I

MASTER EQUATION METHODS IN QUANTUM OPTICS*

BY

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§ 1. Introduction

In recent years increasing use has been made of methods of quantum statistical mechanics and stochastic processes in treatments of various problems in quantum optics. This is seen from a large number of publications dealing with the theory of lasers (Gordon [1967], Haken [1970], Lax [1966c, 1968a], Scully and Lamb [1967]), with superradiance (Agarwal [1970; 1971b, c, e], Bonifacio et al. [1971a, b]), with problems in nonlinear optics such as parametric oscillators (Graham [1968]). Moreover some of the methods were specifically developed to treat the problems in quantum optics. These include the well known phase space methods (Glauber [1963, 1965], Sudarshan [1963], Cahill and Glauber [1969a, b], Agarwal and Wolf [1968, 1970a, b, c], Lax [1968b]). In phase space methods the $c$-number distribution functions for quantum systems are introduced, which in many physical situations are found to obey equations of the Fokker-Planck type. One may then use the language of stochastic processes to study various quantum systems. In quantum optics, one is usually concerned with the study of a subsystem which is a part of a large system, for example, in case of the laser one is mainly interested in the statistical properties of the emitted radiation. In this context master equation methods have played a very important role.

We start by giving a brief history of the subject. A master equation was first obtained by Pauli [1928]. He obtained an equation of motion for the diagonal elements $\rho_{nn}$ of the density operator by making the statistical hypothesis of random phases at all times. Pauli's master equation has the form

$$\frac{\partial \rho_{nn}}{\partial t} = \sum_{m \neq n} \gamma_{nm} \rho_{mn} - \sum_{m \neq n} \gamma_{mn} \rho_{nn},$$

where $\gamma_{nm}$ is the transition probability per unit time for the system to make a transition from the state $|m\rangle$ to the state $|n\rangle$. It should be noted that $\rho_{nn}$ is the probability that the system be found in the state $|n\rangle$ and, therefore, (1.1) is of the form of a rate equation. An equation of the type (1.1) for $\rho_{nn}$ is also expected from the principle of detailed balance.

We will now outline Pauli's derivation. Our presentation follows that of Van Hove [1962]. We write the Hamiltonian of the system as

$$H = H_0 + gH_1,$$

where $H_0$ is the unperturbed Hamiltonian and $gH_1$ is a small perturbation. We work in a representation in which $H_0$ is diagonal, with eigenfunctions $|\psi_n\rangle$ and eigenvalues $E_n$. The state of the system at time $t+\Delta t$ is related to the state at time $t$ by the unitary transformation

$$|\psi(t+\Delta t)\rangle = \exp \{-i(H_0+gH_1)\Delta t\} |\psi(t)\rangle,$$

where $\hbar$ has been put equal to unity and this we do throughout this article. On expanding $|\psi(t+\Delta t)\rangle$ and $|\psi(t)\rangle$ in a complete set of states

$$|\psi(t+\Delta t)\rangle = \sum_n c_n(t+\Delta t) |\psi_n\rangle, \quad |\psi(t)\rangle = \sum_n c_n(t) |\psi_n\rangle,$$

we find that the probability that the system be found in a state $|\psi_n\rangle$ at time $t+\Delta t$ is given by

$$\rho_{nn}(t+\Delta t) \equiv p_n(t+\Delta t) = |c_n(t+\Delta t)|^2$$

$$= \sum_{l,m} c_m(t)c^*_l \langle \psi_l | \exp \{-i(H_0+gH_1)\Delta t\} |\psi_m\rangle \langle \psi_m | \exp \{i(H_0+gH_1)\Delta t\} |\psi_n\rangle.$$  

Pauli made the statistical assumption of random phase at all times. This assumption enables us to ignore all the terms in (1.5) with $m \neq l$. In fact the assumption of random phases implies that the terms in (1.5) with $m \neq l$ oscillate very rapidly and so they average out to zero. Eq. (1.5) then reduces to

$$p_n(t+\Delta t) = \sum_m |\langle \psi_n | |\psi_m\rangle|^2 p_m(t).$$

(1.6)
Since $\Delta t$ is small enough the expression $|\langle \psi_n | \exp \{-i(H_0 + gH_I)\Delta t\} | \psi_m \rangle|^2$ for $n \neq m$ is easily evaluated by first order perturbation theory and one finds that

$$|\langle \psi_n | \exp \{-i(H_0 + gH_I)\Delta t\} | \psi_m \rangle|^2 = \gamma_{nm} \Delta t, \quad (n \neq m),$$

where

$$\gamma_{nm} = 2\pi g^2 \delta(E_n - E_m)|\langle \psi_n | H_I | \psi_m \rangle|^2.$$  

Equation (1.6) in the limit $\Delta t \to 0$ leads to the Pauli master equation (1.1) if use is also made of the following identity

$$\sum_m |\langle \psi_n | \exp \{-i(H_0 + gH_I)\Delta t\} | \psi_m \rangle|^2 = 1.$$  

In the derivation it has also been assumed that the spectrum is continuous. Pauli's equation is valid for times such that $\tau_c \ll t \ll t_{\text{relax}}$, where $\tau_c$ is the interaction time and $t_{\text{relax}}$ is the relaxation time. The 'interaction time' for the problem of anharmonic interaction in solids, for example, is of the order of $(\omega_D)^{-1}$, where $\omega_D$ is the Debye frequency (see e.g. PRIGOGINE [1962]).

In the past two decades, there has been revival of interest in deriving the master equations by making far less reaching assumptions than Pauli did. The work on the derivation of the master equations can mainly be divided into three groups: (i) Van Hove [1955, 1957, 1962], (ii) Prigogine and his co-workers (see e.g. PRIGOGINE [1962]) and (iii) Zwanzig [1961a, 1964]. Van Hove appears to have been the first to give a rigorous derivation of the Pauli equation. He also derived master equations to all orders in perturbation. Prigogine et al. made extensive use of diagrammatic methods to derive master equations. Zwanzig developed very elegant projection operator methods for deriving the master equations to all orders in perturbation and also established the identity of various other master equations (MONTROLL [1961], PRIGOGINE and RESIBOIS [1961]).

In the present article we will not be concerned with rigorous derivation of master equations but rather in reviewing research that demonstrates the power of the master equation approach in the study of many problems in quantum optics (for rigorous derivation of master equations see for example, Van Kampen [1954], Van Hove [1955, 1957], Emch and Sewell [1968]). We will show how these techniques may be employed in studies of the relaxation of oscillators and spin systems (two-level atomic systems), of lasers and of superradiance. We will make use of Zwanzig's projection operator techniques to obtain master equations for a wide variety of systems.

The density operator $\rho$ characterizing the state of a quantum mechanical system satisfies the equation of motion
\[
\frac{\partial \rho}{\partial t} = -i \mathcal{L} \rho,
\]
where the Liouville operator \( \mathcal{L} \) is given by
\[
\mathcal{L} \equiv [H, \quad].
\]

Zwanzig noted that the part of the density operator which is of interest can be obtained from the total density operator by suitably projecting it. To obtain an equation of motion for the diagonal elements of \( \rho \) Zwanzig [1961a, 1964] introduced the following projection operator
\[
\mathcal{P}_{mnm'nm'} = \delta_{mn} \delta_{mm'} \delta_{nn'}.
\]
Zwanzig regarded \( \mathcal{L} \) and \( \mathcal{P} \) tetradics. However, there is no need for introducing the notion of a tetradic, one may very well work instead with the projection operator
\[
\mathcal{P} \ldots = \sum_m \mathcal{P}_m \text{Tr} \{ \mathcal{P}_m, \ldots \},
\]
where \( \mathcal{P}_m \) is the projection operator onto the state \(|m\rangle\), i.e.
\[
\mathcal{P}_m = |m\rangle \langle m|.
\]
It is seen from (1.13) that \( \mathcal{P} \) projects out the diagonal elements of \( \rho \), since the off diagonal elements of the operator \( \mathcal{P} \rho \) are identically equal to zero.

We can treat both classical and quantum systems by using the projection operator methods. In classical statistical mechanics the distribution function \( \Phi_N(\{q\}, \{p\}; t) \) satisfies the Liouville equation of motion
\[
\frac{\partial \Phi_N}{\partial t} = \sum_i \left( \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} \right) \Phi_N,
\]
which can be rewritten in the form
\[
\frac{\partial \Phi_N}{\partial t} = -i \mathcal{L} \Phi_N,
\]
where the Liouville operator \( \mathcal{L} \) is given by
\[
\mathcal{L} = i \sum_i \left( \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} \right).
\]
In many problems one is only interested in the momentum distribution function \( \Phi_N(\{p\}; t) \) which is obtained from \( \Phi_N(\{q\}, \{p\}; t) \) by integrating over the volume \( V \) of the system:
\[
\Phi_N(\{p\}; t) = \int \ldots \int \Phi_N(\{q\}, \{p\}; t) d\{q\}.
\]
The relation (1.18) can be written as
\[ \Phi_N(\{p\}; t) = V^N \mathcal{P} \Phi_N(\{q\}, \{p\}; t), \]
where \( \mathcal{P} \) is the projection operator
\[ \mathcal{P} \dots = \frac{1}{V^N} \int \ldots \int d(\{q\}) \ldots . \]

In other problems, such as anharmonic interaction in solids, one is interested in calculating the energy distribution. In this case one introduces the action-angle variables \( J_i \) and \( \varphi_i \) and regards the distribution as a function of \( \{J_i\} \) and \( \{\varphi_i\} \). Then the energy distribution is given by
\[ \Phi_N(\{J_i\}; t) = (2\pi)^N \mathcal{P} \Phi_N(\{J_i\}, \{\varphi_i\}; t), \]
where
\[ \mathcal{P} \ldots = \frac{1}{(2\pi)^N} \int \ldots \int d(\{\varphi_i\}) \ldots . \]

In case of a system interacting with stochastic perturbations, such as a randomly modulated harmonic oscillator, the density operator of the distribution function becomes a random function. It is then preferable to work with the ensemble average of the density operator. The appropriate projection operator for this problem is
\[ \mathcal{P} G = \bar{G}, \]
where the bar denotes the ensemble average with respect to the distribution of stochastic perturbations. Other examples of the projection operator will be given in subsequent sections.

We begin in § 2 with a brief account of the phase space methods. We introduce the concept of phase space distribution functions for quantum systems, a concept which was originally introduced by Wigner [1932] and later studied in great detail and generalized by a large number of workers (Moyal [1949], Groenewold [1946], Glauber [1963], Sudarshan [1963], Mehta and Sudarshan [1965], Cahill and Glauber [1969a, b], Agarwal and Wolf [1968, 1970a, b, c]). We also give in § 2 the equations of motion for these phase space distribution functions. In §§ 3–5 master equations for both classical and quantum systems are obtained. The application of these master equations to various problems in quantum optics is considered in § 4 and §§ 6–12. The problems treated in detail include relaxation of an oscillator and an atom, Brownian motion of a quantum oscillator, super-radiance and relaxation of spin systems, theory of a single mode laser, systems interacting with intense external fields, parametric frequency conver-
sion, parametric oscillator and a related problem of anharmonic interaction in solids. Finally an appendix is devoted to the study of the properties of the Fokker-Planck equations.

§ 2. Phase Space Methods

In this section we present a summary of phase space methods in quantum mechanics. Consider a quantum mechanical system with one degree of freedom. Let $a$ and $a^+$ be the annihilation and the creation operators* satisfying the boson commutation relations, i.e.

$$[a, a^+] = 1, \quad [a, a] = [a^+, a^+] = 0. \quad (2.1)$$

Let $\rho$ be the density operator characterizing the state of the system under consideration. It is possible to introduce a $c$-number distribution function, which we call the phase space distribution function, corresponding to the given density operator. The c-number distribution function is not unique. It depends on the rule of mapping that is adopted to map the operators onto c-numbers and vice versa. For a given rule of mapping, there is one and only one distribution function corresponding to a given density operator. In what follows, we will be considering only the normal, the antinormal and the Weyl rules of mapping (for a general theory see AGARWAL and WOLF [1968, 1970a, b, c]; see also LAX [1968b], CAHILL and GLAUBER [1969a, b]).

Let $\Omega$ be a linear mapping operator that transforms an arbitrary function of the c-numbers $z$ and $z^*$ ($z^*$ being the complex conjugate of $z$) onto an operator function of $a$ and $a^+$ and let $\Theta$ be the linear mapping operator, inverse to $\Omega$, which transforms an arbitrary function of the operators $a$ and $a^+$ onto a c-number function of $z$ and $z^*$. Here $z$ and $z^*$ are the c-numbers onto which the operators $a$ and $a^+$ are mapped, i.e.

$$z \mapsto a, \quad z^* \mapsto a^+. \quad (2.2)$$

The mapping operators for the normal, the antinormal and the Weyl rules are now given by

$$z^* m^* z \overset{\Omega^{(N)}}{\longrightarrow} a^m a^n \overset{\Theta^{(N)}}{\longrightarrow} z^* m^* z, \quad (2.3)$$

$$z^* m^* z \overset{\Omega^{(A)}}{\longrightarrow} a^m a^n \overset{\Theta^{(A)}}{\longrightarrow} z^* m^* z, \quad (2.4)$$

$$z^* m^* z \overset{\Omega^{(W)}}{\longrightarrow} (a^m a^n)_W \overset{\Theta^{(W)}}{\longrightarrow} z^* m^* z, \quad (2.5)$$

where the superscripts $N$, $A$ and $W$ on $\Omega$ and $\Theta$ stand for the normal, the antinormal and the Weyl rules of mapping. In eq. (2.5) $(a^m a^n)_W$ denotes

* The superscript + denotes the Hermitian adjoint throughout this article.
the Weyl symmetrized product, i.e. \((a^+ a^n)_W\) is equal to the sum of all possible products involving \(m\) \(a^+\)'s and \(n\) \(a\)'s divided by the total number of such terms. It is thus seen from (2.3) and (2.4) that the normal (antinormal) rule brings all the creation (annihilation) operators to the left of all the annihilation (creation) operators.

Let \(\Phi^{(\Omega)}(z, z^*)\) be the phase space distribution function associated with the density operator according to a particular rule. It is obtained from \(\rho/\pi\) by the \(\Theta\) rule of mapping, i.e.

\[
\frac{\rho}{\pi} \xrightarrow{\Theta} \Phi^{(\Omega)}(z, z^*) \xrightarrow{\Omega} \frac{\rho}{\pi}.
\]

(2.6)

The phase space distribution function so obtained is not necessarily non-negative and may even be singular, and must, in general, be regarded as a function in the sense of generalized function theory. We give below explicit expressions for the distribution functions \(\Phi^{(W)}\), \(\Phi^{(A)}\) and \(\Phi^{(N)}\).

The distribution function \(\Phi^{(W)}\), which is obtained from \(\rho\) via the Weyl rule of mapping, is known as the Wigner distribution function. In coordinate-momentum representation it was first introduced by WIGNER [1932] and studied extensively by MOYAL [1949] and GROENEWOLD [1946] (see also KUBO [1964]). For a given density operator the Wigner distribution function always exists. The following explicit expression in terms of coherent states was given by AGARWAL and WOLF [1970a]

\[
\Phi^{(W)}(z, z^*) = \frac{2}{\pi^2} \exp \left(2|z|^2\right) \int \langle -\alpha | \rho | \alpha \rangle \exp \{-2(\alpha z^* - \alpha^* z)\} d^2 \alpha, \quad (2.7)
\]

where the integration extends over the whole complex \(\alpha\)-plane. In eq. (2.7) \(|\alpha\rangle\) is a coherent state (for properties of coherent states, see GLAUBER [1963], KLAUDER and SUDARSHAN [1968]); it is an eigenstate of the annihilation operator \(a\) with the eigenvalue \(\alpha\). Coherent states \(|\alpha\rangle\) are known to form a complete set (KLAUDER [1960]) but are not orthogonal, more precisely

\[
\frac{1}{\pi} \int |\alpha\rangle \langle \alpha | d^2 \alpha = 1, \quad \langle \alpha' | \alpha \rangle = \exp \{\alpha^* \alpha - \frac{1}{2} |\alpha|^2 - \frac{1}{2} |\alpha'|^2\}. \quad (2.8)
\]

We will refer to the distribution function \(\Phi^{(A)}\), obtained from \(\rho\) via the antinormal rule of mapping, as the Sudarshan-Glauber distribution function (SUDARSHAN [1963], GLAUBER [1963]). It is often referred to as the \(P\)-representation of the density operator. In terms of \(\Phi^{(A)}\) the density operator admits the following representation

\[
\rho = \int \Phi^{(A)}(z, z^*) |z\rangle \langle z| d^2 z. \quad (2.9)
\]
The relation (2.9) has been inverted by MEHTA [1967] to obtain the following formula for $\phi^{(A)}$

$$\phi^{(A)}(z, z^*) = \frac{1}{\pi^2} \exp (|z|^2) \int \langle -z|\rho|z \rangle \exp \{|z|^2 - (az^* - \alpha^* z)\} d^2z. \quad (2.10)$$

The distribution function $\phi^{(N)}$, obtained from $\rho$ via the normal rule of mapping is given by (MEHTA and SUDARSHAN [1965], GLAUBER [1965])

$$\phi^{(N)}(z, z^*) = \frac{1}{\pi} \langle z|\rho|z \rangle. \quad (2.11)$$

This function has the interesting properties that it exists for all class of density operators and is non-negative. These two properties make it very attractive to use in physical problems. This distribution function is thus closest to the classical distribution functions.

In calculations, we also need the $c$-number functions associated with any general operator $G$. Let $F^{(O)}_G$ be the "\(\Omega\)-equivalent" of $G$, obtained from $G$ via the $\Theta$ rule of mapping i.e.

$$G \overset{\Theta}{\rightarrow} F^{(O)}_G \overset{\Omega}{\rightarrow} G. \quad (2.12)$$

The functions $F^{(O)}_G$, $F^{(A)}_G$ and $F^{(N)}_G$ are given in terms of the operator $G$ by the relations (2.7), (2.10) and (2.11) with $\rho/\pi$ replaced by $G$.

The phase space distribution functions are very useful in computing the expectation values. It is easily shown that the expectation value of an operator may be expressed in the forms (AGARWAL and WOLF [1970b]; see also LAX and LOUISELL [1967])

$$\langle G \rangle = \text{Tr} (\rho G) = \int \phi^{(W)}(z, z^*) F^{(W)}_G(z, z^*) d^2z \quad (2.13)$$

$$= \int \phi^{(A)}(z, z^*) F^{(N)}_G(z, z^*) d^2z \quad (2.14)$$

$$= \int \phi^{(N)}(z, z^*) F^{(A)}_G(z, z^*) d^2z. \quad (2.15)$$

We have thus expressed the quantum mechanical expectation values in the same form as the averages occurring in classical statistical mechanics, i.e. averages are given as integrals over the phase space. In quantum optics, one often wishes to compute the expectation values of operators that are already in some ordered form for example the normally ordered correlation functions of the form, $\langle a^{+m}a^n \rangle$. In this case we have from (2.14)

$$\langle a^{+m}a^n \rangle = \int \phi^{(A)}(z, z^*) z^m z^n d^2z. \quad (2.16)$$
This formula brings into evidence even more clearly the formal analogy between the phase space representations and classical statistical mechanics. It should be noted that if we use (2.13) or (2.15), then we obtain instead of (2.16) (for \( n = m \))

\[
\langle a^{+m} a^m \rangle = (-1)^m \int \Phi^{(N)}(z, z^*) L_m(|z|^2) d^2z, \tag{2.17}
\]

\[
= (-\frac{1}{2})^m \int \Phi^{(W)}(z, z^*) L_m(2|z|^2) d^2z, \tag{2.18}
\]

where \( L_m \) is the Laguerre polynomial of order \( m \). We emphasize that for practical calculations, one may use any of the phase space distribution functions and in some cases we may be naturally led to use one particular distribution function.

We now consider the dynamic aspects of phase space methods (AGARWAL and WOLF [1968, 1970b, c]). The phase space distribution functions \( \Phi^{(Q)} \) are now time dependent and an equation of motion for \( \Phi^{(Q)} \) is obtained from eq. (1.10) by applying the mapping operator \( \Theta \) to both sides of (1.10), i.e.

\[
\frac{\partial}{\partial t} \Phi^{(Q)} = \Theta \left( \frac{\partial^\rho}{\partial t^\rho} \right) = -i\Theta([H, \rho]]. \tag{2.19}
\]

The right hand side of (2.19) may be expressed in terms of \( \Phi^{(Q)} \) and \( F^{(Q)}_H \) and their derivatives as follows

\[
\frac{\partial}{\partial t} \Phi^{(Q)} = -iF^{(Q)}(\mathcal{D}^{(1)} - \mathcal{D}^{(2)}) \Phi^{(Q)}, \tag{2.20}
\]

where the operators \( \mathcal{D}^{(1)} \) and \( \mathcal{D}^{(2)} \) are given by (cf. AGARWAL and WOLF [1968, 1970b])

\[
\mathcal{D}^{(1)} = \exp \left( (\lambda + \frac{1}{2}) \hat{\frac{\partial}{\partial z}} \hat{\frac{\partial}{\partial z^*}} + (\lambda - \frac{1}{2}) \frac{\partial}{\partial z^*} \frac{\partial}{\partial z} \right), \tag{2.21}
\]

\[
\mathcal{D}^{(2)} = \exp \left( (\lambda + \frac{1}{2}) \hat{\frac{\partial}{\partial z^*}} \hat{\frac{\partial}{\partial z}} + (\lambda - \frac{1}{2}) \frac{\partial}{\partial z} \frac{\partial}{\partial z^*} \right). \tag{2.22}
\]

In eqs. (2.21) and (2.22) the arrow pointing to the left (right) indicates that the differential operator below it operates on quantities on the left (right) of the operators \( \mathcal{D}^{(1)} \) and \( \mathcal{D}^{(2)} \). In eq. (2.20) \( \Omega \) refers to any of the three rules of mapping and \( \lambda \) takes values 0, \( \frac{1}{2} \), and \( -\frac{1}{2} \) corresponding to the Weyl, the normal and the antinormal rules of mapping. Eq. (2.20) can also be written in the form
\[ \frac{\partial \Phi^{(A)}}{\partial t} = -i \mathcal{L}^{(A)} \Phi^{(A)}, \quad (2.23) \]

where the Liouville operator $\mathcal{L}^{(A)}$ is now given by
\[ \mathcal{L}^{(A)} = \mathcal{H}^{(A)} \{ \mathcal{D}^{(1)} - \mathcal{D}^{(2)} \}. \quad (2.24) \]

Let $K^{(A)}(z, z^*, t|z_0, z^*_0, t_0)$ be the Green's function associated with eq. (2.23). The Green's function $K^{(A)}$ is the solution of (2.23) with the initial condition
\[ K^{(A)}(z, z^*, t_0|z_0, z^*_0, t_0) = \delta^{(2)}(z - z_0). \quad (2.25) \]

The Green's function is useful in the calculation of the multitime correlation functions (LAX [1968b], GRAHAM et al. [1968], AGARWAL and WOLF [1970c]). In particular the normally-ordered time-ordered correlation functions defined by
\[ \Gamma_T^{(N)} = \langle [a^+(t_1)]^{i_1} \cdots [a^+(t_n)]^{i_n} [a(t_n)]^{j_n} \cdots [a(t_1)]^{j_1} \rangle, \quad (t_n \geq t_{n-1} \geq \ldots \geq t_1), \quad (2.26) \]
where $a(t)$ and $a^+(t)$ are operators in the Heisenberg picture, may be shown to be expressible in the form
\[ \Gamma_T^{(N)} = \int \cdots \int \Phi^{(A)}(z_0, z^*_0, t_0) \times \prod_{\lambda=1}^{n} \{ (z^*_\lambda)^{j_\lambda} (z_\lambda)^{i_\lambda} K^{(A)}(z_\lambda, z^*_\lambda, t_\lambda|z_{\lambda-1}, z^*_{\lambda-1}, t_{\lambda-1}) d^2z_\lambda \} d^2z_0. \quad (2.27) \]

Next we give some identities which are useful in obtaining the operator form of the phase space equations of motion. Some of these identities, which are easily proved by using the general theory developed by AGARWAL and WOLF [1970b], are given by
\[ \frac{\partial}{\partial z} \left( z\Phi^{(A)} \right) \xrightarrow{\mathcal{D}(A)} - \frac{1}{\pi} [a^+, \rho a], \quad \frac{\partial}{\partial z} \left( z^*\Phi^{(A)} \right) \xrightarrow{\mathcal{D}(A)} - \frac{1}{\pi} [a^+, \rho a^+], \quad (2.28) \]
\[ \frac{\partial}{\partial z} \left( z\Phi^{(N)} \right) \xrightarrow{\mathcal{D}(N)} - \frac{1}{\pi} [a^+, \rho a], \quad \frac{\partial}{\partial z} \left( z^*\Phi^{(N)} \right) \xrightarrow{\mathcal{D}(N)} - \frac{1}{\pi} [a^+, a^+ \rho]. \quad (2.29) \]

The results, which we have so far presented, are easily generalized to boson systems with many degrees of freedom (AGARWAL and WOLF [1970b]). One may develop a phase space theory for a system of fermions along similar lines. AGARWAL [1969] for example, has made use of the Schwinger's boson representation to deal with two-level atoms. The phase space theory for a system of fermions will not be discussed here.
Finally we introduce the action and angle variables $J$ and $\varphi$ defined by

$$J = |z|^2, \quad \varphi = \frac{1}{2i} \ln\left(\frac{z}{z^*}\right). \quad (2.30)$$

The transformation laws are given by

$$\frac{\partial}{\partial z} = J^4 e^{-i\varphi} \left( \frac{\partial}{\partial J} - \frac{i}{2J} \frac{\partial}{\partial \varphi} \right), \quad \frac{\partial}{\partial z^*} = J^4 e^{i\varphi} \left( \frac{\partial}{\partial J} + \frac{i}{2J} \frac{\partial}{\partial \varphi} \right), \quad (2.31)$$

and

$$\int d^2z \ldots = \frac{1}{2} \int_0^\infty dJ \int_0^{2\pi} d\varphi \ldots \quad (2.32)$$

In the case when the Hamiltonian $H$ is equal to $\omega a^+ a$ the Liouville operator $\mathcal{L}^{(o)}$ is found, from (2.24) and (2.31), to be given by

$$\mathcal{L}^{(o)} = i\omega \frac{\partial}{\partial \varphi}. \quad (2.33)$$

For a system with many degrees of freedom and with $H$ equal to $\sum_k \omega_k a_k^+ a_k$, the Liouville operator is

$$\mathcal{L}^{(o)} = i \sum_k \omega_k \frac{\partial}{\partial \varphi_k}. \quad (2.34)$$

The operator $\mathcal{L}^{(o)}$ has the following eigenfunctions and eigenvalues

$$\psi(\{v_k\}) = (2\pi)^{-N/2} \exp\left(-i \sum_k v_k \varphi_k\right) \quad E(\{v_k\}) = \sum_k v_k \omega_k, \quad (2.35)$$

where $v_k$ vary over all positive and negative integers. The eigenfunctions $\psi(\{v_k\})$ are orthogonal and play a very important role in the perturbation expansions of the distribution functions in interaction problem (cf. § 12; for the use of eigenfunctions $\psi(\{v_k\})$ in problems in classical statistical mechanics, see PRIGOGINE [1962]).

