formulation of the frictionless thermoelastic unilateral contact problem between a body and a rigid foundation is the following.

Problem P_{thel-S}. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^3$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^3$, *and a temperature field* $\theta : \Omega \times [0,T] \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma} = \mathcal{B}_{el}\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{M}\theta \qquad \text{in } \Omega_T, \tag{10.4.2}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \qquad \text{in } \Omega_T, \tag{10.4.3}$$

$$\dot{\theta} - \operatorname{div}(\mathcal{K}\nabla\theta) = -\mathcal{M}\cdot\nabla\dot{\mathbf{u}} + q_{th} \qquad \text{in } \Omega_T, \tag{10.4.4}$$

$$\theta = 0 \qquad \text{on } \Gamma \times (0,T), \tag{10.4.5}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_D \times (0,T), \tag{10.4.6}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \qquad \text{on } \Gamma_N \times (0,T), \tag{10.4.7}$$

$$u_n \le g,\ \sigma_n \le 0,\ \sigma_n(u_n - g) = 0,\ \boldsymbol{\sigma}_\tau = \mathbf{0} \quad \text{on } \Gamma_C \times (0,T), \tag{10.4.8}$$

$$\theta(0) = \theta_0 \qquad \text{in } \Omega. \tag{10.4.9}$$

Here, g represents the gap between the reference configuration and the rigid foundation (see Fig. 1 on page 11) and θ_0 is the initial temperature field; moreover 'div' and ∇ denote the divergence and the gradient operators for vector-valued and scalar functions, respectively, while $\mathcal{M}\cdot\nabla\dot{\mathbf{u}} = m_{ij}\dot{u}_{i,j}$.

We now proceed to a weak formulation of the problem. To that end we introduce the following function spaces

$$V_{th} = \{\mathbf{u} \in W^{1,2}(0,T;V) : u_{i,jk},\ \dot{u}_{i,j} \in L^2(0,T;L^2(\Omega))\},$$

and

$$H_{th} = \{\theta \in W^{1,2}(0,T;H_0^1(\Omega)) : \Delta\theta \in L^2(0,T;L^2(\Omega))\},$$

where we set $u_{i,jk} = \partial^2 u_i/\partial x_j\partial x_k$ and $\dot{u}_{i,j} = \partial^2 u_i/\partial x_j \partial t$. Also, Δ denotes the Laplace operator and V is given in (6.2.3). Next, we define

$$W = V_{th} \times H_{th}.$$

These three spaces are real Hilbert spaces, with respective norms

$$\|\mathbf{u}\|^2_{V_{th}} = \int_{\Omega_T}\left(\|\dot{\mathbf{u}}\|^2 + \sum_{i=1}^3 \|\nabla u_i\|^2 + \sum_{i,j=1}^3 (\dot{u}_{i,j})^2 + \sum_{i,j,k=1}^3 (u_{i,jk})^2\right) dxdt,$$

$$\|\theta\|^2_{H_{th}} = \int_{\Omega_T}\left(\dot{\theta}^2 + \|\nabla\theta\|^2 + (\Delta\theta)^2 + \theta^2\right) dxdt,$$

and

$$\|(\mathbf{v},\xi)\|^2_W = \int_{\Omega_T}\left(\|\dot{\mathbf{v}}\|^2 + \sum_{i=1}^3 \|\nabla v_i\|^2 + \sum_{i,j=1}^3 (\dot{v}_{i,j})^2 + \dot{\xi}^2 + (\Delta\xi)^2\right) dxdt.$$

Next, we need the following convex subset of W,

$$K_T = \{(\mathbf{v}, \xi) \in W : \xi(0) = \theta_0 \ \text{ in } \Omega, \quad v_n \le g \ \text{ on } \Gamma_C \times (0,T)\},$$

and let W' denote the dual of the space W. We also define the operator A_{thel} by

$$\langle A_{thel}(\mathbf{u}, \theta), (\mathbf{v}, \xi)\rangle = \int_{\Omega_T} \Big(\dot\theta\xi - (k_{ij}\theta_{,j})_{,i}\xi + m_{ij}\dot u_{j,i}\xi \\ + b_{ijkl}u_{k,l}v_{i,j} - m_{ij}\theta v_{i,j}\Big)\, dxdt.$$

In this section $\langle\cdot,\cdot\rangle$ represents the duality pairing between W' and W.

The weak formulation of the problem is as follows.

Problem P^V_{thel-S}. *Find a pair* $(\mathbf{u}, \theta) \in K_T$ *such that*

$$\langle A_{thel}(\mathbf{u}, \theta), (\mathbf{v}, \xi)\rangle \ge \int_{\Omega_T} (\mathbf{f}_B \cdot (\mathbf{v} - \mathbf{u}) + q_{th}(\xi - \theta))\, dxdt \\ + \int_{\Gamma_N \times (0,T)} \mathbf{f}_N \cdot (\mathbf{v} - \mathbf{u})\, dSdt, \quad (10.4.10)$$

for all $(\mathbf{v}, \xi) \in K_T$.

We make the following assumptions on the problem data:

$$b_{ijkl} \in L^\infty(\Omega), \quad m_{ij} \in W^{1,\infty}(\Omega), \quad k_{ij} \in W^{1,\infty}(\Omega). \quad (10.4.11)$$

$$b_{ijkl} = b_{jikl} = b_{klij}, \\ b_{ijkl}\xi_{kl}\xi_{ij} \ge b^*\xi_{ij}\xi_{ij} \quad \forall \boldsymbol{\xi} = (\xi_{ij}) \in \mathbb{S}^3. \quad (10.4.12)$$

$$m_{ij} = m_{ji}, \quad 0 \le m_{ij} \le m^*. \quad (10.4.13)$$

$$k_{ij} = k_{ji}, \quad k_{ij}z_jz_i \ge k^* z_iz_i \quad \forall \mathbf{z} = (z_i) \in \mathbb{R}^3. \quad (10.4.14)$$

$$\mathbf{f}_B \in W^{1,2}(0,T; L^2(\Omega)^3). \quad (10.4.15)$$

$$\mathbf{f}_N \in W^{1,2}(0,T; L^2(\Gamma_N)^3). \quad (10.4.16)$$

$$q_{th} \in L^2(0,T; L^2(\Omega)). \quad (10.4.17)$$

$$g \in L^2(\Gamma_C), \quad g \ge 0 \ \text{ a.e. on } \Gamma_C. \quad (10.4.18)$$

$$\theta_0 \in H^1_0(\Omega). \quad (10.4.19)$$

Here, b^*, m^*, and k^* are positive constants.

The following existence result has been established in [39].

Theorem 10.4.1. *Under the assumptions* (3.4.13–23) *problem* P^V_{thel-S} *has a solution, provided* m^* *is sufficiently small.*

The proof was based on the Schauder fixed-point theorem, which led to the size restriction on the thermal expansion bound m^*. The size restriction on m^* has been removed in [40]. We conclude by Theorem 10.4.1 that Problem P_{thel-S} has at least one weak solution, however, the uniqueness remains an unresolved question.

10.5 Thermoviscoelastic Bilateral Contact

We consider, following [130], a thermoviscoelastic body in frictional bilateral contact with a rigid moving foundation. We assume that the contact is maintained at all times, which is the case in many engineering systems. The rigid foundation moves with a prescribed tangential velocity $\mathbf{v}^* = \mathbf{v}^*(t)$, and this motion is accompanied by frictional heat generation on the part of the contact surface where relative slip takes place. As above, it is assumed that the body is acted upon by volume forces $\mathbf{f}_B$ and surface tractions $\mathbf{f}_N$, and may have a volume heat source of density q_{th}.

We assume that the material is linearly thermoviscoelastic, with constitutive relation

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{M}\theta,$$

in components,

$$\sigma_{ij} = a_{ijkl}\dot{u}_{k,l} + b_{ijkl}u_{k,l} - m_{ij}\theta.$$

Here, the a_{ijkl} are the components of the viscosity tensor $\mathcal{A}_{ve}$, the b_{ijkl} are the components of the elasticity tensor $\mathcal{B}_{ve}$ and the m_{ij} are the components of the thermal expansion tensor $\mathcal{M}$.

The contact is assumed bilateral, thus $u_n = 0$ on $\Gamma_C \times (0, T)$. We employ the space V_1, (6.2.7), and denote by V_1' its dual. To describe the friction condition we use the smoothing operator $\mathcal{R}$ (which was described in Sect. 8.5), so that $\mathcal{R}\sigma_n$ makes sense on Γ_C. The friction bound $H(\sigma_n)$ is given by

$$H(\sigma_n) = \mu|\mathcal{R}\sigma_n|(1 - \delta|\mathcal{R}\sigma_n|)_+, \tag{10.5.1}$$

where the friction coefficient μ is assumed to be a constant, and δ is a small surface material parameter. Then, we describe friction on Γ_C by

$$\|\boldsymbol{\sigma}_\tau\| \le H(\sigma_n),$$

and

$$\boldsymbol{\sigma}_\tau = -H(\sigma_n)\frac{\dot{\mathbf{u}}_\tau - \mathbf{v}^*}{\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|} \qquad \text{if} \quad \dot{\mathbf{u}}_\tau \ne \mathbf{v}^*.$$

We assume that the temperature on the part of the surface $\Gamma_D \cup \Gamma_N$ is given, while on Γ_C the heat flux condition, which includes frictional heat generation, is specified. The latter depends on the friction bound H and on the relative slip speed, and is in the form

$$k_{ij}\theta_{,i}\, n_j = H(\sigma_n)\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\| - k_e(\theta - \theta_R),$$

where the first term on the right-hand side describes the heat generated by friction, and the second term represents heat exchange between the body and the foundation. The coefficient of heat exchange k_e is assumed a positive constant, and the foundation's temperature θ_R is assumed to be known.

For technical reasons we replace the term $\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|$ with a regularized term $s_c(\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|)$, where s_c is a bounded Lipschitz function which may depend on the position $\mathbf{x} \in \Gamma_C$. The mathematical reason for this is the need to control the term when $\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\| \to \infty$, and therefore, in each specific application we may use $s_c(\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|) = \|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|$, up to a sufficiently high limit, and then assume it to be constant. This restriction is purely for mathematical reasons. From the applied point of view there is no real loss of generality by such a choice, since one can set the limit so high that the system under investigation cannot reach it.

The classical model for the process of quasistatic thermoviscoelastic bilateral contact with friction and frictional heat generation is as follows.

Problem P_{thve-b}. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *and a temperature field* $\theta : \Omega \times [0,T] \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{M}\theta \quad \text{in} \quad \Omega_T, \tag{10.5.2}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \quad \text{in} \quad \Omega_T, \tag{10.5.3}$$

$$\dot{\theta} - \operatorname{div}(\mathcal{K}\theta) = -\mathcal{M}\cdot\nabla\dot{u} + q_{th} \quad \text{in} \quad \Omega_T, \tag{10.5.4}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_D \times (0,T), \tag{10.5.5}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \quad \text{on} \quad \Gamma_N \times (0,T), \tag{10.5.6}$$

$$\theta = \theta_b \quad \text{on} \quad (\Gamma_D \cup \Gamma_N) \times (0,T), \tag{10.5.7}$$

$$\left.\begin{array}{l} u_n = 0, \\ \|\boldsymbol{\sigma}_\tau\| \le H(\sigma_n), \\ \boldsymbol{\sigma}_\tau = -H(\sigma_n)\dfrac{\dot{\mathbf{u}}_\tau - \mathbf{v}^*}{\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|} \quad \text{if} \quad \dot{\mathbf{u}}_\tau \neq \mathbf{v}^*, \end{array}\right\} \quad \text{on} \quad \Gamma_C \times (0,T), \tag{10.5.8}$$

$$\begin{aligned} k_{ij}\theta_{,i}n_j = H(\sigma_n)s_c(\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|) \\ -k_e(\theta - \theta_R) \quad \text{on} \quad \Gamma_C \times (0,T), \end{aligned} \tag{10.5.9}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \theta(0) = \theta_0 \quad \text{in} \quad \Omega. \tag{10.5.10}$$

Recall that $H(\sigma_n)$ is the function given by (10.5.1). We note that although the process is quasistatic, since the material is viscoelastic we need to specify the initial displacements $\mathbf{u}_0$, in addition to the initial temperature θ_0.

We proceed with the variational formulation of the problem. First, we describe the assumptions on the problem data, then obtain the variational formulation, derive from it an abstract formulation in terms of various operators and then state the existence and uniqueness result.

The need for the normal regularization operator $\mathcal{R}$ has been explained in Sect. 8.5. It is a linear and continuous operator $\mathcal{R} : H^{-1/2}(\Gamma) \to L^2(\Gamma)$ and we use it for mathematical reasons, although there may be good physical reasons to use it, see e.g., [213]. The results below do not depend on the the particular form of $\mathcal{R}$.

We use the spaces defined in Chap. 6 and also

$$H_1 = \{\, \eta \in H^1(\Omega) : \eta = 0 \text{ on } \Gamma_D \cup \Gamma_N \,\}.$$

H_1 is the subspace of $H^1(\Omega)$ where we seek the solutions of the thermal problem. It is a real Hilbert space when equipped with the inner product of $H^1(\Omega)$ and its dual will be denoted by H_1'. Moreover, in this section, $\langle\cdot,\cdot\rangle$ denotes either the duality pairing between V_1' and V_1, or H_1' and H_1, as is dictated by the context.

We now describe the assumptions on the data. The problem coefficients of elasticity, viscosity, thermal expansion, and thermal conductivity satisfy:

$$\left.\begin{array}{l}
\text{(a) } a_{ijkl},\ b_{ijkl},\ m_{ij},\ k_{ij} \in L^\infty(\Omega).\\
\text{(b) } a_{ijkl} = a_{jikl} = a_{klij},\\
\qquad a_{ijkl}\xi_{kl}\xi_{ij} \ge c_1\xi_{ij}\xi_{ij} \quad \forall \boldsymbol{\xi} = (\xi_{ij}) \in \mathbb{S}^d.\\
\text{(c) } b_{ijkl} = b_{jikl} = b_{klij},\\
\qquad b_{ijkl}\xi_{kl}\xi_{ij} \ge c_2\xi_{ij}\xi_{ij} \quad \forall \boldsymbol{\xi} = (\xi_{ij}) \in \mathbb{S}^d.\\
\text{(d) } m_{ij} = m_{ji}.\\
\text{(e) } k_{ij} = k_{ji}, \quad k_{ij}z_jz_i \ge c_3 z_i z_i \quad \forall \mathbf{z} = (z_i) \in \mathbb{R}^d.
\end{array}\right\} \tag{10.5.11}$$

Here, c_1, c_2 and c_3 are positive constants.

The body forces and the volume heat sources satisfy

$$\mathbf{f}_B \in L^2(0,T;L^2(\Omega)^d), \qquad q_{th} \in L^2(0,T;H_1'). \tag{10.5.12}$$

The friction coefficient and the velocity of the foundation satisfy

$$\left.\begin{array}{l}
\text{(a) } \mu \in L^\infty(\Gamma_C) \quad \mu \ge 0, \quad \text{a.e. on} \quad \Gamma_C.\\
\text{(b) } \mathbf{v}^* : \Gamma_C \times [0,T] \to \mathbb{R}^d \ \text{ is a continuous function.}
\end{array}\right\} \tag{10.5.13}$$

The function s_c satisfies

$$\left.\begin{array}{l}
\text{(a) } s_c : \Gamma_C \times \mathbb{R} \longrightarrow \mathbb{R}_+.\\
\text{(b) There exists } c_4 > 0 \text{ such that}\\
\qquad |s_c(\mathbf{x}, r_1) - s_c(\mathbf{x}, r_2)| \le c_4|r_1 - r_2|\\
\qquad \forall\, r_1,\, r_2 \in \mathbb{R}, \ \text{a.e. } \mathbf{x} \in \Gamma_C.\\
\text{(c) For all } r \in \mathbb{R},\ \mathbf{x} \mapsto s_c(\mathbf{x}, r) \text{ is measurable on } \Gamma_C.\\
\text{(d) There exists } c_5 > 0 \text{ such that}\\
\qquad s_c(\mathbf{x}, r) \le c_5 \quad \forall\, r \in \mathbb{R}, \ \text{a.e. } \mathbf{x} \in \Gamma_C.
\end{array}\right\} \tag{10.5.14}$$

Here, c_4 and c_5 are positive constants.

The assumptions on the boundary and initial data are:

$$\left.\begin{array}{l}
\text{(a) } \mathbf{f}_N \in L^2(0,T;L^2(\Gamma_N)^d).\\
\text{(b) There exists } \Theta \in W^{1,2}(0,T;H^1(\Omega)) \text{ such that}\\
\qquad \Theta = \theta_b \ \text{ on } \ \Gamma_D \cup \Gamma_N.\\
\text{(c) } \theta_R \in L^2(0,T;L^2(\Gamma_C)).\\
\text{(d) } \mathbf{u}_0 \in V_1, \qquad \theta_0 \in L^2(\Omega).
\end{array}\right\} \tag{10.5.15}$$

For technical reasons, it is convenient to shift the temperature function so that it is zero on $\Gamma_D \cup \Gamma_N$. To that end, we introduce the new shifted temperature ξ, given by

$$\xi = \theta - \Theta,$$

and then $\xi_0 = \theta_0 - \Theta(0)$. To simplify the notation, we will not indicate explicitly the dependence on t.

We can now present the following weak formulation of problem (10.5.2)–(10.5.10), which may be obtained in the usual way.

Find a triple $(\mathbf{u}, \boldsymbol{\sigma}, \xi)$ *such that:*

$$\mathbf{u} \in W^{1,2}(0,T;V_1), \quad \mathbf{u}(0) = \mathbf{u}_0, \tag{10.5.16}$$

$$\xi \in L^2(0,T;H_1), \quad \dot{\xi} \in L^2(0,T;H_1'), \quad \xi(0) = \xi_0, \tag{10.5.17}$$

$$\boldsymbol{\sigma} = (\sigma_{ij}) \in L^2(0,T;Q_1),$$

$$\sigma_{ij} = b_{ijkl}u_{k,l} + a_{ijkl}\dot{u}_{k,l} - m_{ij}(\xi + \Theta), \tag{10.5.18}$$

$$\begin{aligned}
&\int_{\Omega_T} a_{ijkl}\dot{u}_{k,l}(w_{i,j} - \dot{u}_{i,j})\, dxdt + \int_{\Omega_T} b_{ijkl}u_{k,l}(w_{i,j} - \dot{u}_{i,j})\, dxdt \\
&\quad - \int_{\Omega_T} m_{ij}\xi(w_{i,j} - \dot{u}_{i,j}) dxdt \\
&\quad + \int_{\Gamma_C \times (0,T)} \mu|\mathcal{R}\sigma_n|\big(1 - \delta|\mathcal{R}\sigma_n|\big)_+\big(\|\mathbf{w}_\tau - \mathbf{v}^*\| - \|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|\big) dSdt \\
&\quad \ge \int_{\Omega_T} \mathbf{f}_B \cdot (\mathbf{w} - \dot{\mathbf{u}}) dxdt + \int_{\Omega_T} m_{ij}\Theta(w_{i,j} - \dot{u}_{i,j}) dxdt \\
&\quad + \int_{\Gamma_N \times (0,T)} \mathbf{f}_N \cdot (\mathbf{w} - \dot{\mathbf{u}})\, dSdt,
\end{aligned} \tag{10.5.19}$$

for all $\mathbf{w} \in L^2(0,T;V_1)$, *and*

$$\begin{aligned}
&\int_0^T \langle \dot{\xi}, \eta \rangle\, dt + \int_{\Omega_T} k_{ij}\xi_{,i}\eta_{,j}\, dxdt + \int_{\Omega_T} m_{ij}\dot{u}_{i,j}\eta\, dxdt \\
&\quad + \int_{\Gamma_C \times (0,T)} k_e \xi \eta\, dSdt + \int_{\Omega_T} k_{ij}\Theta_{,i}\eta_{,j}\, dxdt \\
&\quad - \int_{\Gamma_C \times (0,T)} \mu|\mathcal{R}\sigma_n|\big(1 - \delta|\mathcal{R}\sigma_n|\big)_+ s_c\big(\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|\big)\eta\, dSdt \\
&= \int_0^T \langle q_{th}, \eta \rangle\, dt - \int_{\Omega_T} \dot{\Theta}\eta\, dxdt - \int_{\Gamma_C \times (0,T)} k_e(\Theta - \theta_R)\eta\, dSdt,
\end{aligned} \tag{10.5.20}$$

for all $\eta \in L^2(0,T;H_1)$.

We write the weak formulation in an abstract form, and to that end we define the operators

$$\begin{aligned} A,\ B : V_1 &\longrightarrow V_1', \\ C_1 : V_1 &\longrightarrow H_1', \\ C_2 : H_1 &\longrightarrow V_1', \\ K_1,\ K_2 : H_1 &\longrightarrow H_1', \\ S : Q_1 \times V_1 &\longrightarrow H_1', \end{aligned}$$

as follows,

$$\langle A\mathbf{v}, \mathbf{w}\rangle = \int_\Omega a_{ijkl} v_{k,l} w_{i,j}\, dx, \tag{10.5.21}$$

$$\langle B\mathbf{u}, \mathbf{w}\rangle = \int_\Omega b_{ijkl} u_{k,l} w_{i,j}\, dx, \tag{10.5.22}$$

$$\langle C_1\mathbf{v}, \eta\rangle = \int_\Omega m_{ij} v_{i,j} \eta\, dx, \tag{10.5.23}$$

$$\langle C_2\xi, \mathbf{w}\rangle = -\int_\Omega m_{ij} \xi w_{i,j}\, dx, \tag{10.5.24}$$

$$\langle K_1\xi, \eta\rangle = \int_{\Gamma_C} k_e \xi \eta dS, \tag{10.5.25}$$

$$\langle K_2\xi, \eta\rangle = \int_\Omega k_{ij} \xi_{,i} \eta_{,j}\, dx, \tag{10.5.26}$$

$$\langle S(\boldsymbol{\sigma}, \mathbf{v}), \eta\rangle = -\int_{\Gamma_C} \mu |\mathcal{R}\sigma_n| \big(1 - \delta|\mathcal{R}\sigma_n|\big)_+ s_c(\|\mathbf{v}_\tau - \mathbf{v}^*\|)\eta\, dS. \tag{10.5.27}$$

We note that each of these operators extends, in a natural way, to an operator defined on the corresponding space of measurable and square-integrable vector-valued functions on $(0, T)$. For example, A extends to an operator from $L^2(0, T; V_1)$ to $L^2(0, T; V_1')$ by setting $(A\mathbf{u})(t) = A(\mathbf{u}(t))$. With a slight abuse of notation, we use below the same symbol to denote both the original operator and its extension, since the meaning will be clear from the context.

We also consider the functions $\mathbf{f} \in L^2(0, T; V_1')$ and $\mathcal{Q} \in L^2(0, T; H_1')$ given by

$$\langle\langle \mathbf{f}, \mathbf{w}\rangle\rangle = \int_{\Omega_T} \mathbf{f}_B \cdot \mathbf{w}\, dxdt + \int_{\Omega_T} m_{ij} \Theta w_{i,j}\, dxdt + \int_{\Gamma_N \times (0,T)} \mathbf{f}_N \cdot \mathbf{w}\, dSdt,$$

$$\begin{aligned} \langle\langle \mathcal{Q}, \eta\rangle\rangle = &\int_0^T \langle q_{th}, \eta\rangle\, dt - \int_{\Omega_T} \dot{\Theta}\eta\, dxdt - \int_{\Gamma_C \times (0,T)} k_e(\Theta - \theta_R)\eta\, dS\, dt \\ &- \int_{\Omega_T} k_{ij} \Theta_{,i} \eta_{,j}\, dxdt, \end{aligned}$$

for all $\mathbf{w} \in L^2(0, T; V_1)$ and $\eta \in L^2(0, T; H_1)$, respectively. Here, $\langle\langle \cdot, \cdot\rangle\rangle$ denotes the duality pairing between $L^2(0, T; V_1')$ and $L^2(0, T; V_1)$, or between $L^2(0, T; H_1')$ and $L^2(0, T; H_1)$, as is dictated by the context. Moreover, in what follows $\partial_2 j(\boldsymbol{\sigma}, \mathbf{v})$ denotes the subdifferential with respect to the argument $\mathbf{v}$ of the functional

$$j(\boldsymbol{\sigma}, \mathbf{v}) = \int_{\Gamma_C \times (0,T)} \mu |\mathcal{R}\sigma_n| \big(1 - \delta |\mathcal{R}\sigma_n|\big)_+ \|\mathbf{v}_\tau - \mathbf{v}^*\| \, dSdt.$$

We can now formulate problem (10.5.16)–(10.5.20) abstractly as follows.

