Introduction to Particle Physics

Sreerup Raychaudhuri

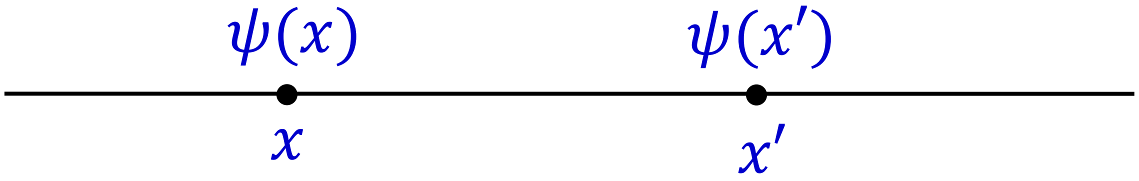
TIFR

Lecture 4

Gauge Symmetry : Manifest and Hidden

Lagrangian Field Theory

Let $\psi(x)$ be a field defined on a Minkowski space with coordinates x i.e. for every value of x there is a value of $\psi(x)$.



If we treat $\psi(x)$ at every point x as a generalised coordinate, then clearly this is a system with *infinite* number of degrees of freedom.

In Lagrangian dynamics, this will be described by a Lagrangian L

$$L = \int d^3\vec{x} \;\; \mathcal{L}\big(\psi(x), \partial_\mu \psi(x)\big)$$

where $\mathcal{L}$ is the Lagrangian density and the integral is over all space.

The action integral will be given by

$$S = \int dt\ L = \int d^4x\ \ \mathcal{L}\big(\psi(x), \partial_\mu \psi(x)\big)$$

The dynamics of this field will be driven by Hamilton's Principle, viz.

$$\text{if}\quad \psi(x) \to \psi(x) + \delta\psi(x) \qquad \text{then} \qquad \delta S = 0$$

This will lead to Euler-Lagrange equations

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial\{\partial_\mu \psi(x)\}}\right] - \frac{\partial \mathcal{L}}{\partial \psi(x)} = 0$$

If there are many fields $\psi_1(x), \psi_2(x), \dots, \psi_n(x)$ the Lagrangian is

$$L = \int d^3\vec{x}\ \ \mathcal{L}\Big(\psi_1(x), \dots, \psi_n(x), \partial_\mu \psi_1(x) \dots, \partial_\mu \psi_n(x)\Big)$$

and there are n sets of Euler-Lagrange equations...

Nature of field	Euler-Lagrange eqs.	Lagrangian density
real scalar $\varphi(x)$	$(\Box + M^2)\varphi = 0$	$\mathcal{L} = \frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2}M^2\varphi^2$
complex scalar $\varphi(x)$	$\left.\begin{array}{l}(\Box + M^2)\varphi = 0 \\ (\Box + M^2)\varphi^* = 0\end{array}\right\}$	$\mathcal{L} = \partial^\mu \varphi^* \partial_\mu \varphi - M^2 \varphi^* \varphi$
Dirac spinor $\psi(x)$	$\left.\begin{array}{l}i\gamma^\mu \partial_\mu \psi - m\psi = 0 \\ i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0\end{array}\right\}$	$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$
e.m. field $A_\mu(x)$	$\partial_\mu F^{\mu\nu} = j^\nu$ $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$	$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j^\nu A_\nu$

These are the standard relativistic fields

Nöther's Theorem (again!)

If, under a transformation $\psi_i(x) \rightarrow \psi_i(x) + \delta\psi_i(x)$, we have $\delta\mathcal{L} = 0$, this will be called a symmetry of the system.

For an infinitesimal change, it follows that

$$\delta\mathcal{L} = \sum_i \frac{\partial\mathcal{L}}{\partial\{\partial_\mu \psi_i\}} \delta\{\partial_\mu \psi_i\} + \frac{\partial\mathcal{L}}{\partial\psi_i} \delta\psi_i$$

As before, substitute the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\psi_i} = \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial\{\partial_\mu \psi_i\}}\right]$$

to get

$$\delta\mathcal{L} = \sum_i \frac{\partial\mathcal{L}}{\partial\{\partial_\mu \psi_i\}} \partial_\mu\{\delta\psi_i\} + \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial\{\partial_\mu \psi_i\}}\right] \delta\psi_i = \sum_i \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial\{\partial_\mu \psi_i\}} \delta\psi_i\right]$$

i.e.

$$\delta\mathcal{L} = \partial_\mu \sum_i \frac{\partial\mathcal{L}}{\partial\{\partial_\mu\psi_i\}}\delta\psi_i = \partial_\mu j^\mu$$

where $j^\mu = \sum_i \frac{\partial\mathcal{L}}{\partial\{\partial_\mu\psi_i\}}\delta\psi_i$ is called the Nöther current.

Now, for a symmetry, $\delta\mathcal{L} = 0 \Rightarrow \partial_\mu j^\mu = 0$

i.e. we get an equation of continuity for the Nöther current.

Written out explicitly, the equation of continuity assumes the usual form, i.e.

$$\partial_\mu j^\mu = 0 \;\Rightarrow\; \partial_t j^0 + \vec{\nabla}.\vec{j} = 0$$

Now, integrating over all space,

$$\partial_t \int d^3\vec{x}\; j^0 + \int d^3\vec{x}\;\vec{\nabla}.\vec{j} = 0 \qquad \Rightarrow \partial_t \int d^3\vec{x}\; j^0 + \oint \vec{j}.\hat{n}\, ds = 0$$

i.e. $\partial_t \int d^3\vec{x}\; j^0 = 0$

We define $Q = \int d^3\vec{x}\, j^0$ as the Nöther charge

Gauge Invariance of a Complex Scalar Field

The Lagrangian density

$$\mathcal{L} = \partial^\mu \varphi^*(x)\partial_\mu \varphi(x) - M^2\varphi^*(x)\varphi(x)$$

is manifestly invariant under a global gauge transformation

$$\varphi(x) \rightarrow \varphi^{'}(x) = e^{-ig\theta}\,\varphi(x)$$

where θ is an arbitrary (real) constant and g is a (real) constant specific to the field...

Also: $\varphi(x) = \frac{1}{\sqrt{2}}[\varphi_1(x) + i\varphi_2(x)]$ and $\varphi^*(x) = \frac{1}{\sqrt{2}}[\varphi_1(x) - i\varphi_2(x)]$

$$\left.\begin{aligned} \varphi'_1(x) &= \varphi_1(x)\cos g\theta - \varphi_2(x)\sin g\theta \\ \varphi_2^{'}(x) &= \varphi_1(x)\sin g\theta + \varphi_2(x)\cos g\theta \end{aligned}\right\}$$

complex rotation

This set of transformations forms an Abelian (commutative) group

Proof:

Group product $\Rightarrow$ successive transformations $\varphi(x) \rightarrow e^{-ig\theta_2} e^{-ig\theta_1} \varphi(x)$

1. *closure* : $e^{-ig\theta_2} e^{-ig\theta_1} = e^{-ig(\theta_2+\theta_1)}$
2. *associativity* : $e^{-ig\theta_3}(e^{-ig\theta_2} e^{-ig\theta_1}) = (e^{-ig\theta_3} e^{-ig\theta_2}) e^{-ig\theta_1}$
   $= e^{-ig(\theta_3+\theta_2+\theta_1)}$
3. *identity* : $\theta = 0 \; ; \; e^0 = 1$
4. *inverse* : $e^{+ig\theta} e^{-ig\theta} = e^0 = 1$
5. *commutativity*: $e^{-ig\theta_2} e^{-ig\theta_1} = e^{-ig\theta_1} e^{-ig\theta_2} = e^{-ig(\theta_1+\theta_2)}$