§ 3. Master Equation for a General System

We have seen that the basic equation of motion for both classical and quantum systems can be written in the form

$$\frac{\partial f}{\partial t} = -i\mathcal{L}f, \quad (3.1)$$

where $\mathcal{L}$ is the appropriate Liouville operator and $f$ represents the state of the system. Table 1 gives the form of the Liouville operator $\mathcal{L}$ and $f$ for both classical and quantum systems. The master equation is an equation
<table>
<thead>
<tr>
<th>Classical systems:</th>
<th>Classical systems:</th>
<th>Quantum systems:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi_N({q_i}, {p_i}, t)$</td>
<td>$i \sum_i \left( \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} \right)$</td>
<td>$\sum_m \mathcal{P}_m \operatorname{Tr} {\mathcal{P}_m \ldots}$</td>
</tr>
<tr>
<td>$\Phi_N({J_i}, {\varphi_i}, t)$</td>
<td>$i \sum_i \left( \frac{\partial H}{\partial \varphi_i} \frac{\partial}{\partial J_i} - \frac{\partial H}{\partial J_i} \frac{\partial}{\partial \varphi_i} \right)$</td>
<td>$\rho(0) \operatorname{Tr}_R \ldots$ (cf. eq. (5.7))</td>
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<tr>
<td>Operator treatment</td>
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<td>$\ldots$ (cf. eq. (1.23))</td>
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<tr>
<td>$\rho$</td>
<td>$[H, \ldots]$</td>
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<tr>
<td>Quantum systems:</td>
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<td>Phase space treatment</td>
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<tr>
<td>$\Phi^{D}({z_i}, {z_i^*}, t)$</td>
<td>$\Phi^{D}(0) \int \ldots d^2(z_R) \ldots$</td>
<td>$\Phi^{D}(0) \int \ldots d^2(z_R) \ldots$ (cf. eq. (5.8))</td>
</tr>
<tr>
<td></td>
<td>$\mathcal{D}^{(1)} = \prod_i \exp \left( (\lambda + \frac{1}{2}) \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_i^<em>} + (\lambda - \frac{1}{2}) \frac{\partial}{\partial z_i^</em>} \frac{\partial}{\partial z_i} \right)$</td>
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<tr>
<td></td>
<td>$\mathcal{D}^{(2)} = \prod_i \exp \left( (\lambda + \frac{1}{2}) \frac{\partial}{\partial z_i^<em>} \frac{\partial}{\partial z_i} + (\lambda - \frac{1}{2}) \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_i^</em>} \right)$</td>
<td>$\left( \frac{1}{2\pi} \right)^N \int \ldots d{\varphi_i} \ldots$</td>
</tr>
</tbody>
</table>
of motion for $\mathcal{P}f$, where $\mathcal{P}$ is the projection operator which projects out the relevant part of $f$. We will refer to $f$ as the distribution function, with the understanding that in the operator treatment of quantum systems it represents the density operator. In Table 1, we have also listed the form of commonly used projection operators. To obtain an equation for $\mathcal{P}f$, we write $f$ in the form

$$ f = \mathcal{P}f + (1 - \mathcal{P})f. \quad (3.2) $$

On taking the Laplace transform of (3.1), we obtain

$$ \beta C(\beta) - f(0) = -i\mathcal{L}C(\beta), \quad (3.3) $$

where $C(\beta)$ is the Laplace transform of $f(t)$, i.e.

$$ C(\beta) = \int_0^\infty e^{-\beta t}f(t)dt, \quad (\text{Re } \beta \geq 0), \quad (3.4) $$

and where we have assumed that $\mathcal{L}$ is explicitly time independent. On multiplying (3.3) by $\mathcal{P}$ and $(1 - \mathcal{P})$ respectively, we obtain the equations

$$ \beta \mathcal{P}C(\beta) - \mathcal{P}f(0) = -i\mathcal{L}\mathcal{P}C(\beta) - i\mathcal{L}(1 - \mathcal{P})C(\beta), \quad (3.5) $$

$$ \beta(1 - \mathcal{P})C(\beta) - (1 - \mathcal{P})f(0) = -i(1 - \mathcal{P})\mathcal{L}\mathcal{P}C(\beta) - i(1 - \mathcal{P})\mathcal{L}(1 - \mathcal{P})C(\beta). \quad (3.6) $$

The solution of (3.6) is

$$ (1 - \mathcal{P})C(\beta) = \{\beta + i(1 - \mathcal{P})\mathcal{L}\}^{-1}\{-i(1 - \mathcal{P})\mathcal{L}\mathcal{P}C(\beta) + (1 - \mathcal{P})f(0)\}. \quad (3.7) $$

We substitute (3.7) in (3.5) and take the inverse Laplace transform. Then we obtain the following equation for $\mathcal{P}f$ (Zwanzig [1961a], Agarwal [1969])

$$ \frac{\partial}{\partial t}[\mathcal{P}f(t)] + i\mathcal{L}[\mathcal{P}f(t)] + i\mathcal{L}\exp\{-i(1 - \mathcal{P})\mathcal{L}t\}(1 - \mathcal{P})f(0) $$

$$ + \int_0^t \mathcal{L}\exp\{-i(1 - \mathcal{P})\mathcal{L}\tau\}(1 - \mathcal{P})\mathcal{L}[\mathcal{P}f(t - \tau)]d\tau = 0. \quad (3.8) $$

Equation (3.8) is the master equation for the relevant part of the distribution function. It should be noted that eq. (3.8) is an integro-differential equation and is an exact consequence of the dynamics of the system. Further simplification can be made depending on the form of the Hamiltonian and the
initial condition. In most of the physical applications the initial condition will be such that

\[(1 - \mathcal{P}) f(0) = 0. \quad (3.9)\]

This condition is usually referred to as the assumption of initial random phase.

We now obtain the explicit form of (3.8) in case when the projection operator \(\mathcal{P}\) is given by eq. (1.13), i.e. we obtain the equation of motion for the diagonal elements of \(\rho\). The condition (3.9) in this case implies that the density operator is initially diagonal. Let the Hamiltonian of the system be written as

\[H = H_0 + gH_1, \quad (3.10)\]

where \(gH_1\) is a small perturbation. We work in a representation in which \(H_0\) is diagonal and \(H_1\) has no diagonal elements. Then the Liouville operator is given by

\[\mathcal{L} = \mathcal{L}_0 + g \mathcal{L}_1, \quad \mathcal{L}_0 = [H_0, \cdot], \quad \mathcal{L}_1 = [H_1, \cdot]. \quad (3.11)\]

It is easily verified from (1.13) and (3.11) that

\[\mathcal{P} \mathcal{L}_0 = \mathcal{L}_0 \mathcal{P} = 0. \quad (3.12)\]

Moreover since \(\mathcal{P} \rho\) has no off-diagonal elements it follows that

\[\mathcal{P} \mathcal{L}_1 [\mathcal{P} \rho(t)] = 0. \quad (3.13)\]

On combining eqs. (3.9)–(3.13), eq. (3.8) reduces to

\[\frac{\partial}{\partial t} [\mathcal{P} \rho(t)] + g^2 \int_0^t K(\tau)[\mathcal{P} \rho(t-\tau)] d\tau = 0, \quad (3.14)\]

where the kernel \(K(\tau)\) is given by

\[K(\tau) = \mathcal{P} \mathcal{L}_1 \exp \{-i(1-\mathcal{P})\mathcal{L}(1-\mathcal{P})\tau\}(1-\mathcal{P})\mathcal{L}_1. \quad (3.15)\]

Equation (3.14) leads to the following equation for \(\rho_{nn}\)

\[\frac{\partial \rho_{nn}}{\partial t} + g^2 \sum_n \int_0^t k_{mn}(\tau) \rho_{nn}(t-\tau) d\tau = 0, \quad (3.16)\]

where

\[k_{mn}(\tau) = \langle m | [H_1, \exp \{-iH_\varphi \tau\} [H_1, \mathcal{P} \rho(t-\mathcal{P}) [H_1, \mathcal{P} \rho(t-\mathcal{P})] m \rangle. \quad (3.17)\]

and where \(H_\varphi\) is given by

\[H_\varphi = H_0 + g(1 - \mathcal{P})H_1(1 - \mathcal{P}). \quad (3.18)\]
From the completeness of states $|n\rangle$, it follows that

$$\sum_n k_{mn}(\tau) = 0. \quad (3.19)$$

On using (3.19), we can write (3.16) in the form

$$\frac{\partial \rho_{mm}}{\partial t} = -g^2 \sum_{n \neq m} \int_0^\tau \{k_{nn}(\tau)\rho_{nn}(t-\tau) - k_{mn}(\tau)\rho_{nm}(t-\tau)\} d\tau. \quad (3.20)$$

It should be noted that so far we have made no assumption about the strength of the perturbation, i.e. (3.20) holds to all orders in perturbation. We only assumed that $\rho$ is initially diagonal. The master equation (3.20) is in the form of the rate equation since the first term on the right hand side represents the transitions from all other states $|n\rangle$ to the state $|m\rangle$ and the second sum represents the transitions from the state $|m\rangle$ to all other states. ZWANZIG [1964] used the expression (3.15) for the kernel $K(\tau)$ to establish the identity of the master equation (3.20) with the master equations of MONTROLL [1962] and of PRIGOGINE and RESIBOIS [1961].

The exact expression for the kernel $k_{mn}(\tau)$ is rather complicated. In the lowest order in perturbation (the Born approximation) we can replace $H^\phi$ by $H_0$, and we then obtain the following expression for $k_{mn}(\tau)$

$$k_{mn}(\tau) = -2\langle m|H_1|n\rangle^2 \cos \left[ (E_n - E_m)\tau \right]. \quad (3.21)$$

To obtain higher order terms, we take the Laplace transform of $K(\tau)$ and expand it in powers of $L_1$. We then obtain the following series expansion (ZWANZIG [1961a])

$$K(\beta) = \mathcal{P} L_1 G_0 (1 - \mathcal{P}) L_1 - ig \mathcal{P} L_1 G_0 (1 - \mathcal{P}) L_1 G_0 (1 - \mathcal{P}) L_1 + \ldots, \quad (3.22)$$

where

$$G_0 = (\beta + i L_0)^{-1}. \quad (3.23)$$

The Pauli equation may be obtained from (3.20) and (3.21) in the following limits (VAN HOVE [1955, 1957], MONTROLL [1961], see also PRIGOGINE [1962]):

(i) $N = \text{number of degrees of freedom of the system} \rightarrow \infty$, and $V = \text{volume of the system} \rightarrow \infty$, such that $N/V = \text{constant}$,

(ii) $g \rightarrow 0$, $t \rightarrow \infty$ such that $(g^2t) = \text{constant}$. In these limits it is possible to replace $\rho_{mn}(t-\tau)$ in eq. (3.20) by $\rho_{mn}(t)$ and extend the upper limit of integration to $\infty$ (for details see MONTROLL [1961]) and we then find that
\[
\frac{\partial \rho_{nm}}{\partial t} = \sum_{n \neq m} \int_0^\infty 2g^2 |\langle m|H_1|n\rangle|^2 \cos [(E_n - E_m)\tau] d\tau [\rho_{nm}(t) - \rho_{mm}(t)].
\] (3.24)

On simplification, eq. (3.24) leads to the Pauli equation (1.1).

The Pauli equation is easily generalized to open systems, i.e. a small system coupled to a large system usually called a reservoir. (Open systems will be discussed in detail in § 5.) For such systems \( n \) and \( m \) in Pauli equation refer to the states of small system and the transition probabilities are no longer symmetric, i.e. \( \gamma_{nm} \neq \gamma_{mn} \). In fact if the relaxation towards thermal equilibrium at temperature \( T \) is assumed, then one finds that

\[
\gamma_{nm}/\gamma_{mn} = \exp \{\beta(E_m - E_n)\},
\] (3.25)

where \( \beta = 1/K_B T \) and \( K_B \) is the Boltzmann constant. Eq. (3.25) is merely a statement of the principle of detailed balance*. General properties of the Pauli equation are discussed in great detail in a recent review by OPPENHEIM et al. [1967]. The Pauli's master equation can be written in the form (MONTROLL [1961])

\[
d\tilde{p}_n/dt = \sum_m b_{nm} \tilde{p}_m,
\] (3.26)

where

\[
\tilde{p}_n = \rho_{nn} \exp \left\{ \frac{1}{2} \beta E_n \right\},
\] (3.27)

\[
b_{nm} = \begin{cases} 
\exp \left\{ \frac{1}{2} \beta (E_n - E_m) \right\} \gamma_{nm} & n \neq m \\
-\sum_{l \neq n} \gamma_{lm} & n = m.
\end{cases}
\] (3.28)

It is seen from the definition of the matrix \( b \) and the fact that \( \gamma_{nm} \) are the transition probabilities, that the matrix \( b \) is a semi-negative definite matrix which is also symmetric (SHULER [1958]). Therefore the eigenvalues \( \lambda_k \) of \( b \) are real and \( \lambda_k \leq 0 \). The solution of the master equation is then given as

\[
p_n(t) = \sum_{m,l} \exp \left\{ \frac{1}{2} \beta (E_m - E_n) \right\} U_{nl} U_{lm}^* p_m(0) \exp \{\lambda_l t\},
\] (3.29)

where \( U \) is the orthogonal matrix which diagonalizes the matrix \( b \). Finally it should be noted that the present analysis is not restricted to the relaxation towards thermal equilibrium and the general case is obtained by the replacement \( \exp \{-\beta E_n\} \rightarrow p_n(\infty) \), where \( p_n(\infty) \) is the steady state solution.


We consider a system which is interacting with stochastic perturbations. The stochastic perturbation may, for example, be the coupling of a spin

* For an explicit proof of the fact that (3.25) follows from microreversibility see AGARWAL [1973b].
with a fluctuating magnetic field or the dipole-dipole coupling between two spins (Slichter [1963], Abragam [1961]). A simpler example of stochastic perturbation is a frequency modulated harmonic oscillator. We will now use the projection operator technique to obtain the master equation for such systems.

Let the Hamiltonian of the system be given by

$$H = H_0 + g H_1,$$  \hspace{1cm} (4.1)

where $H_1$ represents the random perturbation. We assume that $H_0$ is time independent and is not stochastic in nature. Let the perturbation be of the form

$$H_1 = \sum_a F_a(t) G_a,$$  \hspace{1cm} (4.2)

where $F_a(t)$ are the random forces and $G_a$ are the system operators. If $G_a$'s are not hermitian operators, then (4.2) can be made hermitian by the convention

$$F_{-a}(t) = F_a^*(t), \quad G_{-a} = G_a^\dagger.$$  \hspace{1cm} (4.3)

We further assume that the random forces $F_a(t)$ are the stationary random functions with zero ensemble mean value. Let $s_{a\beta}$ be the cross spectral density defined as

$$s_{a\beta}(\omega) = \int_0^\infty e^{i\omega t} F_a(t) F_\beta(t-\tau) d\tau.$$  \hspace{1cm} (4.4)

Since the Hamiltonian is now explicitly time dependent, it is convenient to work in the interaction picture. Various operators and $c$-numbers in the interaction picture will be distinguished from those in the Schrödinger picture by a subscript $I$.

In the interaction picture we have

$$a_{P_i l} = -i g [H_1(t), a_{P_i}].$$  \hspace{1cm} (4.5)

$$H_1(t) = \exp \{i H_0 t\} H_1 \exp \{-i H_0 t\},$$  \hspace{1cm} (4.6)

$$\rho_1 = \exp \{i H_0 t\} \rho \exp \{-i H_0 t\}.$$  \hspace{1cm} (4.6)

Hence the basic equation of motion in the interaction picture is of the form

$$\frac{\partial f_i}{\partial t} = -i g \mathcal{L}_1(t) f_i.$$  \hspace{1cm} (4.7)

To obtain the master equation, we multiply both sides of (4.7) by $\mathcal{P}$ and $(1 - \mathcal{P})$ respectively

$$\frac{\partial}{\partial t} [\mathcal{P} f_i] = -i g \mathcal{P} \mathcal{L}_1(t) \mathcal{P} f_i - i g (1 - \mathcal{P}) \mathcal{L}_1(t) (1 - \mathcal{P}) f_i,$$  \hspace{1cm} (4.8)

$$\frac{\partial}{\partial t} [(1 - \mathcal{P}) f_i] = -i g (1 - \mathcal{P}) \mathcal{L}_1(t) \mathcal{P} f_i - i g (1 - \mathcal{P}) \mathcal{L}_1(t) (1 - \mathcal{P}) f_i.$$  \hspace{1cm} (4.9)
The solution of (4.9) is given by
\[
(1 - \mathcal{P})f_i(t) = \mathcal{U}(t,0)[(1 - \mathcal{P})f_i(0)] - ig \int_0^t \mathcal{U}(t, \tau)(1 - \mathcal{P})\mathcal{L}_i(\tau)[\mathcal{P}f_i(\tau)]d\tau,
\]
where
\[
\mathcal{U}(t, \tau) = T \exp \left\{ -ig \int_\tau^t (1 - \mathcal{P})\mathcal{L}_i(t')dt' \right\},
\]
and where \(T\) is Dyson's time ordering operator. On substituting from (4.10) into (4.8), we obtain the following master equation (Agarwal [1969])
\[
\frac{\partial}{\partial t} [\mathcal{P}f_i(t)] + ig \mathcal{P} \mathcal{L}_i(t)[\mathcal{P}f_i(t)] + ig \mathcal{P} \mathcal{L}_i(t)\mathcal{U}(t,0)[(1 - \mathcal{P})f_i(0)]
\]
\[
+ g^2 \int_0^t \mathcal{P} \mathcal{L}_i(t)\mathcal{U}(t, \tau)(1 - \mathcal{P})\mathcal{L}_i(\tau)[\mathcal{P}f_i(\tau)]d\tau = 0.
\]

For the problem under consideration the projection operator is given by
\[
\mathcal{P}G = \overline{G},
\]
where the bar denotes the ensemble average with respect to the distribution of the random forces \(F_a(t)\). On making use of the condition of initial random phase (eq. (3.9)) and of the property of the random forces \(F_a(t)\), assumed above, eq. (4.12) reduces to
\[
\frac{\partial \overline{f_i(t)}}{\partial t} + g^2 \sum_{z, \beta} \int_0^t \mathcal{L}_a(t)\mathcal{U}(t, t - \tau)F_\beta(t - \tau)\mathcal{L}_\beta(t - \tau)\overline{f_i(t - \tau)}d\tau = 0,
\]
where \(\mathcal{L}_a(t)\) is the Liouville operator corresponding to the operator \(G_a(t)\). We have not made any assumption about the strength of the interaction and eq. (4.14) is therefore a master equation to all orders in the coupling coefficient \(g\) for systems interacting with stochastic perturbations.

We now make the commonly used approximations (Abragam [1961], Bloch [1956], Redfield [1957, 1965], Slischer [1963]): (i) Born approximation in which we replace \(\mathcal{U}(t, \tau)\) by unity, (ii) short memory approximation in which we replace \(\overline{f_i(t - \tau)}\) under the integral sign in (4.14) by \(\overline{f_i(t)}\) and (iii) take the long time limit. The master equation (4.14) then reduces to
\[
\frac{\partial \overline{f_i(t)}}{\partial t} + g^2 \sum_{z, \beta} \int_0^\infty F_\beta(t)\mathcal{L}_a(t)\mathcal{L}_\beta(t - \tau)\overline{f_i(t - \tau)}d\tau = 0.
\]
On taking the matrix elements with respect to the unperturbed states, this equation reduces to Redfield's [1957, 1965] master equation.
We now discuss two applications of the master equation (4.14). We first consider an elementary model of a laser, known as the phase diffusion model (Glauber [1965], Lamb [1965], Klauder and Sudarshan [1968, p. 228]; a more realistic model of the laser will be discussed in § 10). Let us consider a randomly modulated* harmonic oscillator for which the Hamiltonian is given by

$$H = \omega a^+ a + g A(t) a^+ a,$$  \hspace{1cm} (4.16)

where $A(t)$ is the random force. We assume that $A(t)$ is a real delta correlated Gaussian random process (cf. Stratonovich [1963]) with zero mean, i.e., one for which

$$F(t) \equiv 0, \quad F(t_1)F(t_2) = \frac{2D}{g^2} \delta(t_1 - t_2).$$  \hspace{1cm} (4.17)

For the Hamiltonian (4.16), it is easily seen from eq. (2.20) that the Sudarshan-Glauber distribution function in the interaction picture satisfies the equation

$$\frac{\partial \Phi_1}{\partial t} = -i \mathcal{L}_I(t) \Phi_1,$$  \hspace{1cm} (4.18)

where

$$\mathcal{L}_I(t) = g A(t) \left( z^* \frac{\partial}{\partial z^*} - z \frac{\partial}{\partial z} \right),$$  \hspace{1cm} (4.19)

and where we have dropped the superscript $A$ from $\Phi^{(A)}$. On substituting (4.19) in (4.14) and on using the properties of the random force $A(t)$, we find that $\Phi_1$ satisfies the following (exact) master equation

$$\frac{\partial \Phi_1}{\partial t} + D \left( z \frac{\partial}{\partial z} - z^* \frac{\partial}{\partial z^*} \right)^2 \Phi_1 = 0.$$  \hspace{1cm} (4.20)

On introducing the polar coordinates $r$ and $\theta$ ($z = re^{i\theta}$), eq. (4.20) becomes

$$\frac{\partial \Phi_1}{\partial t} = D \frac{\partial^2 \Phi_1}{\partial \theta^2}.$$  \hspace{1cm} (4.21)

It should be noted that (4.21) is just the diffusion equation for diffusion on a circular ring. It is easily seen that the Green's function associated with (4.21) is given by the expression

$$K(r, \theta, t | r_0, \theta_0, 0) = \frac{1}{2\pi r} \delta(r - r_0) \sum_{n = -\infty}^{+\infty} \exp \{ in(\theta - \theta_0) - D n^2 t \}. $$  \hspace{1cm} (4.22)

Various normally-ordered time-ordered correlation functions can be calculated by using (4.22) and (2.27). In particular one finds that

* Generalizations of this kind of stochastic processes have been treated by Fox [1972].
\[ \langle a^+(t)a(0) \rangle = \langle a^+(0)a(0) \rangle e^{i \omega t - D t}, \]  
\text{(4.23)}

which obviously corresponds to a Lorentzian spectrum with half width equal to \( D \). It should also be noted that there are no amplitude fluctuations in this model and it is often used to describe the behavior of a laser beam far above threshold.