Problem P^V_{thve-b}. *Find* $(\mathbf{u}, \boldsymbol{\sigma}, \xi)$ *satisfying* (10.5.16)–(10.5.18) *and*

$$A\dot{\mathbf{u}} + B\mathbf{u} + C_2\,\xi + \partial_2 j(\boldsymbol{\sigma}, \dot{\mathbf{u}}) \ni f \quad \text{in } L^2(0,T;V_1'),$$

$$\dot{\xi} + K_1\xi + K_2\xi + C_1\dot{\mathbf{u}} + S(\boldsymbol{\sigma}, \dot{\mathbf{u}}) = \mathcal{Q} \quad \text{in} \quad L^2(0,T;H_1').$$

The main existence and uniqueness result in [130] is the following.

Theorem 10.5.1. *Assume that* (10.5.11–15) *hold. Then Problem* P^V_{thve-b} *has a unique solution, provided that* $\|\mu\|_{L^\infty(\Gamma_C)}$ *is sufficiently small.*

The proof in [130] was based on an abstract existence theorem of [231] applied to a regularized problem. Then, it was shown that for sufficiently small friction coefficient the solution operator is a contraction on the appropriate Hilbert space. Estimating the allowed size of the friction coefficient, or removing this restriction altogether, remains an open and very interesting problem.

We conclude that problem (10.5.2)–(10.5.10) has a unique weak solution when $\|\mu\|_{L^\infty(\Gamma_C)}$ is sufficiently small.

11 Contact with Wear or Adhesion

In this chapter we present results on models of frictional contact when the wear of the contacting surfaces is taken into account. The importance of the control and minimization of the wear of industrial parts and components cannot be overstated, and therefore, effective models for the prediction of wear in industrial settings are indispensable to the design engineer. We describe three problems of contact with wear for viscoelastic materials of the form (6.4.3). The wear is described by the differential form of Archard's law and the contact is bilateral in Sect. 11.1 and with the normal compliance condition in Sect. 11.2. It is assumed, in both cases, that the wear particles or debris are removed from the system as quickly as they are being produced. This is the case in many settings, and one of the functions of engine oil is to remove such debris. However, in other settings wear particles remain and migrate or diffuse in the gap between the contacting surfaces causing further wear and deterioration of the lubricant or of the surfaces themselves. We describe this in Sect. 11.3, where the diffusion equation is used for the debris.

In the case of bilateral contact with slip it is found that the wear condition leads to a normal damped response form for the contact stresses. When the normal compliance condition is used, wear enters as a modification of the gap between the body and the foundation.

The study of adhesive contact is very recent in the mathematical literature. The novelty lies in the introduction of the adhesion field on the contact surface and deriving an equation for its evolution; and we refer to Sect. 3.3 for the modelling details.

In this chapter we present in detail two problems of frictionless adhesive contact, and mention few others. In Sect. 11.4 we describe a bilateral frictionless contact problem with reversible and with memory or history dependent adhesion. It models a situation where cycles of debonding and rebonding may take place, and in each cycle the adhesive deteriorates slightly. The existence of the unique solution is stated and proved in Sect. 11.5. In Sect. 11.6 we present a recent model for the adhesive contact between a membrane and a rigid obstacle. This model is somewhat different from the others in Part II of this monograph, since it deals with a simpler geometry and a scalar equation.

We use dimensionless variables in this chapter.

M. Shillor, M. Sofonea, J.J. Telega: *Models and Analysis of Quasistatic Contact*, Lect. Notes Phys. **655**, (2001), 183–206
http://www.springerlink.com/

11.1 Bilateral Frictional Contact with Wear

In the problem we describe now the contact between the body and the rigid moving foundation is maintained at all times, and there is only relative sliding. A setting of this type is a conveyer belt or a chain connecting two rotating wheels.

Following [22], let $w : \Gamma_C \times [0, T] \to \mathbb{R}$ be the *wear* function, which is identified as the normal depth of the material that is worn out and immediately removed from the system. It is negative when the foundation is worn out, and positive when the surface of the body wears out. Here, we assume that the foundation is rigid and the body wears out, and therefore w is nonnegative.

Since the body is in bilateral contact with the foundation,

$$u_n = -w. \tag{11.1.1}$$

Since $w \geq 0$ it follows that $u_n \leq 0$, and therefore the effect of the wear is the recession of Γ_C. In this way we describe the evolution of the shape of the contact zone as a result of wear.

The evolution of the wear of the contacting surface is governed by a simplified version of Archard's law (3.2.1) which we now describe. Since the normal stress on the contact surface is nonnegative ($\sigma_n \leq 0$ on Γ_C), the wear rate form of Archard's law is

$$\dot{w} = -k_w \sigma_n \|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|,$$

where $k_w > 0$ is the wear coefficient, $\mathbf{v}^*$ is the tangential velocity of the foundation and $\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|$ represents the relative slip speed between the contact surface and the foundation. For the sake of simplicity we assume in this section that the motion of the foundation is uniform, i.e., $\mathbf{v}^*$ does not vary in time, and so $\alpha^* = \|\mathbf{v}^*\| > 0$ is constant, and k_w is also assumed to be constant. We assume that $\|\mathbf{v}^*\|$ is large, as is the case in metal forming, so that $\|\dot{\mathbf{u}}_\tau\|$ is negligible in comparison to $\|\mathbf{v}^*\|$, and thus we obtain the following version of the wear law,

$$\dot{w} = -k_w \alpha^* \sigma_n. \tag{11.1.2}$$

We can eliminate, by using (11.1.2), the unknown function w from the problem. In this manner the problem decouples, and once the solution of the frictional contact problem has been obtained, the wear of the surface can be obtained by integration of (11.1.2) in time. Indeed, let $\delta_w = 1/(k_w \alpha^*)$, which by the assumptions is constant. Using (11.1.1) and (11.1.2) we find

$$\sigma_n = \delta_w \, \dot{u}_n. \tag{11.1.3}$$

Note that (11.1.2) implies $\dot{w} \geq 0$, that is the wear increases in time, also $\dot{u}_n \leq 0$ and since $\sigma_n \leq 0$, it follows that

$$-\sigma_n = \delta_w\,|\dot{u}_n|. \tag{11.1.4}$$

We note that (11.1.4) has the same form as the normal damped response condition (2.6.4).

Using now the usual law of dry friction, and noting that there is only sliding contact, so that $\dot{\mathbf{u}}_\tau \neq \mathbf{v}^*$, we obtain

$$\sigma_\tau = -\mu\,|\sigma_n|\,\frac{\dot{\mathbf{u}}_\tau - \mathbf{v}^*}{\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|}\,, \tag{11.1.5}$$

where μ is the coefficient of friction. Although in the wear condition we have neglected $\|\dot{\mathbf{u}}_\tau\|$, as compared with $\|\mathbf{v}^*\|$, here we retain it, since otherwise the mathematical problems simplifies considerably. Since $\sigma_n \le 0$, condition (11.1.5) implies

$$\boldsymbol{\sigma}_\tau = \mu\sigma_n\,\frac{\dot{\mathbf{u}}_\tau - \mathbf{v}^*}{\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|}\,. \tag{11.1.6}$$

With (6.4.3) as the constitutive law and (11.1.4), (11.1.6) as the frictional contact condition, the classical formulation of the mechanical problem of bilateral contact with sliding friction and wear is the following.

Problem P_{ve-bw}. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{in } \Omega_T, \tag{11.1.7}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \qquad \text{in } \Omega_T, \tag{11.1.8}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_D \times (0,T), \tag{11.1.9}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \qquad \text{on } \Gamma_N \times (0,T), \tag{11.1.10}$$

$$\left.\begin{aligned} -\sigma_n &= \delta_w\,|\dot{u}_n|, \\ \boldsymbol{\sigma}_\tau &= \mu\sigma_n\frac{\dot{\mathbf{u}}_\tau - \mathbf{v}^*}{\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|} \end{aligned}\right\} \text{ on } \Gamma_C \times (0,T), \tag{11.1.11}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \text{in } \Omega. \tag{11.1.12}$$

Here and below it is assumed that $\dot{\mathbf{u}}_\tau \neq \mathbf{v}^*$, and that (8.5.10), (8.6.13), (8.6.14) hold.

Recall that the space V, (6.2.3), is a real Hilbert space when equipped with the inner product (6.2.5). Denote by $\mathbf{F}$ the element given by (8.3.13) and let $j : V \times V \to \mathbb{R}$ be the surface functional

$$j(\mathbf{u},\mathbf{v}) = \int_{\Gamma_C} \delta_w\,|u_n|(\mu\|\mathbf{u}_\tau - \mathbf{v}^*\| + v_n)\,dS \qquad \forall\,\mathbf{u},\mathbf{v} \in V. \tag{11.1.13}$$

The variational formulation of the mechanical problem (11.1.7)–(11.1.12) is as follows.

Problem P^V_{ve-bw}. *Find a displacement field* $\mathbf{u} : [0,T] \to V$ *and a stress field* $\boldsymbol{\sigma} : [0,T] \to Q_1$ *such that*

$$\mathbf{u}(0) = \mathbf{u}_0, \tag{11.1.14}$$

and for all $t \in [0,T]$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \tag{11.1.15}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\dot{\mathbf{u}}(t), \mathbf{v}) - j(\dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t)) \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \quad \forall\, \mathbf{v} \in V. \tag{11.1.16}$$

The solvability of this problem, when the slip speed is sufficiently large, is stated in the following theorem, taken from [22].

Theorem 11.1.1. *Let conditions* (6.4.4), (6.4.5), (8.5.10), (8.6.13), *and* (8.6.14) *hold. There exists a positive constant* α_0*, which depends only on* Ω, Γ_D, Γ_C, $\mathcal{A}_{ve}$ *and* μ*, such that, if*

$$\delta_w < \alpha_0, \tag{11.1.17}$$

then Problem P^V_{ve-bw} *has a unique solution* $(\mathbf{u}, \boldsymbol{\sigma})$ *which satisfies* (8.5.22).

The proof of Theorem 11.1.1 may be carried out by using the same steps and similar arguments as those in the in the proof of Theorem 8.5.1.

We observe that if α^* is large enough then $\delta_w = 1/(k_w\alpha^*)$ is sufficiently small and, therefore, condition (11.1.17) for the unique solvability of Problem P^V_{ve-bw} is satisfied. We conclude that the mechanical problem (11.1.7)–(11.1.12) has a unique weak solution if the tangential velocity of the foundation is large enough. This is consistent with neglecting the term $\dot{\mathbf{u}}_\tau$ in the wear condition (11.1.2) as compared to $\mathbf{v}^*$, which cannot be justified when the latter is small.

Moreover, having solved the problem (11.1.7)–(11.1.12), we obtain the wear function w by integration of (11.1.2), using the initial condition $w(0) = 0$. The latter means that initially the surface is free from any prior wear.

Details on the variational analysis of Problem P^V_{ve-bw} can be found in [22]. Numerical analysis of the problem, including error estimates for semi-discrete and fully discrete schemes, is provided in [146].

11.2 Frictional Contact with Normal Compliance and Wear

In this section we follow [20] and describe frictional contact with normal compliance when the wear of the contacting surface, due to friction, is taken into account. As in Sect. 11.1, the foundation is assumed to move steadily and only sliding takes place. Let g be the initial gap between the body and the foundation and let p_n and p_τ denote the normal and tangential compliance functions. As in the previous section, we introduce the *wear* function $w : \Gamma_C \times$

$[0,T] \to \mathbb{R}_+$, which measures the accumulated wear of the surface Γ_C. We denote by $\mathbf{v}^*$ and $\alpha^* = \|\mathbf{v}^*\|$ the tangential velocity and the tangential speed of the foundation, respectively. We use the modified version of Archard's law (11.1.2) to describe the evolution of wear. We modify the contact conditions (8.3.5) to take into account the instantaneous material removal that takes place on the contact surface. We assume below that only sliding takes place so that $\dot{\mathbf{u}}_\tau \neq \mathbf{v}^*$. The classical formulation of the problem of sliding frictional contact with wear we consider is the following.

Problem P_{ve-ncw}. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$*, and a wear function* $w : \Gamma_C \times [0,T] \to \mathbb{R}_+$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega_T, \tag{11.2.1}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \quad \text{in } \Omega_T, \tag{11.2.2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \times (0,T), \tag{11.2.3}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \quad \text{on } \Gamma_N \times (0,T), \tag{11.2.4}$$

$$\left.\begin{aligned} -\sigma_n &= p_n(u_n - w - g),\\ \boldsymbol{\sigma}_\tau &= -p_\tau(u_n - w - g)\frac{\dot{\mathbf{u}}_\tau - \mathbf{v}^*}{\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|}, \end{aligned}\right\} \quad \text{on } \Gamma_C \times (0,T), \tag{11.2.5}$$

$$\dot{w} = -k_w\,\alpha^*\sigma_n \quad \text{on } \Gamma_C \times (0,T), \tag{11.2.6}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad w(0) = 0 \quad \text{in } \Omega. \tag{11.2.7}$$

We observe that now the wear appears in the normal compliance condition. Thus, compared to the problem studied in the previous section, problem (11.2.1)–(11.2.7) is coupled and therefore more complicated. Indeed, we may write (11.2.6) as

$$\dot{w} = k_w\,\alpha^* p_n(u_n - w - g),$$

and, clearly, we must solve the problem of wear together with the contact problem. Here, too, we assume that k_w and α^* are positive constants.

In the study of the mechanical problem (11.2.1)–(11.2.7) we assume that the viscosity operator $\mathcal{A}_{ve}$ and the elasticity operator $\mathcal{B}_{ve}$ satisfy the conditions (6.4.4) and (6.4.5) and the compliance functions p_e $(e = n, \tau)$ satisfy condition (8.3.9). We also assume that the force and traction densities satisfy (8.3.10), the gap function satisfies (8.3.11) and the initial displacements satisfy (8.3.12).

We denote by $\mathbf{F}(t)$ the element of V given by (8.3.13), for $\mathbf{v} \in V$, and $t \in [0,T]$, and we use the surface functional $j : V \times V \times L^2(\Gamma_C) \to \mathbb{R}$, given by

$$j(\mathbf{u},\mathbf{v},w) = \int_{\Gamma_C} p_n(u_n - w - g)\,v_n\,dS + \int_{\Gamma_C} p_\tau(u_n - w - g)\,\|\mathbf{v}_\tau - \mathbf{v}^*\|\,dS \tag{11.2.8}$$

for $\mathbf{u},\mathbf{v} \in V$, $w \in L^2(\Gamma_C)$.

The variational formulation of the mechanical problem (11.2.1)–(11.2.7) can be stated as follows.

Problem P^V_{ve-ncw}. *Find a displacement field* $\mathbf{u} : [0,T] \to V$, *a stress field* $\boldsymbol{\sigma} : [0,T] \to Q_1$, *and a wear function* $w : [0,T] \to L^2(\Gamma_C)$ *such that*

$$\mathbf{u}(0) = \mathbf{u}_0, \quad w(0) = 0, \tag{11.2.9}$$

and for all $t \in [0,T]$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \tag{11.2.10}$$

$$\dot{w} = -k_w\, \alpha^* \sigma_n, \tag{11.2.11}$$

$$\begin{aligned}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\mathbf{u}(t), \mathbf{v}, w(t)) - j(\mathbf{u}(t), \dot{\mathbf{u}}(t), w(t))\\ \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \quad \forall\, \mathbf{v} \in V.\end{aligned} \tag{11.2.12}$$

A triple $(\mathbf{u}, \boldsymbol{\sigma}, w)$ which satisfies (11.2.9)–(11.2.12) is called a *weak solution* of the mechanical problem of sliding frictional contact with normal compliance and wear.

The main result in this section is as follows ([20]).

Theorem 11.2.1. *Assume* (6.4.4), (6.4.5), *and* (8.3.9)–(8.3.12). *Then there exists a unique solution* $(\mathbf{u}, \boldsymbol{\sigma}, w)$ *of Problem* P^V_{ve-ncw}. *Moreover, the solution satisfies*

$$\mathbf{u} \in C^1([0,T]; V), \quad \boldsymbol{\sigma} \in C([0,T]; Q_1), \quad w \in C^1([0,T]; L^2(\Gamma_C)). \tag{11.2.13}$$

The proof of Theorem 11.2.1 can be found in ([20]), and was carried out in several steps. In the first step the wear was assumed to be given and the corresponding displacements and stresses were found, by using a version of Theorem 8.3.1. Then, problem P^V_{ve-ncw} was solved by using the Banach fixed-point theorem.

We would like to point out that, unlike the case in the previous section, no smallness assumptions were needed in the proof.

11.3 Frictional Contact with Normal Compliance and Wear Diffusion

In this section we extend the model in the previous section to include the diffusion of the wear particles or debris on the contact surface. We follow [153] and refer the reader to [152] for the full details. Whereas in many applications the wear debris is assumed to be removed immediately from the surface, in some important applications the debris remains on the surface, diffuses and may cause additional, even sever, wear. Such situations arise in orthopedic biomechanics of joint prostheses after arthoplasty (see, e.g., [148, 149] and references therein). Since friction and wear debris influence the quality and long term performance of artificial joints and implants, they need to be taken into account when modelling these processes.

We model the process in which a viscoelastic body is in frictional contact with a moving foundation and, as a result, a part of its surface wears out. The wear particles are assumed to remain and diffuse on the potential contact surface.

The setting is as in the previous section and is depicted in Fig. 1, and the body Ω is three-dimensional. Moreover, we assume that the coordinate system is such that Γ_C occupies a regular domain in the $x_3 = 0$ plane, the foundation is planar and is moving with velocity $\mathbf{v}^*$ in the plane $x_3 = -g \leq 0$. We assume that Γ_C is divided into two subdomains D_d and D_w by a smooth curve γ^*, and wear takes place only on the part D_w, while the diffusion of the wear particles takes place in the whole of Γ_C. The boundary $\gamma = \partial\Gamma_C$ of Γ_C is assumed Lipschitz and is composed of two parts γ_d and γ_w. Thus, $\partial D_w = \gamma_w \cup \gamma^*$ and $\partial D_d = \gamma_d \cup \gamma^*$. The setting is depicted in Fig. 9.

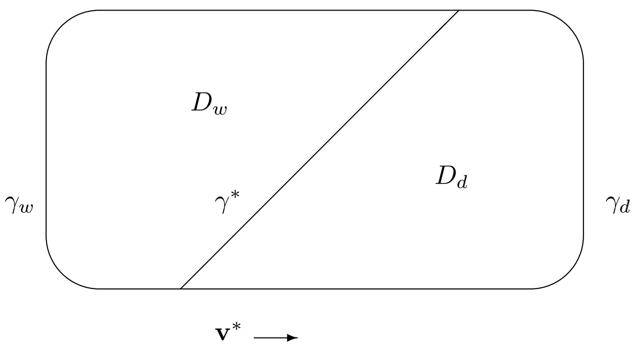

Fig. 9. The contact surface Γ_C; wear is produced only in D_w

The wear of the surface is described by the *wear function* w which is defined on the part D_w of Γ_C, and the diffusion of the wear debris by the *wear particle surface density function* w_d which is defined on the whole of Γ_C. The wear function w measures the volume density of material removed per unit surface area, see, e.g., [97, 98] and references therein. We assume that $w_d = \kappa_w w$ in D_w, where κ_w is a conversion factor from wear depth to wear particles surface density and is assumed to be a positive constant. This assumption simplifies the model since $w = \eta_w w_d$ in D_w, for $\eta_w = 1/\kappa_w$. Next, we extend w by zero to the whole of Γ_C, and thus, $w = \eta_w w_d \chi_{[D_w]}$ on $\Gamma_C \times (0, T)$, where $\chi_{[D_w]}$ is the characteristic function of D_w.

The diffusion of the particles is described by the nonlinear evolutionary equation,

$$\dot{w}_d - \operatorname{div}(k\,\nabla w_d) = \kappa\kappa_w\|\boldsymbol{\sigma}_\tau\|\,R_M(\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|)\chi_{[D_w]}, \tag{11.3.1}$$

in $\Gamma_C \times (0,T)$. Here k denotes the wear particle diffusion coefficient, κ is the wear rate constant, and $R_M : \mathbb{R}_+ \to \mathbb{R}_+$ is the truncation operator: $R_M(r) = r$ if $r \le M$, $R_M(r) = M$ if $r > M$, M being a given positive constant. We need this operator in order to avoid some mathematical difficulties related to very large slip rates, however, from the physical point of view the use of R_M is not restrictive since, in practice, the slip velocity is bounded and no smallness assumption will be made on M. We use $\chi_{[D_w]}$ on the right-hand side of (11.3.1) since the particles are produced only in D_w, and the rate of production is multiplied by κ_w.

We use a version of the normal compliance condition to model the contact, but since the process involves the wear of the contacting surfaces we take into account the change in the geometry by replacing the gap g with $g + w$, as in the previous section. Therefore,

$$-\sigma_n = p_n(u_n - \eta_w w_d \chi_{[D_w]} - g) \quad \text{on } \Gamma_C \times (0,T). \tag{11.3.2}$$

The friction law is chosen as (2.6.6), where $H = \mu|\sigma_n|$, thus,

$$\begin{aligned} &\|\boldsymbol{\sigma}_\tau\| \le \mu|\sigma_n|, \\ &\text{if } \dot{\mathbf{u}}_\tau \ne \mathbf{v}^* \text{ then } \boldsymbol{\sigma}_\tau = -\mu|\sigma_n|\frac{\dot{\mathbf{u}}_\tau - \mathbf{v}^*}{\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|} \end{aligned} \tag{11.3.3}$$

on $\Gamma_C \times (0,T)$. Here, $\mu = \mu(w_d, \|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|)$ is the coefficient of friction which is assumed to depend on the density of the wear particles and on the slip rate.

The classical formulation of the problem of *frictional contact of a viscoelastic body with wear diffusion* is as follows.

Problem $P_{ve-ncwd}$. *Find a displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^3$, a stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^3$, and a wear function $w_d : \Gamma_C \times [0,T] \to \mathbb{R}_+$ such that*

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega_T, \tag{11.3.4}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \quad \text{in } \Omega_T, \tag{11.3.5}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \times (0,T), \tag{11.3.6}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \quad \text{on } \Gamma_N \times (0,T), \tag{11.3.7}$$

$$\left.\begin{aligned} &-\sigma_n = p_n(u_n - \eta_w w_d \chi_{[D_w]} - g), \\ &\|\boldsymbol{\sigma}_\tau\| \le \mu|\sigma_n|, \\ &\boldsymbol{\sigma}_\tau = -\mu|\sigma_n|\frac{\dot{\mathbf{u}}_\tau - \mathbf{v}^*}{\|\dot{u}_\tau - \mathbf{v}^*\|} \quad \text{if } \dot{\mathbf{u}}_\tau \ne \mathbf{v}^* \end{aligned}\right\} \quad \text{on } \Gamma_C \times (0,T), \tag{11.3.8}$$

$$\dot{w}_d - \operatorname{div}(k\,\nabla w_d) = \mu_w p_n R_M(\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|)\chi_{[D_w]} \quad \text{on } \Gamma_C \times (0,T), \tag{11.3.9}$$

$$w_d = 0 \quad \text{on } \gamma \times (0,T), \tag{11.3.10}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \text{ in } \Omega, \quad w_d(0) = w_{d0} \quad \text{on } \Gamma_C. \tag{11.3.11}$$

Here, we set $\mu_w = \kappa\kappa_w\mu$; (11.3.5) represents the equation of equilibrium, since the process is assumed quasistatic; (11.3.6) and (11.3.7) are the displacement and traction boundary conditions, and (11.3.10) is an absorbing boundary condition, since once a wear particle reaches the boundary $\gamma = \partial\Gamma_C$ it dissapears; finally, (11.3.11) represent the initial conditions in which $\mathbf{u}_0$ and w_{d0} are given.