This set of phases $e^{-ig\theta}$ forms the group of unitary 1×1 matrices: U(1)

These are global U(1) gauge transformations

Nöther current corresponding to the global U(1) gauge symmetry:

$$j^{\mu} = \frac{\partial \mathcal{L}}{\partial\{\partial_{\mu}\varphi\}}\delta\varphi + \frac{\partial \mathcal{L}}{\partial\{\partial_{\mu}\varphi^{*}\}}\delta\varphi^{*}$$

If $\mathcal{L} = \partial^{\mu}\varphi^{*}(x)\partial_{\mu}\varphi(x) - M^{2}\varphi^{*}(x)\varphi(x)$, we get

$$\frac{\partial \mathcal{L}}{\partial\{\partial_{\mu}\varphi\}} = \partial^{\mu}\varphi^{*} \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial\{\partial_{\mu}\varphi^{*}\}} = \partial^{\mu}\varphi$$

Now consider an infinitesimal gauge transformation, i.e. $\theta \ll 1$

$$\delta\varphi(x) = \varphi^{'}(x) - \varphi(x) = \left(e^{-ig\theta} - 1\right)\varphi(x) \approx -ig\theta\varphi(x)$$

$$\delta\varphi^{*}(x) = \varphi^{'*}(x) - \varphi^{*}(x) = \left(e^{+ig\theta} - 1\right)\varphi^{*}(x) \approx +ig\theta\varphi^{*}(x)$$

Plugging in these values..

$$j^\mu = \partial^\mu \varphi^* [-ig\theta\varphi(x)] + \partial^\mu \varphi [+ig\theta\varphi^*(x)]$$
$$= -ig\theta[\partial^\mu \varphi^* \ \varphi(x) - \ \varphi^*(x)\, \partial^\mu \varphi\ ]$$

Drop the θ factor:

$$J^\mu = -ig[\partial^\mu \varphi^* \ \varphi(x) - \ \varphi^*(x)\, \partial^\mu \varphi\ ] = -ig\varphi^* \overleftrightarrow{\partial}_\mu \varphi$$

scalar current

Nöther charge:

$$Q = \int d^3\vec{x}\, j^0 = g \int d^3\vec{x}\, (-i)[\partial^\mu \varphi^* \ \varphi(x) - \ \varphi^*(x)\, \partial^\mu \varphi\ ]$$

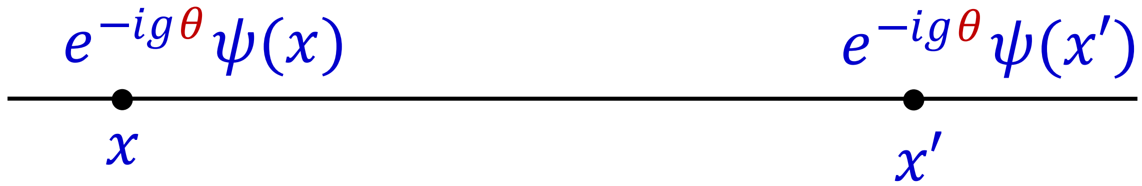
This is nothing but the probability for a Klein-Gordon particle,

i.e. gauge symmetry leads to conservation of probability...

Normalisation: $\int d^3\vec{x}\, (-i)[\partial^\mu \varphi^* \ \varphi(x) - \ \varphi^*(x)\, \partial^\mu \varphi\ ] = 1$

i.e. $Q = g$ U(1) charge of $\varphi(x)$

A global gauge transformation is not compatible with relativity



does not account for finite time of signal propagation

Replace it with a local U(1) gauge transformation:

$$\varphi(x) \rightarrow \varphi'(x) = e^{-ig\theta(x)}\varphi(x)$$

which also forms a U(1) group

(Set of global U(1) gauge transfns $\subset$ set of local U(1) gauge transfns)

Demand: The action *S* should be invariant under this transformation, since it is physically meaningful

Under this local U(1) g.t. the fields change to

$$\varphi(x) \;\; \rightarrow \varphi^{'}(x) \;\; = e^{-ig\theta\,(x)}\varphi(x)$$

$$\varphi^{*}(x) \rightarrow \varphi'^{*}(x) = e^{+ig\theta\,(x)}\varphi^{*}(x)$$

The Lagrangian changes to

$$\mathcal{L}' = \partial^{\mu}\varphi^{'\,*}(x)\partial_{\mu}\varphi'(x) - M^{2}\varphi^{'\,*}(x)\varphi'(x)$$

$$= \partial^{\mu}\left[e^{+ig\theta\,(x)}\varphi^{*}(x)\right]\partial_{\mu}\left[e^{-ig\theta\,(x)}\varphi(x)\right] - M^{2}\varphi^{*}(x)\varphi(x)$$

$$= \; \mathcal{L} \; + ig\partial_{\mu}\theta \; (\varphi^{*} - \varphi) - g^{2}(\varphi^{*}\varphi) \; \partial^{\mu}\theta \; \partial_{\mu}\theta$$

The theory is no longer gauge invariant!!

This is not physically acceptable, because then we would be able to measure phases in quantum mechanics, which we cannot $\Rightarrow$ paradox

Something must be missing...

Take the Lagrangian density

$$\mathcal{L} = [\partial^\mu \varphi(x)]^* \left[\partial_\mu \varphi(x) \right] - M^2 \varphi^*(x)\varphi(x)$$

and rewrite it as

$$\mathcal{L} = [\partial^\mu \varphi(x) + igA^\mu(x)\varphi(x)]^* \left[\partial_\mu \varphi(x) + igA_\mu(x)\varphi(x) \right] - M^2 \varphi^*(x)\varphi(x)$$

where $A_\mu(x)$ is a gauge field introduced to get gauge invariance.

Under local U(1) g.t.:

$$\begin{aligned}
\partial_\mu \varphi + igA_\mu \varphi \rightarrow\ & \partial_\mu \varphi' + igA'_\mu \varphi' \\
&= \partial_\mu \left[e^{-ig\theta} \varphi \right] + igA'_\mu \left[e^{-ig\theta} \varphi \right] \\
&= e^{-ig\theta} \partial_\mu \varphi - ig\partial_\mu \theta e^{-ig\theta} \varphi + igA'_\mu e^{-ig\theta} \varphi \\
&= e^{-ig\theta} \left(\partial_\mu \varphi - ig\partial_\mu \theta \varphi + igA'_\mu \varphi \right) \\
&= e^{-ig\theta} \left[\partial_\mu \varphi - ig\left(\partial_\mu \theta - A'_\mu \right) \varphi \right] \\
&= e^{-ig\theta} \left[\partial_\mu \varphi + igA_\mu \varphi \right] \quad \text{if we have } A'_\mu = A_\mu + \partial_\mu \theta
\end{aligned}$$

<u>Shorter notation</u>: write $\partial_\mu \varphi + igA_\mu \varphi = (\partial_\mu + igA_\mu)\varphi \equiv D_\mu \varphi$

The Lagrangian density becomes

$$\mathcal{L} = [D^\mu \varphi(x)]^* [\, D_\mu \varphi(x)] - M^2 \varphi^*(x)\varphi(x)$$

Under a local U(1) g.t., we have seen that

$$\varphi(x) \to \varphi'(x) \ = e^{-ig\theta(x)}\varphi(x)$$

$$D_\mu \varphi(x) \to \ D'_\mu \varphi'(x) \ = e^{-ig\theta(x)}\, D_\mu \varphi(x)$$

so that the Lagrangian density becomes trivially invariant.