Next we use the master equation (4.15) to obtain Bloch’s phenomenological equations (see, for example, ABRAGAM [1961]). In problems of relaxation Redfield’s equation leads to relaxation at infinite temperature. This is so because the heat bath has been treated classically. The correct relaxation behavior is obtained by replacing \( \tilde{f}_i(t) \) in (4.15) by \( \tilde{f}_i(t) - f_{th} \), where \( f_{th} \) is the equilibrium distribution function at temperature \( T \) and is given by

\[ \rho_{th} = \exp \left( -H_0/K_B T \right) / \text{Tr} \left\{ \exp \left( -H_0/K_B T \right) \right\}. \]  
\text{(4.24)}

The Hamiltonian for the case of spin relaxation can be written as

\[ H_0 = -\omega S^2, \quad gH_1 = -\gamma (S^+ F^+(t) + S^- F^-(t) + S^z F^z(t)). \]  
\text{(4.25)}

We assume that the random forces are such that

\[ s^+ = s^- = s^+ = s^- = 0, \quad s^+ \neq 0, \quad s_{zz} \neq 0, \]  
\text{(4.26)}

where \( s_{\alpha \beta} \) are defined by eq. (4.4). On substituting (4.25) and (4.26) in eq. (4.15), we find the following master equation

\[ \frac{\partial \tilde{p}_i}{\partial t} = -\gamma^2 s_{zz}(\omega)[S^+, [S^-, \tilde{p}_i - \rho_{th}]] 
-\gamma^2 s_{zz}^*(\omega)[S^-, [S^+, \tilde{p}_i - \rho_{th}]] - \gamma^2 s^z(0)[S^z, [S^z, \tilde{p}_i]]. \]  
\text{(4.27)}

Equation (4.27) leads to the following equations of motion for the macroscopic mean values

\[ \frac{\partial}{\partial t} \langle S^+ \rangle = -i\omega \langle S^+ \rangle - \{2s^*_{zz}(\omega) + s_{zz}(0)\}\gamma^2 \langle S^+ \rangle, \]  
\text{(4.28)}

\[ \frac{\partial}{\partial t} \langle S^z \rangle = -2\gamma^2 (s^*_{zz}(\omega) + s_{zz}(\omega)) \{ \langle S^z \rangle - \langle S^z \rangle_{th} \}, \]  
\text{(4.29)}

where

\[ \langle S^z \rangle_{th} = \text{Tr} \{ S^z \rho_{th} \}. \]  
\text{(4.30)}

In deriving (4.28) and (4.29) we have also assumed that the spin system under consideration is a spin-\( \frac{1}{2} \) system. Let \( T_1 \) and \( T_2 \) be the longitudinal and transverse relaxation coefficients. Then eqs. (4.28), (4.29) are the Bloch’s phenomenological equations, with
\[ T_1^{-1} = 4\gamma^2 \text{Re} \{ s_{+ -}(\omega) \}, \quad T_2^{-1} = \gamma^2 \{ 2 \text{Re} s_{+ -}(\omega) + s_{zz}(0) \}. \] 

It should be noted that the imaginary part of \( s_{+ -}(\omega) \) causes the frequency shift. Bloch equations are extensively used in optical problems such as in connection with self-induced transparency (Mccall and Hahn [1969]).

\section*{5. Master Equations for Open Systems}

We have already defined an open system as a system which is coupled to another large system, usually called reservoir. Most of the systems which we encounter in physics are open systems. Some of the problems, involving open systems, in quantum optics are those of lasers, relaxation of oscillators and two-level atoms, superradiance, parametric oscillators. In problems like superradiance the radiation field plays the role of the reservoir. In this section we obtain the master equation for the reduced density operator (phase space distribution function) of the sub-system of interest.

We denote the system of interest by \( S \) and the reservoir by \( R \). We write the total Hamiltonian of the open system as

\[ H = H_S + H_R + H_{RS}, \]  

where \( H_S \) and \( H_R \) are the unperturbed Hamiltonians of the system \( S \) and of the reservoir \( R \) respectively and \( H_{RS} \) is the interaction Hamiltonian between the system and the reservoir. We assume, for the sake of simplicity, that the Hamiltonian is explicitly time independent. Let \( \rho_{S+R}(\{Z_S\}, \{Z^*_S\}; \{z_R\}, \{z_R^*\}; t) \) be the density operator (phase space distribution function) for the combined system. The reduced density operator corresponding to the system \( S \) is given by

\[ \rho_S(t) = \text{Tr}_R \{ \rho_{S+R}(t) \}, \]  

where \( \text{Tr}_R \) indicates the trace over the reservoir variables. The reduced phase space distribution function is given by

\[ \phi_S(\{Z_S\}, \{Z^*_S\}; t) = \int \phi_{S+R}(\{Z_S\}, \{Z^*_S\}; \{z_R\}, \{z_R^*\}; t) d^2(\{z_R\}). \]  

We make the following two assumptions:

(i) The reservoir is initially in a state of thermal equilibrium at temperature \( T \) i.e. \( \rho_R(0) \) is given by

\[ \rho_R(0) = \exp \{ -H_R/K_B T \} / \text{Tr} \exp \{ -H_R/K_B T \}, \]  

or, more generally, \( \rho_R(0) \) is a function of \( H_R \) only.
(ii) The system $S$ and the reservoir $R$ are initially uncorrelated, i.e.

$$\rho_{S+R}(0) = \rho_S(0)\rho_R(0).$$  \hspace{1cm} (5.5)

The first assumption is not necessary but has been made for the sake of simplicity (for a rigorous treatment of open systems see, for example, Emch and Sewell [1968]).

It is clear from (5.2) that the reduced density operator can be obtained from the total density operator by suitably projecting it, i.e.

$$\mathcal{P}\rho_{S+R}(t) = \rho_R(0)\rho_S(t),$$  \hspace{1cm} (5.6)

where the projection operator $\mathcal{P}$ is given by (Argyres and Kelley [1964]; Agyres [1966])

$$\mathcal{P} \ldots = \rho_R(0)\text{Tr}_R \ldots.$$  \hspace{1cm} (5.7)

The corresponding projection operator for the phase space distribution functions is given by (Agarwal [1969])

$$\mathcal{P} \ldots = \Phi_R(0)\int d^2\{z_R\} \ldots,$$  \hspace{1cm} (5.8)

where $\Phi_R(0)$ is the phase space distribution function corresponding to the density operator $\rho_R(0)$. The Liouville operator $\mathcal{L}$ may be written as

$$\mathcal{L} = \mathcal{L}_S + \mathcal{L}_R + \mathcal{L}_{RS}.$$  \hspace{1cm} (5.9)

It is easily verified from (5.5)–(5.9) that the projection operator $\mathcal{P}$ has the properties

$$\mathcal{P}\mathcal{L}_S = \mathcal{L}_S\mathcal{P}, \quad \mathcal{P}\mathcal{L}_R = \mathcal{L}_R\mathcal{P} = 0, \quad (1-\mathcal{P})f_{S+R}(0) = 0.$$  \hspace{1cm} (5.10)

On using (3.8), (5.9) and (5.10), we find that $\mathcal{P}f$ satisfies the master equation

$$\frac{\partial}{\partial t} [\mathcal{P}f(t)] + i(\mathcal{L}_S + \mathcal{P}\mathcal{L}_{RS}\mathcal{P})[\mathcal{P}f(t)]$$

$$+ \int_0^t \mathcal{P}\mathcal{L} \exp[-i\tau(1-\mathcal{P})\mathcal{L}](1-\mathcal{P})\mathcal{L}[\mathcal{P}f(t-\tau)]d\tau = 0.$$  \hspace{1cm} (5.11)

Eq. (5.11) may be transformed in the standard fashion (see for example, Emch and Sewell [1968]) to the following form

$$\frac{\partial}{\partial t} [\mathcal{P}f(t)] = -i\mathcal{L}_S[\mathcal{P}f(t)] - i\mathcal{P}\mathcal{L}_{RS}[\mathcal{P}f(t)]$$

$$- \int_0^t K(t-\tau)[\mathcal{P}f(\tau)]d\tau,$$  \hspace{1cm} (5.12)
where the kernel $K(\tau)$ is given by
\[
K(\tau) = \mathcal{P} \mathcal{L}_{RS} U_0(\tau)(1 - \mathcal{P}) U(\tau) \mathcal{L}_{RS}.
\] (5.13)

In eq. (5.13) $U_0(t)$ and $U(t)$ are the time development operators defined by
\[
U_0(t) = \exp \left\{ -i(\mathcal{L}_R + \mathcal{L}_S)t \right\},
\] (5.14)
\[
U(t) = T \exp \left\{ -i \int_0^t (1 - \mathcal{P}) U_0(-\tau) \mathcal{L}_{RS} U_0(\tau)(1 - \mathcal{P}) d\tau \right\},
\] (5.15)

where $T$ is Dyson's time ordering operator.

Equation (5.12) is the desired master equation for the reduced density operator (or the phase space distribution function). This master equation is exact and is an integro-differential equation. The first term on the right hand side of (5.12) describes the unperturbed motion of the system $S$. The second term describes the effect of the reservoir $R$ to first order in the interaction $H_{RS}$ and its effect can generally be taken into account by a simple renormalization of the energy levels of the system $S$. The last term describes the effect of the reservoir to all other orders in perturbation $H_{RS}$.

In the case when (5.12) is the master equation for the phase space distribution function, it is a c-number differential equation and as we will see later, its Markovian form is useful in establishing connection with the Fokker-Planck equations of the stochastic processes.

In applications it is preferable to employ the master equation for the distribution function $f$ in the interaction picture. On transforming (5.12) to the interaction picture we obtain the equation
\[
\frac{\partial}{\partial t} \left[ \mathcal{P} f(t) \right] + i \mathcal{P} \mathcal{L}_{RS} (t) \left[ \mathcal{P} f(t) \right] \\
+ \int_0^t \mathcal{P} \mathcal{L}_{RS} (t) \mathcal{U}(t, \tau)(1 - \mathcal{P}) \mathcal{L}_{RS} (\tau) [\mathcal{P} f(\tau)] d\tau = 0,
\] (5.16)

where
\[
\mathcal{U}(t, \tau) = T \exp \left\{ -i \int_\tau^t d\tau' (1 - \mathcal{P}) \mathcal{L}_{RS} (\tau') (1 - \mathcal{P}) \right\}.
\] (5.17)

In eq. (5.16) $\mathcal{L}_{RS} (t)$ is the Liouville operator $\mathcal{L}_{RS}$ in the interaction picture, i.e.
\[
\mathcal{L}_{RS} (t) = \exp \left\{ i(\mathcal{L}_R + \mathcal{L}_S)\tau \right\} \mathcal{L}_{RS} \exp \left\{ -i(\mathcal{L}_R + \mathcal{L}_S)\tau \right\}.
\] (5.18)

We now consider the case of the system $S$ interacting weakly with the reservoir $R$. In the lowest Born approximation we replace $\mathcal{U}(t, \tau)$ by unity.

We further assume that the properties of the reservoir are such that
\[ \mathcal{L}_{\text{RS}}(t) \mathcal{P} f_l(\tau) = 0. \]  

(5.19)

In most of the applications which we will be considering this condition will be satisfied (if it is not, its effect can be taken into account by renormalization). Then (5.16) reduces to

\[ \frac{\partial}{\partial t} [\mathcal{P} f(t)] + \int_0^t \mathcal{L}_{\text{RS}}(t) \mathcal{L}_{\text{RS}}(\tau) [\mathcal{P} f(\tau)] d\tau = 0, \]  

(5.20)

where, for the sake of brevity, we have suppressed the subscript \( I \). The Markovian form of the approximate master equation (5.20) will be the basis for all the applications which we consider in the rest of the article.

We write the explicit form of (5.20) when \( f \) is the phase space distribution function and when \( H_{\text{RS}} \) is of the form

\[ H_{\text{RS}} = \sum_k (G_{Rk} G_{Sk}^\dagger + \text{H.C.}), \]  

(5.21)

where \( G_{Rk}(G_{Sk}) \) is an operator acting on the reservoir (system) variables alone. In the interaction picture we have

\[ H_{\text{RS}}(t) = \sum_k (G_{Rk}(t) G_{Sk}^\dagger(t) + \text{H.C.}). \]  

(5.22)

The Liouville operator \( \mathcal{L}_{\text{RS}}(t) \) is then given by (cf. eq. (2.24))

\[ \mathcal{L}_{\text{RS}}(t) = \sum_k \{ \psi_{Rk}(t) \mathcal{D}_S^{(1)} \psi_{Sk}(t) \mathcal{D}_S^{(1)} + \psi_{Rk}^*(t) \mathcal{D}_S^{(1)} \psi_{Sk}(t) \mathcal{D}_S^{(1)} + \text{C.C.} \}, \]  

(5.23)

where \( \psi_{Rk}(t) \) is the \( \Omega \)-equivalent of \( G_{Rk}(t) \) and \( \psi_{Sk}(t) \) is the \( \Omega \)-equivalent of \( G_{Sk}(t) \). On substituting from (5.23) in (5.20), we obtain the equation

\[ \frac{\partial \varphi_S^{(P)}}{\partial t} + \int_0^t K(t, \tau) d\tau = 0, \]  

(5.24)

where

\[ K(t, \tau) = \sum_{kli} \{ \Gamma_{11k}^{li}(t, \tau) [\psi_{Si}(t) \mathcal{D}_S^{(1)}(\psi_{Si}(\tau) \mathcal{D}_S^{(1)} \varphi_S^{(P)}(\tau))] \\
+ \Gamma_{12k}^{li}(t, \tau) [\psi_{Si}(t) \mathcal{D}_S^{(1)}(\psi_{Si}(\tau) \mathcal{D}_S^{(1)} \varphi_S^{(P)}(\tau))] \\
- \Gamma_{11k}^{li}(t, \tau) [\psi_{Si}(t) \mathcal{D}_S^{(1)}(\psi_{Si}(\tau) \mathcal{D}_S^{(2)} \varphi_S^{(P)}(\tau))] \\
- \Gamma_{12k}^{li}(t, \tau) [\psi_{Si}(t) \mathcal{D}_S^{(1)}(\psi_{Si}(\tau) \mathcal{D}_S^{(2)} \varphi_S^{(P)}(\tau))] \\
+ \Gamma_{21k}^{li}(t, \tau) [\psi_{Si}(t) \mathcal{D}_S^{(1)}(\psi_{Si}(\tau) \mathcal{D}_S^{(1)} \varphi_S^{(P)}(\tau))] \\
+ \Gamma_{22k}^{li}(t, \tau) [\psi_{Si}(t) \mathcal{D}_S^{(1)}(\psi_{Si}(\tau) \mathcal{D}_S^{(1)} \varphi_S^{(P)}(\tau))] \\
- \Gamma_{21k}^{li}(t, \tau) [\psi_{Si}(t) \mathcal{D}_S^{(1)}(\psi_{Si}(\tau) \mathcal{D}_S^{(2)} \varphi_S^{(P)}(\tau))] \\
- \Gamma_{22k}^{li}(t, \tau) [\psi_{Si}(t) \mathcal{D}_S^{(1)}(\psi_{Si}(\tau) \mathcal{D}_S^{(2)} \varphi_S^{(P)}(\tau))] + \text{C.C.} \}. \]  

(5.25)
In eq. (5.25) $I_{j_1, R}(t, \tau)$ are the elements of the reservoir correlation matrix defined by

$$
I_{j_1, R}(t, \tau) = \left( \begin{array}{c}
tr (\rho_R(0)G_{R_R}(t)G_{R_1}(\tau)) \\
tr (\rho_R(0)G_{R_R}^+(t)G_{R_1}(\tau)) \\
tr (\rho_R(0)G_{R_R}(t)G_{R_R}(\tau)) \\
tr (\rho_R(0)G_{R_R}^+(t)G_{R_R}(\tau))
\end{array} \right). \quad (5.26)
$$

We now discuss the effect of the weak external field on the system $S$ which is interacting with a reservoir. Let the effect of the external field be represented by the Hamiltonian $H_{\text{ext}}(t)$. Then the master equation (5.24) is modified, having the form

$$
\frac{\partial \Phi^{(\Omega)}_S}{\partial t} + \int_0^t K(t, \tau)d\tau + i\mathcal{L}_{\text{ext}}(t)\Phi^{(\Omega)}_S = 0, \quad (5.27)
$$

where

$$
\mathcal{L}_{\text{ext}}(t) = [H_{\text{ext}}(t), \ ] , \quad (5.28)
$$

and all the quantities are in the interaction picture.

§ 6. Relaxation of a Harmonic Oscillator

We first apply the theory developed in § 5 to the problem of a harmonic oscillator interacting with a reservoir. We present a fairly detailed treatment of this problem because this elementary example illustrates very clearly the techniques developed in previous sections. Our presentation follows closely that of Agarwal [1969] (for various other treatments see Montroll and Shuler [1957], Senitzky [1960, 1961], Louisell and Walker [1965], Weidlich and Haake [1965], Louisell and Marburger [1967], Oppenheim, Shuler and Weiss [1967], Lax [1966]).

The Hamiltonian of the total system consisting of the oscillator $S$ and the reservoir $R$ is assumed to be given by

$$
H = \omega a^+a + \sum_k \omega_k a_k^+a_k + \sum_k (g_k a_k^+a + \text{H.C.}), \quad (6.1)
$$

where we have also assumed, for the sake of simplicity, that the reservoir is made up of harmonic oscillators. The operators $a, a^+, a_k$ and $a_k^+$ satisfy the commutation relations

$$
[a, a^+] = 1, \quad [a_k, a_k^+] = 1, \quad (6.2)
$$

and all other commutators vanish. We first obtain the master equation for the reduced phase-space distribution function which is obtained from the density operator by the normal rule of mapping. The functions $\psi_{kR}$ and $\psi_{ks}$ for the problem under consideration are given by

$$
\psi_{kR}(t) = g_k z_k \exp \{-i\omega_k t\}, \quad \psi_{ks}(t) = z \exp \{-i\omega t\}. \quad (6.3)
$$
The operators $\mathcal{D}_s^{(1)}$ and $\mathcal{D}_s^{(2)}$ (defined by eqs. (2.21) and (2.22) respectively) for the normal rule of mapping are given by

$$\mathcal{D}_s^{(1)} = \exp \left( \frac{\hat{a}^+\hat{a}}{\partial z} \frac{\hat{a}^+\hat{a}}{\partial z^*} \right), \quad \mathcal{D}_s^{(2)} = \exp \left( \frac{\hat{a}^+\hat{a}}{\partial z} \frac{\hat{a}^+\hat{a}}{\partial z^*} \right),$$

and the correlation matrix $\Gamma_k^{(i)}(t, \tau)$ is easily seen to be given by

$$\Gamma_k^{(i)}(t, \tau) = g_k^2 \begin{pmatrix} 0 & \langle n_k \rangle + 1 \exp(-i\omega_k(t-\tau)) \\ \langle n_k \rangle \exp(i\omega_k(t-\tau)) & 0 \end{pmatrix} \delta_{k1}. \quad \text{(6.5)}$$

In eq. (6.5) $\langle n_k \rangle$ is the average occupation number for the $k$th mode of the oscillator of the heat bath and is equal to

$$\langle n_k \rangle = (\exp \{ \beta \omega_k \} - 1)^{-1}, \quad \beta = 1/K_B T, \quad \text{(6.6)}$$

where $K_B$ is the Boltzmann constant and $T$ is the temperature of the heat bath. On substituting eqs. (6.3)-(6.6) in eq. (5.24), we find that $\Phi^{(N)}(z, z^*)$ satisfies the following master equation

$$\frac{\partial \Phi^{(N)}(t)}{\partial t} = \int_0^t \left\{ \sum_k g_k^2 \exp \left[ -i(\omega_k - \omega)\tau \right] \times \left[ \frac{\partial}{\partial z} (z\Phi^{(N)}(t-\tau)) + (1 + \langle n_k \rangle) \frac{\partial^2}{\partial z \partial z^*} \Phi^{(N)}(t-\tau) \right] \right\} \mathrm{d}\tau, \quad \text{(6.7)}$$

where $\Phi^{(N)}(t)$ corresponds to the density operator in the interaction picture and where, for the sake of brevity, we have dropped the subscript $S$ from $\Phi^{(N)}$. It is seen that (6.7) is a non-Markovian equation, i.e. the time rate change of $\Phi^{(N)}$ at time $t$ depends on the values of $\Phi^{(N)}$ at all earlier times. The equation (6.7) can be solved exactly by taking its Laplace transform. However, in what follows we consider only the Markovian behavior of an oscillator interacting with the heat bath (for some of the non-Markovian effects associated with (6.7) see HAAKE [1969a] and for a general discussion of non-Markovian effects see ZWANZIG [1961b], AGARWAL [1973c]).

We assume that the bath oscillators are closely spaced in frequency so that $\sum_k$ can be replaced by $\int d\omega_k h(\omega_k) \ldots$, where $h(\omega_k)d\omega_k$ is the number of oscillators with frequencies lying between $\omega_k$ and $\omega_k + d\omega_k$. Moreover we assume that the reservoir correlations have a short correlation time $\tau_c$ so that for $t \gg \tau_c$ we can replace $\Phi^{(N)}(t-\tau)$ in (6.7) by $\Phi^{(N)}(t)$ and extend the upper limit of integration to infinity (short memory approximation). Then eq. (6.7) reduces to
\[
\frac{\partial \Phi^{(N)}}{\partial t} = (\gamma + i\Delta \omega) \frac{\partial}{\partial z} (z\Phi^{(N)}) + \gamma (1 + \langle n(\omega) \rangle) \frac{\partial^2 \Phi^{(N)}}{\partial z \partial z^*} + \text{C.C.}, \tag{6.8}
\]

where
\[
\gamma = \pi h(\omega) |g(\omega)|^2, \quad \Delta \omega = P \int d\omega_k h(\omega_k) |g(\omega_k)|^2 (\omega - \omega_k)^{-1}, \tag{6.9}
\]

and where \( P \) denotes the Cauchy principal value of the integral. The distribution function \( \Phi^{(N)} \) in (6.8) corresponds to the density operator in the interaction picture and on transforming to the Schrödinger picture (cf. Agarwal [1971d, Appendix A]) we find that \( \Phi^{(N)} \) satisfies the following master equation
\[
\frac{\partial \Phi^{(N)}}{\partial t} = \{ i(\omega + \Delta \omega) + \gamma \} \frac{\partial}{\partial z} (z\Phi^{(N)}) + \gamma (1 + \langle n(\omega) \rangle) \frac{\partial^2 \Phi^{(N)}}{\partial z \partial z^*} + \text{C.C.}. \tag{6.10}
\]

It is seen from (6.10) that \( \Delta \omega \) causes the frequency shift and its effect can be taken into account by frequency renormalization*. In what follows we will ignore the effect of this term. We may similarly obtain the master equation for the Wigner distribution function and the Sudarshan-Glauber distribution function. We write collectively these master equations as follows
\[
\frac{\partial \Phi^{(D)}}{\partial t} = (i\omega + \gamma) \frac{\partial}{\partial z} (z\Phi^{(D)}) + \gamma (\lambda + \frac{1}{2} + \langle n(\omega) \rangle) \frac{\partial^2 \Phi^{(D)}}{\partial z \partial z^*} + \text{C.C.}, \tag{6.11}
\]

where we recall that \( \lambda \) is +\( \frac{1}{2} \) for the normal rule of mapping, −\( \frac{1}{2} \) for the antinormal rule of mapping and is zero for the Weyl rule of mapping.

It should be noted that the master equation under the approximations discussed above, is of the form of a Fokker-Planck equation and we can now use the methods of the stochastic processes to study the dynamics of our system. The stochastically equivalent Langevin equations corresponding to the Fokker-Planck process (6.11) are given by (cf. eq. (A.35))
\[
\dot{z} = -i\omega z - \gamma z + F(t), \quad \dot{z}^* = i\omega z^* - \gamma z^* + F^*(t), \tag{6.12}
\]

where \( F(t) \) is a delta correlated complex Gaussian random process with the properties
\[
\langle F(t) \rangle = \langle F^*(t) \rangle = \langle F(t)F(t') \rangle = \langle F^*(t)F^*(t') \rangle = 0,
\]
\[
\langle F(t)F^*(t') \rangle = 2\gamma (\lambda + \frac{1}{2} + \langle n(\omega) \rangle) \delta(t - t'), \tag{6.13}
\]

and all the higher order linked moments (cumulants) of \( F(t) \) vanish. For the calculation of the moments one can use either the Fokker-Planck equation (6.11) or the Langevin equations (6.12). Since the Langevin

* Cf. footnote on p. 45.
equations (6.12) correspond to the quantum system, it is clear that the moments calculated from (6.12) would correspond to normally ordered moments $\langle a^{+m}a^n \rangle$ for $\lambda = -\frac{1}{2}$, to anti-normally ordered moments $\langle a^n a^{+m} \rangle$ for $\lambda = \frac{1}{2}$ and to Weyl ordered moments $\langle (a^{+m}a^n)_{\text{W}} \rangle$ for $\lambda = 0$. The solution of (6.12) is given by

$$z(t) = z(0)e^{-i\omega t - \gamma t} + \int_0^t F(t-\tau)e^{-i\omega \tau - \gamma \tau}d\tau.$$  (6.14)

On using (6.13) and (6.14) it follows that

$$\langle z^*(t)z(t) \rangle = \langle z^*(0)z(0) \rangle e^{-2\gamma t} + (\lambda + \frac{1}{2} + \langle n(\omega) \rangle)(1 - e^{-2\gamma t}),$$  (6.15)

and, therefore, for $\lambda = -\frac{1}{2}$, (6.15) leads to

$$\langle a^+(t)a(t) \rangle = \langle a^+(0)a(0) \rangle e^{-2\gamma t} + \langle n(\omega) \rangle(1 - e^{-2\gamma t}).$$  (6.16)

It is seen from (6.16) that the average occupation number at time $t$ is a sum of two terms: a term which depends on the initial occupation number and a term which depends on the temperature of the reservoir and which vanishes as the temperature $\to 0$.