To obtain a variational formulation for problem $P_{ve-ncwd}$ we proceed as above. For the surface particle density function we use the space $H_0^1(\Gamma_C)$. We denote by $H^{-1}(\Gamma_C)$ the dual of $H_0^1(\Gamma_C)$ and $\langle\cdot,\cdot\rangle$ represents the duality pairing between $H^{-1}(\Gamma_C)$ and $H_0^1(\Gamma_C)$.

To study the mechanical problem $P_{ve-ncwd}$ we make the following assumptions on the problem data.
The *viscosity operator* $\mathcal{A}_{ve} : \Omega\times\mathbb{S}^3 \longrightarrow \mathbb{S}^3$ satisfies: there exist two constants $L_\mathcal{A} > 0$ and $m_\mathcal{A} > 0$ such that

$$\begin{aligned}
&(a)\ \|\mathcal{A}_{ve}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}_{ve}(\mathbf{x},\boldsymbol{\varepsilon}_2)\| \le L_\mathcal{A}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|,\ \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^3,\ \text{a.e. } \mathbf{x}\in\Omega;\\
&(b)\ (\mathcal{A}_{ve}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}_{ve}(\mathbf{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_\mathcal{A}\,\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2,\\
&\qquad\qquad\forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^3,\ \text{a.e. } \mathbf{x}\in\Omega;\\
&(c)\ \mathbf{x}\longmapsto \mathcal{A}_{ve}(\mathbf{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega,\ \forall\,\boldsymbol{\varepsilon}\in\mathbb{S}^3;\\
&(d)\ \mathbf{x}\longmapsto \mathcal{A}_{ve}(\mathbf{x},\mathbf{0}) \in Q.
\end{aligned} \tag{11.3.12}$$

The *elasticity operator* $\mathcal{B}_{ve} : \Omega\times\mathbb{S}^3 \longrightarrow \mathbb{S}^3$ satisfies: there exists a constant $L_\mathcal{B} > 0$ such that

$$\begin{aligned}
&(a)\ \|\mathcal{B}_{ve}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathcal{B}_{ve}(\mathbf{x},\boldsymbol{\varepsilon}_2)\| \le L_\mathcal{B}\ \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|,\ \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^3,\ \ \text{a.e. } \mathbf{x}\in\Omega;\\
&(b)\ \mathbf{x}\longmapsto \mathcal{B}_{ve}(\mathbf{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega,\ \forall\,\boldsymbol{\varepsilon}\in\mathbb{S}^3;\\
&(c)\ \mathbf{x}\longmapsto \mathcal{B}_{ve}(\mathbf{x},\mathbf{0}) \in Q.
\end{aligned} \tag{11.3.13}$$

The *normal compliance function* $p_n : \Gamma_C\times\mathbb{R}\to\mathbb{R}_+$ satisfies: there exist two constants $L_n > 0$ and $p_n^* > 0$ such that

$$\begin{aligned}
&(a)\ |p_n(\mathbf{x},u_1) - p_n(\mathbf{x},u_2)| \le L_n|u_1 - u_2|,\ \forall\,u_1,\,u_2\in\mathbb{R},\ \text{a.e. } \mathbf{x}\in\Gamma_C;\\
&(b)\ \mathbf{x}\longmapsto p_n(\mathbf{x},u)\ \text{ is Lebesgue measurable on } \Gamma_C,\ \forall\,u\in\mathbb{R};\\
&(c)\ \mathbf{x}\longmapsto p_n(\mathbf{x},u) = 0\ \text{ for }\ u\le 0,\ \text{a.e. } \mathbf{x}\in\Gamma_C;\\
&(d)\ p_n^*(\mathbf{x},u) \le p_n^*\ \forall\,u\in\mathbb{R},\ \text{a.e. } \mathbf{x}\in\Gamma_C.
\end{aligned} \tag{11.3.14}$$

The *coefficient of friction* $\mu : \Gamma_C\times\mathbb{R}\to\mathbb{R}_+$ satisfies: there exist two constants $L_\mu > 0$ and $\mu^* > 0$ such that

$$\begin{aligned}
&(a)\ |\mu(\mathbf{x},r_1,s_1) - \mu(\mathbf{x},r_2,s_2)| \le L_\mu(|r_1 - r_2| + |s_1 - s_2|),\\
&\qquad\qquad\forall\,r_1,r_2,s_1,s_2\in\mathbb{R},\ \text{a.e. } \mathbf{x}\in\Gamma_C;\\
&(b)\ \mathbf{x}\longmapsto \mu(\mathbf{x},r,s) \text{ is Lebesgue measurable on } \Gamma_C,\ \forall\,r,\,s\in\mathbb{R};\\
&(c)\ \mu(\mathbf{x},r,s) \le \mu^*,\ \forall\,r,\,s\in\mathbb{R},\ \text{a.e. } \mathbf{x}\in\Gamma_C.
\end{aligned} \tag{11.3.15}$$

The *forces*, *tractions*, *particle diffusion coefficient*, *wear rate constant* and the *initial data* satisfy, respectively:

$$\begin{aligned}
&(a)\ \mathbf{f}_B \in C([0,T];L^2(\Omega)^3), \quad \mathbf{f}_N \in C([0,T];L^2(\Gamma_N)^3);\\
&(b)\ k \in L^\infty(\Gamma_C), \quad k \ge k^* > 0 \text{ a.e. on } \Gamma_C.\\
&(c)\ \kappa \in L^\infty(\Gamma_{D_w}), \quad \kappa \ge 0 \text{ a.e. on } \Gamma_{D_w}.\\
&(d)\ \mathbf{u}_0 \in V, \quad w_{d0} \in L^2(\Gamma_C).
\end{aligned} \tag{11.3.16}$$

Next, we define the function $\mathbf{F} : [0,T] \to V$, the functional $j : L^2(\Gamma_C) \times V^3 \to \mathbb{R}$, the bilinear form $a : H_0^1(\Gamma_C) \times H_0^1(\Gamma_C) \to \mathbb{R}$ and the operator $\Lambda : H_0^1(\Gamma_C) \times V^3 \to H^{-1}(\Gamma_C)$ by

$$\begin{aligned}
(\mathbf{F}(t), \mathbf{v})_V &= \int_\Omega \mathbf{f}_B(t) \cdot \mathbf{v}\, dx + \int_{\Gamma_N} \mathbf{f}_N(t) \cdot \mathbf{v}\, dS,\\
j(w_d, \mathbf{u}, \mathbf{v}, \mathbf{w}) &= \int_{\Gamma_C} p_n(u_n - \eta_w w_d \chi_{[D_w]} - g) w_n\, dS\\
&\quad + \int_{\Gamma_C} \mu(w_d, \|\mathbf{v}_\tau - \mathbf{v}^*\|) p_n(u_n - \eta_w w_d \chi_{[D_w]} - g)\\
&\qquad \times \|\mathbf{w}_\tau - \mathbf{v}^*\|\, dS,\\
a(w_d, \xi) &= \int_{\Gamma_C} k\, \nabla w_d \cdot \nabla \xi\, dS,\\
\langle \Lambda(w_d, \mathbf{u}, \mathbf{v}, \mathbf{w}), \xi \rangle &= \int_{D_w} \mu_w(w_d, \|\mathbf{v}_\tau - \mathbf{v}^*\|) p_n(u_n - \eta_w w_d - g)\\
&\qquad \times R_M(\|\mathbf{w}_\tau - \mathbf{v}^*\|) \xi\, dS,
\end{aligned}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, w_d, $\xi \in H_0^1(\Gamma_C)$ and $t \in [0,T]$.

Using Green's formula leads to the following variational formulation of problem $P_{ve-ncwd}$.

Problem $P^V_{ve-ncwd}$. *Find a displacement field* $\mathbf{u} : [0,T] \longrightarrow V$ *and a surface particle density field* $w_d : [0,T] \longrightarrow H_0^1(\Gamma_C)$ *such that*

$$\begin{aligned}
&(\mathcal{A}_{ve}(\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t))), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + (\mathcal{B}_{ve}(\boldsymbol{\varepsilon}(\mathbf{u}(t))), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q\\
&\quad + j(w_d(t), \mathbf{u}(t), \dot{\mathbf{u}}(t), \mathbf{v}) - j(w_d(t), \mathbf{u}(t), \dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t))\\
&\quad \ge (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \qquad \forall\, \mathbf{v} \in V,\ t \in [0,T],
\end{aligned} \tag{11.3.17}$$

$$\begin{aligned}
&\langle \dot{w}_d(t), \xi \rangle + a(w_d(t), \xi) = \langle \Lambda(w_d(t), \mathbf{u}(t), \dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t)), \xi \rangle\\
&\qquad \forall\, \xi \in H_0^1(\Gamma_C), \quad \text{a.e. } t \in (0,T),
\end{aligned} \tag{11.3.18}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad w_d(0) = w_{d0}. \tag{11.3.19}$$

The following result on the existence and uniqueness of the solution to problem $P^V_{ve-ncwd}$ was established in [153].

Theorem 11.3.1. *Assume that* (11.3.12)–(11.3.16) *hold. Then, there exists a constant* $c_0 > 0$*, which depends on* Ω*,* Γ_D*,* Γ_C*,* $m_{\mathcal{A}}$*,* L_n*,* L_μ*,* $\|\kappa\|_{L^\infty(D_w)}$*,* κ_w *and* M *such that, if* $p_n^* < c_0$ *and* $\mu^* < c_0$*, there exists a unique solution of problem* $P_{ve-ncwd}^V$*. Moreover, the solution satisfies*

$$\mathbf{u} \in C^1([0,T];V),\ w_d \in L^2(0,T;H_0^1(\Gamma_C)) \cap C([0,T];L^2(\Gamma_C)), \tag{11.3.20}$$

$$\dot{w}_d \in L^2(0,T;H^{-1}(\Gamma_C)). \tag{11.3.21}$$

The proof of the theorem was based on elements from the theories of equations of evolution and time-dependent elliptic variational inequalities, and fixed point arguments. The full details can be found in [153].

Let now $\{\mathbf{u}, w_d\}$ denote a solution of Problem $P_{ve-ncwd}^V$ and let $\boldsymbol{\sigma}$ be the stress field given by (11.3.4). Using (11.3.12) and (11.3.13) it follows that $\boldsymbol{\sigma} \in C([0,T];Q)$.

A triple of functions $\{\mathbf{u}, \boldsymbol{\sigma}, w_d\}$ which satisfies (11.3.4), (11.3.17)– (11.3.19) is called a *weak solution* of the mechanical problem $P_{ve-ncwd}$. We conclude that, if the normal compliance function p_n and the coefficient of friction μ are sufficiently small, then problem $P_{ve-ncwd}$ has a unique weak solution.

The size of the allowed bounds p_n^* and μ^*, and whether they have physical significance or are only artifacts caused by the use of the fixed-point argument are open and intricate questions. In view of their important applications, the investigation of such problems is very likely to expand rapidly.

11.4 Adhesive Viscoelastic Bilateral Contact

We now describe a model for frictionless contact with adhesive on the contacting surfaces.

We assume that the contact is bilateral, so there is no separation between the body and the foundation during the process, and then the normal displacement vanishes on Γ_C. We follow the presentation in [162], and note that similar results for the model with normal compliance have been obtained recently in [160, 168].

Let β denote the bonding or adhesion field, which represents the fractional density of the active glue bonds on the contact surface (see Sect. 3.3 for full details). We assume that the resistance to tangential motion is generated by the glue, in comparison to which the frictional traction can be neglected. A different assumption, taking friction into account, can be found in [148, 149, 158]. Therefore, the tangential contact traction depends only on the bonding field and the tangential displacement, thus,

$$-\boldsymbol{\sigma}_\tau = p_\tau(\beta, \mathbf{u}_\tau).$$

Here, p_τ is a general prescribed function. In particular, we may consider the case

$$p_\tau(\beta, \mathbf{r}) = \begin{cases} q_\tau(\beta)\,\mathbf{r} & \text{if} \quad \|\mathbf{r}\| \le L_b, \\ q_\tau(\beta)\,\dfrac{L_b}{\|\mathbf{r}\|}\,\mathbf{r} & \text{if} \quad \|\mathbf{r}\| > L_b, \end{cases} \tag{11.4.1}$$

where $L_b > 0$ is a characteristic length of the bonds (see, e.g., [158], or Sect. 3.3), and q_τ is a prescribed, nonnegative tangential stiffness function. A more general condition may be used in the three-dimensional case, when the surface has intrinsic directions, such as grooves. Then, one needs to replace q_τ with a two-dimensional tensor. The results below can be easily extended to such cases. In two dimensions the surface is only a curve and q_τ is just a number.

As in [161], the evolution of the adhesion field is assumed to depend in a general manner on β and $\mathbf{u}_\tau$. The process is assumed reversible, and we do not impose sign restrictions on it, and thus, cycles of debonding and rebonding may take place, as a result of imposed periodic forces or displacements. In addition, we include the possibility that, as the cycles of bonding and debonding go on, there is a deterioration of the glue and, thus, a decrease in the bonding effectiveness. Therefore, the process is also assumed to be with memory so that it depends on the bonding history, which we denote by

$$\psi_\beta(x,t) = \int_0^t \beta(x,s)\,ds. \tag{11.4.2}$$

The process is assumed to be governed by the differential equation

$$\dot{\beta} = H_{ad}(\beta, \psi_\beta, R_{L_b}(\|\mathbf{u}_\tau\|)).$$

Here, H_{ad} is a general adhesive rate function discussed below, which vanishes when its first argument vanishes. The function $R_{L_b} : \mathbb{R}_+ \to \mathbb{R}_+$ is the truncation operator defined as

$$R_{L_b}(s) = \begin{cases} s & \text{if} \quad 0 \le s \le L_b, \\ L_b & \text{if} \quad s > L_b. \end{cases}$$

We use it in H_{ad} assuming that when the glue is stretched beyond the limit L_b, it does not contribute more to the bond strength.

An example of such a function is

$$H_{ad}(\beta, r) = -\gamma_n \beta r^2, \tag{11.4.3}$$

where γ_n is the bonding energy coefficient, and then $\gamma_n L$ is the maximal tensile normal traction that the adhesive can provide. We note that in this case the process is irreversible and only debonding is allowed. Another example, in which H_{ad} depends on all three variables is

$$H_{ad}(\beta, \psi_\beta, r) = -\gamma_1 \beta\, r^2 + \gamma_2 \frac{\beta_+(1-\beta)_+}{1 + d^* \psi_\beta^2}\,, \tag{11.4.4}$$

where γ_1, γ_2 and d^* are positive coefficients, see also (3.3.4). Here, the magnitude of the tangential displacement $r = \|\mathbf{u}_\tau\|$ causes debonding, and is represented by the first term on the right-hand side. There is also a natural tendency to rebond, which is described by the second term on the right-hand side. However, the bonding field cannot exceed $\beta = 1$, and moreover, the rebonding becomes weaker as the process goes on, which is represented by the factor $1+d^*\psi_\beta^2$ in the denominator, where d^* is the history weight factor.

Let $\mathbf{u}_0$ be the initial displacement and β_0 the initial bonding field. Then, the classical formulation of the mechanical problem of viscoelastic, frictionless, bilateral and adhesive contact may be stated as follows.

Problem $P_{ve-badh}$. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{R}^d$*, and an adhesion field* $\beta : \Gamma_C \times [0,T] \to [0,1]$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in} \quad \Omega_T, \tag{11.4.5}$$

$$\mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \quad \text{in} \quad \Omega_T, \tag{11.4.6}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_D \times (0,T), \tag{11.4.7}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \quad \text{on} \quad \Gamma_N \times (0,T), \tag{11.4.8}$$

$$u_n = 0 \quad \text{on} \quad \Gamma_C \times (0,T), \tag{11.4.9}$$

$$-\boldsymbol{\sigma}_\tau = p_\tau(\beta, \mathbf{u}_\tau) \quad \text{on} \quad \Gamma_C \times (0,T), \tag{11.4.10}$$

$$\dot{\beta} = H_{ad}(\beta, \psi_\beta, R_{L_b}(\|\mathbf{u}_\tau\|)) \quad \text{on} \quad \Gamma_C \times (0,T), \tag{11.4.11}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in} \quad \Omega, \tag{11.4.12}$$

$$\beta(0) = \beta_0 \quad \text{on} \quad \Gamma_C. \tag{11.4.13}$$

To obtain a variational formulation of problem (11.4.5)–(11.4.13), in terms of the displacement field, we need the space V_1 defined by (6.2.7). Recall that V_1 is a real Hilbert space endowed with the inner product (6.2.5) and the associated norm (6.2.6). Also, for the stress field $\boldsymbol{\sigma}$ we need the spaces Q and Q_1, defined by (6.2.2) and (6.2.10), respectively.

We assume that the tangential contact function satisfies

$$\left.\begin{array}{l}
\text{(a) } p_\tau : \Gamma_C \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d. \\[4pt]
\text{(b) There exists } \mathcal{L}_\tau > 0 \text{ such that} \\
\quad |p_\tau(\mathbf{x}, \beta_1, \mathbf{r}_1) - p_\tau(\mathbf{x}, \beta_2, \mathbf{r}_2)| \le \mathcal{L}_\tau\,(|\beta_1 - \beta_2| + \|\mathbf{r}_1 - \mathbf{r}_2\|) \\
\quad \forall\, \beta_1,\ \beta_2 \in \mathbb{R},\ \mathbf{r}_1,\ \mathbf{r}_2 \in \mathbb{R}^d, \quad \text{a.e. } \mathbf{x} \in \Gamma_C. \\[4pt]
\text{(c) For any } \beta \in \mathbb{R} \text{ and } \mathbf{r} \in \mathbb{R}^d,\ \mathbf{x} \mapsto p_\tau(\mathbf{x}, \beta, \mathbf{r}) \\
\quad \text{is measurable on } \Gamma_C. \\[4pt]
\text{(d) The mapping } \mathbf{x} \mapsto p_\tau(\mathbf{x}, 0, \mathbf{0}) \in L^\infty(\Gamma_C)^d. \\[4pt]
\text{(e) } p_\tau(\mathbf{x}, \beta, \mathbf{r}) \cdot \mathbf{n}(\mathbf{x}) = 0\ \ \forall \mathbf{r} \in \mathbb{R}^d \text{ such that } \mathbf{r} \cdot \mathbf{n}(\mathbf{x}) = 0, \\
\quad \text{a.e. } \mathbf{x} \in \Gamma_C.
\end{array}\right\} \tag{11.4.14}$$

Clearly, if the function $q_\tau : \mathbb{R} \to \mathbb{R}$ in (11.4.1) is bounded and Lipschitz continuous, then the corresponding tangential contact function p_τ satisfies condition (11.4.14). We conclude that the results below are valid for the corresponding contact problems.

Next, the adhesion rate function H_{ad} is assumed to satisfy

$$\left.\begin{array}{l}
\text{(a) } H_{ad} : \Gamma_C \times \mathbb{R} \times \mathbb{R} \times [0,L] \to \mathbb{R}. \\
\text{(b) There exists } \mathcal{L}_{ad} > 0 \text{ such that} \\
\qquad |H_{ad}(\mathbf{x}, b_1, z, r) - H_{ad}(\mathbf{x}, b_2, z, r)| \le \mathcal{L}_{ad}\, |b_1 - b_2| \\
\qquad \forall\, b_1,\, b_2 \in \mathbb{R},\ z \in \mathbb{R},\ r \in [0,L], \quad \text{a.e. } \mathbf{x} \in \Gamma_C \quad \text{and} \\
\qquad |H_{ad}(\mathbf{x}, b_1, z_1, r_1) - H_{ad}(\mathbf{x}, b_2, z_2, r_2)| \\
\qquad \le \mathcal{L}_{ad}\,(|b_1 - b_2| + |z_1 - z_2| + |r_1 - r_2|) \\
\qquad \forall\, b_1,\, b_2 \in [0,1],\ z_1,\, z_2 \in \mathbb{R},\ r_1,\, r_2 \in [0,L], \quad \text{a.e. } \mathbf{x} \in \Gamma_C. \\
\text{(c) For any } b,\, z \in \mathbb{R} \text{ and } r \in [0,L],\ \mathbf{x} \mapsto H_{ad}(\mathbf{x}, b, z, r) \\
\qquad \text{is measurable on } \Gamma_C. \\
\text{(d) The mapping } (b,z,r) \mapsto H_{ad}(\mathbf{x}, b, z, r) \text{ is continuous on} \\
\qquad \mathbb{R} \times \mathbb{R} \times [0,L], \quad \text{a.e. } \mathbf{x} \in \Gamma_C. \\
\text{(e) } H_{ad}(\mathbf{x}, 0, z, r) = 0 \quad \forall\, z \in \mathbb{R},\ r \in [0,L], \quad \text{a.e. } \mathbf{x} \in \Gamma_C. \\
\text{(f) } H_{ad}(\mathbf{x}, b, z, r) \ge 0 \quad \forall\, b \le 0,\ z \in \mathbb{R},\ r \in [0,L], \quad \text{a.e. } \mathbf{x} \in \Gamma_C, \\
\qquad H_{ad}(\mathbf{x}, b, z, r) \le 0 \quad \forall\, b \ge 1,\ z \in \mathbb{R},\ r \in [0,L], \quad \text{a.e. } \mathbf{x} \in \Gamma_C.
\end{array}\right\} \quad (11.4.15)$$

We observe that if $\beta \in L^\infty(\Gamma_C)$, $z \in L^\infty(\Gamma_C)$ and $r : \Gamma_C \to \mathbb{R}$ is a measurable function, then conditions (11.4.15) imply that the mapping $\mathbf{x} \mapsto H_{ad}(\mathbf{x}, \beta(\mathbf{x}), z(\mathbf{x}), R_{L_b} r(\mathbf{x}))$ belongs to $L^\infty(\Gamma_C)$. It is straightforward to see that if the adhesion coefficient $\gamma_n \in L^\infty(\Gamma_C)$ satisfies $\gamma_n \ge 0$ a.e. on Γ_C then the function H_{ad} in example (11.4.3) satisfies (11.4.15); moreover, if the coefficients γ_1, γ_2 and d^* are positive functions which belong to $L^\infty(\Gamma_C)$, then the function H_{ad} in example (11.4.4) satisfies (11.4.15). We conclude that all the results below are valid for this choice of H_{ad}.