The construction $D_\mu \varphi$ transforms in the same way as the $\varphi(x)$, so we call it a <u>covariant derivative</u>.

Write out the Lagrangian density in full...

$$\mathcal{L} = [\partial^\mu \varphi + igA^\mu \varphi]^* \left[\partial_\mu \varphi + igA_\mu \varphi \right] - M^2 \varphi^* \varphi$$

$$= (\partial^\mu \varphi)^* \, \partial_\mu \varphi - ig\left(\varphi^* \partial_\mu \varphi - \partial_\mu \varphi^* \varphi\right) A^\mu + g^2 \varphi^* \varphi A^\mu A_\mu - M^2 \varphi^* \varphi$$

$$= \underbrace{(\partial^\mu \varphi)^* \, \partial_\mu \varphi - M^2 \varphi^* \varphi}_{\text{free scalar}} + \underbrace{ig\left(\varphi^* \overleftrightarrow{\partial}_\mu \varphi\right) A^\mu}_{gJ_\mu A^\mu} + \underbrace{g^2 \varphi^* \varphi \, A^\mu A_\mu}_{\text{‘seagull’ term}}$$

Do we understand this A_μ field?

Consider its Euler-Lagrange equation: $\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \{\partial_\mu A_\nu\}} \right] - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$

$$gJ_\nu + g^2 \varphi^* \varphi \, A_\nu = 0 \quad \Rightarrow \quad A_\nu = \frac{J_\nu}{g \varphi^* \varphi} = \frac{1}{g} \frac{\varphi^* \overleftrightarrow{\partial}_\mu \varphi}{\varphi^* \varphi}$$

$\Rightarrow$ nonlinear Lagrangian... nonlinear wave equations... no quantum theory

Again, something must be missing....

The A_ν fields must have some <u>dynamics</u>,

i.e. there must be a term with $\partial_\mu A_\nu$

This term must be both Lorentz-invariant and gauge-invariant

Under a local U(1) g.t., we know that $A_\nu \rightarrow A_\nu + \partial_\nu \theta$

Then, $\partial_\mu A_\nu \rightarrow \partial_\mu A_\nu + \partial_\mu \partial_\nu \theta$

and $\partial_\nu A_\mu \rightarrow \partial_\nu A_\mu + \partial_\nu \partial_\mu \theta$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \rightarrow \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$$ field strength tensor

Lorentz-invariant construction: $F_{\mu\nu} F^{\mu\nu}$

Full Lagrangian:

$$\mathcal{L} = (\partial^\mu \varphi)^* \, \partial_\mu \varphi - M^2 \varphi^* \varphi + ig\left(\varphi^* \overleftrightarrow{\partial}_\mu \varphi\right) A^\mu + g^2 \varphi^* \varphi \, A^\mu A_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

The Euler-Lagrange equation becomes:

$$\partial_\mu F^{\mu\nu} = gJ^\nu - g^2 \varphi^* \varphi \, A^\nu$$

For small g, this reduces to

$$\partial_\mu F^{\mu\nu} = gJ^\nu$$

i.e. identical with Maxwell’s equations...

It follows that the A_μ must be the electromagnetic field and $g = qe$.

The quantum mechanics of a complex scalar field has no physical meaning unless we couple it to an electromagnetic field...

electromagnetism $\Leftrightarrow$ inability to measure phase of a wavefunction

Gauge Invariance of a Dirac Field

The Lagrangian density

$$\mathcal{L} = i\,\bar{\psi}\,\gamma^{\mu}\,\partial_{\mu}\psi - m\,\bar{\psi}\psi$$

is manifestly invariant under a global U(1) gauge transformation

$$\psi(x) \to \psi^{'}(x) = e^{-ie\theta}\,\psi(x)$$

where θ is an arbitrary (real) constant and g is a (real) constant specific to the field...

Easy to show that the Nöther current corresponding to this symmetry is the Dirac current $J^{\mu} = e\,\overline{\psi}\,\gamma^{\mu}\psi$ and the Nöther charge is just

$$Q = \int d^3\vec{x}\ j^0 = e\int d^3\vec{x}\ \bar{\psi}\,\gamma^0\psi = g\int d^3\vec{x}\,\psi^{\dagger}\psi = e$$

For local U(1) gauge invariance, replace $\mathcal{L} = i\,\bar{\psi}\,\gamma^{\mu}\,\partial_{\mu}\psi - m\,\bar{\psi}\psi$ by

$$\mathcal{L} = i\,\bar{\psi}\,\gamma^{\mu}\,D_{\mu}\psi - m\,\bar{\psi}\psi - \tfrac{1}{4}F_{\mu\nu}\,F^{\mu\nu}$$

where, as before, $D_{\mu} = \partial_{\mu} + ieA_{\mu}$ and $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$.

Expanding the covariant derivative, we get

$$\mathcal{L} = i\,\bar{\psi}\,\gamma^{\mu}\,\partial_{\mu}\psi - e\,\bar{\psi}\,\gamma^{\mu}\,\psi A_{\mu} - m\,\bar{\psi}\psi - \tfrac{1}{4}F_{\mu\nu}\,F^{\mu\nu}$$

$$= \underbrace{i\,\bar{\psi}\,\gamma^{\mu}\,\partial_{\mu}\psi - m\,\bar{\psi}\psi}_{\text{free fermion}} - \underbrace{\tfrac{1}{4}F_{\mu\nu}\,F^{\mu\nu}}_{\text{free e.m.}} \underbrace{- e\,\bar{\psi}\,\gamma^{\mu}\,\psi A_{\mu}}_{-eJ^{\mu}A_{\mu}}$$

We will also get Maxwells’ equations: $\partial_{\mu}F^{\mu\nu} = eJ^{\nu}$

For local U(1) gauge invariance, replace $\mathcal{L} = i\,\bar{\psi}\,\gamma^{\mu}\,\partial_{\mu}\psi - m\,\bar{\psi}\psi$ by

$$\mathcal{L} = i\,\bar{\psi}\,\gamma^{\mu}\,D_{\mu}\psi - m\,\bar{\psi}\psi - \tfrac{1}{4}F_{\mu\nu}\,F^{\mu\nu}$$

where, as before, $D_{\mu} = \partial_{\mu} + ieA_{\mu}$ and $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$.