The equation of motion for the diagonal elements $\rho_{nn}$ of the reduced density operator can be obtained by using the relation (which can be obtained from eq. (2.9) by taking the diagonal matrix elements with respect to the Fock state $|n\rangle$)

$$\rho_{nn} = \int \phi^{(A)}(z, z^*) \frac{|z|^2 e^{-|z|^2}}{n!} d^2z,$$  (6.17)

and the eq. (6.11) for $\lambda = -\frac{1}{2}$. A straightforward calculation shows that the diagonal elements satisfy the master equation

$$\frac{\partial \rho_{nn}}{\partial t} = 2\gamma \langle n(\omega) \rangle n \rho_{n-1, n-1} + 2\gamma (1 + \langle n(\omega) \rangle)(n+1) \rho_{n+1, n+1}$$

$$- 2\gamma (n+2\langle n(\omega) \rangle + \langle n(\omega) \rangle) \rho_{nn}.$$  (6.18)

The master equation (6.18) is of the form of the Pauli equation. The flow of the probability is schematically represented in Fig. 1. It is seen from this figure that the steady state solution is given by

$$2\gamma (n+1)\langle n(\omega) \rangle \rho_{nn} = 2\gamma (n+1)(1 + \langle n(\omega) \rangle) \rho_{n+1, n+1},$$  (6.19)

which leads to the Bose-Einstein distribution for $\rho_{nn}$

$$\rho_{nn} = \langle n(\omega) \rangle^n (1 + \langle n(\omega) \rangle)^{n+1}.$$  (6.20)

The operator form of the master equation is easily obtained by using (6.11)
for $\lambda = -\frac{1}{2}$ and by using the identities (2.28). The reduced density operator satisfies the master equation

$$\frac{\partial \rho}{\partial t} = -i\omega [a^+, a, \rho] - \gamma\{a^+ a \rho - 2a\rho a^+ + \rho a^+ a\} - 2\gamma\langle n(\omega)\rangle [a^+, [a, \rho]].$$

(6.21)

We now present the solution of the relaxation eq. (6.11). It should be noted that (6.11) is of the form of a linearized Fokker-Planck equation. From the solution (A.24) of the linearized Fokker-Planck equation it can be shown that the Green's function $K^{(Q)}(z, z^*, t|z_0, z_0^*, 0)$ associated with eq. (6.11) is given by

$$K^{(Q)}(z, z^*, t|z_0, z_0^*, 0) = \frac{\pi(1-e^{-2\gamma\tau})(\lambda + \frac{1}{2} + \langle n(\omega)\rangle)^{-1}}{\sqrt{\pi(1-e^{-2\gamma\tau})(\lambda + \frac{1}{2} + \langle n(\omega)\rangle)^{-1}}} \times \exp \left[-|z - z_0 e^{-\gamma\tau}| \right].$$

(6.22)

The distribution function $\Phi^{(Q)}$ is then obtained from the relation

$$\Phi^{(Q)}(z, z^*, t) = \int K^{(Q)}(z, z^*, t|z_0, z_0^*, 0) \Phi^{(Q)}(z_0, z_0^*, 0) d^2z_0,$$

(6.23)

where $\Phi^{(Q)}(z_0, z_0^*, 0)$ is the distribution function at time $t = 0$ and $K^{(Q)}$ is given by (6.22). If the initial state of the oscillator is a coherent state then $\Phi^{(Q)}(z, z^*, t)$ is given by (6.22) with $\lambda = -\frac{1}{2}$. In addition if the reservoir is at zero temperature $\langle n(\omega)\rangle \rightarrow 0$, then $\Phi^{(Q)}$ is given by

$$\Phi^{(Q)}(z, z^*, t) = \delta^{(2)}(z - z_0 e^{-i\omega t - \gamma\tau}).$$

(6.24)

The result (6.24) shows that if the oscillator is initially in a coherent state then it remains in a coherent state with exponentially decaying amplitude. A similar result was found for closed systems, characterized by the Hamiltonian of the form (6.1), by Mehta and Sudarshan [1966].
The multi-time correlation functions are easily computed by using (6.22) and (2.27). In particular it can be shown that

\[
\langle a^+(t)a(0) \rangle = \langle a^+(0)a(0) \rangle e^{i\omega t - \gamma t},
\]

(6.25)

\[
\langle a^+(0)a^+(t)a(t)a(0) \rangle = \langle a^+(0)a(0) \rangle \langle n(\omega) \rangle (1 - e^{-2\gamma t}) + \langle (a^+(0))^2(a(0))^2 \rangle e^{-2\gamma t}.
\]

(6.26)

One may also compute a time dependent entropy (Agarwal [1971a]) for the oscillator system. It can be shown that the quantum entropy \( S \) for the oscillator system, if initially it was in a coherent state, is given by

\[
S = K_B \{ (\sigma + 1) \ln (\sigma + 1) - \sigma \ln \sigma \},
\]

(6.27)

where

\[
\sigma = \langle n(\omega) \rangle (1 - e^{-2\gamma t}).
\]

(6.28)

Thus one may introduce the concept of a time dependent temperature \( T(t) \) by (see also Louisell and Walker [1965])

\[
T(t) = \omega \{ K_B \ln (1 + \sigma^{-1}) \}^{-1}.
\]

(6.29)

In fact the reduced density operator \( \rho(t) \) which is obtained from the solution of (6.21) under the initial condition \( \rho(0) = |z_0\rangle \langle z_0| \) is given by

\[
\rho(t) = \frac{\exp \left[ -\beta(t)(a^+ - z_0^* e^{i\omega t - \gamma t})(a - z_0 e^{-i\omega t + \gamma t}) \right]}{\text{Tr} \exp \left[ -\beta(t)(a^+ - z_0^* e^{i\omega t - \gamma t})(a - z_0 e^{-i\omega t + \gamma t}) \right]},
\]

(6.30)

where

\[
\beta(t) \equiv \omega \{ K_B \ T(t) \}^{-1}.
\]

(6.31)

Thus the oscillator system at time \( t \) is found in thermal equilibrium (in terms of displaced coordinates) at a time dependent temperature \( T(t) \) given by eq. (6.29).

We now present the solution of the Pauli equation (6.18). We introduce the generating function \( Q(x, t) \) defined by

\[
Q(x, t) = \sum_0^\infty (1 - x)^n \rho_{nn}(t).
\]

(6.32)

It is easily shown that \( Q(x, t) \) satisfies the following first order differential equation

\[
\frac{\partial Q}{\partial t} = -2\gamma \langle n(\omega) \rangle xQ - 2\gamma [1 + \langle n(\omega) \rangle x] \partial Q / \partial x.
\]

(6.33)

This equation is easily solved by the method of characteristics (Snellson [1957, Chap. II]) and the solution is

\[
Q(x, t) = \{ 1 + \langle n(\omega) \rangle x(1 - e^{-2\gamma t}) \}^{-1} Q_0 [x e^{-2\gamma t} \{ 1 + \langle n(\omega) \rangle x(1 - e^{-2\gamma t}) \}^{-1}],
\]

(6.34)
The time dependence of the diagonal elements $\rho_{n,n}(t)$, for the problem of harmonic oscillator relaxation, for a class of initial distributions

<table>
<thead>
<tr>
<th>Initial distribution</th>
<th>$\rho_{n,n}(0)$</th>
<th>$Q_{0}(x)$</th>
<th>$\rho_{n,n}(t)$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bose-Einstein</td>
<td>$\bar{n}^{n}/(\bar{n} + 1)^{n+1}$</td>
<td>$(1 + \bar{n}x)^{-1}$</td>
<td>$\delta^n/(1 + \delta)^{n+1}$</td>
<td>$\delta \equiv \langle n(\omega) \rangle + (\bar{n} - \langle n(\omega) \rangle)e^{-2\gamma t}$ The distribution remains Bose-Einstein</td>
</tr>
<tr>
<td>Poisson distribution</td>
<td>$\bar{n}^{n}e^{-\bar{n}}/n!$</td>
<td>$e^{-\bar{n}x}$</td>
<td>$\frac{\mu^{n}}{(1 + \mu)^{n+1}} \exp \left(-\bar{n} e^{-2\gamma t}/(1 + \mu)\right)$ \times $L_{n}(-\bar{n} e^{-2\gamma t}/\mu(1 + \mu))$</td>
<td>$\mu \equiv \langle n(\omega) \rangle(1 - e^{-2\gamma t})$ This distribution corresponds to a $\Phi(A)$ which is equal to the convolution of two probability distributions, one corresponding to a coherent state and the other to thermal equilibrium</td>
</tr>
<tr>
<td>Fock state $</td>
<td>m\rangle$</td>
<td>$\delta_{n,m}$</td>
<td>$(1 - x)^{m}$</td>
<td>$(1 - e^{-2\gamma t})m^{m} \left(\frac{e^{\beta \omega} - 1}{e^{\beta \omega} - e^{-2\gamma t}}\right)^{m+1}$ \times $F\left[-n, -m, 1; \left(\frac{\sinh \frac{1}{2} \beta \omega}{\sinh \gamma t}\right)^{2}\right]$</td>
</tr>
</tbody>
</table>
where $Q_0(x)$ is the generating function at time $t = 0$. The time dependence of the diagonal elements of $\rho$ is then calculated from

$$\rho_{nn}(t) = \left. \frac{(-1)^n}{n!} \frac{\partial^n Q(x, t)}{\partial x^n} \right|_{x = 1}.$$ (6.35)

In Table 2, we give the values of $\rho_{nn}(t)$ for several important initial distributions. At zero temperature ($\langle n(\omega) \rangle = 0$), eqs. (6.34) and (6.35) lead to the following result for the time dependence of $\rho_{nn}(t)$

$$\rho_{nn}(t) = \sum_{m \geq n} \binom{m}{n} e^{-2\gamma t}(1 - e^{-2\gamma t})^{m-n} \rho_{mm}(0).$$ (6.36)

We next consider the effect of a weak external field on the relaxation of the oscillator. The Hamiltonian of our system in the presence of an external field is given by

$$H = \omega a^+ a + \sum_k \omega_k a_k^+ a_k + \sum_k (g_k a_k^+ a + \text{H.C.}) + g_e [e^*(t)a + \text{H.C.}],$$ (6.37)

where $e(t)$ is a $c$-number driving field. The reduced phase space distribution function satisfies the master equation

$$\partial_t \Phi^{(\alpha)} = (i\omega + \gamma) \frac{\partial}{\partial z} (z\Phi^{(\alpha)}) + i e(t) g_e \frac{\partial}{\partial z} \Phi^{(\alpha)} + \gamma(\lambda + \frac{1}{2} + \langle n(\omega) \rangle) \frac{\partial^2 \Phi^{(\alpha)}}{\partial z \partial z^*} + \text{C.C.}$$ (6.38)

The Fokker-Planck equation (6.38) is stochastically equivalent to the Langevin equations

$$\dot{z} = -i\omega z - \gamma z - ie(t) + F(t), \quad \dot{z}^* = i\omega z^* - \gamma z^* + ie^*(t) + F^*(t),$$ (6.39)

where $F(t)$ is a delta correlated complex Gaussian random process with properties given by eq. (6.13) and where we have set $g_e = 1$. Eq. (6.38) has the form of a linearized Fokker-Planck equation and the Green's function associated with (6.38) is, therefore, given by

$$K^{(\alpha)}(z, z^*, t|z_0, z_0^*, 0) = \frac{\exp \{-|z - \langle z(t) \rangle|^2/((\lambda + \frac{1}{2} + \langle n(\omega) \rangle)(1 - e^{-2\gamma t}))\}}{\pi(\lambda + \frac{1}{2} + \langle n(\omega) \rangle)(1 - e^{-2\gamma t})},$$ (6.40)

where

$$\langle z(t) \rangle = z_0 e^{-i\omega t - \gamma t} - i \int_0^t e(t - \tau) e^{-i\omega t - \gamma t} d\tau.$$ (6.41)

On comparing (6.22) and (6.40), we find that the effect of the external field is to displace the center of the Gaussian distribution. The other
dynamical properties may be calculated from (6.40) and we will not discuss them here (they are considered in LOUISELL and MARBURGER [1967]).

SCHWINGER [1961] has discussed the problem of the relaxation of the oscillator from a different standpoint. He considers the augmented Hamiltonian

$$ H = H_0 + K^*(t)a + K(t)a^+, $$

(6.42)

where $H_0$ is given by (6.1) and $K(t)$ is an external force. He introduced the functional (see also SCULLY and WHITNEY [1972])

$$ Z = \text{Tr} \{ U_+^+(t, t_0)U_+(t, t_0)\rho(t_0) \}, $$

(6.43)

where

$$ U_+(t, t_0) = T \exp \left\{ -i \int_{t_0}^{t} (K^*_+(\tau)a(\tau) + K_+(\tau)a^+(\tau)) d\tau \right\}, $$

(6.44)

$$ U_-^+(t, t_0) = \bar{T} \exp \left\{ i \int_{t_0}^{t} (K^*_-(\tau)a(\tau) + K_-(\tau)a^+(\tau)) d\tau \right\}, $$

(6.45)

and $T$ and $\bar{T}$ are the operators for the chronological and the antichronological orderings respectively. The operators $a(\tau), a^+(\tau)$ are in the interaction picture with respect to $H_0$. The variation of the functional $Z$ is given by

$$ \delta \ln Z = -i \int_{t_0}^{t} \left\{ \mathcal{A}^*_+(\tau)\delta K_+(\tau) + \mathcal{A}_+(\tau)\delta K^*_+(\tau) - \mathcal{A}_-(\tau)\delta K_-(\tau) - \mathcal{A}^*_-(\tau)\delta K^*_-(\tau) \right\} d\tau, $$

(6.46)

where

$$ \mathcal{A}_+(\tau) \equiv i \frac{\delta \ln Z}{\delta K_+(\tau)}, \quad \mathcal{A}^*_+(\tau) \equiv i \frac{\delta \ln Z}{\delta K^*_+(\tau)}, $$

(6.47)

$$ \mathcal{A}_-(\tau) \equiv -i \frac{\delta \ln Z}{\delta K_-(\tau)}, \quad \mathcal{A}^*_-(\tau) \equiv -i \frac{\delta \ln Z}{\delta K^*_-(\tau)}. $$

(6.48)

Thus the functional $\mathcal{A}^*_+(t)$ is given by

$$ \mathcal{A}^*_+(t) = \text{Tr} \left\{ U_+^+(t, t_0) T \right\} \times \left[ a^+(t) \exp \left\{ -i \int_{t_0}^{t} (K^*_+(\tau)a(\tau) + K_+(\tau)a^+(\tau)) d\tau \right\} \right] \rho(t_0). $$

(6.49)

The functionals $\mathcal{A}^*_-(\tau), \mathcal{A}_-(\tau), \mathcal{A}_+(\tau)$ are given by expressions similar to (6.49). The functional $Z$ is then calculated by using the equations of motion for the functional $\mathcal{A}_+(\tau), \mathcal{A}^*_+(\tau)$. The correlation functions are obtained from $Z$ by the functional differentiation of $Z$. (The details can be found in a recent review article by SCULLY and WHITNEY [1972].)
The functional $Z$ is closely related to the phase space distribution functions. For example when $K_+(t) = K^*(t) = 0$ then (6.43) reduces to

$$Z = \left\langle T \left[ \exp \left\{ i \int_{t_0}^t K_-(\tau) a(\tau) d\tau \right\} \right] T \left[ \exp \left\{ i \int_{t_0}^t K_+(\tau) a(\tau) d\tau \right\} \right] \right\rangle,$$

(6.50)

which is seen to be the generating functional for the normally-ordered time-ordered correlation functions of the form (2.26). Moreover, in the special case when

$$K_-(\tau) = -i\alpha_0 \delta(\tau - t), \quad K^*_+(\tau) = i\alpha^* \delta(\tau - t),$$

(6.51)

(6.50) reduces to

$$Z = \langle \exp (\alpha a^+) \exp (-\alpha^* a) \rangle.$$

(6.52)

and the Fourier transform of (6.52) is just the Sudarshan-Glauber distribution function.

The model which we have discussed in this section is of great value in quantum optics. It is used to describe the damping of the field mode. Consider a nonlinear process in which several field modes are interacting with each other and let this interaction be described by the Hamiltonian $H$. The equation of motion for the Sudarshan-Glauber distribution function in the absence of losses is of the form (2.23). The losses are taken into account by assuming that each field mode is interacting with its own reservoir characterized by the parameters $\gamma_i$ and $\langle n_i \rangle$. Then the equation of motion for $\Phi^{(A)}$, when losses are taken into account, is of the form

$$\frac{\partial \Phi^{(A)}}{\partial t} = -i \mathcal{L}^{(A)} \Phi^{(A)} + \sum_i \left\{ (i\omega_i + \gamma_i) \frac{\partial}{\partial z_i} (z_i \Phi^{(A)}) + \gamma_i \langle n_i \rangle \frac{\partial^2 \Phi^{(A)}}{\partial z_i \partial z_i^*} + C.C. \right\}.$$

(6.53)

We apply this procedure to obtain the Fokker-Planck equation for the parametric oscillator. The Hamiltonian for this system, in the interaction picture, is given by

$$H = (i\chi a^+_1 a^+_2 a_3 + \text{H.C.}) - i(F^*_p(t)a_3 - \text{H.C.}),$$

(6.54)

where we have ignored the effects of detuning. The parameter $\chi$ is related to the nonlinear susceptibility of the medium (for the form of $\chi$ see Graham [1968]). In (6.54) $F_p(t)$ is the external pump field. On using (6.54) and (2.24) generalized to the case of many degrees of freedom (see Table 1) $\mathcal{L}^{(A)}$ is easily calculated. On using this value of $\mathcal{L}^{(A)}$ and (6.53), we obtain the Fokker-Planck equation.
\[
\frac{\partial \Phi^{(A)}}{\partial t} = \left\{ \frac{\partial}{\partial z_1} (\gamma_1 z_1 - \chi z_2^* z_3) + \frac{\partial}{\partial z_2} (\gamma_2 z_2 - \chi z_1^* z_3) \right. \\
+ \left. \frac{\partial}{\partial z_3} (\gamma_3 z_3 - F_p(t) + \chi z_1 z_2) + \chi \frac{\partial^2}{\partial z_1 \partial z_2} z_3 + \sum_{i=1}^{3} \gamma_i \langle n_i \rangle \frac{\partial^2}{\partial z_i \partial z_i^*} + \text{C.C.} \right\} \Phi^{(A)}.
\]

(6.55) is the basic equation of Graham [1968]. Below and above threshold, this equation can be solved by linearization and by quasilinearization procedures respectively.

\section*{§ 7. Brownian Motion of a Quantum Oscillator}

In this section we treat the problem of the Brownian motion of a quantum oscillator. Our approach is based on the master equation satisfied by the distribution function corresponding to the oscillator system alone. The treatment presented here follows closely the one given by Agarwal [1971d] (for various other approaches see Schwinger [1961], Ullersma [1966a, b]). The Brownian motion of a classical oscillator is discussed in great detail in the papers of Kramers [1940], Chandrasekhar [1943] and Wang and Uhlenbeck [1945].

The displacement \( q(t) \) of the oscillator satisfies the equation of motion

\[
d\frac{d^2 q}{dt^2} + 2\gamma \frac{dq}{dt} + \omega^2 q = F(t)/m,
\]

where \( 2\gamma \) is the phenomenological damping coefficient, \( \omega \) is the natural frequency of oscillation and \( m \) is the mass of the particle. \( F(t) \) is a random force which is assumed to be a delta correlated real Gaussian process with zero mean, i.e.

\[
\langle F(t) \rangle = 0, \quad \langle F(t)F(t') \rangle = 2D\delta(t-t'),
\]

where \( D \) is the diffusion coefficient. Eq. (7.1) is equivalent to the following two first order differential equations

\[
\dot{q} = p/m, \quad \dot{p} = -2\gamma p - m\omega^2 q + F(t),
\]

where \( p \) is the momentum of the particle. Equations (7.3) are the Langevin equations describing the Brownian motion of a classical oscillator and the corresponding random process is a two dimensional Gaussian Markov process (cf. Appendix and also Stratonovich [1963]).

We now discuss the Brownian motion of a quantum oscillator. The Brownian motion is described by a model Hamiltonian which is taken to be the one describing the interaction between the oscillator and the reservoir at temperature \( T \). This Hamiltonian is given by
$H = \omega a^+ a + \sum_j \omega_j a_j^+ a_j + \sum_j \{g_j(a + a^+)a_j^+ + \text{H.C.}\}$, \hspace{1cm} (7.4)

where contrary to the case discussed in § 6, we have retained the terms of the form $aa_j$ and $a^+ a_j^+$. We use the Hamiltonian (7.4) and make approximations similar to the ones made in § 6. We then find that $\Phi^{(\omega)}(z, z^*; t)$, which is the reduced phase space distribution function, satisfies the master equation (Agarwal [1971d])

$$
\frac{\partial \Phi^{(\omega)}}{\partial t} = \left\{ i\omega \frac{\partial}{\partial z} z + \gamma \frac{\partial}{\partial z} (z - z^*) - \frac{1}{2} \gamma (\langle n(\omega) \rangle + \lambda + \frac{1}{2}) \right. \times \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial z^*} \right)^2 + \text{C.C.} \right\} \Phi^{(\omega)}.
$$

(7.5)

The parameters $\gamma$ and $\langle n(\omega) \rangle$ are defined by (6.9) and (6.6) respectively. We also recall that the parameter $\lambda$ is $-\frac{1}{2}$ for the Sudarshan-Glauber distribution function, $\frac{1}{2}$ for the distribution function obtained with the normal rule of mapping and is zero for the Wigner distribution function. We make the transformation to the real variables $q$ and $p$ defined by

$$z = \left( \frac{m\omega}{2} \right)^{\frac{1}{4}} q + i \left( \frac{1}{2m\omega} \right)^{\frac{1}{4}} p, \hspace{1cm} z^* = \left( \frac{m\omega}{2} \right)^{\frac{1}{4}} q - i \left( \frac{1}{2m\omega} \right)^{\frac{1}{4}} p.
$$

(7.6)

Then the Fokker-Planck equation (7.5) transforms into the equation

$$
\frac{\partial \Phi^{(\omega)}}{\partial t} = \left\{ - \frac{\partial}{\partial q} \left( p/m \right) + \frac{\partial}{\partial p} \left( m\omega^2 q + 2\gamma p \right) + 2m\omega\gamma (\langle n(\omega) \rangle + \lambda + \frac{1}{2}) \frac{\partial^2}{\partial p^2} \right\} \Phi^{(\omega)}.
$$

(7.7)

The Fokker-Planck equation (7.7) is stochastically equivalent to the Langevin equations (cf. Appendix)

$$
\dot{q} = \frac{p}{m}, \hspace{1cm} \dot{p} = -2\gamma p - m\omega^2 q + F(t),
$$

(7.8)

where $F(t)$ is a real Gaussian random process with the properties

$$
\langle F(t) \rangle = 0, \hspace{1cm} \langle F(t)F(t') \rangle = 2D\delta(t - t').
$$

(7.9)

Here the diffusion coefficient $D$ is given by

$$D = 2m\omega\gamma (\langle n(\omega) \rangle + \lambda + \frac{1}{2}).
$$

(7.10)

It should be noted that the Langevin equations (7.8) are of the same form as the equations (7.3) describing the Brownian motion of a classical oscillator. It should be borne in mind that in the classical case $q$ and $p$ are the position and momentum variables whereas in the quantum case $q$ and $p$
are the c-number variables onto which the operators $q$ and $p$ are mapped by the $\Omega$-rule of mapping. Since we are dealing with a quantum system certain care should be exercised in the proper interpretation of moments calculated from (7.8). It is convenient to transform to the complex variables $z$ and $z^*$. We then obtain the following Langevin equations (which are stochastically equivalent to the Fokker-Planck equation (7.5))

$$
\dot{z} = -i\omega z - \gamma(z - z^*) + \mathcal{F}(t), \quad \dot{z}^* = i\omega z^* - \gamma(z^* - z) + \mathcal{F}^*(t),
$$

(7.11)

where $\mathcal{F}(t)$ is a complex Gaussian random process with the properties

$$
\langle \mathcal{F}(t) \rangle = \langle \mathcal{F}^*(t) \rangle = 0, \quad \langle \mathcal{F}(t) \mathcal{F}^*(t') \rangle = -\langle \mathcal{F}(t) \mathcal{F}(t') \rangle = 2\gamma \langle n(\omega) \rangle + \lambda + \frac{1}{2} \delta(t - t').
$$

(7.12)

As remarked earlier in § 6, the moments calculated from (7.11) correspond to the normally ordered moments $\langle a^+ a^m \rangle$ for $\lambda = -\frac{1}{2}$, to antinormally ordered moments $\langle a^a a^+ a^m \rangle$ for $\lambda = +\frac{1}{2}$ and to Weyl ordered moments $\langle (a^+ a^m)w \rangle$ for $\lambda = 0$. It should also be noted that the diffusion coefficient $D$, given by (7.10), in the high temperature limit goes over to the classical value, viz.

$$
D_{ct} = 2\gamma m K_B T.
$$

(7.13)

The Green's function $K^{(\Omega)}(z, z^*, t|z_0, z_0^*, 0)$ corresponding to the Fokker-Planck equation (7.5), which is in the form of a linearized Fokker-Planck equation, is easily seen to be given by

$$
K^{(\Omega)}(z, z^*, t|z_0, z_0^*, 0) = (\pi^2 A_0)^{-\frac{1}{4}}
\times \exp \{ A_0^{-1} \left[ \mu(z^* - \langle z^* \rangle)^2 + \mu^*(z - \langle z \rangle)^2 - \tau|z - \langle z \rangle|^2 \right] \},
$$

(7.14)

where

$$
\langle z(t) \rangle = \left( \cos \omega_0 t - \frac{i\omega}{\omega_0} \sin \omega_0 t \right) z_0 e^{-\gamma t} + \frac{\gamma}{\omega_0} \sin \omega_0 t z_0^* e^{-\gamma t},
$$

(7.15)

$$
\mu = -\left( \frac{\gamma}{\omega_0} \right) \left( \langle n(\omega) \rangle + \lambda + \frac{1}{2} \right) \sin \omega_0 t \left( \cos \omega_0 t - \frac{i\omega}{\omega_0} \sin \omega_0 t \right) e^{-2\gamma t},
$$

(7.16)

$$
\tau = \left( \langle n(\omega) \rangle + \lambda + \frac{1}{2} \right) \left[ 1 - \left( 1 + \frac{2\gamma^2}{\omega_0^2} \sin^2 \omega_0 t \right) e^{-2\gamma t} \right],
$$

(7.17)

and

$$
\omega_0 = (\omega^2 - \gamma^2)^{\frac{1}{4}}, \quad A_0 = (\tau^2 - 4\mu \mu^*).
$$

(7.18)

The Green's function (7.14) is in the form of complex Gaussian distribution which is centered at $z = \langle z(t) \rangle$. The steady state solution of (7.7) is given by
\[ \Phi_{st}^{(\omega)} = \{ \pi(\langle n(\omega) \rangle + \lambda + \frac{1}{2}) \}^{-1} \exp \{ -|z|^2(\langle n(\omega) \rangle + \lambda + \frac{1}{2})^{-1} \}, \] (7.19)

which corresponds to an oscillator in thermal equilibrium at temperature \( T \).

