Test-3 Solution

Ans 1(a)

x_k	0	1	2	4
y_k	1	1	2	5

Lagrange's formula is

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$f(x) = \frac{(x-1)(x-2)(x-4)}{(-1)(-2)(-4)}(1) + \frac{x(x-2)(x-4)}{(1)(-1)(-3)}(1)$$

$$+ \frac{x(x-1)(x-4)}{(2)(1)(-2)}(2) + \frac{x(x-1)(x-2)}{(4)(3)(2)}(5)$$

$$= (x^3-7x^2+14x-8)\left(\frac{-1}{8}\right) + (x^3-6x^2+8x)\left(\frac{1}{3}\right)$$

$$+ (x^3-5x^2+4x)\left(-\frac{1}{2}\right) + (x^3-3x^2+2x)\left(\frac{5}{24}\right)$$

$$= \frac{-3x^3+21x^2-42x+24+8x^3-48x^2+64x-12x^3+60x^2-48x+5x^3-15x^2+10x}{24}$$

$$= \frac{-2x^3+18x^2-16x+24}{24}$$

The polynomial is $p(x) = \frac{1}{12}(-x^3+9x^2-8x+12)$

Ans.1 (b) $k=-1, 0, 1$

$$y_k = y_0 + \binom{k}{1}\delta y_{-1/2} + \binom{k+1}{2}\delta^2 y_0$$

$k=-1$

$$\binom{-k}{i} = \frac{(-k)(-k-1)\cdots(-k-i+1)}{i!} \quad \Big| \quad \binom{-1}{1} = \frac{-1}{1!} = -1$$

$RHS = y_0 - \delta y_{-1/2} = (1 - \delta E^{-1/2})\, y_0 = [1 - (1 - E^{-1})]\, y_0 = E^{-1} y_0 = y_{-1} = LHS$

$k=0$ $\quad RHS = y_0 = LHS$

$k=1$ $\quad RHS = y_0 + \delta y_{-1/2} + \delta^2 y_0 = [1 + \delta E^{-1/2} + \delta^2]\, y_0 = [1 + (1-E^{-1}) + (E - 2 + E^{-1})]\, y_0$

$= E y_0 = y_1 = LHS$

Ans. 2

$$[x_0, x_1, x_2] = \frac{\begin{vmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix}}{\begin{vmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix}}$$

$$RHS = \frac{\begin{vmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix}}{\begin{vmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix}} = \frac{1(y_2 x_1 - x_2 y_1) - x_0 (y_2 - y_1) + y_0 (x_2 - x_1)}{1(x_2^2 x_1 - x_1^2 x_2) - x_0 (x_2^2 - x_1^2) + x_0^2 (x_2 - x_1)}$$

$$= \frac{y_2 x_1 - x_2 y_1 - x_0 y_2 + x_0 y_1 + y_0 x_2 - y_0 x_1}{-x_1 x_2 (x_1 - x_2) + x_0 (x_1^2 - x_2^2) - x_0^2 (x_1 - x_2)}$$

$$= \frac{-y_0 (x_1 - x_2) - y_1 (x_2 - x_0) - y_2 (x_0 - x_1)}{(x_1 - x_2)[-x_1 x_2 + x_0 x_1 + x_0 x_2 - x_0^2]}$$

$$= \frac{-y_0 (x_1 - x_2) - y_1 (x_2 - x_0) - y_2 (x_0 - x_1)}{(x_1 - x_2)(x_0 - x_1)(x_2 - x_0)}$$

$$= -\frac{y_0}{(x_0 - x_1)(x_2 - x_0)} - \frac{y_1}{(x_1 - x_2)(x_0 - x_1)} - \frac{y_2}{(x_1 - x_2)(x_2 - x_0)} \quad \text{--- } (1)$$

$$LHS = [x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{(x_2 - x_0)} = \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{(x_2 - x_0)}$$

$$LHS = \frac{y_2}{(x_2-x_1)(x_2-x_0)} - \frac{y_1}{(x_2-x_1)(x_2-x_0)} - \frac{y_1}{(x_1-x_0)(x_2-x_0)} + \frac{y_0}{(x_1-x_0)(x_2-x_0)}$$

$$= -\frac{y_0}{(x_0-x_1)(x_2-x_0)} - \frac{y_1}{(x_1-x_2)(x_0-x_1)} - \frac{y_2}{(x_1-x_2)(x_2-x_0)} \quad \text{——(2)}$$

From (1) and (2), LHS = RHS

Ans. 3 $f(x) = 2x^3 - 6x^2 + 2x - 1$

$f(2)$ is $-ve$

$f(3)$ is $+ve$

The root is between 2 and 3.

2	-6	2	-1
0	4	-4	-4
2	-2	-2	-5000
0	4	4	3792
2	2	200	-1208000
0	4	432	
2	60	632	
0	12	504	
2	72	113600	
0	12		
2	84		
0	12		
2	960		
0			
2			
0			
2			

(2.69

Diminish by 2

$f_1(x) = 2x^3 + 6x^2 + 2x - 5$ ——(1)

Multiply the root by 10, so effectively multiply second term by 10, third by 100 and fourth by 1000 in (1).