Expanding the covariant derivative, we get

$$\mathcal{L} = i\,\bar{\psi}\,\gamma^{\mu}\,\partial_{\mu}\psi - e\,\bar{\psi}\,\gamma^{\mu}\,\psi A_{\mu} - m\,\bar{\psi}\psi - \tfrac{1}{4}F_{\mu\nu}\,F^{\mu\nu}$$

$$= \underbrace{i\,\bar{\psi}\,\gamma^{\mu}\,\partial_{\mu}\psi - m\,\bar{\psi}\psi}_{\text{free fermion}} - \underbrace{\tfrac{1}{4}F_{\mu\nu}\,F^{\mu\nu}}_{\text{free e.m.}} \underbrace{- e\,\bar{\psi}\,\gamma^{\mu}\,\psi A_{\mu}}_{-eJ^{\mu}A_{\mu}}$$

We will also get Maxwells' equations: $\partial_{\mu}F^{\mu\nu} = eJ^{\nu}$

$\boxed{\text{Quantum Electrodynamics (QED)}}$

Once we have Maxwell's equations, we can write

$$\partial_\mu F^{\mu\nu} = eJ^\nu$$

or,

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = eJ^\nu$$

or,

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \left(\partial_\mu A^\mu\right) = eJ^\nu$$

or,

$$\Box A^\nu - \partial^\nu (\partial_\mu A^\mu) = eJ^\nu$$

Choose the Lorentz gauge $\partial_\mu A^\mu = 0$ and we recover

$$\Box A^\nu = eJ^\nu$$

In static limit, this leads to Coulomb's law and a long-range interaction

Can the photon have a mass?

Then we would have a Klein-Gordon equation: $\left(\Box + M_\gamma^2\right) A^\nu = eJ^\nu$

coming from a Maxwell equation: $\partial_\mu F^{\mu\nu} + M_\gamma^2 A^\nu = eJ^\nu$

If this is the Euler-Lagrange equation, the Lagrangian density must have an extra *mass term*

$$\mathcal{L}_M = \tfrac{1}{2} M_\gamma^2 A^\nu A_\nu$$

Under a gauge transformation, $A_\nu \rightarrow A_\nu + \partial_\nu \theta$, and it follows that

$$\mathcal{L}_M \rightarrow \tfrac{1}{2} M_\gamma^2 (A^\nu + \partial^\nu \theta)(A_\nu + \partial_\nu \theta) \neq \tfrac{1}{2} M_\gamma^2 A^\nu A_\nu$$

For gauge invariance, we must set $M_\gamma = 0$, i.e. the photon must be massless

gauge invariance $\Leftrightarrow$ long range interactions

The electromagnetic interaction is not always long-range...

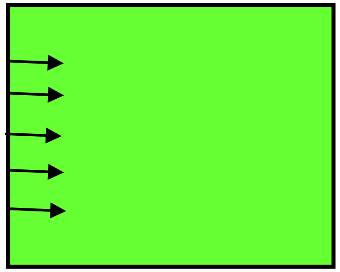
Consider a superconductor:

at $T < T_c$, it exhibits the following behaviour:

1. perfect conductor, i.e. electric field is thrown out (K.-Onnes)
2. perfect diamagnet, i.e, magnetic field is thrown out (Meissner)

i.e. no component of the electromagnetic field can propagate inside a superconductor...

However, at the edges, the fields fall off exponentially:



i.e. $A^{\mu} \sim e^{-x/\ell}$

ℓ is the penetration depth

The electromagnetic interaction is not always long-range...

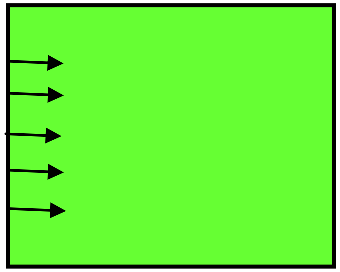
Consider a superconductor:

at $T < T_c$, it exhibits the following behaviour:

1. perfect conductor, i.e. electric field is thrown out (K.-Onnes)
2. perfect diamagnet, i.e, magnetic field is thrown out (Meissner)

i.e. no component of the electromagnetic field can propagate inside a superconductor...

However, at the edges, the fields fall off exponentially:



i.e. $A^{\mu} \sim e^{-x/\ell} \sim e^{-Mx}$

ℓ is the penetration depth ; $M = \ell^{-1}$ is the mass

$\Rightarrow$ the photon is massive inside a superconductor

Somehow, the local U(1) gauge symmetry breaks down inside the superconductor...

Do we understand this phenomenon?

Yes.

- It was first explained by Landau and Ginzburg in 1937 for a non-relativistic theory (which applies to superconductors).
- It was extensively applied in condensed matter systems by Philip Anderson in the 1950's
- It was worked out for a relativistic theory by Englert & Brout (1964) and independently, by Peter Higgs (1964).

The phenomenon is called spontaneous symmetry-breaking, but a better name (Coleman) is hidden symmetry...

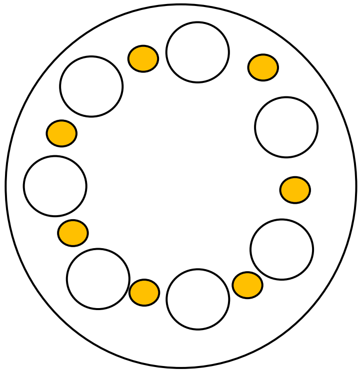
Hidden symmetry

This arises when the action of a system has a particular symmetry, but the ground state does not...

Hidden symmetry

This arises when the action of a system has a particular symmetry, but the ground state does not...

Example 1: Salam's banquet



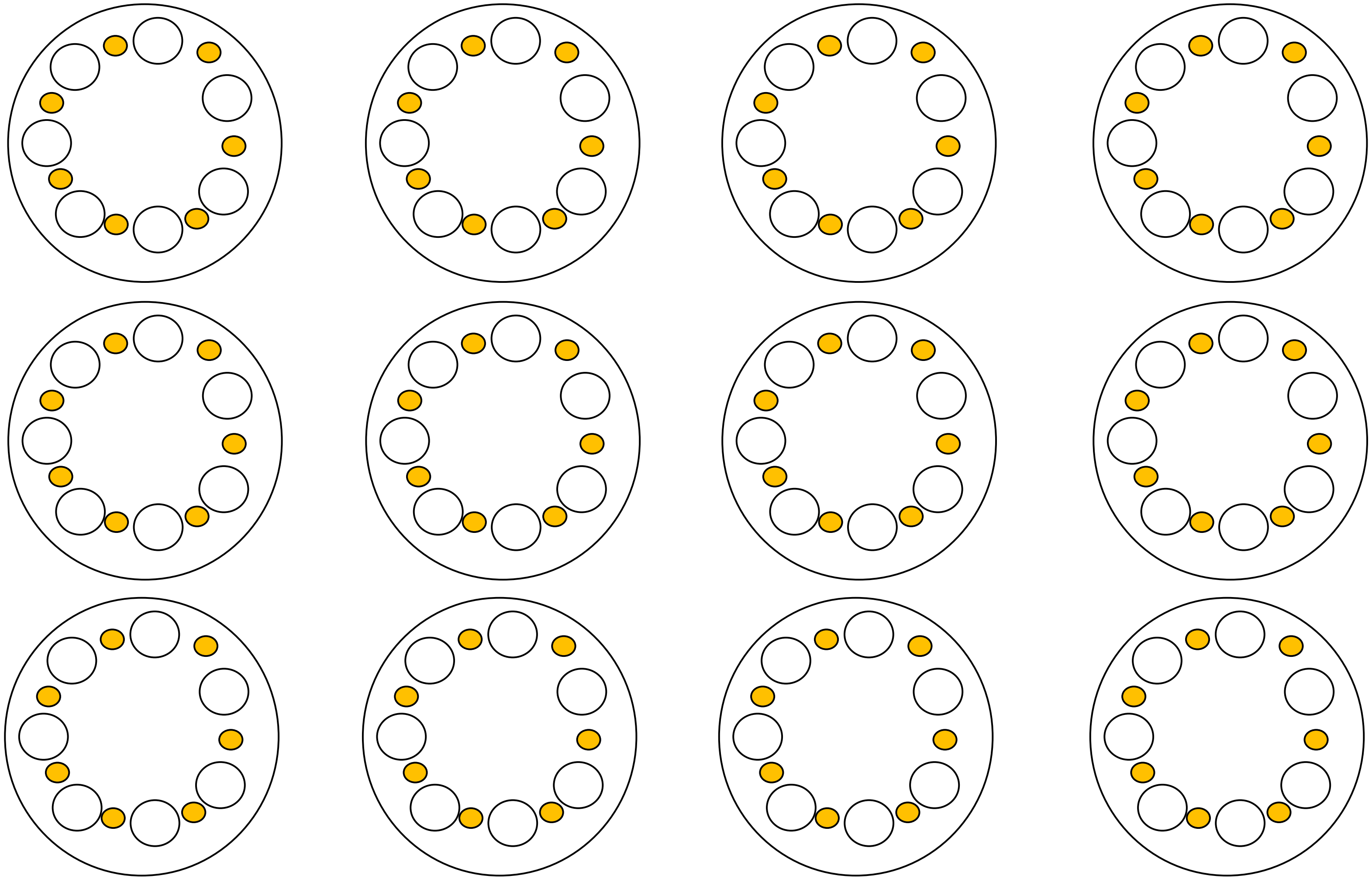
People are sitting to dinner at a round table. Each has a plate in front and a glass on either hand.
Before the meal, there is perfect symmetry between left glasses and right glasses.