We conclude this section by giving the operator form of the master equation (7.5). This is easily obtained from (7.5) by the substitution \( \lambda = -\frac{1}{2} \) and by using the identities (2.28). The result is

\[ \frac{\partial \rho}{\partial t} = -i\omega[a^+a, \rho] - \gamma(a^+a \rho - 2a \rho a^+ + \rho a^+a + a^2 \rho - a \rho a - a^+ \rho a^+ - \rho a^+^2) \]

\[ -\gamma\langle n(\omega) \rangle(2[a^+, [a, \rho]] + [a^+, [a^+, \rho]] + [a, [a, \rho]]). \] (7.20)

### § 8. Relaxation of an Atom

We next consider the relaxation of an atom with unperturbed energy eigenvalues \( E_i \) and eigenfunctions \( |i\rangle \). We write the Hamiltonian of the atom interacting with the reservoir in the form

\[ H = \sum_i E_i A_i + H_R + \sum_{kl} v_{kl} A_{kl}, \] (8.1)

where the operators \( A_{kl} \) are given by

\[ A_{kl} = |k\rangle \langle l|. \] (8.2)

The \( v_{kl} \)'s in eq. (8.1) are the functions of the reservoir operators. We leave \( v_{kl} \) arbitrary. The Liouville operator in the interaction picture is given by the expression

\[ \mathcal{L}_{RS}(t) = \sum_{kl} v_{kl}(t) A_{kl} \exp (i\omega_{kl}t), \] (8.3)

where

\[ \omega_{kl} = (E_k - E_l), \quad v_{kl}(t) = \exp \{iH_R t\} v_{kl} \exp \{-iH_R t\}. \] (8.4)

We assume that the condition of the initial random phase and the condition (5.19) are satisfied. Then on combining (8.3) and (5.20), we find that the reduced density operator corresponding to the atomic system alone satisfies the master equation

\[ \frac{\partial \rho}{\partial t} = \sum_{klmn} \{(A_{mn} \rho A_{kl} - A_{kn} \rho \delta_{lm}) \gamma_{klmn}^+ + (A_{kl} \rho A_{mn} - \rho A_{ml} \delta_{nk}) \gamma_{mnkl}^- \} \exp \{i(\omega_{kl} + \omega_{mn})t\}, \] (8.5)

where we have, for the sake of brevity, suppressed the subscript \( S \) from \( \rho \). The density operator \( \rho \), in (8.5), is in the interaction picture. \( \gamma_{klmn}^+ \) and \( \gamma_{mnkl}^- \) are given by (LOUISELL [1969], LAX [1966c])
\[
\gamma^+_{klnm} = \int_0^\infty \langle v_{kl}(t)v_{mn}(0) \rangle \exp \{-i\omega_{mn}t\} dt, \quad (8.6)
\]

\[
\gamma^-_{mnkl} = \int_0^\infty \langle v_{mn}(0)v_{kl}(t) \rangle \exp \{-i\omega_{mn}t\} dt \quad (8.7)
\]

\[
= (\gamma^+_{lknm})^*. \quad (8.8)
\]

In eqs. (8.6) and (8.7) \(\langle v_{kl}(t)v_{mn}(0) \rangle\) and \(\langle v_{mn}(0)v_{kl}(t) \rangle\) are the reservoir correlation functions. In deriving (8.5) we also made the Markovian approximation which is justified for times \(t \gg \tau_c\) where \(\tau_c\) is the reservoir correlation time which is assumed to be very short. We also make the rotating wave approximation, i.e. we drop the rapidly oscillating terms in (8.5). Eq. (8.5) then reduces to

\[
\frac{\partial \rho}{\partial t} = \sum_{klmn} \{ (A_{mn} \rho A_{kl} - A_{kn} \rho \delta_{lm}) \gamma^+_{klnm} + (A_{kl} \rho A_{mn} - \rho A_{ml} \delta_{nk}) \gamma^-_{mnkl} \}, \quad (8.9)
\]

where summation in (8.9) is over those values of \(k, l, m, n\) which satisfy the relation

\[
\omega_{mn} + \omega_{kl} = 0, \quad \text{i.e.} \quad E_m - E_n + E_k - E_l = 0. \quad (8.10)
\]

We assume that the energy levels of the atom are non-degenerate and are unevenly spaced (the case of evenly spaced energy levels is treated in § 9). Then (8.10) will be satisfied in the following three cases

(i) \(k = n, m = l\);  (ii) \(k = l, m = n\);  (iii) \(k = l = m = n\). \quad (8.11)

On grouping the terms in eq. (8.9) according to (8.11), we find that (8.9) reduces to

\[
\frac{\partial \rho}{\partial t} = \sum_{k \neq l} \gamma_{lk} A_{lk} \rho A_{kl} + \sum_{kl} (\gamma^+_{lk} + \gamma^-_{lk}) A_{kk} \rho A_{kl} - \sum_{kl} \{ \gamma^+_{kl} A_{kk} \rho + \gamma^-_{kl} \rho A_{kk} \}, \quad (8.12)
\]

where we have set

\[
\gamma_{lk} = \gamma^+_{kl} + \gamma^-_{kl} \quad (8.13)
\]

\[
= \int_{-\infty}^{+\infty} \langle v_{kl}(t)v_{lk}(0) \rangle \exp \{-i\omega_{lk}t\} dt, \quad (8.14)
\]

where the last relation follows on combining eqs. (8.6), (8.7) and (8.13). Eq. (8.12) is the desired master equation for the reduced density operator \(\rho\) (in the interaction picture) describing the relaxation of the atom. On taking the matrix elements of both sides of (8.12), we obtain the equation
\[
\frac{\partial \rho_{ij}}{\partial t} = \delta_{ij} \sum_{k \neq i} \gamma_{ik} \rho_{kk} - \Gamma^c_{ji} \rho_{ij},
\]
(8.15)

where \( \Gamma^c_{ij} \) is defined by
\[
\Gamma^c_{ij} = -(\gamma^+_{ij} + \gamma^-_{ij}) + \sum_i (\gamma^+_{ji} + \gamma^-_{ii}).
\]
(8.16)

It is easily seen from (8.14) that \( \gamma_{ik} \) is the transition probability per unit time that the atom makes a transition from the state \(|k\rangle\) to the state \(|l\rangle\).

For a reservoir in thermal equilibrium at temperature \( T \), one can further show that
\[
\gamma_{kl} = \gamma_{lk} \exp(\beta \omega_{lk}).
\]
(8.17)

Let us denote by \( \Gamma_{ij} \) the real part of \( \Gamma^c_{ij} \). Then it is easy to show (LOUISELL [1969]) that \( \Gamma_{ii} \) is the transition probability that the reservoir causes the atom to make the transition from a state \(|i\rangle\) to all other states, i.e.
\[
\Gamma_{ii} = \sum_{k \neq i} \gamma_{ki}.
\]
(8.18)

Moreover from the definition (8.16), the following relation is easily established
\[
\Gamma_{ij} = \frac{1}{2}(\Gamma_{ii} + \Gamma_{jj}) + \Gamma_{ij}^{ph},
\]
(8.19)

where \( \Gamma_{ij}^{ph} \) is given by (LOUISELL [1969])
\[
\Gamma_{ij}^{ph} = \pi \sum_{E_R} \langle E_R | v_{ii} - v_{jj} | E_R \rangle^2 \rho_R(E_R),
\]
(8.20)

and where \( E_R \) and \( |E_R\rangle \) are the energy eigenvalues and the energy eigenfunctions of \( H_R \) and the \( \rho_R(0) \) is assumed to be a function of \( H_R \) only.

It should be noted that \( \Gamma_{ij}^{ph} \) appears only in the equations of motion for the off-diagonal elements and this term describes the damping by virtual processes. On putting \( i = j \) in (8.15), we obtain the Pauli type equation
\[
\frac{\partial \rho_{ii}}{\partial t} = \sum_{k \neq i} (\gamma_{ik} \rho_{kk} - \gamma_{ki} \rho_{ii}),
\]
(8.21)

where use has been made of the relation (8.18).

So far our discussion for the relaxation of an atom has been quite general. We now specialize to the case of a two-level atom. Let \(|1\rangle\) and \(|2\rangle\) be the ground and the excited states of the atom respectively. The operators \( A_{kl} \) are then related to the spin angular momentum operators for spin \( \frac{1}{2} \)-value (cf. FEYNMAN, VERNON and HELFWARTH [1957]) by the equation
\[
A_{21} \equiv |2\rangle \langle 1| = S^+, \quad A_{12} \equiv |1\rangle \langle 2| = S^-, \quad \frac{1}{2}(A_{22} - A_{11}) = S^z.
\]
(8.22)
The relaxation equation (8.12) in case of a two-level atomic system can be written in the form (see also Risken [1970])

\[
\frac{d\rho}{dt} = \frac{1}{2} \gamma_{12} (2S^- \rho S^+ - \rho S^+ S^- - S^+ S^- \rho) + \frac{1}{2} \gamma_{21} (2S^+ \rho S^- - \rho S^- S^+ - S^- S^+ \rho) + \Gamma_{12}^{ph} (2S^+ \rho S^- - \rho S^- S^+ - S^- S^+ \rho).
\]  
(8.23)

Equation (8.23) leads to the following equations for the macroscopic mean values

\[
\frac{\partial}{\partial t} \langle S^+ \rangle = i\omega \langle S^+ \rangle - \langle S^+ \rangle / T_2, \tag{8.24}
\]

\[
\frac{\partial}{\partial t} \langle S^- \rangle = -(\langle S^\downarrow \rangle - \langle S^\downarrow \rangle) / T_1, \tag{8.25}
\]

where

\[
\omega = E_2 - E_1, \tag{8.26}
\]

and \(T_1\) and \(T_2\) are the longitudinal and transverse relaxation times defined by

\[
T_1^{-1} = (\gamma_{12} + \gamma_{21}), \quad T_2^{-1} = (\frac{1}{2} \gamma_{12} + \frac{1}{2} \gamma_{21} + \Gamma_{12}^{ph}). \tag{8.27}
\]

In eq. (8.25) \(\langle S^- \rangle_{st}\) is the steady state value of \(\langle S^- \rangle\) and is given by

\[
\langle S^- \rangle_{st} = -\frac{1}{2} (\gamma_{12} - \gamma_{21}) / (\gamma_{12} + \gamma_{21}), \tag{8.28}
\]

which in case of thermal reservoir reduces to

\[
\langle S^- \rangle_{st} = -\frac{1}{2} \tanh (\frac{1}{2} \beta \omega), \tag{8.29}
\]

where use of (8.17) has also been made. Equations (8.24) and (8.25) are recognized to be the familiar Bloch equations which predict the correct steady state behavior, i.e. relaxation at a finite temperature \(T\) rather than at infinite temperature which was found in the semiclassical treatment given in § 4.

Finally we mention that if the reservoir is made up of harmonic oscillators and is initially in a state of thermal equilibrium, then one finds the following expression for the transition probabilities (Agarwal [1969], Louisell [1969])

\[
\gamma_{12} = 2\pi h(\omega) |g(\omega)|^2 (\langle n(\omega) \rangle + 1), \tag{8.30}
\]

\[
\gamma_{21} = 2\pi h(\omega) |g(\omega)|^2 \langle n(\omega) \rangle, \tag{8.31}
\]

and \(\Gamma_{12}^{ph}\) is zero since the atom is assumed to have no permanent dipole moment. \(\langle n(\omega) \rangle\) is given by eq. (6.6) and \(g(\omega), h(\omega)\) have the same meaning as in § 6.
We have considered here the relaxation of an atom whose energy levels are unevenly spaced and are non-degenerate*. For the case of a two-level atom, the problem was equivalent to the relaxation of a spin $-\frac{1}{2}$ system. In the next section we consider the problem of relaxation of $N$ two-level atoms. The problem of relaxation of general spin systems is treated in papers of BLOCH [1956, 1957], REDFIELD [1957, 1965] and in the book of ABRAGAM [1961].

§ 9. Incoherent and Coherent (Superradiance) Spontaneous Emission

In this section we consider the spontaneous emission from a collection of $N$ identical two-level atoms, using the master equation techniques. This system was originally studied by DICKE [1954], who found that under certain conditions the radiation rate is proportional to the square of the number of atoms. This coherent emission of radiation is known as superradiance and has recently been the subject of many investigations (DILLARD and ROBL [1969], LEHMBERG [1970a], DIALETIS [1970], AGARWAL [1970, 1971b, c, e], REHLER and EBERLY [1971], BONIFACIO, SCHWENDIMAN and HAAKE [1971a, b], WILLIS and PICARD [1973]).

The Hamiltonian for a collection of $N$ identical two-level atoms interacting with a quantized radiation field can be shown to be given by (see e.g. HAKEN [1970])

$$H = \omega \sum_j S_j^z + \sum_{ks} \omega_{ks} a^*_k a_k + \sum_{ks} \sum_j (S_j^+ a_k g_{jks} + \text{H.C.}).$$

(9.1)

In deriving (9.1), the nonresonant terms have been ignored, i.e. we have made the rotating wave approximation. In eq. (9.1) $a_k$ and $a^*_k$ are the annihilation and the creation operators associated with the $ks$ mode of the radiation field and $S_j^\pm, S_j^z$ are the components of the spin angular momentum operator (corresponding to spin $-\frac{1}{2}$ value) associated with the $j$th atom (cf. eq. (8.22); FEFYMAN, VERNON and HELLWARTH [1957]). $g_{jks}$ is the coupling constant and is given by

$$g_{jks} = (-\frac{i\omega}{c}(2\pi e/L^3)^{\frac{1}{2}} k^{-\frac{1}{2}}(e_{ks} \cdot d) \exp (ik \cdot R_j),$$

(9.2)

where $R_j$ is the position vector of the $j$th atom and $d$ is the dipole moment matrix element. All other symbols have the usual meaning.

Since we are studying spontaneous emission, the initial state of the system is given by

$$\rho_{S+R}(0) = \rho_S(0)|\{0\}\rangle\langle\{0\}|,$$

(9.3)

* The equation (8.15) at zero temperature and its generalizations describe spontaneous emission from multi-level atoms, AGARWAL [1973a].
where \(|\{0\}\rangle\) is the vacuum state of the radiation field and \(\rho_s(0)\) is the initial state of the atomic system which we leave arbitrary. It is clear that the radiation field in the vacuum state interacting with atoms behaves as a reservoir at zero temperature interacting with the atomic system. The problem of spontaneous emission thus reduces to that of relaxation at zero temperature. We can now use the master equation techniques of § 5 to obtain the equation of motion for the reduced density operator corresponding to the atomic system alone. It is found that, in the Born approximation and in the Markovian approximation, the reduced density operator satisfies the master equation (Agarwal [1970])

\[
\frac{\partial \rho}{\partial t} = -i \sum_{ij} A_{ij} [S_i^+ S_j^-, \rho] - \sum_{ij} \gamma_{ij} \{S_i^+ S_j^- \rho - 2 S_j^\rho S_i^+ + \rho S_i^+ S_j^\},
\]

(9.4)

where *

\[
\gamma_{ij} = \left( \frac{2\pi^2 \omega^2}{c} \right) \left( \frac{1}{2\pi} \right)^3 \int d^3k \exp \{i \mathbf{k} \cdot (\mathbf{R}_i - \mathbf{R}_j)\} \left( |d|^2 - \frac{|d \cdot k|^2}{|k|^2} \right) \delta(c \omega - \omega) - \frac{k}{k},
\]

(9.5)

\[
A_{ij} = \left( \frac{2\pi \omega}{c} \right) \left( \frac{1}{2\pi} \right)^3 \times P \int d^3k \exp \{i \mathbf{k} \cdot (\mathbf{R}_i - \mathbf{R}_j)\} \left( |d|^2 - \frac{|d \cdot k|^2}{k^2} \right) k^{-1}(\omega - k \omega)^{-1},
\]

(9.6)

and where \(P\) denotes the Cauchy principal part. In (9.4) \(\rho\) refers to the reduced density operator in the interaction picture and we have dropped the subscript \(S\) on \(\rho_s\). The master equation (9.4) is valid both for small and for large systems. By a small system we mean a system whose linear dimensions are small compared to a wavelength. In what follows we consider mainly small systems. For such systems \(\gamma_{ij} \rightarrow \gamma\), independent of the indices \(i\) and \(j\). On evaluation \(\gamma\) is found to be

\[
\gamma = (2\omega^3 |d|^2)/(3c^3),
\]

(9.7)

and the coefficient \(A_{ij}\) is related to the frequency shifts; its value may be obtained by renormalization. The various results on coherent and incoherent spontaneous emission follow from the master equation (9.4). In what follows, we will also ignore the effect of the frequency shift terms involving \(A_{ij}\).

For one two-level atom, eq. (9.4) is easily solved and one finds that

\[
\langle S^+ (t) \rangle = \langle S^+ (0) \rangle e^{i \omega t - \eta}, \quad \langle S^\tau (t) \rangle = -\frac{1}{2} + \langle S^\tau (0) \rangle + \frac{1}{2} e^{-2\eta}. \quad (9.8)
\]

* The correct value of \(A_{ij}\) is obtained by making rotating wave approximation on the master equation rather than the Hamiltonian itself (Agarwal [1973a]).
It should be noted that the results (9.8) correspond to the well known `exponential decay' (Weisskopf and Wigner [1930]). $2\gamma$ is equal to the inverse life time of a single atom. The density operator $\rho$ is given by

$$\rho = \frac{1}{2} + 2\langle S^z \rangle S^z + \langle S^+ \rangle S^- + \langle S^- \rangle S^+.$$  \hfill (9.9)

It is easily deduced from (9.8) and (9.9) that $\rho^2 \neq \rho$, i.e. because of spontaneous emission the atom is left in a mixed state unless $t = \infty$.

For the case of two atoms the master equation (9.4) can be easily solved. Here we only give the time dependence of the total energy $W(t) = \sum_{i=1}^{N} \langle S_i^z \rangle$ (in units of $\omega$)

$$W(t) = (W(0) + \frac{1}{4} + \langle S_1 \cdot S_2 \rangle_0 + 4\gamma t \langle S_1^+ S_1^- S_2^+ S_2^- \rangle_0) e^{-4\gamma t} - (\frac{1}{4} + \langle S_1 \cdot S_2 \rangle_0),$$ \hfill (9.10)

where $\langle \cdot \rangle_0$ refers to the mean value at time $t = 0$. It is seen from (9.10) that

$$W(t) \xrightarrow{t \to \infty} - (\frac{1}{4} + \langle S_1 \cdot S_2 \rangle_0),$$ \hfill (9.11)

which shows that the atoms are left in a state which is determined from the initial state. If the atoms are initially excited to a permutationally symmetric state $\langle S_1 \cdot S_2 \rangle_0 = \frac{1}{4}$, then only each atom will be found in its ground state.

We now consider the general problem of $N$ two-level atoms. It is easily seen from (9.4) that the mean energy $\langle S_i^z \rangle$ of the $i$th atom obeys the equation

$$\frac{\partial}{\partial t} \langle S_i^z \rangle + \gamma \sum_j (\langle S_i^+ S_j^- \rangle + \text{C.C.}) = 0.$$ \hfill (9.12)

The radiation rate, which we denote by $I(t)$, is defined as the time rate change of the total energy of the system. Evidently

$$I(t) = -\omega \sum_i \frac{\partial}{\partial t} \langle S_i^z \rangle = 2\omega \sum_{ij} \langle S_i^+ S_j^- \rangle,$$ \hfill (9.13)

where the correlation $\langle S_i^+ S_j^- \rangle$ is to be obtained from the solution of the master equation (9.4). A perturbation theoretic result is obtained by replacing $\rho(t)$ in (9.13) by $\rho(0)$ and is given by

$$I_0 = 2\gamma \omega \sum_{ij} \text{tr} (\rho(0) S_i^+ S_j^-).$$ \hfill (9.14)

We now assume that the system was initially excited to a state of the form

$$\rho(0) = \prod_i | \theta_0, \varphi_0 \rangle \langle \theta_0, \varphi_0 |,$$ \hfill (9.15)

where
\[ |\theta_0, \varphi_0\rangle_i = \cos \left( \frac{\theta_0}{2} \right) e^{i\varphi_0/2} |+\rangle_i + \sin \left( \frac{\theta_0}{2} \right) e^{-i\varphi_0/2} |\rangle_i, \quad (9.16) \]

and where \(|+\rangle_i\) and \(|-\rangle_i\) are the excited and ground states of the \(i\)th atom. The state \((9.15)\) is characterized by two parameters \(\theta_0\) and \(\varphi_0\) and can be obtained by exciting the atomic system by an external field (see e.g. Dialetis [1970]). On substituting \((9.15)\) and \((9.16)\) in \((9.14)\), we find that \(I_0\) is given by

\[ I_0 = 2\gamma\omega N \sin^2 \left( \frac{\theta_0}{2} \right) \{1 + (N - 1) \cos^2 \left( \frac{\theta_0}{2} \right)\}, \quad (9.17) \]

which in the special cases leads to

\[ I_0 \propto \begin{cases} N & \text{if } \theta_0 = \pi, \\ N^2 & \text{if } \theta_0 = \frac{\pi}{2}. \end{cases} \quad (9.18) \]

A system excited to a state of the form \((9.16)\) with \(\theta_0 = \frac{\pi}{2}\) gives, therefore, rise to superradiant emission (Dicke [1954]) even though there are no correlations among different atoms. One should note that in this case the dipole moment is maximum. The author has referred to this type of superradiance as the superradiance of first kind (Agarwal [1971c]).

To study the collective behavior of the system, we introduce, following Dicke [1954], the operators \(S^\pm, S^z\) defined by

\[ S^\pm = \sum_i S_i^\pm, \quad S^z = \sum_i S_i^z. \quad (9.19) \]

In terms of the collective variables, the master equation \((9.4)\) becomes

\[ \frac{\partial \rho}{\partial t} = -\gamma(S^+S^-\rho - 2S^-\rho S^+ + \rho S^+S^-). \quad (9.20) \]

Starting from a different standpoint the master equation \((9.20)\) has also been obtained by Bonifacio, Schwendimann and HaaKe [1971a]. The perturbation theoretic result \((9.14)\) in terms of the collective variables may be written as

\[ I_0 = 2\gamma\omega \text{tr} \{\rho(0)S^+S^-\}. \quad (9.21) \]

If we assume that the system was initially excited to the state \(|\frac{1}{2}N, m\rangle\), (Dicke state, which is the simultaneous eigenstate of the operators \(S^2\) and \(S^z\) with eigenvalues \(\frac{1}{2}N(\frac{1}{2}N + 1)\) and \(m\) respectively) then \((9.21)\) reduces to

\[ I_0 = 2\gamma\omega \{\frac{1}{2}N(\frac{1}{2}N + 1) - m^2 + m\}. \quad (9.22) \]

It is seen from \((9.22)\) that for large \(N\)

\[ I_0 \propto \begin{cases} N & \text{if } m = \frac{1}{2}N, \\ N^2 & \text{if } m = 0. \end{cases} \quad (9.23) \]

We conclude from \((9.23)\) that a system excited to a state \(|\frac{1}{2}N, 0\rangle\) leads
to the superradiant emission (Dicke [1954]). We refer to this type of
superradiance as the superradiance of second kind (Agarwal [1971c]).
One may further show that \(|\frac{1}{2}N, 0\rangle\) is a state in which the dipole moment
is zero and in which there are correlations among different atoms, one has, for example,
\[
\langle S_i^z S_j^z \rangle - \langle S_i^z \rangle \langle S_j^z \rangle = -\{4(N-1)\}^{-1}.
\] (9.24)

It is, therefore, clear that the atomic correlations play an important role
in the superradiant emission of the second kind. The radiation from the
state \(|\frac{1}{2}N, \frac{1}{2}N\rangle\) is incoherent to start with but, as we will see, it becomes
superradiant as the system develops in time. This is another example of the
superradiant emission of the second kind, since the dipole moment of the
system is found to be zero (cf. eq. (9.31)).