We suppose that the body forces and surface tractions satisfy

$$\mathbf{f}_B \in L^\infty(0,T; L^2(\Omega)^d), \qquad \mathbf{f}_N \in L^\infty(0,T; L^2(\Gamma_N)^d), \tag{11.4.16}$$

and the initial data satisfy

$$\mathbf{u}_0 \in V_1, \quad \beta_0 \in L^\infty(\Gamma_C) \text{ and } 0 \le \beta_0 \le 1 \text{ a.e. on } \Gamma_C. \tag{11.4.17}$$

Next, we define the element $\mathbf{F}(t) \in V_1$ by (7.3.11), for all $\mathbf{v} \in V_1$, a.e. $t \in (0,T)$, and let $j : L^\infty(\Gamma_C) \times V_1 \times V_1 \to \mathbb{R}$ be the adhesion functional

$$j(\beta, \mathbf{u}, \mathbf{v}) = \int_{\Gamma_C} p_\tau(\beta, \mathbf{u}_\tau) \cdot \mathbf{v}_\tau \, dS \qquad \forall \beta \in L^\infty(\Gamma_C),\ \forall\, \mathbf{u},\, \mathbf{v} \in V_1. \tag{11.4.18}$$

We have the following variational formulation of the problem $P_{ve-badh}$.

Problem $P^V_{ve-badh}$. *Find a displacement field* $\mathbf{u} : [0,T] \to V_1$*, a stress field* $\boldsymbol{\sigma} : [0,T] \to Q_1$*, and an adhesion field* $\beta : [0,T] \to L^\infty(\Gamma_C)$*, such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \tag{11.4.19}$$

$$\dot{\beta}(t) = H_{ad}(\beta(t), \psi_\beta(t), R_{L_b}(\|\mathbf{u}_\tau(t)\|)), \quad 0 \le \beta(t) \le 1, \tag{11.4.20}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + j(\beta(t), \mathbf{u}(t), \mathbf{v}) = (\mathbf{F}(t), \mathbf{v})_V \quad \forall\, \mathbf{v} \in V_1, \tag{11.4.21}$$

for a.e. $t \in (0,T)$*, and*

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \beta(0) = \beta_0. \tag{11.4.22}$$

The following existence and uniqueness result was established in [162].

Theorem 11.4.1. *Assume that* (6.4.4)*,* (6.4.5)*,* (11.4.14)–(11.4.17) *hold. Then there exists a unique solution* $(\mathbf{u}, \boldsymbol{\sigma}, \beta)$ *of problem* $P^V_{ve-badh}$ *and it satisfies*

$$\mathbf{u} \in W^{1,\infty}(0,T;V_1), \tag{11.4.23}$$

$$\boldsymbol{\sigma} \in L^\infty(0,T;Q_1), \tag{11.4.24}$$

$$\beta \in W^{1,\infty}(0,T;L^\infty(\Gamma_C)). \tag{11.4.25}$$

The proof of Theorem 11.4.1 will be presented in the next section. We conclude that problem P_{ve-bad} has a unique weak solution.

It may be of interest to generalize the adhesion evolution rule (11.4.11) to a general Lipschitz function, but then it would be necessary to add subdifferential terms to it to guarantee that $0 \le \beta \le 1$, which is dictated by the interpretation of β.

11.5 Proof of Theorem 11.4.1

The proof of the Theorem will be carried out in several steps, provided in the lemmas below. The assumption of Theorem 11.4.1 hold in this section. Let $\boldsymbol{\eta} \in L^\infty(0,T;V_1)$ be given, which means that the elastic part ($\boldsymbol{\eta} = \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u})$) of the stress is prescribed. In the first step we consider the following purely viscous problem.

Problem P^η_V**.** *Find a displacement field* $\mathbf{u}_\eta : [0,T] \to V_1$ *such that*

$$(\mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)), \boldsymbol{\varepsilon}(\mathbf{v}))_Q + (\boldsymbol{\eta}(t), \mathbf{v})_V = (\mathbf{F}(t), \mathbf{v})_V \quad \forall\, \mathbf{v} \in V_1, \quad \text{a.e. } t \in (0,T), \tag{11.5.1}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0. \tag{11.5.2}$$

We have the following result for this problem.

Lemma 11.5.1. *There exists a unique solution for problem P_V^η, and it satisfies* (11.4.23).

Proof. We define the operator $A : V_1 \to V_1$ by

$$(A\,\mathbf{u}, \mathbf{v})_V = (\mathcal{A}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q \qquad \forall\,\mathbf{u},\ \mathbf{v} \in V_1. \tag{11.5.3}$$

Using (6.2.5) and (6.4.4) it follows that

$$\|A\,\mathbf{u} - A\,\mathbf{v}\|_V \le \mathcal{L}_{\mathcal{A}}\|\mathbf{u} - \mathbf{v}\|_V \qquad \forall\,\mathbf{u},\ \mathbf{v} \in V_1,$$

$$(A\,\mathbf{u} - A\,\mathbf{v}, \mathbf{u} - \mathbf{v})_V \ge m_{\mathcal{A}}\,\|\mathbf{u} - \mathbf{v}\|_V^2 \qquad \forall\,\mathbf{u},\ \mathbf{v} \in V_1,$$

i.e., A is a strongly monotone Lipschitz continuous operator. Since $\mathbf{F} - \boldsymbol{\eta} \in L^\infty(0,T;V_1)$ it follows from Corollary 6.3.4 that there exists a unique function $\mathbf{v}_\eta \in L^\infty(0,T;V_1)$, which satisfies

$$A\,\mathbf{v}_\eta(t) + \boldsymbol{\eta}(t) = \mathbf{F}(t) \quad \text{a.e. } t \in (0,T). \tag{11.5.4}$$

Let $\mathbf{u}_\eta : [0,T] \to V_1$ be defined by

$$\mathbf{u}_\eta(t) = \int_0^t \mathbf{v}_\eta(s)ds + \mathbf{u}_0 \qquad \forall\,t \in [0,T]. \tag{11.5.5}$$

It follows from (11.5.3)–(11.5.5) that $\mathbf{u}_\eta$ is a solution of the variational problem P_V^η, and it satisfies (11.4.23). This concludes the existence part of the lemma. The uniqueness of the solution follows from the unique solvability of the time dependent equation (11.5.4), which concludes the proof. □

We denote by $\mathbf{u}_\eta$ the solution of problem P_V^η obtained in Lemma 11.5.1, for $\boldsymbol{\eta} \in L^\infty(0,T;V_1)$. In the next step, we solve equation (11.4.20) for the adhesion field under the assumption that $\mathbf{u} = \mathbf{u}_\eta$. Thus, we consider the following evolution problem.

Problem Q_V^η. *Find an adhesion field $\beta_\eta : [0,T] \to L^\infty(\Gamma_3)$ such that*

$$\dot{\beta}_\eta(t) = H_{ad}(\beta_\eta(t), \psi_{\beta_\eta}(t), R_{L_b}(\|\mathbf{u}_{\eta\,\tau}(t)\|)) \quad \text{a.e.} \quad t \in (0,T), \tag{11.5.6}$$

$$\beta_\eta(0) = \beta_0. \tag{11.5.7}$$

The following result asserts that, for each given $\mathbf{u} = \mathbf{u}_\eta$, this problem has a unique solution β_η.

Lemma 11.5.2. *There exists a unique solution for problem Q_V^η, and it satisfies* (11.4.25) *and also*

$$0 \le \beta_\eta(t) \le 1 \qquad \forall t \in [0,T],\ \text{a.e. on } \Gamma_C. \tag{11.5.8}$$

Proof. For the sake of simplicity we suppress the dependence of various functions on $\mathbf{x} \in \Gamma_C$, and note that the equalities and inequalities below are valid a.e. $\mathbf{x} \in \Gamma_C$. Let $\psi \in L^\infty(0,T;L^\infty(\Gamma_C))$ and define the map $\mathcal{F}_{\eta\,\psi}(t,\cdot) : L^\infty(\Gamma_C) \to L^\infty(\Gamma_C)$, a.e. on $(0,T)$, by

$$\mathcal{F}_{\eta\,\psi}(t,\beta) = H_{ad}(\beta, \psi(t), R_{L_b}(\|\mathbf{u}_{\eta\,\tau}(t)\|)).$$

It is easy to check that $\mathcal{F}_{\eta\,\psi}$ is Lipschitz continuous with respect to the second variable, uniformly in time, and for all $\beta \in L^\infty(\Gamma_C)$ the mapping $t \mapsto \mathcal{F}_{\eta\,\psi}(t,\beta)$ belongs to $L^\infty(0,T;L^\infty(\Gamma_C))$. Thus, using Theorem 6.3.6 we deduce that there exists a unique function $\beta_{\eta\,\psi} \in W^{1,\infty}(0,T;L^\infty(\Gamma_C))$ such that

$$\dot\beta_{\eta\,\psi}(t) = H_{ad}(\beta_{\eta\,\psi}(t), \psi(t), R_{L_b}(\|\mathbf{u}_{\eta\tau}(t)\|)) \quad \text{a.e.} \quad t \in (0,T), \tag{11.5.9}$$

$$\beta_{\eta\,\psi}(0) = \beta_0. \tag{11.5.10}$$

We prove next that $\beta_{\eta\,\psi}$ satisfies condition (11.5.8). To this end we suppose that $\beta_{\eta\,\psi}(t_0) < 0$ for some $t_0 \in [0,T]$. By assumption (11.4.17) we have that $0 \le \beta_{\eta\,\psi}(0) \le 1$, and since the mapping $t \mapsto \beta(t) : [0,T] \to \mathbb{R}$ is continuous, we can find $t_1 \in [0,t_0)$, such that $\beta_{\eta\,\psi}(t_1) = 0$. Now, let $t_2 = \sup\{t \in [t_1,t_0), : \beta_{\eta\,\psi}(t) = 0\}$, then $t_2 < t_0$, $\beta_{\eta\,\psi}(t_2) = 0$ and $\beta_{\eta\,\psi}(t) < 0$, for $t \in (t_2,t_0]$. Assumptions (11.4.15)(f) and (11.5.9) imply that $\dot\beta_{\eta\,\psi}(t) \ge 0$ for $t \in (t_2,t_0]$, therefore $\beta_{\eta\,\psi}(t_0) \ge \beta_{\eta\,\psi}(t_2) = 0$, which is a contradiction. A similar argument shows that $\beta_{\eta\,\psi}(t) \le 1$ for all $t \in [0,T]$. We conclude that

$$0 \le \beta_{\eta\,\psi}(t) \le 1 \qquad \forall t \in [0,T], \text{ a.e. on } \Gamma_C. \tag{11.5.11}$$

Let the operator $\Lambda_\eta : L^\infty(0,T;L^\infty(\Gamma_C)) \to L^\infty(0,T;L^\infty(\Gamma_C))$, which associates to ψ the integral of the solution $\beta_{\eta\,\psi}$, be given by

$$\Lambda_\eta\psi(t) = \int_0^t \beta_{\eta\,\psi}(s)\,ds \qquad \forall t \in [0,T]. \tag{11.5.12}$$

We prove that it has a unique fixed point. Indeed, let $\psi_1,\ \psi_2 \in L^\infty$ $(0,T;L^\infty(\Gamma_C))$ and let $s \in [0,T]$. It follows from (11.5.9), (11.5.10) for $i = 1,2$ that

$$\beta_{\eta\,\psi_i}(s) = \beta_0 + \int_0^s H_{ad}(\beta_{\eta\,\psi_i}(\theta), \psi_i(\theta), R_{L_b}(\|\mathbf{u}_{\eta\,\tau}(\theta)\|))\,d\theta,$$

and, using (11.5.11), (11.4.15)(b), we find

$$|\beta_{\eta\,\psi_1}(s) - \beta_{\eta\,\psi_2}(s)| \le$$
$$\mathcal{L}_{ad}\int_0^s |\beta_{\eta\,\psi_1}(\theta) - \beta\,\psi_{\eta_2}(\theta)|\,d\theta + \mathcal{L}_{ad}\int_0^s |\psi_1(\theta) - \psi_2(\theta)|\,d\theta.$$

Applying Gronwall's inequality, yields

$$|\beta_{\eta\,\psi_1}(s) - \beta_{\eta\,\psi_2}(s)| \le c \int_0^s |\psi_1(\theta) - \psi_2(\theta)|\, d\theta,$$

and so we obtain

$$\|\beta_{\eta\,\psi_1}(s) - \beta_{\eta\,\psi_2}(s)\|_{L^\infty(\Gamma_C)} \le c \int_0^s \|\psi_1(\theta) - \psi_2(\theta)\|_{L^\infty(\Gamma_C)}\, d\theta. \quad (11.5.13)$$

Here and below, we denote by c a positive constant which may depend on the data but is independent of time, and whose value may change from place to place.

From (11.5.12) and (11.5.13) we find, $\forall t \in [0,T]$, that

$$\|\Lambda_\eta\psi_1(t) - \Lambda_\eta\psi_2(t)\|_{L^\infty(\Gamma_C)} \le c \int_0^t \int_0^s \|\psi_1(\theta) - \psi_2(\theta)\|_{L^\infty(\Gamma_C)}\, d\theta.$$

Reiterating this inequality n times yields

$$\|\Lambda_\eta^n\, \psi_1 - \Lambda_\eta^n\, \psi_2\|_{L^\infty(0,T;L^\infty(\Gamma_C))} \le \frac{(cT)^{2\,n}}{(2\,n)\,!}\, \|\psi_1 - \psi_2\|_{L^\infty(0,T;L^\infty(\Gamma_C))},$$

and, since

$$\lim_n \frac{(cT)^{2n}}{2n!} = 0,$$

it follows that for a sufficiently large n the mapping Λ_η^n is a contraction in the Banach space $L^\infty(0,T;L^\infty(\Gamma_C))$. Therefore, Theorem 6.3.9 implies that there exists a unique $\psi_\eta \in L^\infty(0,T;L^\infty(\Gamma_C))$ such that $\Lambda_\eta^n\, \psi_\eta = \psi_\eta$ and, moreover, ψ_η is the unique fixed point of Λ_η.

Let $\beta_\eta = \beta_{\eta\,\psi_\eta}$ be the solution of (11.5.9), (11.5.10) for $\psi = \psi_\eta$. Using (11.5.12) and (11.4.2) we obtain

$$\psi_\eta(t) = \Lambda_\eta\psi_\eta(t) = \int_0^t \beta_{\eta\,\psi_\eta}(s)\, ds = \int_0^t \beta_\eta(s)\, ds = \psi_{\beta_\eta}(t) \quad \forall t \in [0,T],$$

and keeping in mind (11.5.9)–(11.5.11), it follows that β_η is a solution of problem Q_V^η and it also satisfies (11.4.25) and (11.5.8). This concludes the existence part of Lemma 11.5.2. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ_η. □

Now, given $\boldsymbol{\eta} \in L^\infty(0,T;V_1)$, we denote by β_η the solution of Problem Q_V^η obtained in Lemma 11.5.2. We also denote by $\Lambda\,\eta(t)$ the element of V_1 defined by

$$(\Lambda\boldsymbol{\eta}(t), \mathbf{v})_V = (\mathcal{B}_{ve}\,\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \boldsymbol{\varepsilon}(\mathbf{v}))_Q + j(\beta_{u_\eta}(t), \mathbf{u}_\eta(t), \mathbf{v}), \quad (11.5.14)$$

for $\mathbf{v} \in V_1$ and $t \in [0,T]$. We have the following result.

Lemma 11.5.3. *For $\boldsymbol{\eta} \in L^\infty(0,T;V_1)$, the function $\Lambda\,\boldsymbol{\eta} : [0,T] \to V_1$ is continuous. Moreover, there exists a unique element $\boldsymbol{\eta}^* \in L^\infty(0,T;V_1)$ such that $\Lambda\,\boldsymbol{\eta}^* = \boldsymbol{\eta}^*$.*

Proof. Let $\boldsymbol{\eta} \in L^\infty(0,T;V_1)$ and let $t_1, t_2 \in [0,T]$. Using (11.5.14), (11.4.18) and (6.2.9) we obtain

$$\|\Lambda\boldsymbol{\eta}(t_1) - \Lambda\boldsymbol{\eta}(t_2)\|_V \le \|\mathcal{B}_{ve}\,\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t_1)) - \mathcal{B}_{ve}\,\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t_2))\|_Q$$
$$+ c_B\,\|p_\tau(\beta_\eta(t_1), \mathbf{u}_{\eta\tau}(t_1)) - p_\tau(\beta_\eta(t_2), \mathbf{u}_{\eta\tau}(t_2))\|_{L^2(\Gamma_C)},$$

and, then (6.4.5) and (11.4.14) imply that

$$\|\Lambda\boldsymbol{\eta}(t_1) - \Lambda\boldsymbol{\eta}(t_2)\|_V \le c\,\|\mathbf{u}_\eta(t_1) - \mathbf{u}_\eta(t_2)\|_V$$
$$+c\,\|\beta_\eta(t_1) - \beta_\eta(t_2)\|_{L^2(\Gamma_C)}. \tag{11.5.15}$$

Now, since $t \mapsto \mathbf{u}(t) : [0,T] \to V_1$ and $t \mapsto \beta_\eta(t) : [0,T] \to L^\infty(\Gamma_C)$ are continuous functions, we deduce from (11.5.15) that $\Lambda\boldsymbol{\eta} : [0,T] \to V_1$ is a continuous function, too.

Let $t \in [0,T]$ be fixed. Let $\boldsymbol{\eta}_1,\ \boldsymbol{\eta}_2 \in L^\infty(0,T;V_1)$, and we use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i,\ \dot{\mathbf{u}}_{\eta_i} = \mathbf{v}_i,\ \beta_{\eta_i} = \beta_i$ for $i = 1,2$. Arguments similar to those in the proof of (11.5.15) yield

$$\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|_V^2 \le c\,\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + c\,\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_C)}^2. \tag{11.5.16}$$

Moreover, from (11.5.6) and (11.5.7) we find

$$\beta_i(t) = \beta_0 + \int_0^t H_{ad}(\beta_i(s), \psi_{\beta_i}(s), R_{L_b}(\|\mathbf{u}_{i\,\tau}(s)\|))\,ds, \tag{11.5.17}$$

for $i = 1,2$. We use (11.5.17), (11.5.8), (11.4.15)(b) and the definition of R_{L_b} (page 194) to obtain

$$|\beta_1(t) - \beta_2(t)| \le \mathcal{L}_{ad}\int_0^t |\beta_1(s) - \beta_2(s)|\,ds + \mathcal{L}_{ad}\int_0^t |\psi_{\beta_1}(s) - \psi_{\beta_2}(s)|\,ds$$
$$+\mathcal{L}_{ad}\int_0^t \|\mathbf{u}_{1\,\tau}(s) - \mathbf{u}_{2\,\tau}(s)\|\,ds. \tag{11.5.18}$$

Using now (11.4.2) yields

$$\int_0^t |\psi_{\beta_1}(s) - \psi_{\beta_2}(s)|\,ds \le c\int_0^t |\beta_1(s) - \beta_2(s)|\,ds, \tag{11.5.19}$$

and then (11.5.18), (11.5.19) and Gronwall's inequality yield

$$|\beta_1(t) - \beta_2(t)|^2 \le c\int_0^t \|\mathbf{u}_{1\,\tau}(s) - \mathbf{u}_{2\,\tau}(s)\|^2\,ds.$$

Integrating the last inequality over Γ_C and using (6.2.9) we obtain

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_C)}^2 \le c\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2\,ds \quad \forall t \in [0,T]. \tag{11.5.20}$$

Then, by (11.5.16) and (11.5.20) after some algebraic manipulation we find

$$\begin{aligned}\|\Lambda\,\boldsymbol{\eta}_1(t) - \Lambda\,\boldsymbol{\eta}_2(t)\|_V^2 &\le c\,\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + c\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2\,ds\\ &\le c\int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V^2\,ds.\end{aligned}\tag{11.5.21}$$

Moreover, from (11.5.1) we obtain

$$(\mathcal{A}_{ve}\boldsymbol{\varepsilon}(\mathbf{v}_1) - \mathcal{A}_{ve},\boldsymbol{\varepsilon}(\mathbf{v}_2), \boldsymbol{\varepsilon}(\mathbf{v}_1) - \boldsymbol{\varepsilon}(\mathbf{v}_2))_Q + (\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2, \mathbf{v}_1 - \mathbf{v}_2)_V = 0$$
a.e. on $(0,T)$.

Integrating this inequality with respect to time and using the properties of the viscosity operator $\mathcal{A}_{ve}$ leads to

$$m_{\mathcal{A}}\int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V^2\,ds \le -\int_0^t (\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s), \mathbf{v}_1(s) - \mathbf{v}_2(s))_V\,ds,$$

which implies

$$\int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V^2\,ds \le c\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_V^2\,ds.\tag{11.5.22}$$

Now, from (11.5.21) and (11.5.22) we have

$$\|\Lambda\,\boldsymbol{\eta}_1(t) - \Lambda\,\boldsymbol{\eta}_2(t)\|_V^2 \le c\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_V^2\,ds.$$

Reiterating this inequality n times yields

$$\|\Lambda^n\,\boldsymbol{\eta}_1 - \Lambda^n\,\boldsymbol{\eta}_2\|_{L^\infty(0,T;V_1)}^2 \le \frac{(cT)^n}{n!}\,\|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{L^\infty(0,T;V_1)}^2,$$

which implies that for a sufficiently large n the mapping Λ^n is a contraction in the Banach space $L^\infty(0,T;V_1)$. Therefore, there exists a unique $\boldsymbol{\eta}^* \in L^\infty(0,T;V_1)$ such that $\Lambda^n\,\boldsymbol{\eta}^* = \boldsymbol{\eta}^*$ and, moreover, $\boldsymbol{\eta}^*$ is the unique fixed point of Λ. □

We now have the ingredients to prove Theorem 11.4.1.

Proof (Theorem 11.4.1). Let $\boldsymbol{\eta}^* \in L^\infty(0,T;V_1)$ be the fixed point of Λ and let $\mathbf{u}$, β be the respective solutions of the variational problems P_V^η and Q_V^η, for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$, i.e. $\mathbf{u} = \mathbf{u}_{\eta^*}$, $\beta = \beta_{\eta^*}$. We denote by $\boldsymbol{\sigma}$ the function given by

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}\boldsymbol{\varepsilon}(\mathbf{u}).$$

Clearly, (11.4.19), (11.4.20), and (11.4.22) hold. Since $\Lambda\,\boldsymbol{\eta}^* = \boldsymbol{\eta}^*$, from (11.5.1) and (11.5.14) we find that (11.4.21) holds too. Moreover, $\mathbf{u} \in$

$W^{1,\infty}(0,T;V_1)$, so it follows from (6.4.4) and (6.4.5) that $\boldsymbol{\sigma} \in L^\infty(0,T;Q)$. Choosing now $\mathbf{v} = \boldsymbol{\varphi} \in C_0^\infty(\Omega)^d$ in (11.4.21) yields

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \mathbf{f}_B(t) = \mathbf{0} \quad \text{a.e. } t \in (0,T).$$

Now, (11.4.16) implies that $\operatorname{Div} \boldsymbol{\sigma} \in L^\infty(0,T;L^2(\Omega)^d)$ which yields (11.4.24). We conclude that the triple $(\mathbf{u}, \boldsymbol{\sigma}, \beta)$ is a solution of Problem $P^V_{ve-badh}$ and it satisfies (11.4.23)–(11.4.25). This concludes the existence proof . The uniqueness of the solution follows from arguments similar to those used in the proof of Theorem 9.1.1 (page 137). These, in turn follow from the uniqueness of solution of the Cauchy problems P^η_V and Q^η_V and the uniqueness of the fixed-point of the operator Λ. □

11.6 Membrane in Adhesive Contact

We describe a new version of the classical obstacle problem for a stretched membrane, when the obstacle is covered with an adhesive or a glue. We consider the setting where the membrane is attached to a rigid rim, its displacements are restricted to lie on or above a rigid obstacle, and it is in adhesive contact with the obstacle. We follow the exposition in [164], where the model was constructed, the existence of the unique solution proved, convergence of a numerical method established, and some simulations presented.

The problem has interest by and of itself, but it also contains all the ingredients of the models for adhesion in a setting that is mathematically simpler to analyze, and is much easier to visualize.

Let Ω denote the projection of the membrane on the xy plane, let $z = \phi(x,y)$ represent the location of the rigid obstacle, and let $\Omega_T = \Omega \times (0,T)$.