The first person to pick up a glass makes a random choice, say the left glass...
...now everyone must pick up the left glass...

During the meal, there is no symmetry...

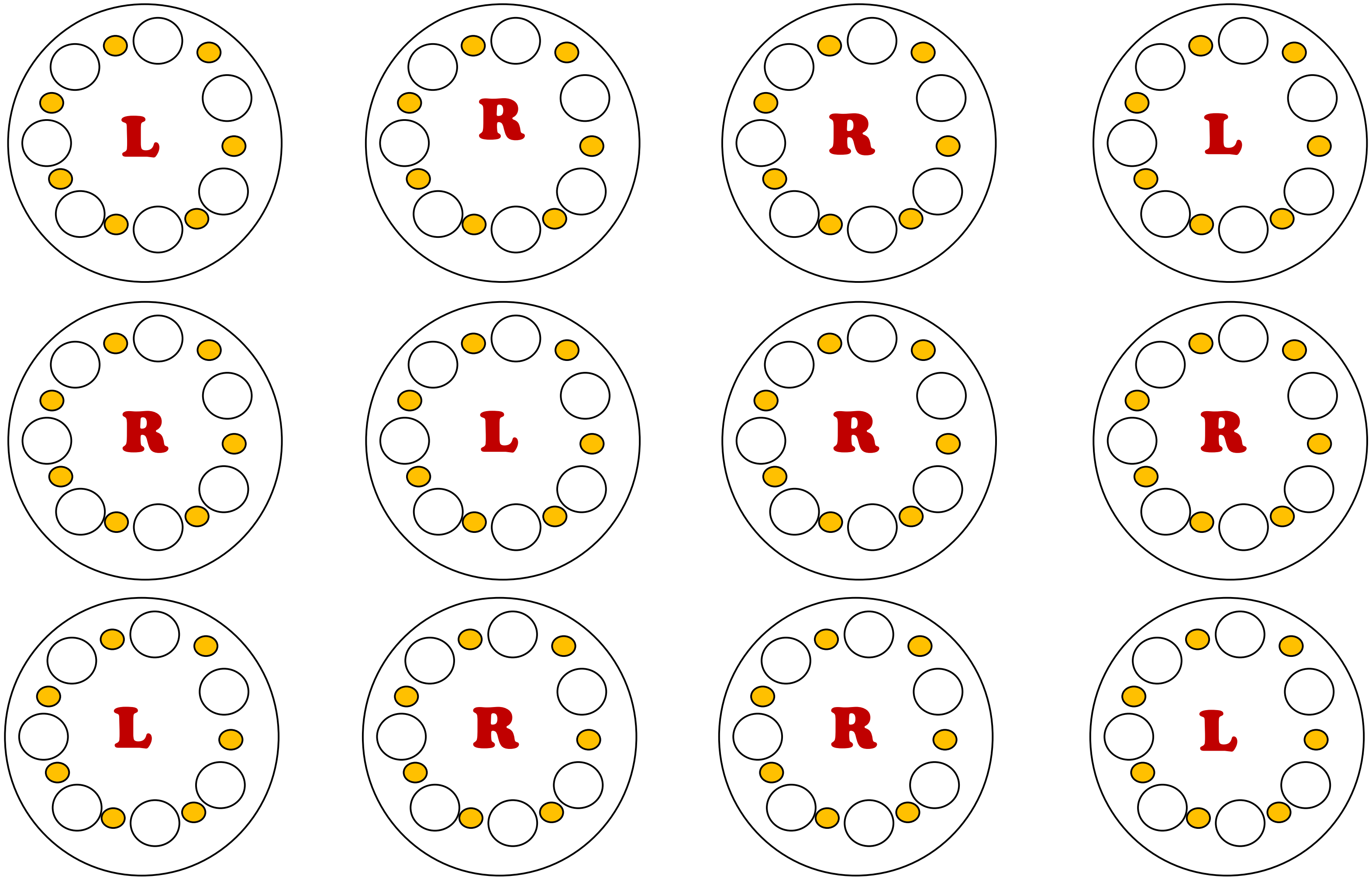
Has the symmetry really been destroyed?

No: if we consider an ensemble of systems...



Has the symmetry really been destroyed?

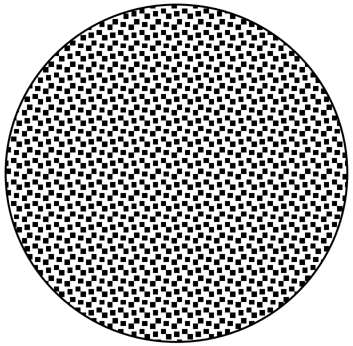
No: if we consider an ensemble of systems... the symmetry reappears!



Hidden symmetry

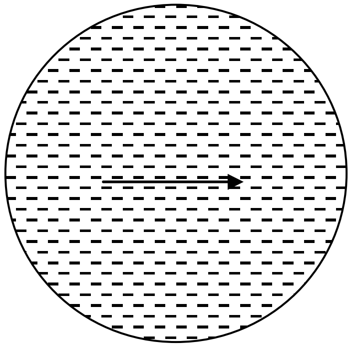
This arises when the action of a system has a particular symmetry, but the ground state does not...

Example 2: Ferromagnet below Curie temperature



Above the Curie temperature, all the domains are in random directions... obeys rotation invariance...