To obtain the time dependence of the radiation rate one must solve
either the master equation (9.4) or (9.20). For small systems \(S^2\) is a constant
of motion. On taking the matrix elements of both sides of (9.20), we obtain
the equation
\[
\begin{aligned}
\partial \rho_{mn}/\partial t &= 2\gamma \{ (\nu_{m+1} \nu_{n+1})^2 \rho_{m+1,n+1} - \frac{1}{2}(\nu_m + \nu_n)\rho_{mn} \}, \\
\rho_{mn} &= \langle S, m|\rho|S, n\rangle,
\end{aligned}
\] (9.25)

where \(\rho_{mn}\) is the eigenstate of the operators \(S^2\) and \(S^z\), with eigenvalues
\(S(S+1)\) and \(m\) respectively. In the theory of superradiance \(S\) is referred to as
\(\text{'cooperation number'}.\) In eq. (9.25) \(2\gamma \nu_m\) is the transition probability that
the atomic system makes a transition from the state \(|m\rangle\) to the state \(|m-1\rangle\)
and is given by
\[
\nu_m = (S-m+1)(S+m).
\] (9.27)

For the diagonal elements eq. (9.25) is an equation of the Pauli type. The
exact solution of (9.25) is given by (Agarwal [1970])
\[
\rho_{m,n}(\beta) = \sum_{l \geq 0} \{ \prod_{k=0}^{l} (\nu_{m+k} \nu_{n+k})^{|l|} \prod_{k=0}^{l} \{ \beta + \frac{1}{2}(\nu_{m+k} + \nu_{n+k}) \}^{-1} \} \rho_{m+l,n+l}(0),
\] (9.28)

where \(\rho_{m,n}(\beta)\) is the Laplace transform of \(\rho_{m,n}(t)\) defined by a relation of
the form (3.4). It is seen from (9.28) that the steady state solution is given by
\[
\rho_{m,n}(\infty) = \delta_{m,n} \delta_{m,-S},
\] (9.29)

where \(S\) is the cooperation number. Thus the steady state value of the
energy is
\[
W(\infty) = -S.
\] (9.30)
A special case of (9.30) is given by eq. (9.11). It should be noted that the cooperation number takes only the integer or half integer values.

From (9.28) one may deduce that

$$\langle S^\pm(t) \rangle = 0 \quad \text{if} \quad \rho(0) = |S, m \rangle \langle S, m|,$$

(9.31)
i.e., a system which starts in a state with zero dipole moment remains in a state with zero dipole moment and such a system will, therefore, not show superradiance of the first kind.

The analytic solution (9.28) is too involved for practical calculations, unless $N$ is a small number. For large values of $N$, we may resort to approximate methods. Before discussing the approximate methods, we will discuss another exactly soluble model. We will consider what happens if each two-level atom is replaced by a harmonic oscillator. The superradiant emission from a system of harmonic oscillators has been studied in detail by Agarwal [1970, 1971b].

For studying the oscillator system it is convenient to work with the phase space distribution functions. The spontaneous emission from a collection of identical harmonic oscillators is described by the following master equation for Sudarshan-Glauber distribution function

$$\frac{\partial \Phi^{(A)}}{\partial t} = \gamma \sum_{ij} \left\{ \frac{\partial}{\partial z_j} (z_j \Phi^{(A)}) + \text{C.C.} \right\}.$$  

(9.32)
The Green's function corresponding to eq. (9.32) is given by

$$K^{(A)}([z_1], [z^*_1], t| [z^0_1], [z^{0*}_1], 0) = \prod_{i=1}^{N} \delta^{(2)}(z_i - \bar{z}_i),$$  

(9.33)
where

$$\bar{z}_i = z_i^0 - \frac{1}{N} (1 - e^{-\gamma t}) \sum_{j=1}^{N} z_j^0.$$  

(9.34)

We assume that each of the oscillators was initially excited to a coherent state $|z_i^0\rangle$. Then the radiation rate from such a system is given by

$$I(t) = 2\gamma \omega e^{-2\gamma t} \sum_{i=1}^{N} z_i^0 |^2,$$

(9.35)
and is proportional to $N^2$ if $z_i^0 = z_0$ and the decay constant is $N$ times larger than that due to a single oscillator. The coherent state $|\{z_0\}\rangle$ is thus a superradiant state for the oscillator system and this is an example of the superradiance of the first kind. The enhanced decay rate for the case of two classical oscillators has been observed in a recent experiment by Lama, Jodoin and Mandel [1972]. The steady state solution is given by
\[ \Phi^{(A)}(\{z_i\}, \{z_i^*\}, \infty) = \prod_{i=1}^{N} \delta^{(2)} \left[ z_i - \left( z_i^0 - \frac{1}{N} \sum_{j=1}^{N} z_j^0 \right) \right], \quad (9.36) \]

which shows that, in the steady state, each oscillator is not necessarily left in its ground state.

It should also be noted that for the initial coherent state excitation, no correlations are induced among different oscillators. However, if the system is initially excited to an incoherent state, such as a Fock state, then correlations are induced among different oscillators because of spontaneous emission (Agarwal [1971b]).

We now discuss approximate expressions for the radiation rate from a collection of two-level atoms. We have seen that the radiation rate can be calculated if the time dependence of the correlation function \( \langle S_i^+ S_j^- \rangle \) is known. The equation of motion for \( \langle S_i^+ S_j^- \rangle \) is coupled to the higher order correlation function of the form \( \langle S_i^+ S_j^- S_k^- \rangle \) (see Agarwal [1970]). So in order to obtain the radiation rate one has to solve the whole hierarchy of equations. The approximate result is obtained by closing the hierarchy of equations, making suitable approximations on the higher order correlation functions. Various approximate procedures are discussed in Agarwal [1971c, e]. From the permutation symmetry of the problem it follows that \( \langle S_i \cdot S_j \rangle = \frac{1}{2} \), if the system is initially excited to a permutationally symmetric state. This relation enables us to express \( \langle S_i^+ S_j^- \rangle \) in terms of \( \langle S_i^+ S_j^- S_k^- \rangle \), i.e.

\[ \langle S_i^+ S_j^- \rangle + \text{C.C.} = 2\{ \langle S_i^+ S_j^- \rangle - \langle S_i^+ S_j^- S_k^- \rangle \}, \quad (i \neq j). \quad (9.37) \]

On using (9.37), eq. (9.12) may be written as

\[ \frac{\partial}{\partial t} \langle S_i^+ \rangle + 2\gamma N \langle S_i^+ S_i^- \rangle - 2\gamma (N - 1) \langle S_i^+ S_j^+ S_i^- S_j^- \rangle = 0, \quad (i \neq j). \quad (9.38) \]

Our approximate procedure consists of expressing \( \langle S_i^+ S_i^- S_j^+ S_j^- \rangle \) in terms of the one-particle mean values. The nature of the approximation depends on the initial excitation.

We assume that the atomic system was initially excited to a state of the form (9.15) with \( \theta_0 < \pi \). We make an "Hartree type" of approximation on the two-particle mean value

\[ \langle S_i^+ S_i^- S_j^+ S_j^- \rangle \approx \langle S_i^+ S_i^- \rangle \langle S_j^+ S_j^- \rangle, \quad (i \neq j). \quad (9.39) \]

On substituting (9.39) in (9.38), we obtain a simple equation for \( \langle S_i^+ \rangle \) which is easily solved. We then find the following expression for the radiation rate.
I(t) = \frac{\omega_0 N^3}{2(N-1)} \text{sech}^2\{N\gamma(t-\tau)\}, \quad \tau = (2N\gamma)^{-1} \ln \frac{N-1}{N \cot^2 \left(\frac{\pi}{2} + \frac{\pi}{2}\right)} + 1. \quad (9.40)

A similar result has been obtained by REHLER and EBERLY [1971]. The result (9.40) agrees well with the numerical solution of the master equation (9.20) (BONIFACIO, SCHWENDIMAN and HAAKE [1971b]). A possible experimental observation of the superradiant decay has been made by COMPAAN and ABELLA [1971]. The approximation (9.39) was analyzed in AGARWAL [1971e], where an improved result for the radiation rate was also given.

Superradiance of the second kind is much more complicated because one must take into account the atomic correlations. The two cases corresponding to the initial excitations \(|\frac{1}{2}N, \frac{1}{2}N\rangle\) and \(|\frac{1}{2}N, 0\rangle\) are discussed by AGARWAL [1971c, e] and by BONIFACIO, SCHWENDIMAN and HAAKE [1971b].

We conclude this section by making some remarks about the relaxation of an atom with equidistant energy levels. It is clear that (9.20) describes the relaxation at zero temperature of a spin (spin value \(S\)) or the relaxation of an atom with \((2S+1)\) equidistant energy levels. It can be easily shown that the relaxation, at finite temperature \(T\), of an atom with \((2S+1)\) equidistant energy levels is described by the equation

\[
\frac{\partial \rho}{\partial t} = -\gamma(1 + \langle n(\omega) \rangle)\{S^+ S^- \rho - 2S^- \rho S^+ + \rho S^+ S^- \} - \gamma\langle n(\omega) \rangle\{S^- S^+ \rho - 2S^+ \rho S^- + \rho S^- S^+ \}, \quad (9.41)
\]

where \(\langle n(\omega) \rangle\) is defined by (6.6). The master equation for the diagonal elements is given by

\[
\frac{\partial \rho_{m,m}}{\partial t} = 2\gamma(1 + \langle n(\omega) \rangle)(v_{m+1} \rho_{m+1,m+1} + v_m \rho_{m,m} - v_{m+1} \rho_{m+1,m+1}) - 2\gamma\langle n(\omega) \rangle(v_{m+1} \rho_{m,m} + v_m \rho_{m-1,m-1}). \quad (9.42)
\]

* This master equation may be used to study spontaneous emission in presence of black body radiation (AGARWAL [1973a]).
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This rate equation is schematically represented in Fig. 2 and has the following steady state solution

$$
\rho_{m+1,m+1}(\infty) = \frac{\langle n(\omega) \rangle}{(1+\langle n(\omega) \rangle)} \rho_{m,m}(\infty) = e^{-\beta \omega} \rho_{m,m}(\infty). \quad (9.43)
$$

§ 10. Laser Master Equation

We use the techniques developed in previous sections to obtain the master equation for a single mode laser (for various theories of lasers see, for example, LAX [1966c, 1968a], LAX and LOUISELL [1969], LAMB [1964, 1965], LAMB and SCULLY [1967], HAKEN [1970], RISKEN [1970]). A laser system typically consists of active atoms, electromagnetic field and pump and loss mechanisms. We assume that each atom is a two-level system and that each atom and the field mode is coupled to its own reservoir. The reservoirs describe the effects of loss mechanism and the pump. We have already treated the effect of reservoirs on each atom and on an oscillator (which is the electromagnetic field mode in the present case) in §§ 6, 8 (see eqs. (6.21), (8.23)). The density operator $\rho$ corresponding to the coupled atoms-field system satisfies the equation

$$
\frac{\partial \rho}{\partial t} = -i \mathcal{L} \rho, \quad (10.1)
$$

where $\mathcal{L}$ is the effective Liouville operator and is non-Hermitian since we have already taken into account the effects of loss mechanism and the pump. $\mathcal{L}$, therefore, consists of two parts: a reversible part and an irreversible part. The reversible part of $\mathcal{L}$ may be written as

$$
\mathcal{L}_{\text{rev}} = \mathcal{L}_A + \mathcal{L}_F + \mathcal{L}_{AF}, \quad (10.2)
$$

where $\mathcal{L}_A + \mathcal{L}_F$ describe the unperturbed motion of the atoms and the field and $\mathcal{L}_{AF}$ describes the interaction between the active atoms and the field. The corresponding Hamiltonians are given by

$$
H_A = \sum_{j=1}^{N} \omega_j S_j^z, \quad H_F = \omega a^+ a, \quad H_{AF} = \sum_{j=1}^{N} g_j (S_j a^+ + \text{H.C.}). \quad (10.3)
$$

The irreversible part of the Liouville operator is given by

$$
\mathcal{L}_{\text{irr}} = i A_F + i A_A, \quad (10.4)
$$

where the operators $A_F$ and $A_A$ are given by (see eqs. (6.21), (8.23))
\( \Lambda_F G = \gamma[[aG, a^+] + [a, Ga^+] + 2\langle n(o) \rangle[[a, G], a^+]], \)  
\( (10.5) \)

\[ \Lambda_A G = \sum_{j=1}^{N} \left\{ \frac{1}{2} \gamma_{12} \{ [S_j^-, G S_j^+] + [S_j^-, G S_j^+] \} \right. 
\left. + \frac{1}{2} \gamma_{21} \{ [S_j^+, G S_j^-] + [S_j^+, G S_j^-] \} + \Gamma_{12}^{ph} \{ [S_j^x, G S_j^-] + [S_j^y G, S_j^-] \} \} \right\}. \quad (10.6) \]

Let \( \gamma_{11} \) and \( \gamma_{1} \) be the inverses of the longitudinal and transverse relaxation times \( T_1 \) and \( T_2 \) respectively, given by

\[ \gamma_{11} = \gamma_{12} + \gamma_{21}, \quad \gamma_{1} = \frac{1}{2}(\gamma_{12} + \gamma_{21} + 2\Gamma_{12}^{ph}), \quad (10.7) \]

and let \( \sigma \) be the unsaturated inversion defined by

\[ \sigma = (\gamma_{21} - \gamma_{12})/(\gamma_{12} + \gamma_{21}). \quad (10.8) \]

The master equation (10.1) contains both the atomic and the field variables. Many of the experiments on lasers concern the statistical properties of the laser light and it is, therefore, appropriate to obtain an equation of motion involving the field variables only. Let \( \rho_F \) be the reduced density operator corresponding to the field alone. It is defined in the usual way, viz.

\[ \rho_F = \text{Tr}_A \rho, \quad (10.9) \]

where \( \text{Tr}_A \) denotes the trace over the atomic variables. Various methods have been developed to eliminate the atomic variables from the original master equation (see LAX and LOUISELL [1967, 1969], ARZT et al. [1966], GORDON [1967], and HAAKE [1969b]). Here we discuss the method due to Haake who used the projection operator techniques to eliminate the atomic variables from the master equation (10.1).

It is easily seen that the derivation given in § 3 goes through even if \( \mathcal{L} \) is non-Hermitian or non-real so that \( \mathcal{P} \rho \) satisfies equations of the form given by (3.8), viz.

\[ \frac{\partial}{\partial t} [\mathcal{P} \rho(t)] + i\mathcal{L} \mathcal{P} \rho(t)] + i\mathcal{L} \mathcal{P} \exp [-i(1-\mathcal{P})\mathcal{L} t](1-\mathcal{P})\rho(0) \]

\[ + \int_{0}^{t} \mathcal{L} \mathcal{P} \exp [-i\tau(1-\mathcal{P})\mathcal{L}](1-\mathcal{P})\mathcal{L} \mathcal{P} \rho(t-\tau)d\tau = 0, \quad (10.10) \]

where \( \mathcal{L} \) is given by the sum of (10.2) and (10.4). For the purpose of eliminating the atomic variables, the projection operator is given by

\[ \mathcal{P} \ldots = G_A \text{Tr}_A \ldots, \quad (10.11) \]

where the operator \( G_A \) will be chosen such that

\[ \mathcal{L}_A G_A = \Lambda_A G_A = 0; \quad \text{Tr} G_A = 1. \quad (10.12) \]
It is easily checked that a possible choice of $G_A$ is
\begin{equation}
G_A = \prod_{i=1}^{N} (\frac{1}{2} + \sigma S_i^z), \tag{10.13}
\end{equation}
where $\sigma$ is the unsaturated inversion given by (10.8). The choice of the operator $G_A$ of the form (10.13) is motivated by the fact that for a reservoir, in thermal equilibrium, and made of two-level atoms, the appropriate $G_A$ is equal to $G_A = \prod_{i=1}^{N} (\frac{1}{2} - \tanh(\frac{1}{2} \beta \omega) S_i^z)$. The reduced density operator $\rho_F$ is then given by
\begin{equation}
\rho_F = \text{Tr}_A(\mathcal{P}\rho). \tag{10.14}
\end{equation}

It is also easily verified that the projection operator $\mathcal{P}$ defined by (10.11) and (10.13) satisfies the identities
\begin{align}
\mathcal{P} L_F &= L_F \mathcal{P}, & \mathcal{P} A_F &= A_F \mathcal{P}, & \mathcal{P} L_A &= \mathcal{P} A_A = 0, \\
L_A \mathcal{P} &= A_A \mathcal{P} = 0, & \mathcal{P}(L_{AF})^\dagger \mathcal{P} &= 0, \tag{10.15}
\end{align}
where $n$ is an odd integer. On using (10.15), eq. (10.10) reduces to
\begin{equation}
\frac{\partial \rho_F}{\partial t} + i(\mathcal{L}_F + iA_F)\rho_F + I(t) + \int_0^t K(\tau)\rho_F(t-\tau)d\tau = 0, \tag{10.16}
\end{equation}
where the inhomogeneous term $I(t)$ and the kernel $K(t)$ are given by
\begin{align}
I(t) &= i \text{Tr}_A \mathcal{L}_{AF} \exp \{-i(1-\mathcal{P})\mathcal{L} t\}(1-\mathcal{P})\rho(0), \tag{10.17} \\
K(t) &= \text{Tr}_A \mathcal{L}_{AF} \exp \{-i(1-\mathcal{P})\mathcal{L} t\} \mathcal{L}_{AF} G_A. \tag{10.18}
\end{align}
The inhomogeneous term $I(t)$ in (10.16) depends on how the laser system has been initially turned on and is important only in the study of its transient behavior. In what follows we consider only the steady state properties and therefore we ignore the term $I(t)$. The kernel $K(t)$ can be simplified in the standard manner (cf. § 5) and may be written in the form
\begin{equation}
K(t) = \text{Tr}_A \mathcal{L}_{AF} U_0(t)(1-\mathcal{P})U(t)\mathcal{L}_{AF} G_A, \tag{10.19}
\end{equation}
where
\begin{align}
U_0(t) &= \exp \{-i(\mathcal{L}_F + iA_F + \mathcal{L}_A + iA_A)t\}, \tag{10.20} \\
U(t) &= T \exp \left\{-i \int_0^t d\tau U_0(-\tau)(1-\mathcal{P})\mathcal{L}_{AF}(1-\mathcal{P})U_0(\tau)\right\}. \tag{10.21}
\end{align}
On using the identities (10.15), the $K(t)$ may be rewritten as (cf. eq. (3.22))
\[ K(t) = \text{Tr}_A \sum_{n=0}^{\infty} (-i)^{2n} \int_0^t \cdots \int_0^t \cdots \int_0^t d\tau_1 \cdots d\tau_{2n} \times \{ U_0(t-\tau_1)(1-\mathcal{P}) \mathcal{L}_A U_0(\tau_1-\tau_2)(1-\mathcal{P}) \mathcal{L}_A \cdots \cdots \mathcal{L}_A U_0(\tau_{2n})(1-\mathcal{P}) \mathcal{L}_A G_A \}. \] (10.22)

Eq. (10.16) is still exact, i.e. it contains the effect of the coherent interaction \( H_{\text{AF}} \) to all orders. Further simplification can be made by terminating the series (10.22). It is well known in the theory of lasers that the Born approximation is inadequate (Lamb [1964]), since in the Born approximation one is taking into account the effect of the atoms on the field but not the effect of the field back on atoms. In order to describe the saturation effects in the atomic system, which are of great importance in the operation of lasers, it turns out sufficient to retain the first two terms in (10.22). As remarked by Haake, this approximation contains the hitherto known theories of the laser.

For the sake of simplicity, we consider only the case of homogeneous line at resonance i.e. \( g_j = g, \omega_j = \omega \). We also assume that the field damping \( \gamma \) is much smaller than \( \gamma \parallel \) and \( \gamma \perp \). This assumption is justified for most gas lasers. We apply the mapping operator \( \Theta^{(A)} \) to both sides of (10.16) and use eqs. (10.3)-(10.6), and (10.22) (which is terminated at \( n = 1 \)). We then find that the Sudarshan-Glauber distribution function satisfies the equation (Haake [1969b])

\[ \frac{\partial \Phi^{(A)}}{\partial t} = i\omega \left( \frac{\partial}{\partial z} z - \frac{\partial}{\partial z^*} z^* \right) \Phi^{(A)} + \gamma \left( \frac{\partial}{\partial z} z + \frac{\partial}{\partial z^*} z^* + 2 \langle n(\omega) \rangle \frac{\partial^2}{\partial z \partial z^*} \right) \Phi^{(A)} \]

\[ + \int_0^t k(\tau) \Phi^{(A)}(t-\tau) d\tau, \] (10.23)

where the kernel \( k(\tau) \) is given by

\[ k(\tau) = \left( \frac{\partial}{\partial z} z + \frac{\partial}{\partial z^*} z^* \right) \left( -\alpha_1 \psi_1(\tau) + \alpha_{nl} \psi_2(\tau) z^* z + 4q \psi_1(\tau) \frac{\partial^2}{\partial z \partial z^*} \right). \] (10.24)

In eq. (10.24) \( \alpha_1, \alpha_{nl} \) and \( q \) are the linear gain, nonlinear gain and fluctuation respectively, defined by

\[ \alpha_1 = \frac{N g^2 \sigma}{\gamma \perp}, \quad \alpha_{nl} = \frac{4 N g^4 \sigma}{\gamma \perp^2 \gamma \parallel}, \quad g = \frac{N g^2 (1+\sigma)}{4 \gamma \perp}, \] (10.25)

and \( \psi_1(\tau) \) and \( \psi_2(\tau) \) are the retardation functions defined by

\[ \psi_1(\tau) = \gamma \parallel \exp (-\gamma \perp \tau), \] (10.26)

\[ \psi_2(\tau) = \frac{\gamma \parallel^2 \gamma \parallel}{(\gamma \perp - \gamma \parallel)^2} \left[ \exp (-\gamma \parallel \tau) - \{ 1 + \tau (\gamma \perp - \gamma \parallel) \} \exp (-\gamma \perp \tau) \right]. \] (10.27)
In writing (10.24) we have also ignored certain correction terms. These terms are of the order of $(\alpha_{nl}/q)^2$, which is quite small in the threshold region. These correction terms are of purely quantum mechanical origin and are discussed in detail in a recent review article by Risken [1970]. Since we ignore such quantum mechanical corrections, we will from now on regard the field variable as a c-number variable and the distribution $\Phi^{(A)}$ as a classical distribution, denoted by $\Phi$. The master equation (10.23) is still an integro-differential equation. We now make the Markovian approximation (adiabatic approximation), which is made in most theories of the laser. Under this approximation (10.23) reduces to

$$\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial z} \left\{ z(\gamma + i\omega - \alpha_t + \alpha_{nl} z^* z) \Phi \right\} + (\gamma \langle n(\omega) \rangle + 2q) \frac{\partial^2 \Phi}{\partial z \partial z^*} + C.C. \quad (10.28)$$

We introduce the normalized variables, defined by

$$b = \left( \frac{\alpha_{nl}}{q} \right)^{1/2} z, \quad \tau = (q\alpha_{nl})^{1/2} t, \quad p = -\gamma + \alpha_t \quad (10.29)$$

and we ignore the relatively small term $\langle n(\omega) \rangle \gamma$, eq. (10.28) then reduces to

$$\frac{\partial \Phi}{\partial t} = -\frac{\partial}{\partial b} \left\{ [(p - b^* b)b^* \Phi] - \frac{\partial}{\partial b^*} \left\{ [(p - b^* b)b^* \Phi] + 4 \frac{\partial^2 \Phi}{\partial b \partial b^*} \right\} \right\} \quad (10.30)$$

where $\Phi$ refers to the distribution function in the interaction picture. Eq. (10.30) is the well known Fokker-Planck equation describing the behavior of a single mode laser. This Fokker-Planck equation is very basic in the theory of the laser and has been studied in great detail (Risken [1970], Lax and Louisell [1967], Risken and Vollmer [1967a, b], Hempstead and Lax [1967]). The Fokker-Planck equation (10.30) is stochastically equivalent to the Langevin equations

$$b = (p - b^* b) + F(t), \quad b^* = (p - b^* b) + F^*(t), \quad (10.31)$$

where $F(t)$ is a delta correlated Gaussian random process with the properties

$$\langle F(t) \rangle = \langle F^*(t) \rangle = 0, \quad \langle F(t) F(t') \rangle = \langle F^*(t) F^*(t') \rangle = 0,$$

$$\langle F(t) F^*(t') \rangle = 4\delta(t-t'). \quad (10.32)$$

It should be noted that the eq. (10.31), without fluctuations, is the same as that for a rotating wave Van der Pol oscillator (see e.g. Davis [1962]) and that is why this theory is also referred to as the Van der Pol oscillator model of laser light. For this model it is easily verified that the detailed balance
condition (A11) is satisfied and, therefore, the steady state solution is given by

\[ \Phi_n(b, b^*) = \left( \frac{\mathcal{N}}{2\pi} \right) \exp \left\{ -\frac{1}{2} |b|^2 + \frac{1}{2} |b|^2 p \right\}, \]

(10.33)

where \( \mathcal{N} \) is the normalization constant. Thus the mean amplitude of the field is zero. This is contrary to the result obtained by ignoring the fluctuations (LAMB [1964]). The stationary distribution (10.33) is Gaussian centered at \( |b|^2 = p \), in the variable \( |b|^2 \) and is truncated at \( |b|^2 = 0 \). From (10.33) it is easily shown that the cumulants \( K_n(p) \) of the intensity distribution are given by (RISKEN [1970])

\[ K_{n+1}(p) = 2 \frac{d}{dp} K_n(p), \quad K_1(p) = p + \frac{2}{\sqrt{\pi}} e^{-kp^2} (1 + \text{erf}(\frac{1}{2} p))^{-1}, \]

(10.34)

where \( \text{erf}(x) \) is the error function of \( x \).