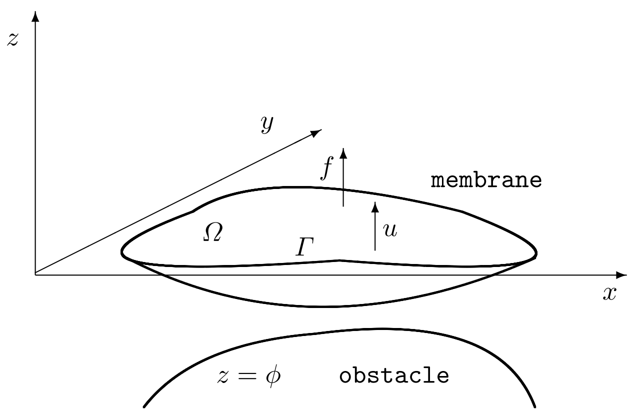

Fig. 10. The membrane above the obstacle

The membrane is being acted upon by a vertical force $f = f(x, y, t)$, and the contact between the membrane and the obstacle involves adhesion. The setting is depicted in Fig. 10.

We let $u = u(x, y, t)$ represent the vertical displacement of the membrane, and let $\xi = \xi(x, y, t)$ be the reaction force of the obstacle, both positive when directed upwards.

The process is assumed to be quasistatic, so we neglect the inertial term in the equation of motion. The membrane is restricted to lie above the obstacle,

$$u \geq \phi \quad \text{in} \;\; \Omega_T. \tag{11.6.1}$$

When contact between the membrane and the obstacle takes place the obstacle's reaction force ξ is directed upward and exactly cancels the applied force,

$$u = \phi \quad \text{implies} \quad \xi \geq 0 \quad \text{in} \;\; \Omega_T. \tag{11.6.2}$$

The reaction force vanishes when there is no contact, thus

$$u > \phi \quad \text{implies} \quad \xi = 0 \quad \text{in} \;\; \Omega_T. \tag{11.6.3}$$

We may write (11.6.1)–(11.6.3) into the the following equivalent complementarity condition

$$u \geq \phi, \quad \xi \geq 0, \quad \xi(u - \phi) = 0 \quad \text{in} \;\; \Omega_T. \tag{11.6.4}$$

We assume that the glue is spread over the whole of the obstacle. The adhesive restoring force $\eta = \eta(x, y, t)$ is directed downward, trying to prevent the separation of the membrane from the obstacle, and is assumed proportional to the distance from the obstacle and to β^2 (see, e.g., [160]),

$$\eta = -\kappa(u - \phi)\beta^2 \quad \text{in} \;\; \Omega_T,$$

where κ is the bonding coefficient or interface stiffness, $\beta = \beta(x, y, t)$ is the adhesion field, and $\kappa\beta^2$ is the surface adhesive spring constant. When the membrane is in contact there is no adhesive restoring force, i.e., $\eta = 0$, and it follows from (11.6.1) that $\eta \leq 0$.

The process is assumed irreversible and the evolution of the adhesion field is given by

$$\dot{\beta} = -\gamma\kappa(u - \phi)^2\beta \quad \text{in} \;\; \Omega_T.$$

Here $\gamma > 0$ is the adhesion rate constant. Initially

$$\beta(x, y, 0) = \beta_0(x, y) \quad \text{in} \;\; \Omega,$$

where β_0 is a given glue distribution.

We turn to describe the obstacle's reaction force ξ. To this end let $I_{(-\infty,0]}$ denote the indicator function of the interval $(-\infty, 0]$, and its subdifferential is given in (4.2.10), Fig. 7. Using it we may rewrite condition (11.6.4) in the form

$$\xi \in \partial I_{(-\infty,0]}(\phi - u).$$

Let f be a given force acting on the membrane, such as gravity. The elastic force in the membrane is $-\Delta u$, where Δ is the Laplace operator, and the balance of forces is

$$-\Delta u - f - \eta = \xi.$$

Then, using the expression for η and the inclusion for ξ, the quasistatic equation of motion of the membrane can be written as the inclusion

$$-\Delta u - f + \kappa(u - \phi)\beta^2 \in \partial I_{(-\infty,0]}(\phi - u) \quad \text{in} \ \ \Omega_T.$$

To complete the model we specify the displacements $u = g_R$ on the boundary $\Gamma = \partial\Omega$, for $0 \le t \le T$. The function g_R just describes the hight of the rigid rim above the xy plane.

Collecting the equations and conditions above, the classical formulation of the problem of quasistatic adhesive contact between a membrane and a rigid obstacle is as follows.

Problem P_{me-adh}. *Find a displacement field* $u : \Omega \times [0,T] \to \mathbb{R}$, *and an adhesion field* $\beta : \Gamma_C \times [0,T] \to [0,1]$ *such that*

$$-\Delta u - f + \kappa(u - \phi)\beta^2 \in \partial I_{(-\infty,0]}(\phi - u) \ \text{in} \ \Omega_T, \qquad (11.6.5)$$

$$\dot{\beta} = -\gamma\kappa(u - \phi)^2\beta \qquad \text{in} \ \Omega_T, \qquad (11.6.6)$$

$$u = g_R \qquad \text{on} \ \Gamma \times (0,T), \qquad (11.6.7)$$

$$\beta(0) = \beta_0 \qquad \text{in} \ \Omega. \qquad (11.6.8)$$

The classical obstacle problem for the membrane is obtained when $\beta \equiv 0$.

We observe that the problem is, as most contact problems are, a free boundary problem. Indeed, let

$$\Lambda(t) = \{(x,y) \in \Omega : u(x,y,t) = \phi(x,y)\}$$

be the contact set, then its boundary $\Gamma^* = \Gamma^*(t) = \partial\Lambda(t)$ is the free boundary separating the contact set from the set where the membrane is above the obstacle. However, the free boundary aspects of the problem, such as the regularity and shape of Γ^* are unresolved, yet.

We assume that the domain Ω is Lipschitz so we can apply the Sobolev embedding theorem. We make the following assumptions on the problem data:

$$f \in W^{1,\infty}(0,T; L^2(\Omega)), \qquad (11.6.9)$$

$$\kappa > 0, \quad \gamma > 0, \qquad (11.6.10)$$

$$\phi \in C(\overline{\Omega}), \qquad \phi \le g_R \ \text{on} \ \overline{\Omega}, \qquad (11.6.11)$$

$$g_R \in H^{1/2}(\Gamma), \qquad (11.6.12)$$

$$\beta_0 \in L^\infty(\Omega), \qquad 0 < \beta_0 \le 1 \quad \text{a.e. on} \ \Omega. \qquad (11.6.13)$$

We proceed to obtain a variational formulation of problem (11.6.5) – (11.6.8). For the sake of simplicity we assume that $g_R \equiv 0$, since otherwise, we need only to make a simple change of the variable u. Therefore, we use the space $H_0^1(\Omega)$ for the displacement field and let the set of admissible displacements be given by

$$K = \{v \in H_0^1(\Omega) \ : \ v \geq \phi \quad \text{in } \Omega\}.$$

It follows from (11.6.11) that $0 \in K$, so the set K is not empty. Also, K is a closed convex subset of $H_0^1(\Omega)$.

Next, let $t \in [0,T]$ be fixed, and for the sake of simplicity we write $u(t)$ instead of $u(x,y,t)$, for $(x,y) \in \Omega$. Then, we multiply both sides of (11.6.5) by $(v - u(t))$, where $v \in K$ is a test function, and integrate over Ω. Using the divergence theorem and the boundary condition $u = 0$, after some manipulations we obtain the following variational formulation of problem (11.6.5)–(11.6.8)

Problem P^V_{me-adh}. *Find a displacements field* $u : [0,T] \to H_0^1(\Omega)$ *and an adhesion field* $\beta : [0,T] \to L^\infty(\Omega)$ *such that*

$$u(t) \in K, \quad \int_\Omega \nabla u(t) \cdot \nabla (v - u(t))\, dx + \kappa \int_\Omega (u(t) + g - \phi)\beta^2 (v - u(t))\, dx$$
$$\geq \int_\Omega f(v - u(t))\, dx \quad \forall v \in K,\ t \in [0,T], \tag{11.6.14}$$

$$\dot\beta(t) = -\gamma\kappa(u(t) + g - \phi)^2 \beta(t), \quad 0 \leq \beta(t) \leq 1, \quad \text{a.e. } t \in (0,T), \tag{11.6.15}$$

$$\beta(0) = \beta_0. \tag{11.6.16}$$

We have the following existence and uniqueness result ([164]).

Theorem 11.6.1. *Under the assumptions* (11.6.9) – (11.6.13)*, there exists a unique solution* (u, β) *of Problem* P^V_{me-adh}. *Moreover, the solution satisfies*

$$u \in W^{1,\infty}(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; H^2(\Omega)), \quad \beta \in W^{1,\infty}(0,T; L^\infty(\Omega)).$$

The proof of Theorem 11.6.1 can be found in [164], and is based on standard results for the classical membrane problem together with a fixed-point argument for the adhesion field. There, a numerical algorithm for the problem was described and implemented, and numerical simulations of the process presented.

We conclude that the mechanical problem P_{me-adh} has a unique weak solution, which solves (11.6.14)–(11.6.16).

The dynamic obstacle problem for the membrane, where the acceleration is taken into account, and with a general adhesive evolution law, has been considered in [165] where the existence of a weak solution was proved.

12 Contact with Damage

We describe new results dealing with contact problems for materials that may undergo internal damage, resulting from strains and stresses which lead to the opening and growth of microscopic cracks. The damage measures the deterioration of the strength of the material in that it reduces the load carrying capacity of the body. A novel way to model material damage was proposed in [171, 172], where the damage field was introduced, and the system evolution, including that of the damage field, was derived from the principle of virtual work. Although the engineering literature dealing with damage and cracks is extensive, mathematical publications on problems with damage, using the notion of damage field, are few. However, their number and scope are on the increase.

In Sect. 12.1 we present a model for the contact of a viscoelastic material with damage and with the normal compliance contact condition. The system now contains, in addition, a parabolic inclusion for the evolution of the damage field. A variational formulation is presented and the theorem on the existence of the unique weak solution stated. The proof of the theorem is given in Sect. 12.2, and is accomplished in a number of steps based on fixed point arguments.

In Sect. 12.3 the problem of contact of a viscoelastic material with damage, with the normal damped response contact condition is described. The existence of the unique solution, under a smallness condition on the contact data, is stated. An outline of the steps in the proof is provided.

Finally, in Sect. 12.4 we describe the problem for a viscoplastic material with damage, with a general dissipative frictional potential. The existence of the unique weak solution is stated, and a short description of the steps of the proof are provided.

We note that the existence results described below are global in time. However, this is a consequence of the fact that we use truncated damage source functions. These results may be considered as local existence results (in time), valid on the time interval on which the damage is strictly greater than the truncation value, since on such an interval the original and the truncated source functions coincide.

In this chapter we use dimensionless variables.

M. Shillor, M. Sofonea, J.J. Telega: *Models and Analysis of Quasistatic Contact*, Lect. Notes Phys. **655**, (2001), 207–222
http://www.springerlink.com/

12.1 Viscoelastic Contact with Normal Compliance and Damage

Most of the results presented in Chap. 8 may be extended to include the evolution of the damage of the viscoelastic material. As an example, we rewrite problem P_{ve-nc}, studied in Sect. 8.3, to include material damage. We assume that the damage does not affect the viscosity of the material, but only its elastic behavior, and therefore we use (6.4.11) as constitutive law.

The classical formulation of the viscoelastic contact problem with normal compliance, friction and damage is as follows.

Problem P_{ve-ncd}. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$, *and a damage field* $\zeta : \Omega \times [0,T] \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\, \boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}(\boldsymbol{\varepsilon}(\mathbf{u}), \zeta) \quad \text{in } \Omega_T, \tag{12.1.1}$$

$$\dot{\zeta} - k_{Dam} \triangle \zeta + \partial I_{[0,1]}(\zeta) \ni \phi(\boldsymbol{\varepsilon}(\mathbf{u}), \zeta) \quad \text{in } \Omega_T, \tag{12.1.2}$$

$$\mathrm{Div}\, \boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \quad \text{in } \Omega_T, \tag{12.1.3}$$

$$\frac{\partial \zeta}{\partial n} = 0 \quad \text{on } \Gamma \times (0,T), \tag{12.1.4}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \times (0,T), \tag{12.1.5}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \quad \text{on } \Gamma_N \times (0,T), \tag{12.1.6}$$

$$\left.\begin{array}{l} -\sigma_n = p_n(u_n - g), \\ \|\boldsymbol{\sigma}_\tau\| \le p_\tau(u_n - g), \\ \boldsymbol{\sigma}_\tau = -\mu p_\tau(u_n - g) \dfrac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if } \dot{\mathbf{u}}_\tau \ne \mathbf{0} \end{array}\right\} \text{on } \Gamma_C \times (0,T), \tag{12.1.7}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \zeta(0) = \zeta_0 \quad \text{in } \Omega. \tag{12.1.8}$$

Here, (12.1.2) is the differential parabolic inclusion for the evolution of the damage field, (3.4.1), in which $k_{Dam} > 0$ is the microcrack diffusion constant; $\partial\zeta/\partial n = \nabla\zeta \cdot \mathbf{n}$ is the normal derivative of ζ on Γ; and ζ_0 is a prescribed initial damage field, chosen as zero in a damage-free material. We recall that the subdifferential term $\partial I_{[0,1]}(\zeta)$ in (12.1.2) guarantees that ζ remains in the interval $[0,1]$, to preserve its interpretation as a fraction.

The choice of the homogeneous Neumann boundary condition for ζ follows [171, 172], and means that there is no influx of microcracks across the boundary. Other boundary conditions may be used too, however, when using the Dirichlet condition, one has to make it clear how is the damage of the boundary controlled.

Next, we derive a weak formulation for Problem P_{ve-ncd}. We introduce the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ given by

$$a(\xi, \eta) = k_{Dam} \int_\Omega \nabla\xi \cdot \nabla\eta \, dx, \tag{12.1.9}$$

and let $\mathcal{K}$ denote the convex set of admissible damage functions

$$\mathcal{K} = \{\xi \in H^1(\Omega) : \xi \in [0,1] \text{ a.e. in } \Omega\}. \tag{12.1.10}$$

We assume that the viscosity operator $\mathcal{A}_{ve}$ satisfies condition (6.4.4), the compliance functions satisfy (8.3.9) and the data $\mathbf{f}_B, \mathbf{f}_N, g$ and $\mathbf{u}_0$ satisfy conditions (8.3.10)–(8.3.12). Recall that examples of compliance functions which satisfy assumptions (8.3.9) have been presented in Sect. 8.3.

The elasticity operator $\mathcal{B}_{ve}$, which in this problem depends on the damage, the damage source function ϕ and the initial damage ζ_0 satisfy:

$$\left.\begin{array}{l}
\text{(a) } \mathcal{B}_{ve} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d.\\[1ex]
\text{(b) There exists an } \mathcal{L}_\mathcal{B} > 0 \text{ such that}\\
\qquad \|\mathcal{B}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}_1, \zeta_1) - \mathcal{B}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}_2, \zeta_2)\| \le\\
\qquad \mathcal{L}_\mathcal{B}\,(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + |\zeta_1 - \zeta_2|)\\
\qquad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ \zeta_1, \zeta_2 \in \mathbb{R},\ \text{a.e. } \mathbf{x} \in \Omega.\\[1ex]
\text{(c) For any } \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } \zeta \in \mathbb{R}\,, \mathbf{x} \mapsto \mathcal{B}_{ve}(\mathbf{x}, \boldsymbol{\varepsilon}, \zeta)\\
\qquad \text{is measurable on } \Omega.\\[1ex]
\text{(d) The mapping } \mathbf{x} \mapsto \mathcal{B}_{ve}(\mathbf{x}, \mathbf{0}, 0) \in Q.
\end{array}\right\} \tag{12.1.11}$$

$$\left.\begin{array}{l}
\text{(a) } \phi : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}.\\[1ex]
\text{(b) There exists an } \mathcal{L}_\phi > 0 \text{ such that}\\
\qquad |\phi(\mathbf{x}, \boldsymbol{\varepsilon}_1, \zeta_1) - \phi(\mathbf{x}, \boldsymbol{\varepsilon}_2, \zeta_2)| \le\\
\qquad \mathcal{L}_\phi\,(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + |\zeta_1 - \zeta_2|)\\
\qquad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ \zeta_1, \zeta_2 \in \mathbb{R},\ \text{a.e. } \mathbf{x} \in \Omega.\\[1ex]
\text{(c) For any } \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } \zeta \in \mathbb{R}\,, x \mapsto \phi(\mathbf{x}, \boldsymbol{\varepsilon}, \zeta)\\
\qquad \text{is measurable on } \Omega.\\[1ex]
\text{(d) The mapping } \mathbf{x} \mapsto \phi(x, \mathbf{0}, 0) \in L^2(\Omega).
\end{array}\right\} \tag{12.1.12}$$

$$\zeta_0 \in \mathcal{K}. \tag{12.1.13}$$

We note that assumptions (6.4.4) and (12.1.11) are rather routine. On the other hand, assumptions (12.1.12) on ϕ are more delicate. Indeed, the damage source function ϕ_{Fr} given in (3.4.3) does not satisfy them. The issue is that once the damage is complete and $\zeta = 0$, the mechanical system may break down and may not support any load. Mathematically, we have the 'quenching' of the solution, and some of its derivatives become unbounded.

To overcome this difficulty, one can consider ϕ as a truncated version of ϕ_{Fr}, valid as long as $0 < \zeta_* \le \zeta$, for some small ζ_*. Therefore, we consider the damage problems studied in this chapter as approximations of damage problems of the type represented by ϕ_{Fr}. The solutions of these models coincide as long as $\zeta_* \le \zeta$. However, as was already noted in Sect. 3.4, the models themselves become inappropriate when the damage is close to zero, since the underlying assumptions become invalid. Moreover, we note that there is a mathematical need to truncate the quadratic strain term in ϕ_{Fr}, too.

Thus, we may consider ϕ as

$$\phi(\boldsymbol{\varepsilon}(\mathbf{u}),\zeta) = \begin{cases} \phi_{\mathrm{Fr}}(\boldsymbol{\varepsilon}(\mathbf{u}),\zeta) & \text{if } \zeta_* \le \zeta \le 1 \text{ and } \|\boldsymbol{\varepsilon}(\mathbf{u})\| \le e_* \\ \phi_{\mathrm{Fr}}(e_*,\zeta) & \text{if } \ \zeta_* \le \zeta \le 1 \text{ and } \|\boldsymbol{\varepsilon}(\mathbf{u})\| > e_* \\ \phi_{\mathrm{Fr}}(e_*,\zeta_*) & \text{if } \ 0 \le \zeta < \zeta_* \text{ and } \|\boldsymbol{\varepsilon}(\mathbf{u})\| > e_* \\ \phi_{\mathrm{Fr}}(\boldsymbol{\varepsilon}(\mathbf{u}),\zeta_*) & \text{if } \ 0 \le \zeta < \zeta_* \text{ and } \|\boldsymbol{\varepsilon}(\mathbf{u})\| \le e_* \end{cases},$$

where e_* is a given strain energy bound. It follows that

$$\phi(\boldsymbol{\varepsilon}(\mathbf{u}),\zeta) = \lambda_D\left(\frac{1-\zeta}{\zeta}\right) - \frac{1}{2}\lambda_E\,\boldsymbol{\varepsilon}(\mathbf{u})\cdot\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda_w,$$

when $\zeta_* \le \zeta \le 1$ and $\|\boldsymbol{\varepsilon}(\mathbf{u})\| \le e_*$.

These mathematical difficulties are of little applied interest since in applications truncating ϕ and restricting ζ and $\boldsymbol{\varepsilon}(\mathbf{u})$ to $0 < \zeta_* \le \zeta$ and $\|\boldsymbol{\varepsilon}(\mathbf{u})\| \le e_*$, for appropriate choices of ζ_* and e_*, will provide sufficiently accurate description of the system evolution.

The mathematical problem without truncation is very difficult; generally, it is likely to lead to only local solutions, and it remains an open and challenging problem. We expand on this issue in Chap. 14.

If we model the damage source with the function ϕ which is given in (3.4.4) we do not encounter the difficulty mentioned above concerning complete damage, i.e., $\zeta = 0$, since the function is Lipschitz when $0 \le \zeta \le 1$, and so we may use it without the restriction $0 < \zeta_* \le \zeta$. However, we do have the need to truncate the quadratic strain term by using the truncation $\|\boldsymbol{\varepsilon}(\mathbf{u})\| \le e_*$. Therefore, all the results below apply to the problem with such a damage source function with the strain energy truncation.

Using standard arguments we obtain the following variational formulation of the problem.

Problem P^V_{ve-ncd}. *Find a displacement field* $\mathbf{u} : [0,T] \to V$*, a stress field* $\boldsymbol{\sigma} : [0,T] \to Q_1$*, and a damage field* $\zeta : [0,T] \to H^1(\Omega)$*, such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}_{ve}(\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t))) + \mathcal{B}_{ve}(\boldsymbol{\varepsilon}(\mathbf{u}(t)),\zeta(t)), \tag{12.1.14}$$

$$\begin{aligned}(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\mathbf{w}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\mathbf{u}(t),\mathbf{w}) - j(\mathbf{u}(t),\dot{\mathbf{u}}(t))& \\ \ge (\mathbf{F}(t),\mathbf{w} - \dot{\mathbf{u}}(t))_V \qquad \forall \mathbf{w} \in V,&\end{aligned} \tag{12.1.15}$$

for all $t \in [0,T]$;

$$\begin{aligned}\zeta(t) \in \mathcal{K},\quad (\dot{\zeta}(t),\xi - \zeta(t))_{L^2(\Omega)} + a(\zeta(t),\xi - \zeta(t))& \\ \ge (\phi(\boldsymbol{\varepsilon}(u(t)),\zeta(t)),\xi - \zeta(t))_{L^2(\Omega)} \qquad \forall \xi \in \mathcal{K},&\end{aligned} \tag{12.1.16}$$

for almost any $t \in (0,T)$*, and*

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \zeta(0) = \zeta_0. \tag{12.1.17}$$

Here, $\mathbf{F}$ is given in (8.3.13), j in (8.3.14), and V and Q_1 denote the spaces given by (6.2.3) and (6.2.10), respectively.

The following existence and uniqueness result for the problem has been established in [178].

Theorem 12.1.1. *Assume that* (6.4.4), (8.3.9)–(8.3.12), (12.1.11)–(12.1.13) *hold. Then Problem* P^V_{ve-ncd} *has a unique solution* $(\mathbf{u}, \boldsymbol{\sigma}, \zeta)$*. Moreover, the solution satisfies*

$$\mathbf{u} \in C^1([0,T]; V), \quad \boldsymbol{\sigma} \in C([0,T]; Q_1),$$

$$\zeta \in W^{1,2}(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)).$$

The proof of Theorem 12.1.1 is presented in the next section.

We conclude from this theorem that, under the previous assumptions, Problem P_{ve-ncd} has a unique weak solution $(\mathbf{u}, \boldsymbol{\sigma}, \zeta)$.

12.2 Proof of Theorem 12.1.1

The proof is based on classical results for elliptic and parabolic variational inequalities and fixed point arguments. It is carried out in several steps. We assume in the following that the assumptions of Theorem 12.1.1 hold and we denote by c a generic positive constant which depends on the problem data, but does not depend on time. Let $\boldsymbol{\eta} \in C([0,T]; Q)$ and $\theta \in C([0,T]; L^2(\Omega))$ be given. The first is a prescribed stress due to the elastic part of the constitutive relation and the second is a given source of damage. In the first step we consider the following two auxiliary problems.