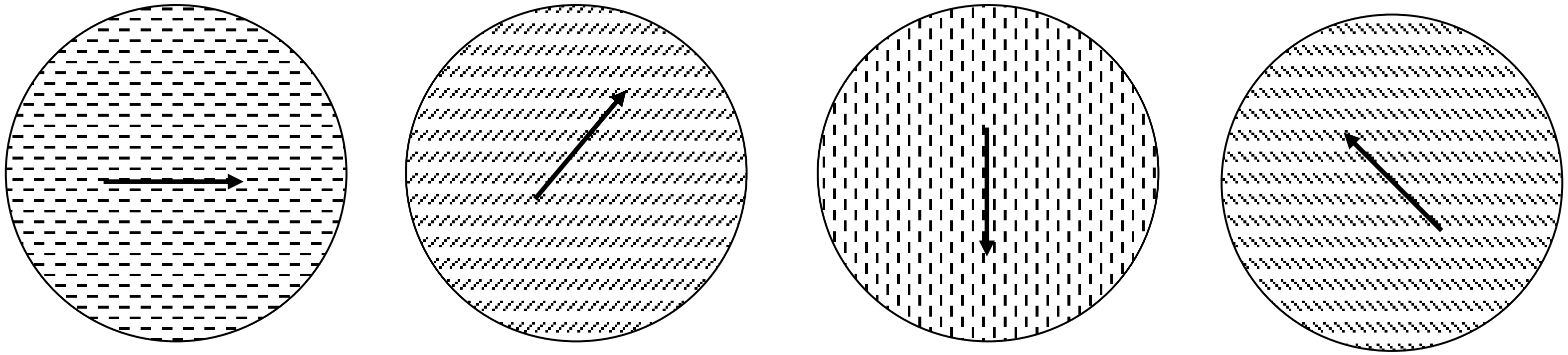
$$H = \sum_{\langle ij \rangle} J_{ij}\, \vec{S}_i . \vec{S}_j$$



Below the Curie temperature, all the domains are aligned parallel to a particular direction... magnetic field measurement will show a preferred direction, i.e. rotation invariance is lost

Has the symmetry really been destroyed?

No: if we consider an ensemble of systems... the symmetry reappears!



If we confine ourselves to the inside of a ferromagnet (Coleman’s demon), then rotation invariance will always be violated...

This is always associated with a phase transition:

i.e. at some high temperature, the symmetry exists

at low temperature the symmetry disappears

in between a flip occurs... critical temperature... $\Rightarrow$ phase transition

How does the superconducting phase break the electromagnetic U(1) gauge invariance?

We discuss the relativistic model, because we will apply the same idea to particle physics problems...

Imagine the interior of the superconductor to have, in addition to the electromagnetic field, a charged scalar field $\varphi(x)$. We have already seen that this leads to a Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + [D^{\mu}\varphi]^{*}\left[D_{\mu}\varphi\right] - M^{2}\varphi^{*}\varphi$$

In addition to this, let the scalar field have a self-interaction term

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + [D^{\mu}\varphi]^{*}\left[D_{\mu}\varphi\right] - M^{2}\varphi^{*}\varphi - \lambda(\varphi^{*}\varphi)^{2}$$

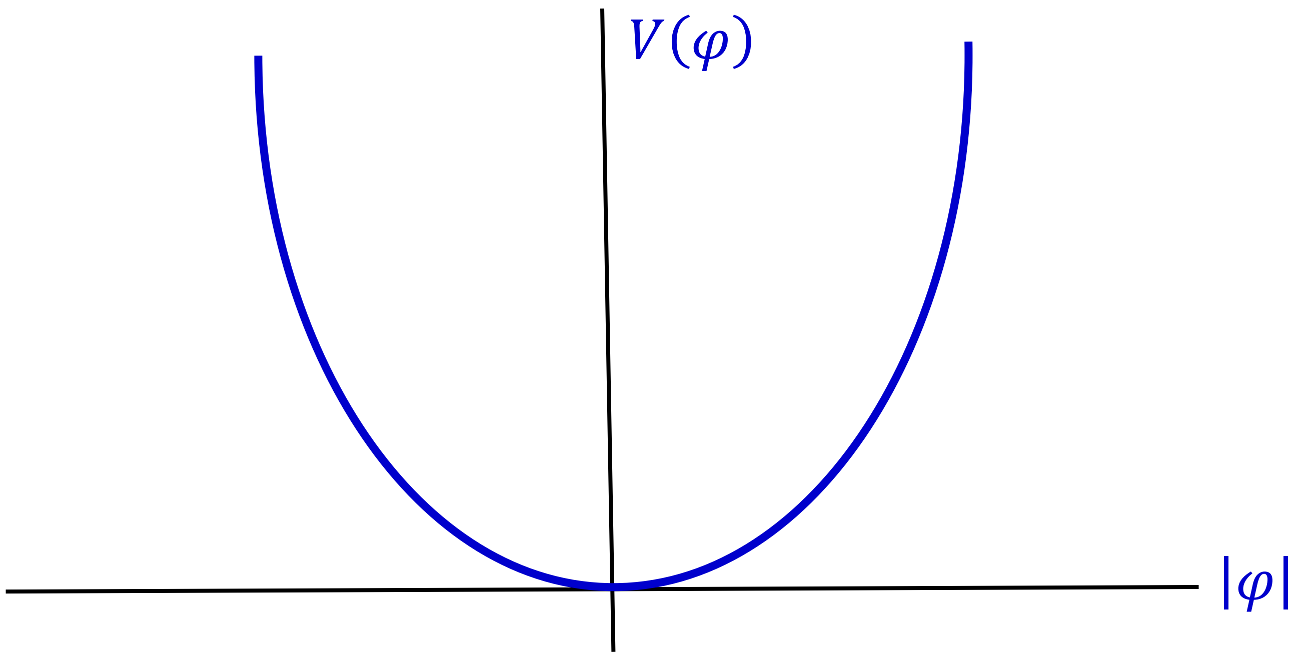
The last two terms can then be thought of as a gauge-invariant potential, i.e. we rewrite

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + [D^{\mu}\varphi]^{*}\big[D_{\mu}\varphi\big] - V(\varphi)$$

where

$$V(\varphi) = M^2\varphi^*\varphi + \lambda(\varphi^*\varphi)^2 = M^2|\varphi|^2 + \lambda|\varphi|^4$$

If we plot this potential as a function of $|\varphi|$, we will get



with a minimum at $|\varphi| = 0$, i.e. at $\varphi = 0$. No symmetries are broken.

But now, let us consider another variant of this theory, viz.

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + [D^\mu \varphi]^* \left[D_\mu \varphi \right] + M^2 \varphi^* \varphi - \lambda (\varphi^* \varphi)^2$$

If we try to treat $+M^2 \varphi^* \varphi$ as a mass term, the scalar particle will become a tachyon. Don't try this. Just let $+M^2 \varphi^* \varphi$ be an interaction term. Now, rewrite

