Random spread on the family of small-world networks

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We present analytical and numerical results of a random walk on the family of small-world graphs. The average access time shows a crossover from regular to random behavior with increasing distance from the starting point of the random walk. We introduce an independent step approximation, which enables us to obtain analytic results for the average access time. We observe a scaling relation for the average access time in the degree of the nodes. The behavior of the average access time as a function of \( p \) shows striking similarity with that of the characteristic length of the graph. This observation may have important applications in routing and switching in networks with a large number of nodes.

I. INTRODUCTION

The small-world network exhibits unusual connection properties. On one hand it shows strong clustering, like regular graphs, and on the other hand it shows a very small average shortest path between any two nodes, like random graphs. Watts and Strogatz have proposed a simple model to describe small-world networks [1]. The model gives a prescription for generating a one-parameter family of graphs, ranging from highly clustered (regular) graphs to random graphs.

Various properties of this model have been studied [2–14]. The spread and percolation properties investigated in Refs. [2–5] deal with the spread of information (disease) along the shortest path in the graph or the spread along the spanning tree. In Ref. [14] the diffusion process and the dispersion relations on small-world networks are studied.

In this paper, we study random walks on the family of small-world networks. Such a random walk corresponds to a random spread of information on the network. In any realistic application of the spread on a graph, we expect the spread to be somewhere in between the two extremes, viz., the shortest path and the random walk. For example, in Milgram’s experiment [15], which studies the connection properties of social networks, the path of a letter from a randomly chosen point to a fixed target is traced. The only condition imposed on the transfer of letter is that the letter should be given to a person whom the sender knows by first name. The path followed by such a letter would have both random and shortest path elements in it. Another example is the path of an internet protocol packet which follows a similar algorithm for forwarding the packet [16]. Most of the earlier work has concentrated on properties based on the shortest path and hence we address the other extreme in this paper [14].

The determination of the shortest path between two nodes is prohibitively expensive for networks with a large number of connections or networks where nodes and connections are added or removed dynamically. Examples of such networks are social networks, telephone networks, the internet, etc.

Another problem with the determination of the shortest path is the incomplete knowledge of the network. Hence, it is clear that an alternative method of generating a path (which need not be the shortest) becomes necessary in these networks. Our analysis of the random walk shows that the average access time (or the first passage time) between two nodes varies as \( O(n) \), for small-world geometry, where \( n \) is the number of nodes. It is thus beneficial to consider a network with random routing or switching, particularly if it has small-world properties. Thus the random routing emerges as both a practical and a computationally cheaper method for large networks.

It is interesting to note that the normal practice of sending packets through the internet does have a random element in it. If a computer has \( k \) connections, depending on the address to which the packet is to be sent, some of the connections are chosen in a deterministic way, while if the packet is not addressed to one of these deterministic values it is sent via one of the remaining connections randomly. Such a choice is normally based on the path of least traffic and need not correspond to the shortest path. There is usually an upper limit to the number of such steps that the packet takes [19]; if the node is not reached within these steps the address is treated as untraceable. Our results now show that such a random routing can be effective for small-world geometries.

In Sec. II we discuss analytical and numerical results for the average access time of the random walk on a one-parameter family of graphs ranging from the regular case to the random case. We introduce an independent step approximation that allows us to get analytical expressions for the average access time. We discuss these results in Sec. III. It is found that the random walk results are similar to the shortest path results. Thus from the nature of the outcome of an experiment it may be difficult to conclude whether the spread was random or along the shortest path. An important consequence of this result can be in the routing and switching in very big networks. Random routing is a promising method, particularly if the network has small-world properties. Section IV summarizes the results.

II. RANDOM WALK ON SMALL-WORLD GRAPHS

The random walk on a graph is performed as follows: We start with a fixed node (say \( i \)) and at each step make a jump