Problem DP_η**.** *Find a displacement field* $\mathbf{u}_\eta : [0,T] \longrightarrow V$ *and a stress field* $\boldsymbol{\sigma}_\eta : [0,T] \longrightarrow Q_1$ *such that*

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)) + \boldsymbol{\eta}(t), \tag{12.2.1}$$

$$\begin{aligned}(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\mathbf{w}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)))_Q + j(\mathbf{u}_\eta(t), \mathbf{w}) - j(\mathbf{u}_\eta(t), \dot{\mathbf{u}}_\eta(t))& \\ \geq (\mathbf{F}(t), \mathbf{w} - \dot{\mathbf{u}}_\eta(t))_V \quad \forall\, \mathbf{w} \in V,&\end{aligned} \tag{12.2.2}$$

for all $t \in [0,T]$*, and*

$$\mathbf{u}_\eta(0) = \mathbf{u}_0. \tag{12.2.3}$$

Problem DP_θ**.** *Find a damage field* $\zeta_\theta : [0,T] \longrightarrow H^1(\Omega)$ *such that*

$$\begin{aligned}\zeta_\theta(t) \in \mathcal{K}, \quad (\dot{\zeta}_\theta(t), \xi - \zeta_\theta(t))_{L^2(\Omega)} + a(\zeta_\theta(t), \xi - \zeta_\theta(t))& \\ \geq (\theta(t), \xi - \zeta_\theta(t))_{L^2(\Omega)} \quad \forall\, \xi \in \mathcal{K},&\end{aligned} \tag{12.2.4}$$

for almost any $t \in (0,T)$, *and*

$$\zeta_\theta(0) = \zeta_0. \tag{12.2.5}$$

In this manner the problem is split into two independent problems: the contact problem DP_η, and the one for damage, DP_θ.

To solve Problem DP_η we need the following result.

Lemma 12.2.1. *Let* $\boldsymbol{\xi} \in C([0,T];V)$. *Then, there exists a unique function* $\mathbf{v}_{\eta\xi} \in C([0,T];V)$ *such that for all* $t \in [0,T]$,

$$\begin{aligned}(\mathcal{A}_{ve}\boldsymbol{\varepsilon}(\mathbf{v}_{\eta\xi}(t)), \boldsymbol{\varepsilon}(\mathbf{w}) - \boldsymbol{\varepsilon}(\mathbf{v}_{\eta\xi}(t)))_Q + (\boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(\mathbf{w}) - \boldsymbol{\varepsilon}(\mathbf{v}_{\eta\xi}(t)))_Q \\ +j(\boldsymbol{\xi}(t), \mathbf{w}) - j(\boldsymbol{\xi}(t), \mathbf{v}_{\eta\xi}(t)) \geq (\mathbf{F}(t), \mathbf{w} - \mathbf{v}_{\eta\xi}(t))_V \quad \forall\, \mathbf{w} \in V.\end{aligned} \tag{12.2.6}$$

Proof. It follows from Theorem 6.3.2 that there exists a unique function $\mathbf{v}_{\eta\xi} : [0,T] \to V$ which solves the time-dependent elliptic variational inequality (12.2.6). To establish that $\mathbf{v}_{\eta\xi} \in C([0,T];V)$, let $t_1, t_2 \in [0,T]$ and denote by $\boldsymbol{\eta}_i = \boldsymbol{\eta}(t_i)$, $\boldsymbol{\xi}_i = \boldsymbol{\xi}(t_i)$, $\mathbf{F}_i = \mathbf{F}(t_i)$ and $\mathbf{v}_i = \mathbf{v}_{\eta\xi}(t_i)$, $i = 1, 2$. By using algebraic manipulations from (12.2.6) we obtain

$$\begin{aligned}&(\mathcal{A}_{ve}\boldsymbol{\varepsilon}(\mathbf{v}_1) - \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\mathbf{v}_2), \boldsymbol{\varepsilon}(\mathbf{v}_1) - \boldsymbol{\varepsilon}(\mathbf{v}_2))_Q \\ &\quad \leq (\mathbf{F}_1 - \mathbf{F}_2, \mathbf{v}_1 - \mathbf{v}_2)_V + (\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2, \boldsymbol{\varepsilon}(\mathbf{v}_2) - \boldsymbol{\varepsilon}(\mathbf{v}_1))_Q \\ &\qquad +j(\boldsymbol{\xi}_1, \mathbf{v}_2) - j(\boldsymbol{\xi}_1, \mathbf{v}_1) + j(\boldsymbol{\xi}_2, \mathbf{v}_1) - j(\boldsymbol{\xi}_2, \mathbf{v}_2).\end{aligned} \tag{12.2.7}$$

The left side is bounded from below by (6.4.4),

$$(\mathcal{A}_{ve}\boldsymbol{\varepsilon}(\mathbf{v}_1) - \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\mathbf{v}_2), \boldsymbol{\varepsilon}(\mathbf{v}_1) - \boldsymbol{\varepsilon}(\mathbf{v}_2))_Q \geq m_{\mathcal{A}} \|\mathbf{v}_1 - \mathbf{v}_2\|_V^2.$$

The last four terms on the right-hand side of (12.2.7) are bounded, due to (8.3.9), by

$$j(\boldsymbol{\xi}_1, \mathbf{v}_2) - j(\boldsymbol{\xi}_1, \mathbf{v}_1) + j(\boldsymbol{\xi}_2, \mathbf{v}_1) - j(\boldsymbol{\xi}_2, \mathbf{v}_2) \leq c\, \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_V \|\mathbf{v}_1 - \mathbf{v}_2\|_V.$$

Using these bounds in (12.2.7) yields

$$\|\mathbf{v}_1 - \mathbf{v}_2\|_V \leq c\,\big(\|\mathbf{F}_1 - \mathbf{F}_2\|_V + \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_Q + \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_V\big). \tag{12.2.8}$$

Then, the conclusion that $\mathbf{v}_{\eta\xi} \in C([0,T];V)$ follows from the continuity of $\mathbf{F}, \boldsymbol{\eta}$, and $\boldsymbol{\xi}$ in the respective spaces V, Q and V. □

Using Lemma 12.2.1 we now show the following existence and uniqueness result for Problem DP_η.

Lemma 12.2.2. *There exists a unique solution to Problem* DP_η *such that* $\mathbf{u}_\eta \in C^1([0,T];V)$, $\boldsymbol{\sigma}_\eta \in C([0,T];Q_1)$.

Proof. We consider the operator $\Lambda_\eta : C([0,T];V) \longrightarrow C([0,T];V)$ defined by

$$\Lambda_\eta \boldsymbol{\xi}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}_{\eta\xi}(s)\,ds \qquad \forall\, \boldsymbol{\xi} \in C([0,T];V), \quad t \in [0,T], \tag{12.2.9}$$

where $\mathbf{v}_{\eta\xi}$ is the solution of (12.2.6). We show that this operator has a unique fixed point $\boldsymbol{\xi}_\eta \in C([0,T];V)$. To this end, let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in C([0,T];V)$ and denote by $\mathbf{v}_i = \mathbf{v}_{\eta\xi_i}$, $i = 1, 2$, the corresponding solutions of (12.2.6). Using the definition (12.2.9) we obtain

$$\|\Lambda_\eta \boldsymbol{\xi}_1(t) - \Lambda_\eta \boldsymbol{\xi}_2(t)\|_V \le \int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V ds \qquad \forall\, t \in [0,T]. \tag{12.2.10}$$

Moreover, using estimates similar to those leading to (12.2.8), we have

$$\|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V \le c\,\|\boldsymbol{\xi}_1(s) - \boldsymbol{\xi}_2(s)\|_V \quad \forall\, s \in [0,T].$$

Substituting this inequality in (12.2.10) yields

$$\|\Lambda_\eta \boldsymbol{\xi}_1(t) - \Lambda_\eta \boldsymbol{\xi}_2(t)\|_V \le c \int_0^t \|\boldsymbol{\xi}_1(s) - \boldsymbol{\xi}_2(s)\|_V ds \quad \forall\, t \in [0,T]. \tag{12.2.11}$$

By reiterating this inequality n times we obtain

$$\|\Lambda_\eta^n \boldsymbol{\xi}_1 - \Lambda_\eta^n \boldsymbol{\xi}_2\|_{C([0,T];V)} \le \frac{c^n}{n!}\, \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_{C([0,T];V)}.$$

This shows that for n sufficiently large the operator Λ_η^n is a contraction in $C([0,T],V)$. Therefore, it follows from Theorem 6.3.9 that there exists a unique element $\xi_\eta \in C([0,T],V)$ such that $\Lambda_\eta^n \boldsymbol{\xi}_\eta = \boldsymbol{\xi}_\eta$ and $\boldsymbol{\xi}_\eta$ is also the unique fixed point of Λ_η.

Next, let $\mathbf{v}_\eta \in C([0,T];V)$, $\mathbf{u}_\eta \in C^1([0,T];V)$ and $\boldsymbol{\sigma}_\eta \in C([0,T];Q)$ be given, for all $t \in [0,T]$, by

$$\mathbf{v}_\eta(t) = \mathbf{v}_{\eta\xi_\eta}(t), \tag{12.2.12}$$

$$\mathbf{u}_\eta(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}_\eta(s)\,ds, \tag{12.2.13}$$

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\mathbf{v}_\eta(t)) + \boldsymbol{\eta}(t). \tag{12.2.14}$$

Clearly, (12.2.1) and (12.2.3) are satisfied. Moreover, by (12.2.13), (12.2.12) and (12.2.9) it follows that $\mathbf{u}_\eta = \boldsymbol{\xi}_\eta$ and $\dot{\mathbf{u}}_\eta = \mathbf{v}_\eta$. Therefore, if we let $\boldsymbol{\xi} = \boldsymbol{\xi}_\eta$ in (12.2.6) we obtain (12.2.2).

To establish the regularity of $\boldsymbol{\sigma}_\eta$, we choose $\mathbf{w} = \dot{\mathbf{u}}_\eta \pm \boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in C_0^\infty(\Omega)^d$ in (12.2.2) to obtain

$$(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}))_Q = (\mathbf{F}(t), \boldsymbol{\varphi})_V \qquad \forall\, \boldsymbol{\varphi} \in C_0^\infty(\Omega)^d,\ t \in [0,T].$$

Recalling the definition (8.3.13) of the term $(\mathbf{F}(t), \boldsymbol{\varphi})_V$, we obtain

$$\operatorname{Div} \boldsymbol{\sigma}_\eta(t) + \mathbf{f}_B(t) = \mathbf{0} \qquad \forall t \in [0, T]. \tag{12.2.15}$$

Now, assumption (8.3.10) and equation (12.2.15) imply that $\boldsymbol{\sigma}_\eta \in C([0, T]; Q_1)$.

This establishes the existence claim in Lemma 12.2.2. The uniqueness claim follows directly from (12.2.1)–(12.2.3), using (6.4.4), (8.3.9) and Gronwall's inequality (Lemma 6.3.11). □

We prove next the unique solvability of Problem DP_θ. By an application of Theorem 6.3.7, with $V = H^1(\Omega)$ and $H = L^2(\Omega)$, the following result holds.

Lemma 12.2.3. *There exists a unique solution ζ_θ of Problem DP_θ, and*

$$\zeta_\theta \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

As a consequence of Lemmas 12.2.2 and 12.2.3, and the assumptions (12.1.11) and (12.1.12), we may define the operator

$$\Lambda : C([0, T]; Q \times L^2(\Omega)) \longrightarrow C([0, T]; Q \times L^2(\Omega)),$$

by

$$\Lambda(\boldsymbol{\eta}, \theta) = \Big(\mathcal{B}_{ve}(\boldsymbol{\varepsilon}(\mathbf{u}_\eta), \zeta_\theta),\ \phi(\boldsymbol{\varepsilon}(\mathbf{u}_\eta), \zeta_\theta)\Big), \tag{12.2.16}$$

for all $(\boldsymbol{\eta}, \theta) \in C([0, T]; Q \times L^2(\Omega))$.

The next step is to investigate this operator.

Lemma 12.2.4. *The operator Λ has a unique fixed point $(\boldsymbol{\eta}^*, \theta^*) \in C([0, T]; Q \times L^2(\Omega))$.*

Proof. Let $(\boldsymbol{\eta}_1, \theta_1),\ (\boldsymbol{\eta}_2, \theta_2) \in C([0, T]; Q \times L^2(\Omega))$ and let $t \in [0, T]$. Using (12.2.16), (12.1.11) and (12.1.12) we deduce that

$$\begin{aligned} &\|\Lambda(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda(\boldsymbol{\eta}_2, \theta_2)(t)\|_{Q \times L^2(\Omega)} \\ &\qquad \le c\left(\|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|_V + \|\zeta_{\theta_1}(t) - \zeta_{\theta_2}(t)\|_{L^2(\Omega)}\right). \end{aligned} \tag{12.2.17}$$

Moreover, it follows from (12.2.13) that

$$\|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|_V \le \int_0^t \|\mathbf{v}_{\eta_1}(s) - \mathbf{v}_{\eta_2}(s)\|_V \, ds. \tag{12.2.18}$$

Using (12.2.1), (12.2.2) and estimates similar to those in the proof of Lemma 12.2.1 (see (12.2.8)) we find that for $s \in [0, T]$,

$$\|\mathbf{v}_{\eta_1}(s) - \mathbf{v}_{\eta_2}(s)\|_V \le c\,(\|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q + \|\mathbf{u}_{\eta_1}(s) - \mathbf{u}_{\eta_2}(s)\|_V). \tag{12.2.19}$$

Combining (12.2.18) and (12.2.19), and using Gronwall's inequality, we have

$$\|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|_V \le c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q \, ds. \tag{12.2.20}$$

On the other hand, (12.2.4), (12.2.5) imply that

$$\|\zeta_{\theta_1}(t) - \zeta_{\theta_2}(t)\|_{L^2(\Omega)} \le c \int_0^t \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)} ds. \tag{12.2.21}$$

Using now (12.2.17), (12.2.20) and (12.2.21) we find

$$\begin{aligned} &\|\Lambda(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda(\boldsymbol{\eta}_2, \theta_2)(t)\|_{Q\times L^2(\Omega)} \\ &\qquad \le c \int_0^t \|(\boldsymbol{\eta}_1, \theta_1)(s) - (\boldsymbol{\eta}_2, \theta_2)(s)\|_{Q\times L^2(\Omega)} ds. \end{aligned} \tag{12.2.22}$$

By iterating this inequality n times, for n sufficiently large, Lemma 12.2.4 follows from the Banach fixed-point theorem (Theorem 6.3.9). □

We have now all the ingredients needed to prove Theorem 12.1.1.

Proof (Theorem 12.1.1). Existence. Let $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*})$ be the solution of (12.2.1)–(12.2.3) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and let ζ_{θ^*} be the solution of (12.2.4)–(12.2.5) for $\theta = \theta^*$. Since $\boldsymbol{\eta}^* = \mathcal{B}_{ve}(\boldsymbol{\varepsilon}(\mathbf{u}_{\eta^*}), \zeta_\theta^*)$ and $\theta^* = \phi(\boldsymbol{\varepsilon}(\mathbf{u}_{\eta^*}), \zeta_\theta^*)$, it is straightforward to see that $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*}, \zeta_{\theta^*})$ is a solution of problem (12.1.14)–(12.1.17). The regularity of the solution, i.e., $\mathbf{u}_{\eta^*} \in C^1([0,T]; V)$, $\boldsymbol{\sigma}_{\eta^*} \in C([0,T]; Q_1)$, and $\zeta_{\theta^*} \in W^{1,2}(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$ follow from Lemmas 12.2.2 and 12.2.3.

Uniqueness. Let $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*}, \zeta_{\eta^*})$ be the solution of (12.1.14)–(12.1.17) obtained above and let $(\mathbf{u}, \boldsymbol{\sigma}, \zeta)$ be another solution of the problem such that $\mathbf{u} \in C^1([0,T]; V)$, $\boldsymbol{\sigma} \in C([0,T]; Q_1)$ and $\zeta \in W^{1,2}(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$. We denote by $\boldsymbol{\eta} \in C([0,T]; Q)$ and $\theta \in C([0,T]; L^2(\Omega))$ the functions

$$\boldsymbol{\eta} = \mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}), \zeta), \quad \theta = \phi(\boldsymbol{\varepsilon}(\mathbf{u}), \zeta). \tag{12.2.23}$$

Now, (12.1.14), (12.1.15) and (12.1.17) imply that $(\mathbf{u}, \boldsymbol{\sigma})$ is a solution of Problem DP_η. Using Lemma 12.2.2 it follows that this problem has a unique solution $\mathbf{u}_\eta \in C^1([0,T]; V)$, $\boldsymbol{\sigma}_\eta \in C([0,T]; Q_1)$ and so we conclude that

$$\mathbf{u} = \mathbf{u}_\eta, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}_\eta. \tag{12.2.24}$$

Next, (12.1.16), (12.1.17) and a similar argument yield

$$\zeta = \zeta_\theta. \tag{12.2.25}$$

Using now (12.2.16), (12.2.24), (12.2.25) and (12.2.23) we obtain $\Lambda(\boldsymbol{\eta}, \theta) = (\boldsymbol{\eta}, \theta)$ and by the uniqueness of the fixed point of the operator Λ, ensured by Lemma 12.2.4, we deduce

$$\boldsymbol{\eta} = \boldsymbol{\eta}^*, \qquad \theta = \theta^*. \tag{12.2.26}$$

The uniqueness of the solution is now a consequence of (12.2.24)–(12.2.26). □

12.3 Viscoelastic Contact with Normal Damped Response and Damage

We describe now a contact problem for a viscoelastic material with damage and, following [179], we use the normal damped response contact condition. We note that this is an extension problem of P_{ve-d}, studied in Sect. 8.6, when the damage of the material is taken into account. As in Sect. 12.1, we assume that the damage does not affect the viscosity of the material, only its elastic behavior.

The classical formulation of the viscoelastic contact problem with normal damped response, friction and damage is the following.

Problem P_{ve-dd}. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *and a damage field* $\zeta : \Omega \times [0,T] \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}_{ve}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}_{ve}(\boldsymbol{\varepsilon}(\mathbf{u}), \zeta) \quad \text{in } \Omega_T, \tag{12.3.1}$$

$$\dot{\zeta} - k_{Dam}\triangle\zeta + \partial I_{[0,1]}(\zeta) \ni \phi(\boldsymbol{\varepsilon}(\mathbf{u}), \zeta) \quad \text{in } \Omega_T, \tag{12.3.2}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \quad \text{in } \Omega_T, \tag{12.3.3}$$

$$\frac{\partial\zeta}{\partial n} = 0 \quad \text{on } \Gamma \times (0,T), \tag{12.3.4}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \times (0,T), \tag{12.3.5}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \quad \text{on } \Gamma_N \times (0,T), \tag{12.3.6}$$

$$\left.\begin{array}{l} -\sigma_n = p_n(\dot{u}_n), \\ \|\boldsymbol{\sigma}_\tau\| \le p_\tau(\dot{u}_n), \\ \boldsymbol{\sigma}_\tau = -p_\tau(\dot{u}_n)\dfrac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if } \dot{\mathbf{u}}_\tau \ne \mathbf{0} \end{array}\right\} \quad \text{on } \Gamma_C \times (0,T), \tag{12.3.7}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \zeta(0) = \zeta_0 \quad \text{in } \Omega. \tag{12.3.8}$$

Here, $\partial\zeta/\partial n$ is the normal derivative of ζ on Γ, ζ_0 is a prescribed initial damage field, chosen as one in a damage-free material, and $k_{Dam} > 0$ is is the microcrack diffusion constant.

Next, we derive a weak formulation for Problem P_{ve-dd}. We assume that the viscosity operator $\mathcal{A}_{ve}$ satisfies condition (6.4.4), the normal and tangential damped response functions satisfy (8.6.12) and the data $\mathbf{f}_B$, $\mathbf{f}_N$ and $\mathbf{u}_0$ satisfy conditions (8.6.13) and (8.6.14). The elasticity operator $\mathcal{B}_{ve}$, which depends on the damage, the damage source function ϕ, and initial damage ζ_0 satisfy conditions (12.1.11), (12.1.12) and (12.1.13), respectively.

Examples of normal and tangential damped response functions which satisfy the assumptions (8.6.12) have been presented in Sect. 8.6.

Using standard arguments we obtain the following variational formulation of the problem.

Problem P^V_{ve-dd}. *Find a displacement field* $\mathbf{u} : [0,T] \to V$*, a stress field* $\boldsymbol{\sigma} : [0,T] \to Q_1$*, and a damage field* $\zeta : [0,T] \to H^1(\Omega)$ *, such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}_{ve}(\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t))) + \mathcal{B}_{ve}(\boldsymbol{\varepsilon}(\mathbf{u}(t)), \zeta(t)), \tag{12.3.9}$$

$$\begin{aligned}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{w}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\dot{\mathbf{u}}(t), \mathbf{w}) - j(\dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t))&\\ \geq (\mathbf{F}(t), \mathbf{w} - \dot{\mathbf{u}}(t))_V \qquad \forall \mathbf{w} \in V,&\end{aligned} \tag{12.3.10}$$

for all $t \in [0, T]$,

$$\begin{aligned}\zeta(t) \in \mathcal{K}, \quad \langle \dot{\zeta}(t), \xi - \zeta(t) \rangle_{L^2(\Omega)} + a(\zeta(t), \xi - \zeta(t))&\\ \geq (\phi(\boldsymbol{\varepsilon}(\mathbf{u}(t)), \zeta(t)), \xi - \zeta(t))_{L^2(\Omega)} \qquad \forall \xi \in \mathcal{K},&\end{aligned} \tag{12.3.11}$$

for a.e. $t \in (0, T)$, *and*

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \zeta(0) = \zeta_0. \tag{12.3.12}$$

Here, we used (8.3.13) for the function $\mathbf{F}$, (8.6.15) for the surface functional j, (12.1.9) for the bilinear form a, and (12.1.10) for the set of admissible damage functions $\mathcal{K}$.

The following existence and uniqueness result for the problem has been established in [179].

Theorem 12.3.1. *Assume that* (6.4.4), (8.6.12)–(8.6.14), (12.1.11)–(12.1.13) *hold. Then, there exists a constant* $\mathcal{L}_0 > 0$, *which depends only on* Ω, Γ_D, Γ_C *and* $\mathcal{A}_{ve}$, *such that Problem* P^V_{ve-dd} *has a unique solution* $(\mathbf{u}, \boldsymbol{\sigma}, \zeta)$, *if* $\mathcal{L}_n + \mathcal{L}_\tau < \mathcal{L}_0$. *Moreover, the solution satisfies*

$$\mathbf{u} \in C^1([0, T]; V), \quad \boldsymbol{\sigma} \in C([0, T]; Q_1),$$

$$\zeta \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

We conclude that when the Lipschitz constants of the contact functions p_n and p_τ are sufficiently small, Problem P_{ve-dd} has a unique weak solution $(\mathbf{u}, \boldsymbol{\sigma}, \zeta)$. We note that the critical value $\mathcal{L}_0$ does not depend on the damage data.

Proof. The proof of Theorem 12.3.1 is similar to that of Theorems 12.1.1 and 8.6.1 and is obtained in several steps. Since the modifications are straightforward, we omit most of the details. The steps are as follows.