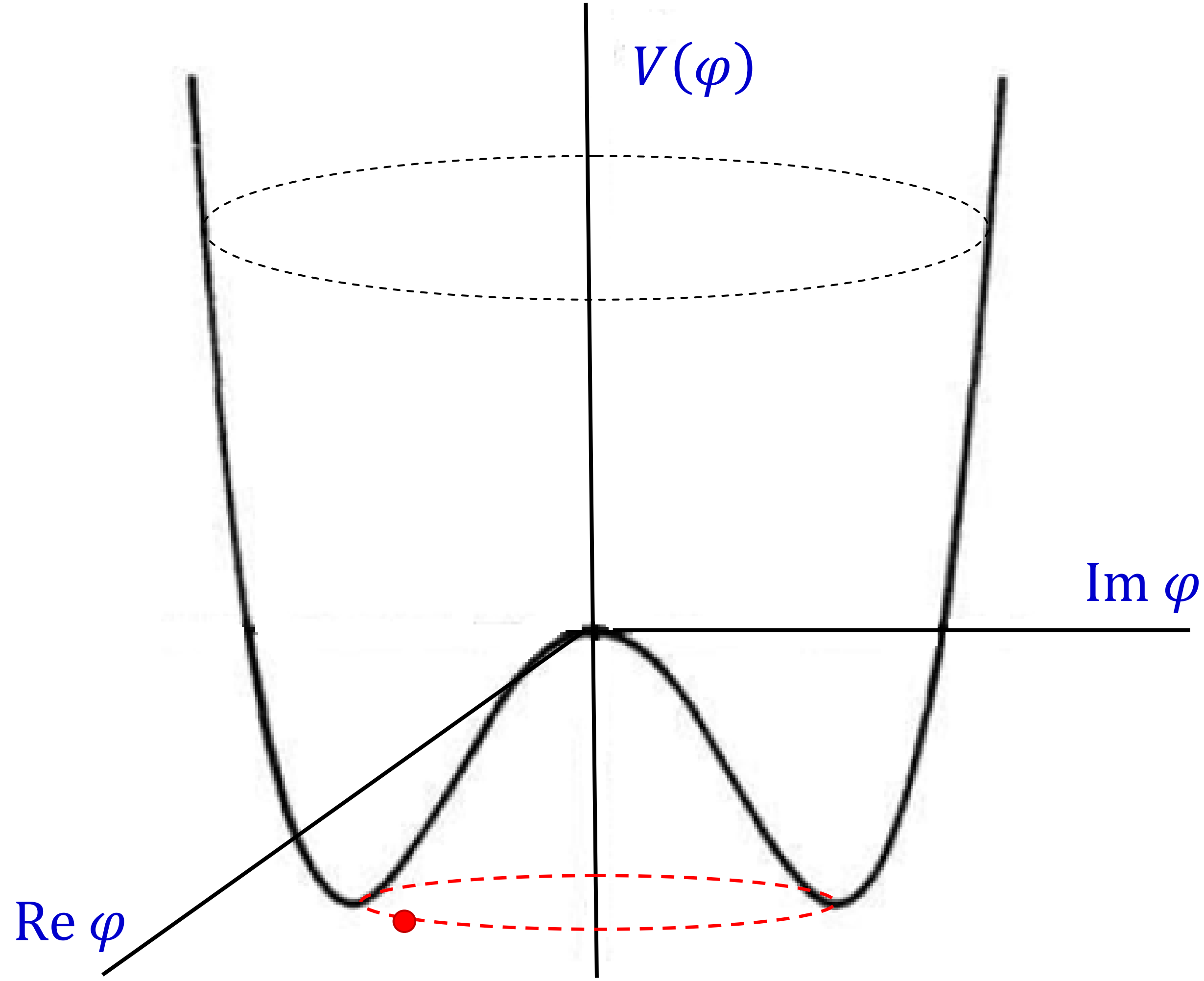
$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + [D^\mu \varphi]^* \left[D_\mu \varphi \right] - V(\varphi)$$

where $V(\varphi) = -M^2 \varphi^* \varphi + \lambda (\varphi^* \varphi)^2 = -M^2 |\varphi|^2 + \lambda |\varphi|^4$

This clearly has extrema at

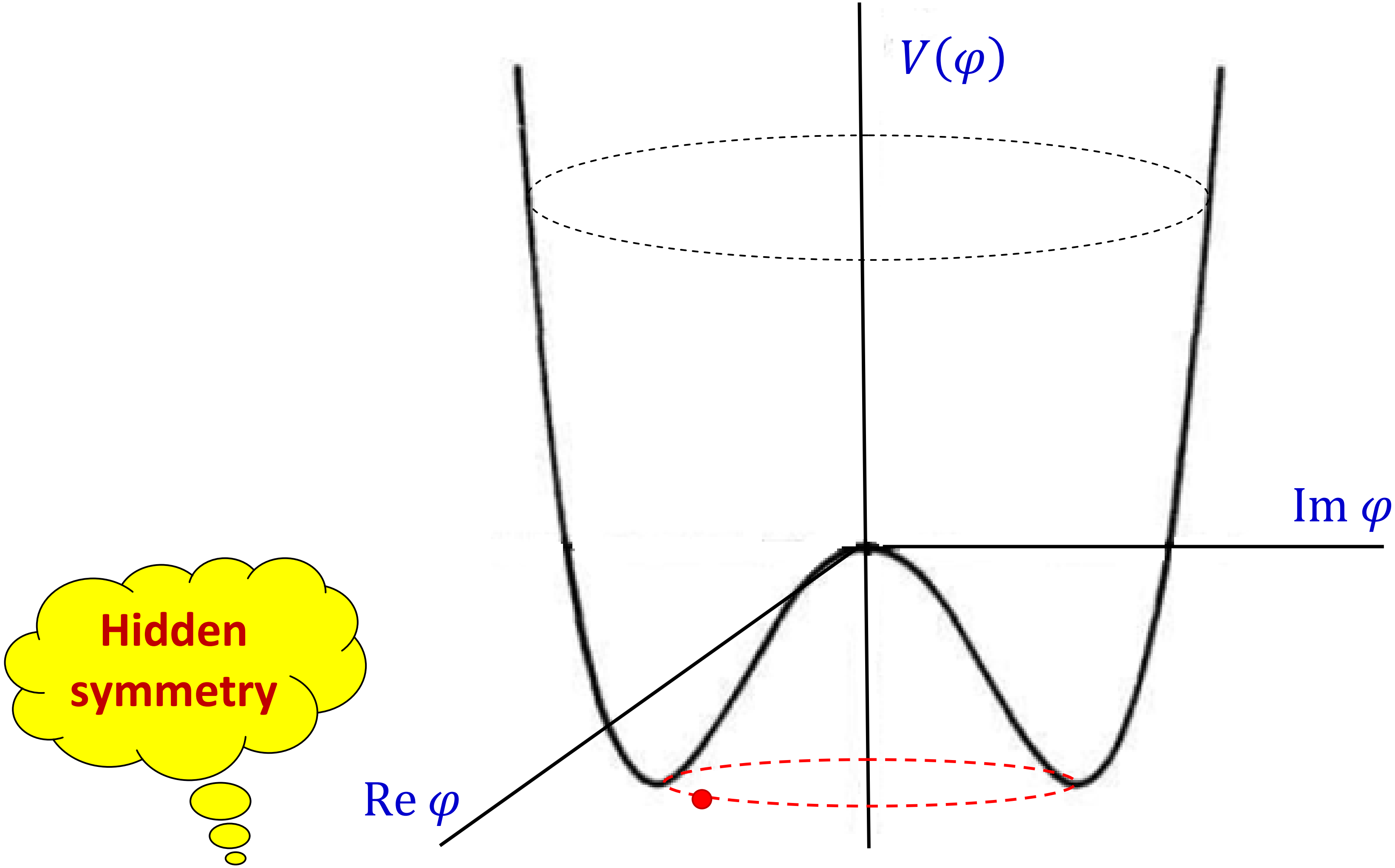
$$\frac{\partial V}{\partial |\varphi|} = 0 \quad \Rightarrow \quad |\varphi| = 0 \ (\mathrm{max}) \, , \sqrt{\frac{M^2}{2\lambda}} \ (\mathrm{min}) \equiv \frac{v}{\sqrt{2}}$$

Thus, there are an infinite number of possible ground states $\varphi = ve^{i\alpha}$



Only one of these can be the true ground state... as in a ferromagnet

Thus, there are an infinite number of possible ground states $\varphi = ve^{i\alpha}$



Only one of these can be the true ground state... as in a ferromagnet

Let us orient the axes in the complex φ plane such that the ground state (wherever it is) falls along the real axis.