$f_2(x) = 2x^3 + 60x^2 + 200x - 5000$ ——(2)

$f_2(6)$ is $-ve$

$f_2(7)$ is $+ve$

Diminish by 6

$f_3(x) = 2x^3 + 96x^2 + 1136x - 1208$ ——(3)

Multiply the root by 10, so effectively multiply second term by 10, Third term by 100 and fourth by 1000 in (3)

$f_4(x) = 2x^3 + 960x^2 + 113600x - 1208000$ ——(4)

$f_4(9)$ is $-ve$

$f_4(10)$ is $+ve$

Diminish by 9

Further calculations are not required as the root is to be found correct upto two decimal places.

Sotue root is 2.69. The calculations are shown above.

Ans. 4

Given equations are $x - 5y = -4$ — (1)

$7x - y = 6$ — (2)

From (1) and (2),

$x = -4 + 5y$ — (3)

$y = -6 + 7x$ — (4)

Start with (0,0) $x_0 = 0,\ y_0 = 0$

First Iteration

From (3) and (4)

$x_1 = -4, \quad y_1 = -6$

Second Iteration

From (3) and (4)

$x_2 = -4 + 5(-6) = -4 - 30 = -34$

$y_2 = -6 + 7(-4) = -6 - 28 = -34$

Third Iteration

From (3) and (4)

$x_3 = -4 + 5(-34) = -4 - 170 = -174$

$y_3 = -6 + 7(-34) = -6 - 238 = -244$

Observation The method diverges.

Now interchange the given equations. The new set of equations is

$7x - y = 6$ — (5)

$x - 5y = -4$ — (6)

From (5),

$x = \frac{1}{7}[6 + y]$ — (7)

From (6),

$y = \frac{1}{5}[4 + x]$ — (8)

$x_0 = 0,\ y_0 = 0$

First Iteration

From (7) $x_1 = \frac{1}{7}(6) = \frac{6}{7}$

From (8) $y_1 = \frac{1}{5}(4) = \frac{4}{5}$

second Iteration

From (7), $x_2 = \frac{1}{7}\left(6 + \frac{4}{5}\right) = \frac{1}{7}\left(\frac{34}{5}\right) = \frac{34}{35}$

From (8), $y_2 = \frac{1}{5}\left(4 + \frac{6}{7}\right) = \frac{1}{5}\left(\frac{34}{7}\right) = \frac{34}{35}$

Third Iteration

From (7), $x_3 = \frac{1}{7}\left(6 + \frac{34}{35}\right) = \frac{1}{7}\left(\frac{244}{35}\right) = \frac{244}{245}$

From (8), $y_3 = \frac{1}{5}\left(4 + \frac{34}{35}\right) = \frac{1}{5}\left(\frac{174}{35}\right) = \frac{174}{175}$

The solution is $x_3 = \frac{244}{245} = 0.9959$

$y_3 = \frac{174}{175} = 0.9943$

Observation The method converges.

If the given system matrix is strictly diagonally dominant, the Jacobi's method converges.

Ans. 5 $I = \int_0^{\pi/2} \sin x \, dx$ The value is required by Simpson's one-third rule using 11 ordinates. Divide the $\frac{\pi}{2}$ interval into 10 sub-intervals.

The following table gives the values of x and y.

x	y	
0	$\sin 0$	$y_0 = 0$
$\frac{\pi}{20}$	$\sin \pi/20$	$y_1 = 0.156$
$\frac{2\pi}{20}$	$\sin 2\pi/20$	$y_2 = 0.309$
$\frac{3\pi}{20}$	$\sin \frac{3\pi}{20}$	$y_3 = 0.454$
$\frac{4\pi}{20}$	$\sin \frac{4\pi}{20}$	$y_4 = 0.588$
$\frac{5\pi}{20}$	$\sin \frac{5\pi}{20}$	$y_5 = 0.707$
$\frac{6\pi}{20}$	$\sin \frac{6\pi}{20}$	$y_6 = 0.809$
$\frac{7\pi}{20}$	$\sin \frac{7\pi}{20}$	$y_7 = 0.891$
$\frac{8\pi}{20}$	$\sin \frac{8\pi}{20}$	$y_8 = 0.951$
$\frac{9\pi}{20}$	$\sin \frac{9\pi}{20}$	$y_9 = 0.988$
$\frac{\pi}{2}$	$\sin \frac{\pi}{2}$	$y_{10} = 1$

The Simpson's one-third formula is given as

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\cdots+y_{n-1}) + 2(y_2+y_4+\cdots+y_{n-2})\right]$$

$n=10$, $h=\frac{\pi}{20}$

$$I = \int_0^{\pi/2} \sin x\,dx = \int_0^{0+10\left(\frac{\pi}{20}\right)} \sin x\,dx$$

$$= \frac{h}{3}\left[(y_0+y_{10}) + 4(y_1+y_3+y_5+y_7+y_9) + 2(y_2+y_4+y_6+y_8)\right]$$

$$= \frac{\pi}{3(20)}\left[(0+1) + 4(0.156+0.454+0.707+0.891+0.988) + 2(0.309+0.588+0.809+0.951)\right]$$

$$= \frac{\pi}{60}\left[1+4(3.196)+2(2.657)\right]$$

$$= \frac{\pi}{60}\left[19.098\right]$$

$$= 0.999968941$$