It does not seem possible to obtain an analytic expression for the Green's function associated with eq. (10.30). Numerical computations of the various statistical properties of the laser light, which follow from (10.30), have been carried out by RISKEN and VOLLMER [1967a, b] and by HEMPSTEAD and LAX [1967]. An excellent treatment of the statistical properties of the laser light is given in a recent review article by RISKEN [1970].

Recently the steady state distribution function (10.33) has attracted a great deal of attention in connection with the analogy between the laser threshold region and the second order phase transitions (GRAHAM and HAKEN [1970], SCULLY and DIGIOGIO [1970], GROSSMAN and RICHTER [1971]), for there is a close correspondence between (10.33) and the logarithm of Ginzburg-Landau energy functional (GINZBURG and LANDAU [1950]). We consider here the effect of a weak external field of the form \( \varepsilon_0 e^{-ivt} \) and make the rotating wave approximation. The Langevin equations (10.31) are then modified to (\( g \) = coupling constant between the laser field and external field)

\[ \dot{x}_1 = -(p-x_1^2-x_2^2)x_1 + F_1 + \text{Re} \, ig \, \varepsilon_0 e^{-ivt + i\omega t}, \]

(10.35)

\[ \dot{x}_2 = -(p-x_1^2-x_2^2)x_2 + F_2 + \text{Im} \, ig \, \varepsilon_0 e^{-ivt + i\omega t}, \]

(10.36)

where we have introduced the real variables defined by \( b = x_1 + ix_2 \), \( F(t) = F_1(t) + iF_2(t) \). We define the linear susceptibility \( \chi_{ij}(v) \) by

\[ \langle x_i \rangle = \text{Re} \sum_j \chi_{ij}(v) e^{ivt} e_{i1}, \]

(10.37)

where \( e_1 = ig\varepsilon_0, \) \( e_2 = -g\varepsilon_0 \). Then it may be shown on using the properties of the Fokker-Planck equation (10.30) that (AGARWAL [1972])
\( \chi_{12}(v) = \chi_{21}(v) = 0, \quad (10.38) \)
\( \chi_{11}(v) = \chi_{22}(v) = \frac{1}{2} K_1(p) \sum_{m=0}^{\infty} V_m^g \lambda_{1m}(\lambda_{1m} + iv)^{-1}, \quad (10.39) \)

where \( V_m^g \) and \( \lambda_{1m} \) are the well known matrix elements and the eigenvalues respectively (RISKEN [1970]). From the behavior of \( V_m^g, \lambda_{1m} \) and \( K_1 \) as a function of the pump parameter \( p \) (see Figs. 2, 4 of RISKEN [1970]) it is found that \( \chi_{11}(v) \) and \( \chi_{22}(v) \) are continuous functions of the pump parameter. This result disagrees with a prediction of the theory of DIGIORGIO and SCULLY [1970], who found that the static susceptibility diverges both from above and from below as \( |p|^{-1} \) at threshold. The reason for this disagreement is that Digiorgio and Scully have ignored the statistical fluctuations and it is the neglect of these fluctuations that leads to the singular behavior of the susceptibility.

### § 11. Master Equations for Strongly Interacting Quantum Systems in Contact with Heat Baths

In this section we obtain the master equations for strongly interacting quantum systems in contact with heat baths. An example of such systems is that of the relaxation of a spin system interacting with strong external fields. The theory for the relaxation of spin systems in presence of external fields has been developed by BLOCH [1957], REDFIELD [1957, 1965] and by others (see e.g. ARGYRES and KELLEY [1964]). This theory also enables to study spontaneous emission in the presence of strong external fields.

We write the total Hamiltonian of the system in the form

\[ H = H_S + H_R + H_{RS} + H_{ext}(t), \quad (11.1) \]

where \( H_{ext}(t) \) is the interaction Hamiltonian due to the external field acting on the system \( S \). Let \( H_{01} \) and \( H_{02} \) be defined by

\[ H_{01} = H_S + H_R, \quad H_{02} = H_S + H_R + H_{ext}(t). \quad (11.2) \]

It is clear from (11.2) that \( H_{01} \) and \( H_{02} \) are the unperturbed parts of \( H \) with respect to the interaction Hamiltonians \( H_{RS} + H_{ext}(t) \) and \( H_{RS} \) respectively. We introduce the operators \( \sigma_i(t) \) and \( \sigma_i(t) \) defined by

\[ \sigma_i(t) = U_{0i}(t, 0) \rho(t) U_{0i}(t, 0), \quad (i = 1, 2), \quad (11.3) \]

where the time evolution operators \( U_{0i} \) are given by

\[ U_{0i}(t, \tau) = T \exp \left\{ -i \int_{\tau}^{t} dt' H_{0i}(t') \right\}, \quad (11.4) \]
and where $H_{01}$ is given by (11.2). The unitary transformations, given by $U_{01}$ and $U_{02}$, lead to two interaction pictures to which we will refer as the interaction picture [1] and the interaction picture [2] respectively. It should be noted that the interaction picture [2], as defined here, is known as the Furry picture in quantum electrodynamics (see e.g. Schwerber [1961; § 15gl]).

It is easily seen that $U_{01}$ and $U_{02}$ are related by

$$U_{02}(t, \tau) = U_{01}(t, 0)V(t, \tau)U_{01}^{+}(\tau, 0),$$

where

$$V(t, \tau) = T \exp \left\{ -i \int_{\tau}^{t} \frac{d\tau'}{i} H_{\text{ext} 1}(\tau') \right\}.$$  \hspace{1cm} (11.5)

In eq. (11.6) $H_{\text{ext} 1}(t)$ is the operator $H_{\text{ext} 1}(t)$ in the interaction picture [1], i.e.

$$H_{\text{ext} 1}(t) = U_{01}^{+}(t, 0)H_{\text{ext}}(t)U_{01}(t, 0).$$  \hspace{1cm} (11.7)

It should also be noted that $\sigma_1(t)$ and $\sigma_2(t)$ are related by

$$\sigma_2(t) = V^{+}(t, 0)\sigma_1(t)V(t, 0).$$

(11.8)

From the definition (11.3), it is clear that $\sigma_2(t)$ satisfies the equation

$$\frac{\partial \sigma_2}{\partial t} = -i[H_{RS2}(t), \sigma_2] = -i \mathcal{L}_{RS2}(t)\sigma_2,$$  \hspace{1cm} (11.9)

where

$$H_{RS2}(t) = U_{02}^{+}(t, 0)H_{RS}U_{02}(t, 0),$$

(11.10)

and where $\mathcal{L}_{RS2}(t)$ is the Liouville operator in the interaction picture [2]. The Liouville equation (11.9) contains both the system variables and the reservoir variables. The master equation is obtained from (11.9) by the elimination of the reservoir coordinates. By applying the projection operator $\mathcal{P}$ to both sides of (11.9) and by using the procedure of § 5, we obtain the master equation (cf. eq. (5.16))

$$\frac{\partial}{\partial t} [\mathcal{P} \sigma_2(t)] + i\mathcal{P} \mathcal{L}_{RS2}(t)[\mathcal{P} \sigma_2(t)]$$

$$+ \int_{0}^{t} d\tau \mathcal{P} \mathcal{L}_{RS2}(t) \mathcal{U}(t, \tau)(1-\mathcal{P})\mathcal{L}_{RS2}(\tau)[\mathcal{P} \sigma_2(\tau)]=0,$$

(11.11)

where

$$\mathcal{U}(t, \tau) = T \exp \left\{ -i \int_{\tau}^{t} \frac{d\tau'}{i} (1-\mathcal{P}) \mathcal{L}_{RS2}(t')(1-\mathcal{P}) \right\},$$

(11.12)

and where the projection operator $\mathcal{P}$ is given by (5.7). The master equation (11.11) contains the effect of the reservoir interaction to all orders. We again consider the approximate master equation obtained from (11.11) by making the Born approximation and the Markovian approximation (for some of
the non-Markovian effects in spin resonance saturation see Argyres and Kelley [1964]). Under these approximations we recover the theory of Bloch and Redfield. The equation so obtained will be valid for $\tau_c < t \leq \tau_r$ where $\tau_c$ and $\tau_r$ are the typical reservoir correlation time and the relaxation time respectively. The master equation (11.11) in these two approximations reduces to

$$\frac{\partial}{\partial t} [\mathcal{P} \sigma_2(t)] + \int_0^\infty \mathcal{P} L_{RS2}(t) L_{RS2}(t-\tau) [\mathcal{P} \sigma_2(t)] d\tau = 0. \quad (11.13)$$

In deriving (11.13) we have also assumed that the properties of the reservoir are such that

$$\mathcal{P} L_{RS2}(t) \mathcal{P} \sigma_2(t) = 0. \quad (11.14)$$

On using (11.8) and (11.13), we obtain the following master equation $\mathcal{P} \sigma_1(t)$

$$\frac{\partial}{\partial t} [\mathcal{P} \sigma_1(t)] + i[H_{ext1}(t), \mathcal{P} \sigma_1] + \rho_R(0) \text{Tr}_R \int_0^\infty [H_{RS1}(t), [V(t, t-\tau) H_{RS1}(t-\tau) V^+(t, t-\tau), \mathcal{P} \sigma_1(t)]] d\tau = 0. \quad (11.15)$$

Eq. (11.15) is a basic equation for the description of strongly driven quantum systems. We now use eq. (11.15) to study spontaneous emission in presence of strong fields.

We assume that the atoms (two-level) are driven by a resonant optical field of the form

$$\epsilon(t) = \epsilon(x \cos \omega t + y \sin \omega t). \quad (11.16)$$

$H_{ext}(t)$ then is given by

$$H_{ext}(t) = -\frac{1}{2} |d| \varepsilon \sum_j (S_j^+ e^{-i\omega t} + H.C.). \quad (11.17)$$

The Hamiltonian $H_S + H_R + H_{RS}$ is given by eq. (9.1). For the system under consideration we have

$$H_{ext1}(t) = -\frac{1}{2} |d| \varepsilon \sum_j (S_j^+ + H.C.), \quad (11.18)$$

$$H_{RS1}(t) = \sum_{j,k} (S_j^+ a_{ks} g_{ks} \exp \{-i\omega_{ks} t + i\omega t\} + H.C.), \quad (11.19)$$

$$V(t, \tau) = \prod_{j=1}^N \exp \{\frac{1}{2} |d| \varepsilon i(S_j^+ + S_j^-)(t-\tau)\}. \quad (11.20)$$
On using the properties of the spin angular momentum operators, it is easy to show that
\[
V(t, t - \tau) H_{RSI}(t - \tau) V^+(t, t - \tau) = \sum_{jks} [g_{ks} a_{ks} \exp \{ -i(t - \tau) \omega_{ks} \}]
\times \{ \cos^2 (\frac{1}{2} \omega_{1} \tau) S_j^+ + \sin^2 (\frac{1}{2} \omega_{1} \tau) S_j^- - 2i \sin (\frac{1}{2} \omega_{1} \tau) \cos (\frac{1}{2} \omega_{1} \tau) S_j^z \} + \text{H.C.},
\]
(11.21)

where
\[
\omega_{1} = |d| z.
\]
(11.22)

On substituting (11.21) and (11.19) in (11.15), we find that the reduced density operator \( \rho(t) \), for the atomic system, in the interaction picture [1] satisfies the master equation
\[
\frac{\partial \rho}{\partial t} - i \sum_{j} (\frac{1}{2} \omega_{1}) [S_j^+ + S_j^-, \rho] + \sum_{ij} \{ \frac{1}{4} (2 \kappa + \kappa_+) [S_i^+ S_j^- \rho] + \frac{1}{4} (2 \kappa - \kappa_+) [S_i^+ S_j^+ \rho] + \frac{1}{2} \kappa - [S_i^+ S_j^-, \rho] + \text{H.C.} \} = 0,
\]
(11.23)

where
\[
\kappa = \gamma - i \Omega, \quad \kappa_\pm = \kappa (\omega + \omega_1) \pm \kappa (\omega - \omega_1),
\]
(11.24)

and where (cf. eqs. (9.5), (9.6))
\[
\gamma(\omega) = \sum_{ks} |g_{ks}|^2 \delta(\omega - \omega_{ks}), \quad \Omega(\omega) = \sum_{ks} |g_{ks}|^2 P(\omega_{ks} - \omega)^{-1}.
\]
(11.25)

The master equation (11.23) should be compared with the one for weak external fields (cf. eqs. (5.27) and eq. (9.4)),
\[
\frac{\partial \rho}{\partial t} - i \sum_{j} (\frac{1}{2} \omega_{1}) [S_j^+ + S_j^-, \rho] + \sum_{ij} \{ \kappa [S_i^+ S_j^-, \rho] + \text{H.C.} \} = 0,
\]
(11.26)

which can be obtained from (11.23) by the approximation \( \kappa_+ \approx 2 \kappa, \) \( \kappa_- \approx 0 \). The master equation (11.23) leads to the following equation for the macroscopic mean value of an operator \( Q \) (in the rotating coordinate frame)
\[
\frac{\partial}{\partial t} \langle Q \rangle + i \sum_{j} (\frac{1}{2} \omega_{1}) [\langle S_j^+ + S_j^- \rangle, Q] + \sum_{ij} \{ -\frac{1}{4} (2 \kappa + \kappa_+) \langle [S_i^+ Q] S_j^- \rangle \\
-\frac{1}{4} (2 \kappa - \kappa_+) \langle [S_i^+ Q] S_j^+ \rangle - \frac{1}{2} \kappa_+ \langle [S_i^+ Q] S_j^- \rangle + \frac{1}{4} (2 \kappa^* + \kappa_+) \langle S_i^+ S_j^- Q \rangle \\
+ \frac{1}{4} (2 \kappa^* - \kappa_+) \langle S_i^- S_j^- Q \rangle + \frac{1}{2} \kappa^* \langle S_i^+ S_j^- Q \rangle \} = 0.
\]
(11.27)

The master equation (11.23) or the mean value eq. (11.27) is too complicated to be solved for the case of \( N \) two-level atoms. We present its solution for the case of one two-level atom. For one atom we have
$$\frac{\partial}{\partial t} \langle S^- \rangle = -i\omega_1 \langle S^\dagger S \rangle - \frac{i}{2} (2\kappa + \kappa_+) \langle S^- \rangle + \frac{i}{2} (2\kappa - \kappa_+) \langle S^+ \rangle + \frac{i}{4} \gamma_-,$$

(11.28)

$$\frac{\partial}{\partial t} \langle S^+ \rangle = \frac{i}{4} \omega_1 \langle S^+ S^- \rangle + \frac{1}{2} \kappa_- \langle S^+ \rangle + \frac{1}{2} \kappa_+ \langle S^- \rangle - \frac{i}{2} (2\gamma + \gamma_+) \langle S^+ S^- \rangle.$$

(11.29)

Eqs. (11.28) and (11.29) have been recently obtained by Lehmerberg [1970b]. Similar equations have been derived by Bloch [1957] and by others in connection with spin resonance saturation. If we ignore the frequency shifts, the steady state solution of eqs. (11.28) and (11.29) is given by

$$\text{Re} \langle S^+ \rangle \approx \frac{1}{2} \frac{\gamma_-}{\gamma_+} \approx \frac{\omega_1}{2} \frac{d}{d\omega} \ln \gamma(\omega); \quad \text{Im} \langle S^+ \rangle \approx -\left( \frac{\omega_1}{2\gamma} \right) \left[ 1 + 2 \left( \frac{\omega_1}{2\gamma} \right)^2 \right]^{-1};$$

$$\langle S^\dagger \rangle \approx -\frac{1}{2} \left[ 1 + 2 \left( \frac{\omega_1}{2\gamma} \right)^2 \right]^{-1},$$

(11.30)

where in obtaining (11.30) we also assumed that $\omega_1 \ll \omega$ so that $\kappa_+ \approx 2\kappa$, $\kappa_- \approx 2\omega_1 \frac{d\kappa}{d\omega}$. Hence $\text{Re} \langle S^+ \rangle \approx (3\omega_1/2\omega)$ and thus the in-phase component continues to increase as the strength of the external field increases. The out of phase components saturates and then decreases. Lehmerberg has noted that this effect can result in enhanced coherent scattering and fluorescence.

We now consider another example of strongly interacting quantum systems. We discuss the parametric frequency conversion in a medium with losses. In the absence of losses, the frequency conversion is described by the model Hamiltonian

$$H = \omega_a a^+ a + \omega_b b^+ b + g(ab^+ + \text{H.C.}),$$

(11.31)

where $a$ and $b$ are the annihilation operators associated with two field modes. We have assumed that the pump field is very strong, so that it can be treated as a $c$-number. The losses in the medium can be taken into account by using the prescription given in § 6 (cf. eq. (6.53)), i.e. by assuming that each field mode is interacting with its own reservoir. From (11.31), (6.53) and (2.24), we obtain the following equation for the Sudarshan-Glauber distribution function $\Phi^{(A)}(\alpha, \alpha^*, \beta, \beta^*; t)$

$$\frac{d\Phi^{(A)}}{dt} = \left\{ (i\omega_a + \gamma_a) \frac{\partial}{\partial \alpha} \alpha + (i\omega_b + \gamma_b) \frac{\partial}{\partial \beta} \beta + ig \frac{\partial}{\partial \beta} \alpha + ig \frac{\partial}{\partial \alpha} \beta + \gamma_a \langle n_a \rangle \frac{\partial^2}{\partial \alpha \partial \alpha^*} \right. \right.$$  

$$+ \gamma_b \langle n_b \rangle \frac{\partial^2}{\partial \beta \partial \beta^*} + \text{C.C.} \right\} \Phi^{(A)}.$$  

(11.32)

In order to keep the analysis as simple as possible, we also assume that
\(\gamma_a = \gamma_b = \gamma, \langle n_a \rangle = \langle n_b \rangle = \langle n \rangle, \omega_a = \omega_b = \omega.\) The Green’s function associated with (11.32) (which is in the form of a linearized Fokker-Planck equation) is given by (WALLS [1970])

\[
K^{(A)}(\alpha, \beta, \beta^*, \alpha^*, t | \alpha_0, \beta_0, \beta_0^*, 0) = \{\pi \langle n \rangle (1 - e^{-2\gamma t})\}^{-2} \times 
\exp \left\{ -\frac{1}{4} \left( (|\alpha + \beta| - (\alpha_0 + \beta_0) e^{-i\omega t} + i\sigma - 2\gamma t^2 + (|\beta - \alpha| - (\alpha_0 - \beta_0) e^{-i\omega t} + i\sigma - 2\gamma t^2)^2) \right) \langle n \rangle (1 - e^{-2\gamma t}) \right\}.
\]

The steady state solution is, therefore, given by

\[
\Phi_{st}^{(A)} = (\pi \langle n \rangle)^{-2} \exp \left\{ -\left( |\alpha|^2 + |\beta|^2 \right) / \langle n \rangle \right\},
\]

which implies that

\[
\rho_{st} = \exp \left\{ -\frac{(\omega a^+ a + \omega b^+ b)}{K_B T} \right\} / \text{Tr} \exp \left\{ -\frac{(\omega a^+ a + \omega b^+ b)}{K_B T} \right\}.
\]

It is seen that (11.35) is not consistent with the Hamiltonian (11.31). This is due to the fact that the interaction between the two modes has not been treated properly when we assumed that each mode is interacting with its own reservoir. The effect of the coupling can be taken into account in the irreversible part of the master equation by using a procedure similar to that used in connection with interactions with strong external fields. We write the total Hamiltonian in the form

\[
H = H_a + H_b + H_{ab} + H_{Ra} + H_{Rb} + H_{aRa} + H_{bRb},
\]

where \(H_{aRa}\) and \(H_{bRb}\) are the interaction Hamiltonian between the mode \(a\) and the reservoir \(R_a\) and the mode \(b\) and the reservoir \(R_b\) respectively. The reduced density operator, in the interaction picture, satisfies the master equation

\[
\frac{d\rho}{dt} + i[H_{ab}(t), \rho] + \text{Tr}_{Ra} \int_0^\infty [H_{aRa}(t), [V(t, t-\tau)H_{aRa}(t-\tau)] 
\times V^+(t, t-\tau), \rho_{Ra}(0)p(t)]]d\tau + \text{Tr}_{Rb} \int_0^\infty [H_{bRb}(t), [V(t, t-\tau)H_{bRb}(t-\tau)] 
\times V^+(t, t-\tau), \rho_{Rb}(0)p(t)]]d\tau = 0,
\]

where

\[
H_{aRa}(t) = \sum_j g_{ja} a^+ a_j \exp \{-i\omega_j t + i\omega_a t\} + \text{H.C.},
\]

\[
H_{bRb}(t) = \sum_j g_{jb} b^+ b_j \exp \{-i\omega_j t + i\omega_b t\} + \text{H.C.},
\]

and

\[
V(t, \tau) = \exp \{-i(t-\tau)g(a^+ b + ab^+)\}.
\]
In deriving (11.37) we made use of our conventional Born and short memory approximations. On simplifying (11.37) and on making the transformation to the Schrödinger picture we obtain the master equation

\[
\frac{\partial \rho}{\partial t} = -i\omega_a[a^+ a, \rho] - i\omega_b[b^+ b, \rho] - ig[ab^+ + a^+ b, \rho] \\
+ \{\gamma_a + [a, \rho a^+] + \gamma_b + [b, \rho b^+] + \gamma_a - [a, \rho b^+] + \gamma_b - [b, \rho a^+] + \text{H.C.} \}
\]

\[
+ \Gamma_{a+} [[a^+, \rho a^+] + [a \rho, a^+] + \Gamma_{b+} [[b^+, \rho b^+] + [b \rho, b^+] \\
+ \frac{1}{2} \Gamma_{a-} [[b^+, \rho], a] + [[b, \rho], a^+] + \frac{1}{2} \Gamma_{b-} [[a^+, \rho], b] + [[a, \rho], b^+],
\]

(11.41)

where

\[
\gamma_{i\pm} = \frac{1}{2}(\gamma(\omega_i + g) \pm \gamma(\omega_i - g)),
\]

\[
\Gamma_{i\pm} = \gamma(\omega_i + g)\langle n(\omega_i + g) \rangle \pm \gamma(\omega_i - g)\langle n(\omega_i - g) \rangle,
\]

(11.42)

and $\gamma(\omega)$ and $\langle n(\omega) \rangle$ are given by eqs. (6.9) and (6.6) respectively. This derivation of (11.41) differs from that of WALLS [1970], who considered the interaction of each of the normal modes of the Hamiltonian (11.31) with the heat bath. The Sudarshan-Glauber distribution function now satisfies the equation

\[
\frac{\partial \Phi^{(A)}}{\partial t} = \left( (\gamma_+ + i\omega) \frac{\partial}{\partial \alpha} + (\gamma_+ + i\omega) \frac{\partial}{\partial \beta} + ig \left( \frac{\partial}{\partial \beta} \alpha + \frac{\partial}{\partial \alpha} \beta \right) \\
+ \frac{1}{2} \Gamma_+ \left( \frac{\partial^2}{\partial \alpha \partial \alpha^*} + \frac{\partial^2}{\partial \beta \partial \beta^*} \right) - \gamma_- \left( \frac{\partial}{\partial \beta} \alpha + \frac{\partial}{\partial \alpha} \beta \right) + \Gamma_- \frac{\partial^2}{\partial \alpha \partial \beta^*} + \text{C.C.} \right) \Phi^{(A)},
\]

(11.43)

where we have again specialized to the case where $\omega_a = \omega_b$. Eq. (11.43) is in the form of a linearized Fokker-Planck equation and is easily solved by using the result given in Appendix. The steady state solution of (11.43) is

\[
\Phi_{st}^{(A)} = \langle \pi^2 \langle n(\omega + g) \rangle \langle n(\omega - g) \rangle \rangle^{-1} \exp \left\{ -\frac{1}{2} \left( \langle n(\omega + g) \rangle^{-1} + \langle n(\omega - g) \rangle^{-1} \right) \\
\times (|\alpha|^2 + |\beta|^2) - \frac{1}{2} \langle n(\omega + g) \rangle^{-1} - \langle n(\omega - g) \rangle^{-1} (|\alpha|^2 + |\beta|^2) \right\},
\]

which implies that

\[
\rho_{st} \propto \exp \left\{ -(\omega_a^+ a + \omega_b^+ b + g(a^+ b + ab^+))(K_b T) \right\}.
\]

(11.45)

This analysis shows that the system relaxes towards the correct steady state which corresponds to the total Hamiltonian $H_a + H_b + H_{ab}$, rather than the free Hamiltonian $H_a + H_b$. Of course in the weak coupling limit $g \to 0$, the master equation (11.43) reduces to (11.32). This result is the analog of the well known result in magnetic resonance, where a strongly driven spin

* The operator eq. (11.41) may be solved directly by using detailed balance (AGARWAL [1973b]).
system tends to relax along the instantaneous total field rather than along the static portion of the field.