(Just a convenient parametrisation – like choosing the *z*-axis along a constant magnetic field)

The ground state is now $\varphi_0 = \frac{v}{\sqrt{2}}$. To construct a viable field theory we must expand around this ground state, i.e. $\varphi(x) = \varphi_0 + \varphi'(x)$.

Rewrite the Lagrangian density in terms of this new field $\varphi'(x)$:

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + [D^{\mu}(\varphi_0 + \varphi')]^{*}\left[D_{\mu}(\varphi_0 + \varphi')\right] - V(\varphi_0 + \varphi')$$

Calculate the terms one by one:

$$D_{\mu}(\varphi_0 + \varphi') = (\partial_{\mu} + ieA_{\mu})(\varphi_0 + \varphi') = \partial_{\mu}\varphi' + ieA_{\mu}\varphi' + ie\varphi_0 A_{\mu}$$

It follows that

$$[D^\mu(\varphi_0+\varphi^{'})]^*[\,D_\mu(\varphi_0+\varphi^{'})]$$

$$=\left[\partial_\mu\varphi^{'}+ieA^\mu\varphi^{'}+ie\varphi_0A^\mu\right]^*\left[\partial_\mu\varphi^{'}+ieA_\mu\varphi^{'}+ie\varphi_0A_\mu\right]$$

$$=\left[\partial_\mu\varphi^{'*}-ieA^\mu\varphi^{'*}-ie\varphi_0A^\mu\right]\left[\partial_\mu\varphi^{'}+ieA_\mu\varphi^{'}+ie\varphi_0A_\mu\right]$$

$$=\cdots+e^2\varphi_0^2\,A^\mu A_\mu$$

Recall $\mathcal{L}_M=\frac{1}{2}M_\gamma^2A^\nu A_\nu$

Inside a superconductor with a potential as assumed here, the photon has become massive!

$$M_\gamma=\sqrt{2}e\varphi_0=\sqrt{2}e\frac{v}{\sqrt{2}}=ev$$

Since $v=\sqrt{M^2/\lambda}$, M_γ is a manifestation of the scalar self-interactions...

Another miracle...

$$V(\varphi_0+\varphi^{'}) = -M^2(\varphi_0+\varphi^{'})^*(\varphi_0+\varphi^{'}) + \lambda[(\varphi_0+\varphi^{'})^*(\varphi_0+\varphi^{'})]^2$$

$$= -M^2\varphi^{'*}\varphi^{'} + .. + \lambda[2\varphi_0^2\varphi'^*\varphi^{'} + (2\varphi_0\mathrm{Re}\varphi^{'})^2 + \cdots]$$

$$= (-M^2 + 2\lambda\varphi_0^2)[(\mathrm{Re}\,\varphi^{'})^2 + (\mathrm{Im}\,\varphi^{'})^2] + 4\lambda\varphi_0^2(\mathrm{Re}\,\varphi^{'})^2 + \cdots$$

Now, $\varphi_0^2 = \sqrt{M^2/2\lambda}$

It follows that

$$V(\varphi_0+\varphi^{'}) = +4\lambda\varphi_0^2(\mathrm{Re}\,\varphi^{'})^2 + \cdots$$

$$= +\frac{1}{2}4M^2(\mathrm{Re}\,\varphi^{'})^2 + \cdots$$

U(1) symmetry is broken
– both global and local

i.e.

$\mathrm{Re}\,\varphi^{'}$ has a (real) mass $2M$

$\mathrm{Im}\,\varphi^{'}$ is massless $\Rightarrow$ Goldstone boson

Goldstone theorem (1962):

To every spontaneously broken continuous global symmetry, there corresponds a massless boson

How to get rid of this massless boson (would induce new long-range interactions otherwise)?

Englert & Brout (1964), Higgs (1964):

Can be done if it is a local symmetry...

Idea is very simple: parametrise $\varphi(x) = \eta(x)\, e^{i\xi(x)}$ (polar form)

Consider the unbroken (i.e. gauge invariant) Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + [D^{\mu}\varphi]^{*}\left[\, D_{\mu}\varphi\right] - V(\varphi)$$

where $V(\varphi) = -M^2\varphi^{*}\varphi + \lambda(\varphi^{*}\varphi)^2 = -M^2|\varphi|^2 + \lambda|\varphi|^4$

At this level, we are free to make any gauge choice we wish...

Make a gauge transformation

$$\varphi(x) \to e^{-ig\theta(x)}\varphi(x) = \eta(x)\, e^{i[\xi(x)-g\theta(x)]}$$

We might as well choose a special gauge, since the gauge symmetry is going to be broken anyway...

Choose the gauge function $\theta(x)$ such that

$$\xi(x) - g\theta(x) = 0$$

This is called the unitary gauge.

In this gauge, $\varphi(x) = \eta(x)$ and the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + [D^{\mu}\eta]^{*}\big[\, D_{\mu}\eta\big] - V(\eta)$$

where $V(\eta) = -M^2\eta^2 + \lambda\eta^4$

The ground state is still at $v/\sqrt{2}$ so we must shift

$$\eta = \frac{v}{\sqrt{2}} + \eta'$$

This will lead to

1. $\mathcal{L}_M = \frac{1}{2} M_\gamma^2 A^\nu A_\nu$ with $M_\gamma = ev$
2. $V(\varphi_0 + \varphi^{'}) = +\frac{1}{2} 4M^2\eta^2 + \cdots$ i.e. $M_\eta = 2M$
3. and there is no Goldstone boson...

if we had kept $\xi(x)$ it would have been the Goldstone boson...

The ground state is still at $v/\sqrt{2}$ so we must shift

$$\eta = \frac{v}{\sqrt{2}} + \eta'$$

This will lead to

1. $\mathcal{L}_M = \frac{1}{2} M_\gamma^2 A^\nu A_\nu$ with $M_\gamma = ev$

2. $V(\varphi_0 + \varphi^{'}) = +\frac{1}{2} 4M^2\eta^2 + \cdots$ i.e. $M_\eta = 2M$

3. and there is no Goldstone boson...

if we had kept $\xi(x)$ it would have been the Goldstone boson...

Looks like magic!!

How can a degree of freedom of the system vanish?

In the unitary gauge, the photon is massive,

i.e. it has <u>three</u> polarisations.

The extra degree of freedom (longitudinal polarisation) which appears here is at the cost of the disappearance of the Goldstone degree of freedom...

Effectively:

the photon ‘swallows up’ the Goldstone boson and becomes massive.

In the unitary gauge, the photon is massive,

i.e. it has <u>three</u> polarisations.

The extra degree of freedom (longitudinal polarisation) which appears here is at the cost of the disappearance of the Goldstone degree of freedom...

Effectively:

the photon ‘swallows up’ the Goldstone boson and becomes massive.

Higgs mechanism