(*i*) Let $\boldsymbol{\eta} \in C([0, T]; Q)$ – the elastic stress, and $\theta \in C([0, T]; L^2(\Omega))$ – the damage source function, be given. In the first step we consider the problem of finding a displacement field $\mathbf{u}_\eta : [0, T] \to V$ and a stress field $\boldsymbol{\sigma}_\eta : [0, T] \to Q_1$ such that

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{A}_{ve}(\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t))) + \boldsymbol{\eta}(t), \tag{12.3.13}$$

$$\begin{aligned}(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\mathbf{w}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)))_Q + j(\dot{\mathbf{u}}_\eta(t), \mathbf{w}) - j(\dot{\mathbf{u}}_\eta(t), \dot{\mathbf{u}}_\eta(t))&\\ \geq (\mathbf{F}(t), \mathbf{w} - \dot{\mathbf{u}}_\eta(t))_V \quad \forall \mathbf{w} \in V,&\end{aligned} \tag{12.3.14}$$

for all $t \in [0, T]$, *and*

$$\mathbf{u}_\eta(0) = \mathbf{u}_0. \tag{12.3.15}$$

We also consider the problem of finding a damage field $\zeta_\theta : [0,T] \longrightarrow H^1(\Omega)$ such that

$$\begin{aligned} \zeta_\theta(t) \in \mathcal{K}, \quad (\dot{\zeta}_\theta(t), \xi - \zeta_\theta(t))_{L^2(\Omega)} + a(\zeta_\theta(t), \xi - \zeta_\theta(t)) \\ \geq (\theta(t), \xi - \zeta_\theta(t))_{L^2(\Omega)} \qquad \forall \xi \in \mathcal{K}, \end{aligned} \tag{12.3.16}$$

for almost any $t \in (0,T)$, and

$$\zeta_\theta(0) = \zeta_0. \tag{12.3.17}$$

We prove that there exists $\mathcal{L}_0 > 0$, which depends only on Ω, Γ_D, Γ_C and $\mathcal{A}_{ve}$, such that problem (12.3.13)–(12.3.15) has a unique solution $(\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta)$, and

$$\mathbf{u}_\eta \in C^1([0,T];V), \qquad \boldsymbol{\sigma}_\eta \in C([0,T];Q_1),$$

if $\mathcal{L}_n + \mathcal{L}_\tau < \mathcal{L}_0$. Moreover, problem (12.3.16)–(12.3.17) has a unique solution ζ_θ such that

$$\zeta_\theta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

(*ii*) Next, we assume that $\mathcal{L}_n + \mathcal{L}_\tau < \mathcal{L}_0$. As a consequence of step (*i*), (12.1.11) and (12.1.12), we may define the operator $\Lambda : C([0,T];Q \times L^2(\Omega)) \longrightarrow C([0,T];Q \times L^2(\Omega))$ by

$$\Lambda(\boldsymbol{\eta}, \theta) = \Big(\mathcal{B}_{ve}(\boldsymbol{\varepsilon}(\mathbf{u}_\eta), \zeta_\theta),\ \phi(\boldsymbol{\varepsilon}(\mathbf{u}_\eta), \zeta_\theta)\Big), \tag{12.3.18}$$

for all $(\boldsymbol{\eta}, \theta) \in C([0,T];Q \times L^2(\Omega))$. Using the Banach theorem we prove that the operator Λ has a unique fixed point $(\boldsymbol{\eta}^*, \theta^*) \in C([0,T];Q \times L^2(\Omega))$.

(*iii*) Let $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*})$ be the solution of (12.3.13)–(12.3.15) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and let ζ_{θ^*} be the solution of (12.3.16)–(12.3.17) for $\theta = \theta^*$. It is straightforward to show that $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*}, \zeta_{\theta^*})$ is a solution of problem (12.3.9)–(12.3.12) such that $\mathbf{u}_{\eta^*} \in C^1([0,T];V)$, $\boldsymbol{\sigma}_{\eta^*} \in C([0,T];Q_1)$ and $\zeta_{\theta^*} \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. The uniqueness of this solution follows from the uniqueness of the fixed point of the operator Λ. □

12.4 Viscoplastic Contact with Dissipative Friction Potential and Damage

Most of the results that were presented in Chap. 9 may be extended to include the damage of a viscoplastic material. As an example, we describe the problem P_{vp-d}, studied in Sect. 9.6, with added material damage.

The results presented in what follows have been obtained in [23]. The variational and numerical analysis of the Signorini frictionless contact problem for viscoplastic materials with damage has been recently performed in [180].

The classical formulation of the viscoplastic contact problem with dissipative frictional potential and damage is the following.

Problem P_{vp-dd}. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$, *and a damage field* $\zeta : [0,T] \to H^1(\Omega)$, *such that*

$$\dot{\boldsymbol{\sigma}} = \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{G}_{vp}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}), \zeta) \quad \text{in } \Omega_T, \tag{12.4.1}$$

$$\dot{\zeta} - k_{Dam}\triangle\zeta + \partial I_{[0,1]}(\zeta) \ni \phi(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}), \zeta) \quad \text{in } \Omega_T, \tag{12.4.2}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \quad \text{in } \Omega_T, \tag{12.4.3}$$

$$\frac{\partial\zeta}{\partial n} = 0 \quad \text{on } \Gamma \times (0,T), \tag{12.4.4}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \times (0,T), \tag{12.4.5}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{f}_N \quad \text{on } \Gamma_N \times (0,T), \tag{12.4.6}$$

$$\mathbf{u} \in U, \quad -\boldsymbol{\sigma}\,\mathbf{n}\cdot(\mathbf{v} - \dot{\mathbf{u}}) \le \varphi(\mathbf{v}) - \varphi(\dot{\mathbf{u}}) \quad \forall\,\mathbf{v} \in U \quad \text{on } \Gamma_C \times (0,T), \tag{12.4.7}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \zeta(0) = \zeta_0 \quad \text{in } \Omega. \tag{12.4.8}$$

Here, (12.4.1) represents the viscoplastic constitutive law with damage, (6.4.12), and (12.4.2) is the evolution equation for the damage field, (3.4.5), in which $k_{Dam} > 0$. Examples of contact and friction laws which lead to an inequality of the form (12.4.7) can be found in Sect. 7.4.

We make the following assumptions on the functions $\mathcal{G}_{vp}$ and ϕ:

$$\left.\begin{array}{l}
\text{(a) } \mathcal{G}_{vp} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d.\\[1ex]
\text{(b) There exists } \mathcal{L}_{vp} > 0 \text{ such that}\\
\quad \|\mathcal{G}_{vp}(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \zeta_1) - \mathcal{G}_{vp}(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \zeta_2)\|\\
\quad \le \mathcal{L}_{vp}\left(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + |\zeta_1 - \zeta_2|\right)\\
\quad \forall\,\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ \zeta_1, \zeta_2 \in \mathbb{R},\ \text{a.e. } \mathbf{x} \in \Omega.\\[1ex]
\text{(c) For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } \zeta \in \mathbb{R}\,, x \mapsto \mathcal{G}_{vp}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \zeta)\\
\quad \text{is measurable on } \Omega.\\[1ex]
\text{(d) The mapping } \mathbf{x} \mapsto \mathcal{G}_{vp}(\mathbf{x}, \mathbf{0}, \mathbf{0}, 0) \in Q.
\end{array}\right\} \tag{12.4.9}$$

$$\left.\begin{array}{l}
\text{(a) } \phi : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}.\\[1ex]
\text{(b) There exists } \mathcal{L}_{\phi} > 0 \text{ such that}\\
\quad |\phi(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \zeta_1) - \phi(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \zeta_2)|\\
\quad \le \mathcal{L}_{\phi}\left(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + |\zeta_1 - \zeta_2|\right)\\
\quad \forall\,\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ \zeta_1, \zeta_2 \in \mathbb{R},\ \text{a.e. } \mathbf{x} \in \Omega.\\[1ex]
\text{(c) For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } \zeta \in \mathbb{R}\,, \mathbf{x} \mapsto \phi(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \zeta)\\
\quad \text{is measurable on } \Omega.\\[1ex]
\text{(d) The mapping } \mathbf{x} \mapsto \phi(\mathbf{x}, \mathbf{0}, \mathbf{0}, 0) \in L^2(\Omega).
\end{array}\right\} \tag{12.4.10}$$

Moreover, the initial damage field satisfies

$$\zeta_0 \in \mathcal{K}. \tag{12.4.11}$$

An explanation and an example of the damage source function ϕ can be found in Sect. 12.1. In particular we may use a truncated source function, described there, but we note that here the function depends on $\boldsymbol{\sigma}$, as well.

Next, we recall that the bilinear form a and the set $\mathcal{K}$ are defined by (12.1.9) and (12.1.10), respectively, $K = [0,1]$, and we use the notation U_1, j introduced on page 109. As usual, for each instant $t \in [0,T]$, we define $\mathbf{F}$ by

$$\langle \mathbf{F}(t), \mathbf{v} \rangle = \int_\Omega \mathbf{f}_B(t) \cdot \mathbf{v}\, dx + \int_{\Gamma_N} \mathbf{f}_N(t) \cdot \mathbf{v}\, dS \quad \forall\, \mathbf{v} \in U_1,$$

and we let

$$\Sigma(t) = \big\{ \boldsymbol{\tau} \in Q : (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + j(\mathbf{v}) \geq \langle \mathbf{F}(t), \mathbf{v} \rangle \quad \forall\, \mathbf{v} \in D(j) \big\}.$$

Then, a variational formulation for Problem P_{vp-dd} is as follows.

Problem P^V_{vp-dd}. *Find a displacement field* $\mathbf{u} : [0,T] \to U_1$*, a stress field* $\boldsymbol{\sigma} : [0,T] \to Q_1$*, and a damage field* $\zeta : [0,T] \to H^1(\Omega)$ *such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{A}_{vp} \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{G}_{vp}(\boldsymbol{\sigma}(t),\, \boldsymbol{\varepsilon}(\mathbf{u}(t)),\, \zeta(t)), \tag{12.4.12}$$

$$\begin{aligned} &(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \\ &\qquad \geq \langle \mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}(t) \rangle \quad \forall \mathbf{v} \in U_1, \end{aligned} \tag{12.4.13}$$

$$\begin{aligned} &\zeta(t) \in \mathcal{K}, \quad (\dot{\zeta}(t), \xi - \zeta(t))_{L^2(\Omega)} + a(\zeta(t), \xi - \zeta(t)) \\ &\qquad \geq (\phi(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \zeta(t)), \xi - \zeta(t))_{L^2(\Omega)} \quad \forall \xi \in \mathcal{K}, \end{aligned} \tag{12.4.14}$$

for a.e. $t \in (0,T)$*, and*

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \quad \zeta(0) = \zeta_0. \tag{12.4.15}$$

We have the following existence and uniqueness result, obtained in [23].

Theorem 12.4.1. *Assume that* (6.4.8)*,* (7.4.9)*–*(7.4.11)*,* (9.6.7*),* (12.4.9)*–*(12.4.11) *hold. Then, there exists a unique solution* $(\mathbf{u}, \boldsymbol{\sigma}, \zeta)$ *of problem* P^V_{vp-dd}*. Moreover, the solution satisfies*

$$\begin{aligned} &\mathbf{u} \in W^{1,2}(0,T;U_1), \quad \boldsymbol{\sigma} \in W^{1,2}(0,T;Q_1), \\ &\zeta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \end{aligned}$$

We conclude that the frictional contact problem for viscoplastic materials with material damage has a unique weak solution $(\mathbf{u}, \boldsymbol{\sigma}, \zeta)$. We note that in this problem we do not have any restrictions on the Lipschitz coefficients of the problem data. This is due to the fact that the friction potential in the boundary condition (12.4.7) depends on the solution only via the velocity and, unlike the functional $j = j(\mathbf{u}, \dot{\mathbf{u}})$ in Sect. 9.3, does not depend on the solution and on its velocity.

Proof. The proof is accomplished in three steps that we outline now.

(i) Let $(\boldsymbol{\eta}, \theta) \in L^2(0,T; Q \times L^2(\Omega))$ and let $\mathbf{z}_\eta \in W^{1,2}(0,T;Q)$ be given by

$$\mathbf{z}_\eta(t) = \int_0^t \boldsymbol{\eta}(s)ds + \boldsymbol{\sigma}_0 - \mathcal{A}_{vp}\boldsymbol{\varepsilon}(\mathbf{u}_0) \quad \forall t \in [0,T]. \tag{12.4.16}$$

It follows that there exists a unique solution $\mathbf{u}_\eta \in W^{1,2}(0,T;U_1)$ and $\boldsymbol{\sigma}_\eta \in W^{1,2}(0,T;Q_1)$ of the variational problem

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{A}_{vp}\, \boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)) + \mathbf{z}_\eta(t) \quad \forall t \in [0,T], \tag{12.4.17}$$

$$\begin{aligned}&(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)))_Q + j(\mathbf{v}) - j(\dot{\mathbf{u}}_\eta(t))\\ &\quad \geq \langle \mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t)\rangle \quad \forall \mathbf{v} \in U_1, \text{ a.e. } t \in (0,T),\end{aligned} \tag{12.4.18}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0. \tag{12.4.19}$$

Using again Theorem 6.3.7 we obtain the existence of the unique function $\zeta_\theta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ such that

$$\begin{aligned}&\zeta_\eta(t) \in \mathcal{K}, \quad (\dot{\zeta}_\theta(t), \xi - \zeta_\theta(t))_{L^2(\Omega)} + a(\zeta_\eta(t), \xi - \zeta_\theta(t))\\ &\quad \geq (\theta(t), \xi - \zeta_\theta(t))_{L^2(\Omega)} \quad \forall \xi \in \mathcal{K}, \text{ a.e. } t \in (0,T),\end{aligned} \tag{12.4.20}$$

$$\zeta_\eta(0) = \zeta_0. \tag{12.4.21}$$

(ii) We consider now the operator

$$\Lambda : L^2(0,T; Q \times L^2(\Omega)) \to L^2(0,T; Q \times L^2(\Omega))$$

defined by

$$\Lambda(\boldsymbol{\eta}, \theta) = \Big(\mathcal{G}_{vp}(\boldsymbol{\sigma}_\eta, \boldsymbol{\varepsilon}(\mathbf{u}_\eta), \zeta_\theta), \phi(\boldsymbol{\sigma}_\eta, \boldsymbol{\varepsilon}(\mathbf{u}_\eta), \zeta_\theta)\Big), \tag{12.4.22}$$

where, for every $(\boldsymbol{\eta}, \theta) \in L^2(0,T; Q \times L^2(\Omega)))$, the triplet $(\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta, \zeta_\theta)$ denotes the solution of the variational problem (12.4.17)–(12.4.21).

Let $(\boldsymbol{\eta}_1, \theta_1)$, $(\boldsymbol{\eta}_2, \theta_2) \in L^2(0,T; Q \times L^2(\Omega))$ and let $t \in [0,T]$. Using (12.4.9) and (12.4.10) we deduce

$$\begin{aligned}&\|\Lambda(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda(\boldsymbol{\eta}_2, \theta_2)(t)\|_Q \leq c\, \|\boldsymbol{\sigma}_{\eta_1}(t) - \boldsymbol{\sigma}_{\eta_2}(t)\|_Q\\ &\qquad + \|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|_V + \|\zeta_{\theta_1}(t) - \zeta_{\theta_2}(t)\|_{L^2(\Omega)},\end{aligned} \tag{12.4.23}$$

and by (12.4.16)–(12.4.21) we obtain

$$\|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|_V^2 \leq c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q^2\, ds,$$

$$\|\boldsymbol{\sigma}_{\eta_1}(t) - \boldsymbol{\sigma}_{\eta_2}(t)\|_Q^2 \leq c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q^2 ds,$$

$$\|\zeta_{\theta_1}(t) - \zeta_{\theta_2}(t)\|_{L^2(\Omega)}^2 \leq c \int_0^t \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)}^2 ds.$$

Here, again, c represents a positive generic constant which does not depend on time. Using the previous inequalities and the Banach fixed-point theorem, we find that the operator Λ has a unique fixed point $(\boldsymbol{\eta}^*, \theta^*) \in L^2(0,T; Q \times L^2(\Omega))$.

(iii) Now, from (12.4.16)–(12.4.21) we obtain that the triplet $(\mathbf{u}, \boldsymbol{\sigma}, \zeta)$ where $\mathbf{u} = \mathbf{u}_{\eta^*}$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\eta^*}$, $\zeta = \zeta_{\theta^*}$ is the unique solution of the variational problem P^V_{vp-dd}. □

As has been mentioned above, problems with material damage in the form described above are new, and their full analysis lies in the future. However, in view of the applied importance of such problems in civil engineering and in industrial design, among others, one can expect rapid development in their understanding in the near future.

13 Dynamic, One-Dimensional and Miscellaneous Problems

The previous part of this monograph makes it clear that the branch of the Mathematical Theory of Contact Mechanics, which deals with quasistatic problems, has been developing and expanding rapidly. Other branches are also undergoing rapid development and in this short chapter, we briefly review a number of recent publications on related dynamic and one-dimensional contact problems with or without thermal effects. Then miscellaneous problems are discussed.

As has been pointed out in the Introduction, these topics already warrant comprehensive surveys of their own. Here, we mention only those which have an emphasis on modelling and mathematical analysis. Our intension is simply to provide a quick, and far from comprehensive, list of recent publications dealing with these and related problems.

A recent review of dynamic models, experiments and applications, with an emphasis on the latter is [232], and references of engineering papers dealing with friction in joints can be found in the review [233].

Section 13.1 provides a review of mathematical publications dealing with dynamic contact in which the inertial terms in the equations of motion cannot be neglected. Some of the results relate to contact problems for viscoelastic materials with friction, thermal effects and frictional heat generation, while others deal with related or parts of such problems. The rapid growth of this part of the theory can be seen from the presented publications.

In Sect. 13.2 we describe recent results on one-dimensional problems with contact. There is clearly a substantial and growing body of results for such problems. Many of these may be used as benchmarks for numerical analysis and simulations, since mathematically they are easier to analyze and numerically easier to simulate. Indeed, some of the difficulties related to two- or three-dimensions are absent, allowing for easier and more transparent treatment.

Section 13.3 presents a collection of various mathematical publications on different aspects of contact that do not directly belong to the previous sectons. These include dual formulations, punch problems, rolling contact, lubrication and other topics.

To a certain degree the lists of publications that follow reflect our knowledge and biases.

M. Shillor, M. Sofonea, J.J. Telega: *Models and Analysis of Quasistatic Contact*, Lect. Notes Phys. **655**, (2001), 225–234
http://www.springerlink.com/

13.1 Dynamic Contact Problems

This monograph is dedicated to quasistatic contact problems. In such problems the process is assumed to evolve slowly so that the accelerations in the system are negligible. That is, the system moves on a trajectory consisting of equilibrium points in the phase space. When this is not the case, the accelerations cannot be neglected and the process becomes dynamic. Mathematically, the system changes character, from being of an elliptic or a parabolic type to a hyperbolic type. In particular, the latter supports waves, while the first two do not. For this reason, any process where the contact is abrupt, with possibly, but not necessarily, discontinuity in the velocity upon contact, has to be described by a dynamic model. In such cases, the waves generated upon contact are likely to be important. In particular, the processes of impact, in which the contacting body has a rigid body velocity and the contact is abrupt, must be described by dynamic models. The waves associated with impact cannot be neglected as anyone interested in earthquakes can attest.

The literature on various aspects of dynamic contact with or without friction is growing rapidly. The first result can be found in [5] and it deals with the Tresca friction condition; see also [91,93]. Moreover, an explanation of the difficulty in obtaining the necessary estimates in dynamic problems can be found in [5]. The existence of a weak solution for a simplified unilateral dynamic contact problem for an elastic material in a special configuration appeared in [234].

However, the existence of a solution to the dynamic contact problem between a body made of an elastic material and a rigid obstacle, described by a unilateral contact condition of the Signorini type, is still an unresolved question.

Dynamic problems with normal compliance were investigated and simulated in [11,12,235]. There, the normal compliance condition (2.6.2), in the form of a power law, was introduced and the existence of the unique weak solution for the problem with a linearly elastic or viscoelastic material was proved by using the Galerkin method. The problem was also simulated numerically.

Dynamic problems with a general normal compliance condition were investigated in [86], where the usual restrictions on the growth of the normal compliance function, dictated by the need to use trace theorems for Sobolev spaces, were removed by the construction of appropriate function spaces.

It is found in these and the publications mentioned below, that the combination of viscoelastic material behavior and the normal compliance condition, in contrast to purely elastic material and the Signorini contact condition, allows for the mathematical analysis of a variety of dynamic problems. Often, the uniqueness of the weak solution can be established, in addition to its existence, and moreover, such solutions have better regularity or smoothness properties. These improved properties are reflected in the behaviour of numerical algorithms for such problems, as they allow for the establishment of convergence results, thus providing confidence in the numerical results.

Indeed, it has been established in [37] that the differentiability of the normal compliance function, assuming that the rest of the data is sufficiently smooth and appropriate assumptions are made on the compatibility of the initial data, controls the regularity, in time, of the solution to the frictionless dynamic contact problem. When the normal compliance function p_n is Lipschitz continuous, the accelerations are continuous upon contact and the acceleration rate may be discontinuous. If p_n is k times differentiable, then the solution may have a jump of the $k+2$ time derivative. This result, in addition to its intrinsic interest, will allow for better numerical estimates and rates of convergence of algorithms for the numerical approximations of these problems.

Dynamic contact problems with normal damped response, or the corresponding limit unilateral condition on the velocities, are to be found in [89, 92, 236, 237]. There, the contact condition was stated in terms of the surface velocity, and it may describe some form of surface damping, a lubricating layer, or a granular material. However, the condition was introduced on an ad-hoc basis, without any clear underlying derivation, such as asymptotic expansion, and in some publications it was investigated only for mathematical reasons. A physical process where such a condition has a clear meaning is the plowing of the ground. There, the rigid plow moves within a viscoplastic material with a known velocity and the ground particles that are in contact with the plow cannot move with a slower velocity than that of the plow.

Various mathematical aspects of dynamic frictional contact problems, in addition to those mentioned above, can be found in [35, 44, 93–95, 108, 161, 166, 169, 238, 239]. In the paper [44] the existence of a weak solution was established when the wear of the contacting surfaces was included in the model. The existence of the weak solution for the frictionless dynamic problem with adhesion was proved in [161], and the model was numerically analyzed and simulated in [166]. Numerical approach to dynamic contact was discussed in [239].

The dynamic obstacle problem for the vibrating membrane, constrained to lie on or above a rigid foundation, was investigated in [165], where the adhesion of the membrane to the obstacle was taken into account. The existence of a weak solution was established which, in particular, establishes the existence of a solution for the dynamic problem without adhesion.

It is commonly assumed in many applications, and there is plenty of experimental support for it (see Sect. 2.7 for further details), that the friction coefficient is a complex quantity, and depends on the slip rate, among other process variables. An important result has been obtained in [106] concerning the slip rate dependence of the friction coefficient. There, a dynamic bilateral contact problem for an elastic material with a rigid support and with slip dependent friction coefficient was investigated in a setting that leads to a one-dimensional mathematical problem. It was shown that, under certain growth assumption on the friction coefficient, the problem possess a continuum of solutions and the *maximum delay principle* was used to choose the 'physical' solution; that is, the one solution that will be actualized in practice. It seems that this solution coincides with the unique solution of the problem for a viscoelastic material, with vanishingly small viscosity coefficient. Thus, one may conjecture that viscosity, even if it is vanishingly small, leads to the solution that is physically realized. This seems to indicate, again, that dynamic problems with purely elastic material constitutive laws and slip rate dependent friction coefficients possesses inherent mathematical difficulties. Clearly, this important topic warrants further investigation.

Models for contact with slip rate dependent friction coefficient were investigated in [32–34,36,240,241]. Although, in applications the friction coefficient is often assumed to depend on the slip rate, as this list indicates, mathematical results for such a coefficient are only now emerging. In addition to its applied interest, the topic deserves further study in view of the possible instabilities related to it. Indeed, some of the unpleasant noise generated by a car while braking may be caused by such instabilities. In [100,106] the coefficient of friction was assumed to depend on the slip $\|\mathbf{u}_\tau\|$ and the existence of a weak solution was proved by using the Galerkin method.

We turn to describe some of the publications in more detail. The dynamic contact problem between a viscoelastic body and a reactive foundation, modelled with the normal compliance condition, with a slip dependent friction coefficient can be found in [32]. There, a general existence and uniqueness theorem for set-valued pseudomonotone operators was proved and applied to the contact problem. This theorem was also used in [33] to establish the existence of a weak solution to the bilateral dynamic contact problem for a viscoelastic material, with a friction coefficient that is slip rate dependent and that has a jump from a static to a dynamic value at the onset of sliding. The friction coefficient was represented by a graph in terms of the slip rate. In [34], a similar result was obtained for the problem with the normal compliance condition. The uniqueness of the solutions in the latter two papers was left open, and since the graph has a jump, they may very well be nonunique. This topic certainly deserves further clarification.

As far as we are aware of [33,34,36] are the only mathematical publications that take into account the possible sharp transition in the friction coefficient, from a stick to a slip state, which is usually modelled with a jump.

A different approach to dynamic contact, based on the introduction of a thin semi-rigid contact layer can be found in [107,108]. It is novel and may lead to some interesting and useful new contact conditions. The idea is in its infancy, and such an approach, using asymptotic expansion to pass to the limit when the thickness of the layer vanishes, deserves further study.

A dynamic problem of frictionless contact with adhesion was recently studied in [242]. There, a nonlinear viscoelastic constitutive law was used to model the material behavior and contact was described with a modified normal compliance condition involving a truncation operator. A variational formulation of the problem was derived and an existence and uniqueness result was obtained. Then, a fully discrete scheme for the problem was introduced and error estimates were derived. Finally, representative simulations, depicting the evolution of the state of the system and, in particular, the evolution of the bonding field, were presented.

A dynamic frictionless contact problem with normal compliance and damage was studied in [243]. There, the existence of the unique weak solution of the model was proved. Numerical approximations of the problem, based on fully discrete schemes, were considered using finite elements to discretize the spatial domain and a forward Euler scheme to discretize the time; error estimates were derived and numerical simulations were presented.

Dynamic Contact Problems with Thermal Effects. There exist recent and rapidly growing mathematical literature on contact problems which include thermal effects. As has been mentioned in few places, thermal effects can be very important in some applications. Indeed, sudden application of the car's brakes causes the rapid decrease in the kinetic energy and releases a large amount of heat, generated by the friction traction, which causes rapid raise of the temperature. This may affect the friction coefficient and also may cause softening or even local melting of the contacting surfaces. The publications below deal mostly with dynamic processes. Publications which deal with quasistatic processes have been mentioned in the appropriate places in this monograph.

General models for contact processes with thermal effects can be found in [65,97,99,140], where the models were derived by using general thermodynamic principles. However, even these do not contain temperature dependent friction coefficient or themomechanical changes of the contacting surfaces.

Dynamic contact with thermal effects in two or three space dimensions can be found in the papers [41,43–45,89,96,129,131–134,139,237,244], while modelling and numerical simulations can be found in [97–99,114,140,141]. We refer the reader to these publications for additional references. These include various models and contact conditions. The main feature is the frictional heat generated during the contact process. However, as far as we are aware of, [131] is the only article where the dependence of the friction coefficient on the temperature is included. Moreover, the possible softening of the surface material has not been addresses in any one of these papers. Clearly, this is a topic of considerable applied and theoretical interest and it is reasonable to expect some progress in its study in the near future.

Finally, we mention the long-term program for the investigation of the stability of thermoelastic contact that has been carried out in [116–122] and references therein. The stability of the steady states was studied in frictionless and frictional contact problems. In [122] the stability of the combined processes of thermal contact and frictional heat generation, when the surface heat exchange coefficient was assumed to be pressure dependent, was analyzed and existence, nonexistence, uniqueness, and nonuniqueness results were obtained. It may be of interest to investigate the influence of viscosity, even in vanishing amounts, on the stability of these problems. Also, it may be of interest to add the slip rate dependence of the friction coefficient to the stability studies. Both issues are of applied importance and deserve an in depth investigation.

13.2 One-Dimensional Dynamic or Quasistatic Contact

We turn now to describe one-dimensional dynamic or quasistatic problems with contact. The interest in such problems lies in the fact that the mathematical analysis is considerably easier and more transparent, as some of the difficulties associated with two or three dimensions are absent. The regularity of the solutions is usually better and the use of trace theorems more convenient. Such problems my provide insight into the possible types of behaviour of the solutions and on occasions lead to decoupling of some of the equations, thus simplifying the analysis even more. Moreover, one may use such models as tests and benchmarks for computer schemes meant for simulation of complicated multidimensional contact problems.

In some cases the one-dimensional problem was solved some time before the multi-dimensional one, leading to the development, in the former, of the tools and ideas needed for the latter.

Dynamic unilateral problems for the vibrating string were studied in [245] and the dynamic rod in [246], where the existence of the weak solutions was established, and in the latter the problem was also numerically simulated.

Models, analysis and simulations of contact problems for beams and rods can be found in [147, 150, 247–253].

Quasistatic frictional contact between a beam and a reactive obstacle under it was investigated in [147] where the wear of the beam resulting from the contact was taken into account. This led to an unusual mathematical problem since the elastic coefficient in the equation became wear dependent. Related problems for frictional contact of a beam with a foundation arising in rail casting can be found in [247].

Dynamic and quasistatic processes of contact with adhesion between an elastic or viscoelastic beam and a foundation were studied in [163, 167]. The contact was modeled with the Signorini condition for a rigid foundation and with the normal compliance condition for a deformable one. The existence and uniqueness of the weak solution for each one of the problems was established using the theory of variational inequalities, fixed point arguments and the existence and uniqueness result in [32]. The numerical approximations of the quasistatic problem with normal compliance were considered, based on semi-discrete and fully-discrete schemes. The convergence of the solutions of the discretized schemes was proved and error estimates for these approximate solutions were derived. These are among the very few publications where both the quasistatic and the dynamic problems were analyzed. However, the relationship between the quasistatic and the dynamic problems was not analyzed and remains an important open problem. Indeed, the unresolved, as yet, question is: ‘When is the quasistatic approximation a reasonable approximation of the dynamic solution?’

In [248] the problem of dynamic frictional contact between the tip of a vibrating beam and a moving surface was analyzed. The contact was assumed at the beam’s tip which simplified the analysis considerably. Nevertheless, the contact shear stress was found to be only a distribution and care was needed in its treatment.

Quasistatic contact of a elastic-perfectly-plastic rod was studied in [249]. This was, to the best of our knowledge, the first result for contact of a material with such a constitutive law.

Dynamic contact of two rods was modelled, analyzed and numerically simulated in [250]. The vibrations of a beam which has one end constrained to move between two stops were analyzed in [251] and numerically simulated in [252]. A related problem in which the stops are not fixed but are mounted on a slider was modelled, analyzed, and numerically simulated in [253].

One-dimensional problems with damage, but without contact, were investigated in [173, 174]. In the first the quasistatic problem was considered and was shown to decouple. Once the damage field was found, the displacements were obtained by integration. In the second, the dynamic problem was shown to possess a local weak solution. A similar problem with contact was investigated in [137] where the existence of the local weak solution was established.

One-Dimensional Contact with Thermal Effects. We turn now to describe results for one-dimensional contact models with thermal effects. It is seen that there are many more results for such problems, and in some cases their investigation paved the way, both mathematically and in terms of insights into the models, to the the recent results on multidimensional thermoviscoelastic contact, many of which were described in this monograph.

Thermoelastic contact of a slender rod was first addressed, mathematically, in [138] where the existence of a weak solution for a quasi-static contact problem in linear thermoelasticity was established.

A comprehensive investigation of the contact problem with the Signorini condition can be found in [123]. The quasistatic problem was shown to decouple, resulting in a nonstandard parabolic equation with a nonlinear and nonlocal source term. The emphasis there was on a comprehensive investigation of the heat exchange condition. It is well known that if one assumes the idealized condition in which the thermal contact is either perfect or there is complete insulation, then the system may exhibit infinitely rapid oscillations which preclude the existence of any solution. Therefore, realistic heat exchange conditions were constructed and analyzed in [123]. The heat exchange coefficient was assumed to depend on the separation distance and on the contact pressure. It was found that in nondimensional variables one can use a natural variable to describe both cases (see Sect. 3.1 for more details).

Two forms of the coefficient were studied, a continuous one and a graph with a jump at the onset of contact.

A decoupled thermoelastic problem for axially symmetric setting can be found in [128] where the existence of a weak solution was established.

Local null controllability of the thermoelastic contact problem in [123] has been established very recently in [126].

The first existence result for dynamic contact of a thermoelastic rod with a rigid obstacle, using the Signorini condition, was obtained in [254]. The analysis there was quite involved and the authors used compensated compactness to prove their result.

Various additional models for one-dimensional thermoviscoelastic contact problems and their analysis can be found in [127, 128, 135–137, 139, 255–262]. The last three references describe numerical approximations to one-dimensional problems.

The wear of the tip of a thermoviscoelastic beam, resulting from frictional contact with a moving surface, was modelled and analyzed in [135], and numerical simulations of the problem were performed in [136].

The dynamic impact of two thermoelastic rods can be found in [257] and the quasistatic contact in [258]. In both papers the heat exchange between the tips was assumed to depend on the distance or the gap when the beams were separated.

13.3 Miscellaneous Results

As was mentioned in the Introduction, once the existence and possible uniqueness or nonuniqueness of solutions for a problem have been established, then the analysis, numerical analysis, computer simulations, and control issues arise. The mathematical literature on these topics is in its infancy, although considerable progress has been made in the numerical analysis and computer simulations (see [51] and references therein).

The stationary unilateral contact problem with phase transition was investigated in [263]. There, the flow of a molten material and its solidification at the boundary were modelled and analyzed. Such problems where molds are used for the manufacturing of parts and components are frequent in industry. The problem is very important because the frictional heat generated during contact may cause local melting, and thus affect, possibly dramatically, the whole process.

Problems of asymptotic decay of solutions for thermoviscoelastic contact models were reported in [45, 139]. These are of applied interest since it is known that thermal decay may be quite slow.

Rolling frictional contact, a very important topic in transportation, can be found in the important monograph [67], and also in [129, 177, 264] and in the references therein. Indeed, the microscopic processes that take place at the contact patch between the wheels of a moving vehicle, a train, or a taxiing airplane and the ground determine the macroscopic behaviour of the entire system.

Mathematical results on contact with lubrication can be found in [68, 265–267] and references therein. Lubricated contact is a topic that occupies many shelves in engineering libraries because of its practical importance to the functioning of machines and moving structures. Mathematical results about it, however, are rather limited. The first steps can be found in the references cited above, but the topic is wide open and much remains to be done.

Dual formulations of contact problems are of considerable applied interest since these are set in terms of the stresses, and contact stresses are the quantity of main interest to the design engineer. Displacements are usually of secondary interest. The equivalence between the primal and the dual formulation is a problem of great interest, since both represent possible variational formulations of the same mechanical problem. Such equivalence results were presented in this monograph (see Theorems 8.5.2, 9.1.3 and 9.5.3); others can be found in [51]. Dual formulations of various contact problems can also be found in [13, 49, 60, 201, 268–270] and references therein. A stress formulation of a frictionless contact problem for an elastic-perfectly-plastic body was investigated in [271].

Various modelling approaches to frictional contact, in addition to the above references, can be found in [272, 273]. An experimental study of the evolution of the friction coefficient with wear can be found in [274]. Indeed, mathematical models that do not have an underlying experimental support are likely to remain just mathematical curiosities.

Results on the stability of frictional contact, in particular when such problems may become unstable, leading to self-induced oscillations or complex behaviour, can be found in [240, 275, 276] and references therein. These are very important both theoretically and numerically. Indeed, if the mathematical problem is unstable, the numerical solutions will have oscillations, or even worse, will produce completely unreliable results. These papers deal with finite dimensional problems that can be obtained from the continuous once by spatial discretization. However, to our knowledge, the mathematical tools needed to investigate the continuous contact problems are not yet available.

Noncoercive quasistatic contact problems which describe contact between a punch and a foundation were addressed in [277, 278], and discretized systems in [279], among others. The mathematical difficulty lies in the simple observation that the punch may move as a rigid body, and, therefore, additional assumptions on the data and more involved methods of analysis are needed for such problems.

Optimal control, optimal shape design and controllability can be found in [57, 126, 280] and references therein. We observe that from the point of view of the design engineer, the control of the process is far more important than its thorough understanding. However, models for contact processes are sufficiently complicated so that simple-minded control strategies that do not involve any analysis of the situation are likely to be of little use in general settings.

Various additional results on plastic, elastic-perfectly-plastic and viscoplastic problems, that may contain contact, were obtained in [281–286] and references therein.

Limit analysis and shakedown theorems, when unilateral conditions are used together with friction, deserve to be mathematically studied in depth. The mechanical settings and problem formulations can be found in [287, 288] and references therein.

Applications of contact problems to animal and human biomechanics were considered very recently in [61, 148, 149, 289] and the references therein. This is a very important topic and it is safe to assume that it will be addressed in many mathematically-oriented publications in the very near future.

Applications of frictional contact in plate tectonics, including the description of earthquake initiation, can be found in [101, 241, 290–296] and in the references therein. The geophysical literature on frictional contact is considerable but deals mainly with modelling and numerical simulations.

Finally, we note that references on the numerical analysis and especially on the computational aspects of contact abound. In addition to the many references above, the interested reader may find a host of additional references in the following [239, 297–317].

14 Conclusions, Remarks and Future Directions

This monograph shows clearly that the branch of the Mathematical Theory of Contact Mechanics which deals with quasistatic processes has made an impressive progress in the last decade. Indeed, from a handful of mathematical results, mainly on static problems and very few on quasistatic and dynamic ones, it has developed into a body of results that encompasses many of the fundamental processes present when deformable bodies come in contact. These include friction, wear, adhesion, thermal effects, and material damage among others. Currently, these results mainly concern the modelling of these processes, their weak or variational formulations, the existence and possibly uniqueness of the weak solutions, and, on occasions, the well-posedness of the models. In the course of analysis of the models, new mathematical results have been obtained, extending the Theory of Variational Inequalities, which were directly motivated by the needs of the analysis. This cross fertilization between modelling and applications on the one-hand and mathematical analysis on the other-hand is one of the important aspects of dealing with contact problems which inherently are nonlinear, diverse, and rather complex.

The Theory is not in its infancy anymore, but it still has a long way to go before it becomes a fully mature discipline. The main progress has been made in the construction of general models for the various processes involved when contact between deformable bodies takes place, and in the proofs of the existence, possible uniqueness and continuous dependence of the solutions of these models. This forms the framework and environment for further development of the Theory. Now, that the models have been shown to make sense, their mathematical analysis is in order, together with numerical analysis and numerical simulations. Here lie some of the open problems that need to be addressed.

Although the progress is impressive, many open problems remain to be investigated and resolved. We now describe some of the open problems which, in our opinion, are urgent because their solution will allow further expansion and growth of the discipline. Some are more technical, or mathematical, others are more general. Any progress in these directions will enhance the Theory, and will open avenues for new advances and ideas.

A major modelling problem exists in the description of frictional contact. Originally, the Amontons or Coulomb friction law was proposed to model

M. Shillor, M. Sofonea, J.J. Telega: *Models and Analysis of Quasistatic Contact*, Lect. Notes Phys. **655**, (2001), 235–239
http://www.springerlink.com/

the frictional contact among rigid bodies. Currently, it is being used for the contact of deformable bodies, in a pointwise sense on the contact surface, which does not have a rigorous physical basis. Indeed, as can be seen from the results quoted in this book and in the references, currently the friction coefficient, which measures the ratio between the pressure and the shear stress on the contacting surfaces, depends on the position, relative slip, on the temperature, on the wear, and on the surface, indeed,

$$\mu = \mu(x, v, \theta, w, \ldots).$$

It is unreasonable to describe so many different and complicated phenomena with a single coefficient function, or even a graph (if we take into account the jump from static to dynamic values at the onset of sliding). No doubt a closer look at the physics of contacting surfaces is needed. That may provide, however, only a partial description of the processes since contacting surfaces have complicated geometries and a wide range of microstructures which change and evolve with the friction process. There are also particles or debris, fluids, oxides, and other materials on the contact surfaces. Therefore, a friction law based on a smooth surface made of the same material as the parent body seems to be useful only in a limited range of applications.

In the short term, there is considerable interest in a general friction condition which behaves as the Coulomb law for small contact pressures and as Tresca's law at large contact pressures, as described in Sect. 2.8. Such seems to be the experimental evidence. However, as yet there is no derivation of such a condition from thermodynamic principles.

On the other hand, this synthesis shows clearly that, mathematically, we can deal with very complicated coefficients of friction, and contact conditions, especially if they are assumed to be Lipschitz continuous, with respect to all of their arguments, and bounded. Using graphs is new and leads to weaker solutions and very likely to loss of uniqueness.

We turn to some open mathematical problems. In evolutionary friction problems for elastic materials, uniqueness results are unavailable and it may be the case that, generally, there is no uniqueness in such problems, as recent examples in finite dimensions show. Moreover, all the existence results were obtained by restricting the size of the (constant) friction coefficient. Whether it is a limitation of the mathematical approaches, which typically were based on fixed-point arguments, or an intrinsic feature of the models is an open question. There seems to be some numerical and theoretical evidence that the difficulties are related to the intrinsic mathematical structure and solutions may bifurcate or blow-up. On the other hand, when viscoelastic materials are considered together with the normal compliance contact condition, both existence and uniqueness have been established without any size restrictions. It was found that even vanishing amounts of viscosity regularize the solutions and allow to prove their uniqueness and well-posedness. These mathematical results strongly suggest, or possibly reflect the fact, that there are no purely elastic materials, certainly not when dynamic contact or impact is involved. Even if one insists on using an elastic material, adding a small amount of viscosity to the model may not change the solution as far as applications are concerned except to make it unique and better behaved. This, in turn, may allow for a more thorough analysis of the solutions since in such cases one deals with functions instead of measures and distributions. We also note that often in numerical codes either viscosity is added, or the algorithm produces the so-called 'numerical viscosity,' which, in either way, allows for better behaved solutions.

There are very few regularity results for contact problems, except for the classical problem of static contact of a membrane with a rigid obstacle and the recent result reported in [38]. Moreover, an almost optimal regularity result for dynamic frictionless contact of a viscoelastic body, when the normal compliance condition is used, has been established in [37]. However, the field is wide open and progress is likely to be slow since we do not have as yet the necessary mathematical tools. While maximum principles and comparison theorems are available for quasistatic problems with a single equation, and allow their in-depth study, such tools are unavailable for systems. The optimal regularity of the solutions is a rather urgent matter, since it is unlikely that significant progress can be made in the mathematical analysis of contact problems without it. We note in passing that almost every contact problem has a regularity ceiling, and unlike many other problems, beyond this regularity ceiling, any additional regularity of the data does not imply additional regularity of the solutions. Indeed, even if the obstacle or foundation is very smooth, one or more of the first or second spatial derivatives will have a jump at the contact/no contact point.

The next topic is intimately associated with the regularity issue. The inclusion of the regularizing operator $\mathcal{R}$ can be traced to [17, 213], and there seems to be some physical justification in considering the normal stress in the friction condition (8.5.5) as averaged over a small surface area which contains many asperities. However, the main motivation for such a choice was mathematical, since in a weak formulation the stress tensor does not have a well defined trace on the boundary. At best, these are complicated distributions. To overcome this difficulty the operator $\mathcal{R}$ has been introduced. This raises the question whether the mathematical difficulties associated with the friction condition, in this setting, have physical origin and the mathematics reflects them, or these are just mathematical difficulties related to the weak formulation and have no underlying physical cause. In particular, this question indicates the urgent need for regularity results for solutions of frictional contact problems. These results will settle, in part, this question.

Mathematical models, not to mention any results, which take into account the random distribution of asperities on contact surfaces are not available, yet. Clearly, such random distributions must be taken into account. It might be that considering frictional contact as a stochastic process may be of use.

The mathematical analysis of the solutions of contact problems, in terms of the description of their behaviour, does not exist yet. It is essential if we wish to understand the details of the contact processes, and there is considerable interest in the description of the contact set, its shape and the shape of its boundary, which is a free boundary, and its evolution in time. Indeed, at each time instant the potential contact surface Γ_C is divided into the part Γ_C^{con} where the body and the foundation are in contact ($u_n \geq g$), and the part Γ_C^{sep} where they are separated ($u_n < g$). The boundary of the set Γ_C^{con} is a free boundary, dictated by the solution of the problem. When the contact is with friction, the part where contact takes place is further divided into $(\Gamma_C^{con})_{slip}$ where relative slip takes place, and $(\Gamma_C^{con})_{stick}$ where the body and the foundation move in tandem. The curve that separates these two sets is also a free boundary. The structure of the sets Γ_C^{con}, $(\Gamma_C^{con})_{slip}$, and $(\Gamma_C^{con})_{stick}$ is of considerable interest, both theoretical and applied, but currently we do not have yet the tools to address these problems.

These issues are important for the design engineer, as they affect the designs of the parts and their reliability and durability. Indeed, the processes that take place on the contact patch between a rail and a wheel of a train engine control the overall performance of the train.

The infinite-dimensional dynamical systems approach to contact problems is virtually nonexistent. In this approach the state of the system is represented by a point in an infinite-dimensional phase space, the space of displacements and velocities. Then, the evolution of the system is described by the trajectory of this point in the phase space. By investigating the structure of all possible trajectories or motions, which are controlled by the attractors of the system, an insight into the dynamics of the system can be obtained. In particular, the asymptotic, or long-time behaviour can be established. To our knowledge, the asymptotic behavior of the solutions, as $t \to \infty$, has been investigated in [45, 139] (see also the references therein). This topic certainly deserves further consideration.

We believe that one of the topics that will be investigated in the near future is the extension of the theory to take into account electric and piezoelectric effects. Indeed, piezoelectric materials, those that have a strong coupling between the mechanical stress and the electric potential, are being used extensively as sensors and switches in engineering applications and are mostly involved in contact. This will lead to the inclusion of electric effects in the models. And, eventually, will deal with the coupling of electric, mechanical, and thermal fields. Moreover, the damage of the material will be included. It is likely that many of the results presented in this monograph will be extended, with appropriate modifications, to such materials, especially the models for contact with or without friction. Moreover, based on our recent experience, these models will lead to new mathematical results for variational inequalities.

Another important topic is that of modelling and analysis of contact within the framework of large deformations. Many of the real world problems involve large deformations, such as in the metal forming processes. But, the mathematical results for such problems are very sketchy. It was our initial intent to dedicate a short part of the book to this topic. However, it became evident, very quickly, that it deserves a monograph of its own. Geometrically nonlinear settings and problems require a separate and very careful study. For instance, even kinematics and the constitutive relations are much more complicated than in the case of small deformations. Therefore, we are currently engaged in writing a monograph dedicated to contact problems for solids and structures undergoing large deformations.

Finally, we observe that in addition to the free boundaries connected with contact sets mentioned above, there exists a whole class of free boundary problems related to grinding, drilling, and polishing of materials. In these processes, the physical boundary of the object changes as a part of the process, and is, thus, a free boundary. Its determination is one of the main objective of any model for such a problem. There are virtually no mathematical models or results dealing with such problems.

Now that the framework is reasonably established, the control of contact processes needs to be addressed. Indeed, in most applications this is the main interest of the design engineer. Related issues are the observability properties of the models and parameter identification. The models described in this monograph include many parameters that have to be determined experimentally. Using reliable parameter identification procedures will help in establishing the validity of the models. This, in turn, will help in the construction of effective and efficient numerical algorithms for the problems with established convergence. Steps in the latter direction can already be found in [51]. As better models for specific applications are obtained, improved mathematical models and numerical simulations will be possible.

In summary, impressive progress has been made and plenty remains to be done.

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Index