§ 12. An Application of Phase Space Techniques to a Problem in Solid State Physics

In this section we consider a related problem namely the problem of anharmonic interaction in solids. This problem belongs to the domain of solid state physics, but we will discuss it here since it may be treated by methods that were specifically developed for treatments of problems in quantum optics. The classical treatment of this problem is given in detail in the book by PRIGOGINE [1962]; our quantum treatment follows closely that of CARRUTHERS and DY [1966] though our derivation of the master equation is different. We will in particular obtain the quantum analog of the BROUT-PRIGOGINE equation [1956].

We write the Hamiltonian in the form

$$ H = \sum_k \omega_k a_k^+ a_k - \sum_{kk'} (\omega_k \omega_{k'} \omega_{k''})^{-\frac{3}{4}} (V_{kk'k''} a_k^+ a_{k'} a_{k''} + H.C.), $$

where we have ignored the energy non-conserving terms and where $V_{kk'k''}$ are the coupling coefficients. For our purpose their specific form is not needed. On using (12.1) and (2.20) (generalized to the case of many degrees of freedom), we find that the distribution function $\Phi^{(N)}(\{z_k\}, \{z^*_k\}; t)$ satisfies the equation

$$ \frac{\partial \Phi^{(N)}}{\partial t} = \left( + i \sum_k \omega_k \left( \frac{\partial}{\partial z_k} - C.C. \right) + i \sum_{kk'k''} (\omega_k \omega_{k'} \omega_{k''})^{-\frac{3}{4}} V_{kk'k''} \right. $$

$$ \times \left[ z_k^* \left( z_{k'} + \frac{\partial}{\partial z_{k'}} \right) \left( z_k + \frac{\partial}{\partial z_k} \right) - \left( z_{k''}^* + \frac{\partial}{\partial z_{k''}} \right) z_k \right] - C.C. \left) \right) \Phi^{(N)}. $$

On transforming to the action angle variables $J_k$, $\varphi_k$ defined by (2.30), eq. (12.2) reduces to

$$ \frac{\partial \Phi^{(N)}}{\partial t} = \sum_k \omega_k \frac{\partial}{\partial \varphi_k} \Phi^{(N)} + \sum_{kk'k''} \left\{ \left( \frac{J_k J_{k'} J_{k''}}{\omega_k \omega_{k'} \omega_{k''}} \right)^{\frac{1}{4}} V_{kk'k''} \exp \{ i(\varphi_k + \varphi_{k'} - \varphi_{k''}) \} \right. $$

$$ \times \left[ i \left( \frac{\partial}{\partial J_k} + \frac{\partial}{\partial J_{k'}} - \frac{\partial}{\partial J_{k''}} \right) - \frac{1}{2} \left( \frac{1}{J_k} \frac{\partial}{\partial \varphi_k} + \frac{1}{J_{k'}} \frac{\partial}{\partial \varphi_{k'}} + \frac{1}{J_{k''}} \frac{\partial}{\partial \varphi_{k''}} \right) \right. $$

$$ + \left. i \left( J_k \frac{\partial^2}{\partial J_k \partial J_{k'}} - 2J_{k'} \frac{\partial^2}{\partial J_k \partial \varphi_{k'}} - \frac{1}{2} \frac{\partial^2}{\partial J_k \partial \varphi_k} - 4J_k J_{k'} \frac{\partial^2}{\partial \varphi_k \partial \varphi_{k'}} \right) \right] $$

$$ + C.C. \right) \Phi^{(N)}. $$

(12.3)
The energy distribution $\Phi^{(N)}(\{J_k\})$ is obtained from $\Phi^{(N)}$ by integrating over the phase variables $\varphi_k$, i.e.

$$\Phi^{(N)}(\{J_k\}; t) = \int_0^{2\pi} \cdots \int \Phi^{(N)}(\{J_k\}, \{\varphi_k\}; t) d(\{\varphi_k\}). \quad (12.4)$$

The projection operator $\mathcal{P}$ is given by eq. (1.22), which can also be written as

$$\mathcal{P} \ldots = \psi(\{0\})(\psi(\{0\}), \ldots), \quad (12.5)$$

where $\psi(\{v_k\})$ are the eigenfunctions given by (2.35). The master equation for $\Phi^{(N)}(\{J_k\}; t)$ in Born and in Markovian approximations, is obtained from (4.12) and is given by

$$\frac{\partial \Phi^{(N)}(\{J_k\}; t)}{\partial t} + \int_0^\infty (\psi(\{0\}), \mathcal{L}_1(t)\mathcal{L}_1(t-\tau)\psi(\{0\}))\Phi^{(N)} d\tau = 0. \quad (12.6)$$

In deriving (12.6) we assumed that the distribution function at time $t = 0$ is independent of the phases $\varphi_k$ so that $(1 - \mathcal{P})\Phi^{(N)}(\{J_k\}; \{\varphi_k\}; 0) = 0$. We also made use of the relation $\mathcal{P} \mathcal{L}_1(t)\mathcal{P} \Phi^{(N)} = 0$, which is easily proved from the form of $\mathcal{L}_1$ as given by the second term on the right-hand side of (12.3). On introducing a complete set of states (12.6) can be written as

$$\frac{\partial \Phi^{(N)}(\{J_k\}; t)}{\partial t} + \sum_{\{v\}} \int_0^\infty d\tau (\psi(\{0\}), e^{+i\omega_0 \tau} \mathcal{L}_1 e^{-i\omega_0 \tau} \psi(\{v_k\}))$$

$$\times (\psi(\{v_k\})e^{+i\omega_0 (t-\tau)} \mathcal{L}_1 e^{-i\omega_0 (t-\tau)} \psi(\{0\}))\Phi^{(N)}(\{J_k\}; t) = 0, \quad (12.7)$$

which on simplification leads to

$$\frac{\partial \Phi^{(N)}}{\partial t} + \sum_{\{v\}} (\psi(\{0\}) \mathcal{L}_1 \psi(\{v_k\})(\psi(\{v_k\}) \mathcal{L}_1 \psi(\{0\}))\delta_{-}(\sum v_k \omega_k) \Phi^{(N)} = 0. \quad (12.8)$$

The equation (12.8) is not valid for time intervals given by $0 \leq t \leq 1/\omega_D$ where $\omega_D$ is the Debye frequency. It is easily shown from (12.3) that the only non vanishing elements of $\mathcal{L}_1$ are $(\psi(\{0\}) \mathcal{L}_1 \psi(\{-1_k, -1_k', 1_k'; 0\}))$, $(\psi(\{0\}) \mathcal{L}_1 \psi(\{1_k, 1_k', -1_k'; 0\}))$ and their complex conjugates. These matrix elements are easily computed and (12.8) then reduces to (CARRUTHERS and DY [1966])*

$$\frac{\partial \Phi^{(N)}}{\partial t} = 2\pi \sum_{kk'} \delta(\omega_k + \omega_{k'} - \omega_0) \left( \frac{|V_{kk' - k'}|^2}{(\omega_k \omega_{k'} \omega_{k'})} \left[ \left( \frac{\partial}{\partial J_k} + \frac{\partial}{\partial J_{k'}} - \frac{\partial}{\partial J_{k'}} + \frac{\partial^2}{\partial J_k \partial J_{k'}} \right) \right] \Phi^{(N)} \right) \times (J_k J_{k'} J_{k'}). \quad (12.9)$$

* One may similarly show that $\Phi^{(A)}$ satisfies (12.9) with the signs of second order derivatives in (\ldots) changed.
This equation should be compared with the Brout-Prigogine equation for the classical distribution function \( \Phi_{cl}(\{J_k\}; t) \):

\[
\frac{\partial}{\partial t} \Phi_{cl} = 2\pi \sum_{kk'k''} \delta(\omega_k + \omega_{k'} - \omega_{k''}) \frac{|V_{kk'k''}|^2}{(\omega_k \omega_{k'} \omega_{k''})} \\
\times \left[ \left( \frac{\partial}{\partial J_k} + \frac{\partial}{\partial J_{k'}} - \frac{\partial}{\partial J_{k''}} \right) (J_k J_{k'} J_{k''}) \left( \frac{\partial}{\partial J_k} + \frac{\partial}{\partial J_{k'}} - \frac{\partial}{\partial J_{k''}} \right) \right] \Phi_{cl}. \tag{12.10}
\]

It is seen that (12.9) differs from (12.10) by the presence of second order derivatives which give rise to spontaneous decay of phonons. From eq. (12.9) we can easily establish the PEIERLS [1929] equation for the rate of change of phonons in the mode \( k \):

\[
\frac{d}{dt} \langle n_k \rangle = \int \frac{\partial}{\partial t} \Phi^{(N)}(\{J_k\}; t) J_k d(\{J_k\})
\]

\[
= 2\pi \sum_{kk'k''} \frac{|V_{kk'k''}|^2}{(\omega_k \omega_{k'} \omega_{k''})} \left\{ 2\delta(\omega_k + \omega_{k'} - \omega_{k''}) \left[ \langle n_k + 1 \rangle \langle n_{k'} + 1 \rangle \langle n_{k''} \rangle \right] \right.
\]

\[
- \left. \left[ \langle n_k \rangle \langle n_{k'} + 1 \rangle \right] \delta(\omega_k - \omega_{k'} - \omega_{k''}) \left[ \langle n_{k'} \rangle \langle n_{k'} + 1 \rangle \langle n_k \rangle \right] \right.
\]

\[
- \left. \left[ \langle n_k \rangle \langle n_k + 1 \rangle \langle n_{k'} \rangle \right] \right\}. \tag{12.11}
\]

Finally we mention that we have not reviewed the related problems of optical pumping and resonant scattering of photons. Optical pumping has been treated by WILLIS [1970] using the master equation methods. In the theory of resonant scattering of photons methods related to master equation techniques have been employed by VDOVIN and GALITSKI [1965].

Appendix

Some Properties of the Fokker-Planck Process

We discuss in this appendix the properties of the Fokker-Planck process (for discussions of the properties of the Fokker-Planck process see, for example STRATONOVICH [1963] or LAX [1966a, b; 1968a]). Consider the Fokker-Planck equation

\[
\frac{\partial P}{\partial t} = -\sum_i \frac{\partial}{\partial x_i} (A_i P) + \sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} (D_{ij} P), \tag{A.1}
\]

where \( A_i \) and \( D_{ij} \) are the drift and the diffusion coefficients respectively, given by
In eqs. (A.2) and (A.3) $P(\{x_{i1}\}, t|\{x_{i0}\}, 0)$ is the conditional distribution function for the Fokker-Planck process and is the solution of (A.1) subject to the initial condition

$$P(\{x_{i1}\}, 0|\{x_{i0}\}, 0) = \prod_{i=1}^{N} \delta(x_{i1} - x_{i0}).$$  

(A.4)

The $N$-time joint probability distributions can, of course, be determined from $P(\{x_{i1}\}, t|\{x_{i0}\}, 0)$ by using the Markoff property of the stochastic process under consideration. The drift coefficients in (A1) give the rate of change of the mean value of the variable $x_i(t)$, i.e.

$$\frac{\partial}{\partial t} \langle x_i(t) \rangle = \langle A_i(\{x_i\}) \rangle.$$  

(A.5)

The diffusion coefficients $D_{ij}$ give rise to the fluctuations in the mean value and form a semi-positive definite matrix. In the present article, we considered mainly quantum systems. The Fokker-Planck equation was then, an equation of motion for the phase space distribution function associated with the density operator. In such cases the diffusion matrix need not be semi-positive definite. However in most of the physical situations, the diffusion matrix is a semi-positive definite matrix and we assume it to be so throughout this appendix.

It is known (Bharucha-Reid [1960]) that the conditional probability distribution function $P(\{x_{i1}\}, t|\{x_{i0}\}, t_0)$ regarded as a function of $\{x_{i0}\}$ and $t_0$ satisfies the following Kolmogorov equation

$$\frac{\partial P}{\partial t_0} = - \sum_i A_i(\{x_{i0}\}) \frac{\partial P}{\partial x_{i0}} - \sum_{ij} D_{ij}(\{x_{i0}\}) \frac{\partial^2 P}{\partial x_{i0} \partial x_{j0}}.$$  

(A.6)

Eq. (A.6) is the adjoint of eq. (A.1). Eq. (A.6) is also known as the “backward equation” and eq. (A.1) as the “forward equation”.

$$A_i(\{x_{i0}\}) = \lim_{t \to 0} \frac{1}{t} \langle x_{i1} - x_{i0} \rangle$$  

$$\equiv \lim_{t \to 0} \frac{1}{t} \int \ldots \int (x_{i1} - x_{i0}) P(\{x_{i1}\}, t|\{x_{i0}\}, 0) d(\{x_{i1}\}),$$  

(A.2)

$$D_{ij}(\{x_{i0}\}) = \frac{1}{2} \lim_{t \to 0} \frac{1}{t} \langle (x_{i1} - x_{i0})(x_{j1} - x_{j0}) \rangle$$  

$$\equiv \frac{1}{2} \lim_{t \to 0} \frac{1}{t} \int \ldots \int (x_{i1} - x_{i0})(x_{j1} - x_{j0}) P(\{x_{i1}\}, t|\{x_{i0}\}, 0) d(\{x_{i1}\}).$$  

(A.3)
Eq. (A.1) can be rewritten in the form
\[ \frac{\partial P}{\partial t} + \sum_i \frac{\partial J_i}{\partial x_i} = 0, \]  
where \( J_i \) is the \( i \)th component of the probability current given by
\[ J_i = \left( A_i - \sum_j \frac{\partial}{\partial x_j} D_{ij} \right) P. \]  

Eq. (A.7) can be interpreted as the equation for the conservation of the probability. The steady state solution of (A.7) is given by \( \frac{\partial P}{\partial t} = 0 \). In many physical problems the principle of detailed balance will be obeyed, i.e. \( J_i = 0 \). Then the steady state solution, denoted by \( P_{\text{st}} \), is given by
\[ \frac{\partial}{\partial x_i} \ln P_{\text{st}} = \sum_j (D^{-1})_{ij} \left( A_j - \sum_k \frac{\partial D_{jk}}{\partial x_k} \right). \]  

From (A.9) and the relation
\[ \frac{\partial^2 P}{\partial x_i \partial x_j} = \frac{\partial^2 P}{\partial x_j \partial x_i}, \]  
it is seen that detailed balance is possible only if
\[ \frac{\partial}{\partial x_j} \left[ \sum_k (D^{-1})_{ik} \left( A_k - \sum_l \frac{\partial D_{kl}}{\partial x_l} \right) \right] = \frac{\partial}{\partial x_i} \left[ \sum_k (D^{-1})_{jk} \left( A_k - \sum_l \frac{\partial D_{kl}}{\partial x_l} \right) \right]. \]  

The condition (A.11) is known as the potential condition (Stratonovich [1963]). The connection of potential condition to the microscopic reversibility is well known* (see, for example, Lax [1968a]). One may show that the potential condition is satisfied for the Van der Pohl oscillator model of laser light and eq. (A.9) then leads to the steady state solution given by eq. (10.33).

One may also prove an "H-theorem" for the Fokker-Planck process. We introduce the "H-function" defined by
\[ H(t) = \int P(\{x_{i1}\}, t|\{x_{i0}\}, 0) \ln \left( \frac{P(\{x_{i1}\}, t|\{x_{i0}\}, 0)}{P_{\text{st}}(\{x_{i1}\})} \right) d(\{x_{i1}\}). \]  

It follows from the positive definiteness of \( P \) and the inequality \( x \ln x - x + y \geq 0 \) that \( H(t) \geq 0 \). Moreover on using (A.12) and (A.9), one may show that the rate of change of the "H-function" is given by
\[ \frac{dH}{dt} = - \sum_{ij} P(\{x_{i1}\}, t|\{x_{i0}\}, 0) D_{ij} \left( \frac{\partial}{\partial x_i} \ln \left( \frac{P(\{x_{i1}\}, t|\{x_{i0}\}, 0)}{P_{\text{st}}(\{x_{i1}\})} \right) \right) \times \left( \frac{\partial}{\partial x_j} \ln \left( \frac{P(\{x_{i1}\}, t|\{x_{i0}\}, 0)}{P_{\text{st}}(\{x_{i1}\})} \right) \right) d(\{x_{i1}\}). \]  

* In our treatment we have assumed that the reversible part of the drift vector is zero (for generalizations see Graham and Haken [1971]).
From eq. (A.13) and from the positive definiteness of the diffusion matrix it follows that
\[ \frac{dH}{dt} \leq 0, \] (A.14)
which is the "H-theorem" for the multidimensional Fokker-Planck process.

We now discuss the eigenfunctions and the eigenvalues of the Fokker-Planck operator
\[ \mathcal{L} = - \sum_i \frac{\partial}{\partial x_i} A_i + \sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} D_{ij}, \] (A.15)
\[ \mathcal{L}\psi_i = -\lambda_i \psi_i. \] (A.16)
In general the operator \( \mathcal{L} \) is not Hermitian. Eq. (A.16) can be changed into a self adjoint equation by the transformation of the eigenfunctions \( \psi_i \)
\[ \psi_i = \exp \{ \chi \} \phi_i. \] (A.17)
We then obtain the eigenvalue equation (LOUISELL [1969])
\[ \mathcal{L}' \phi_i = -\lambda_i \phi_i, \] (A.18)
where
\[ \mathcal{L}' = \sum_{ij} \left[ \frac{\partial}{\partial x_i} D_{ij} \frac{\partial}{\partial x_j} + \frac{\partial^2 D_{ij}}{\partial x_i \partial x_j} \right] + \left( \frac{\partial^2 \chi}{\partial x_i \partial x_j} + \frac{\partial^2 \chi}{\partial x_i \partial x_j} \right) \frac{\partial \chi}{\partial x_i} \frac{\partial \chi}{\partial x_j} - \sum_i \left( \frac{\partial A_i}{\partial x_i} + A_i \frac{\partial \chi}{\partial x_i} \right), \] (A.19)
and the function \( \chi \) is given by
\[ \frac{\partial \chi}{\partial x_i} = -\frac{1}{4} \sum_j (D^{-1})_{ij} \left( \sum_k \frac{\partial D_{jk}}{\partial x_k} - A_{ij} \right). \] (A.20)
Since \( \mathcal{L}' \) is a self-adjoint operator its eigenfunctions and eigenvalues form a complete set:
\[ \sum_i \psi_i(\{x_{1i}\})\psi_i(\{x_{0i}\}) \exp (-2\chi) = \prod_i \delta(x_{1i} - x_{0i}), \] (A.21)
and the conditional distribution function is then given by
\[ P(\{x_{1i}\}, t|\{x_{0i}\}, t_0) = \sum_i e^{-\chi(t-t_0)} \psi_i(\{x_{1i}\})\psi_i(\{x_{0i}\}) \exp (-2\chi). \] (A.22)
Some examples of the eigenfunction expansion are given in the book by STRATONOVICH [1963]. On comparing (A.20) and (A.9) it is seen that \( \chi = \frac{1}{2} \ln P_{st} \).

The conditional distribution function \( P \), for the special case when the drift coefficients are linear in \( \{x_i\} \) and the diffusion coefficients \( D_{ij} \) are
independent of random variables, is well known (Wang and Uhlenbeck [1945]). If we set

$$A_i = \sum_j \beta_{ij} x_j,$$  \hspace{1cm} (A.23)

then the solution is

$$P(x_{i1}, t|x_{i0}, 0) = \left[ (2\pi)^N |\det \sigma(t)| \right]^{-\frac{1}{2}} \times \exp \left\{ -\frac{1}{2}(X - b(t)X_0)^T \sigma^{-1}(t)(X - b(t)X_0) \right\}. \hspace{1cm} (A.24)$$

where $X_0$ and $X$ are the column matrices

$$X_0 = \begin{pmatrix} x_{10} \\ \vdots \\ x_{N0} \end{pmatrix}, \quad X = \begin{pmatrix} x_{11} \\ \vdots \\ x_{N1} \end{pmatrix},$$

and the superscript $T$ denotes the transpose of the matrix. The parameters $b(t)$ and $\sigma(t)$ are given by

$$b(t) = e^{bt}, \quad \sigma(t) = \sigma(\infty) - b(t)\sigma(\infty)b^T(t), \hspace{1cm} (A.25)$$

and $\sigma(\infty)$ is the solution of

$$\beta\sigma(\infty) + \sigma(\infty)b^T = -2D. \hspace{1cm} (A.26)$$

The linearized Fokker-Planck equations, i.e. the eq. (A.1) with $A_i$ given by (A.23) and with the diffusion constants independent of $\{x_i\}$, occur in many physical examples such as in Brownian motion of an oscillator (eq. (7.7); Wang and Uhlenbeck [1945], Agarwal [1971d]); in the theories of relaxation of an oscillator (eq. (6.11); Louisell and Marburger [1967], Agarwal [1969]), parametric frequency conversion (eq. (11.43); Walls [1970]), parametric oscillator (Graham [1968]) oscillating below or above threshold and a laser (Risken et al. [1966]) oscillating below or above threshold.

We next consider the Langevin treatment of the Fokker-Planck process. We will only quote the main result (for details see Lax [1966b], Stratonovich [1963]). Consider a nonlinear Langevin process defined by

$$\frac{dx_i}{dt} = \beta_i + \sum_j \sigma_{ij} F_j(t), \quad (i = 1, 2, \ldots, N), \hspace{1cm} (A.27)$$

where $F_j(t)$ are independent delta correlated Gaussian random processes, i.e.

$$\langle F_j(t) \rangle = 0, \quad \langle F_i(t)F_j(t') \rangle = 2\delta_{ij}\delta(t-t'), \hspace{1cm} (A.28)$$

and all the higher order linked moments (cumulants) vanish. Then the Langevin process (A.27) is equivalent to the Fokker-Planck process (A.1) with
\[ A_i = \beta_i + \sum_{jk} \frac{\partial \sigma_{ij}}{\partial x_k} \sigma_{kj}, \]  
(A.29)

and

\[ D_{ij} = \sum_k \sigma_{ik} \sigma_{jk}. \]  
(A.30)

Conversely if a Fokker-Planck process is given, we can construct a Langevin process since the diffusion matrix is a symmetric and positive definite matrix and necessarily possesses a square root.

It is easily seen that the linearized Fokker-Planck equation is equivalent to the Langevin equations

\[ \frac{dx_i}{dt} = \sum_j \beta_{ij} x_j + F_i(t), \quad (i = 1, 2, \ldots, N), \]  
(A.31)

where

\[ \langle F_i(t) \rangle = 0, \quad \langle F_i(t)F_j(t') \rangle = 2D_{ij} \delta(t-t'), \]  
(A.32)

and all the higher order linked moments of the random force \( F_i(t) \) vanish. It is easily seen from (A.31) that the random variables \( \{x_i\} \) constitute a Gaussian Markov process. It is also clear from (A.31) that the autocorrelation function of the process \( X(t) \) is given by

\[ \langle X(t)X'(0) \rangle = \exp(\beta t)\langle X(0)X'(0) \rangle. \]  
(A.33)

We therefore obtain Doob's theorem (Wang and Uhlenbeck [1945]): A random process that is stationary, Gaussian and Markovian, possesses an autocorrelation function of the form (A.33).

In the text of the present article we have written the Fokker-Planck equations in terms of the complex coordinates \( z \) and \( z^* \) in the form*

\[ \frac{\partial P}{\partial t} = \left\{ \frac{\partial}{\partial z} (A(z)P) + \text{C.C.} \right\} + \left\{ D_{zz} \frac{\partial^2 P}{\partial z^2} + \frac{\partial^2 P}{\partial z \partial z^*} + \text{C.C.} \right\}. \]  
(A.34)

This Fokker-Planck equation is equivalent to the following Langevin equations

\[ \dot{z} = -A(z) + F(t), \quad \dot{z}^* = -[A(z)]^* + F^*(t), \]  
(A.35)

where \( F(t) \) is a complex delta correlated Gaussian random process with zero mean, i.e.

\[ \langle F(t) \rangle = 0, \quad \langle F(t)F(t') \rangle = 2D_{zz} \delta(t-t'), \]  
\[ \langle F(t)F^*(t') \rangle = 2D_{zz^*} \delta(t-t'), \]  
(A.36)

and all the higher order linked moments of the complex force \( F(t) \) vanish.

* We have assumed for the sake of simplicity that \( D \)'s are constants.
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Note added in proof:

Since this article was submitted for publication, much more has been done on the properties of Fokker-Planck equations and their generalizations under the condition that detailed balance is obeyed (Graham and Haken [1971], Risken [1972], Agarwal [1972, 1973b, c], Van Kampen [1971]).

The attention of the author has also been drawn to a number of Russian papers which deal with master equation methods in quantum optical problems:
