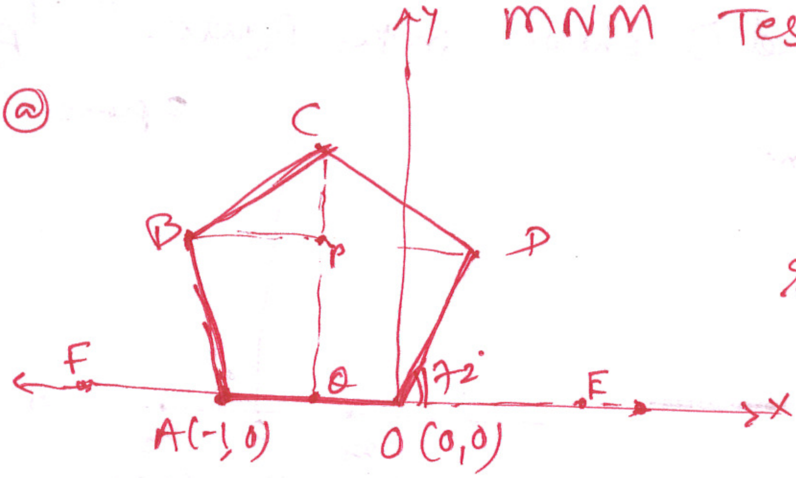
MNM Test-1 Solution

Ans. 1 (a)



Internal angle of a regular pentagon is $\frac{(n-2)\times 180}{n} = 108°$

$OA = 1$, θ is mid point of OA

Each side is of value 1.

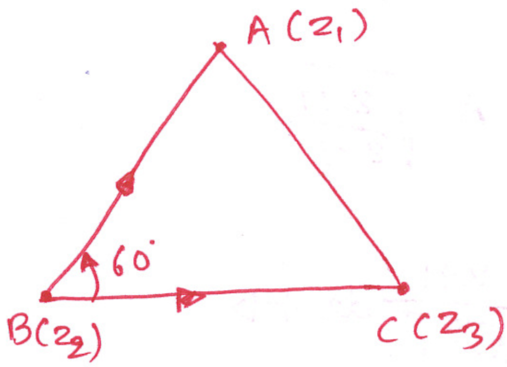
$\angle BAF = \angle DOE = 72°$, $\cos 72° = 0.31$, $\sin 72° = 0.95$

B is $(-1.31, 0.95)$, D is $(0.31, 0.95)$

$\angle BCD$ is $36°$, $CP = BC \sin 36° = 0.59$

So C is $(-0.5, 1.54)$

Ans. 1 (b)



ΔABC is equilateral, so BC when rotated through $60°$ coincides with BA.

To turn the direction of a complex number through an $\angle\theta$, multiply by $\cos\theta + i\sin\theta$

$$\therefore \overrightarrow{BC}\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = \overrightarrow{BA}$$

$$(z_3 - z_2)\left[\frac{1+i\sqrt{3}}{2}\right] = z_1 - z_2$$

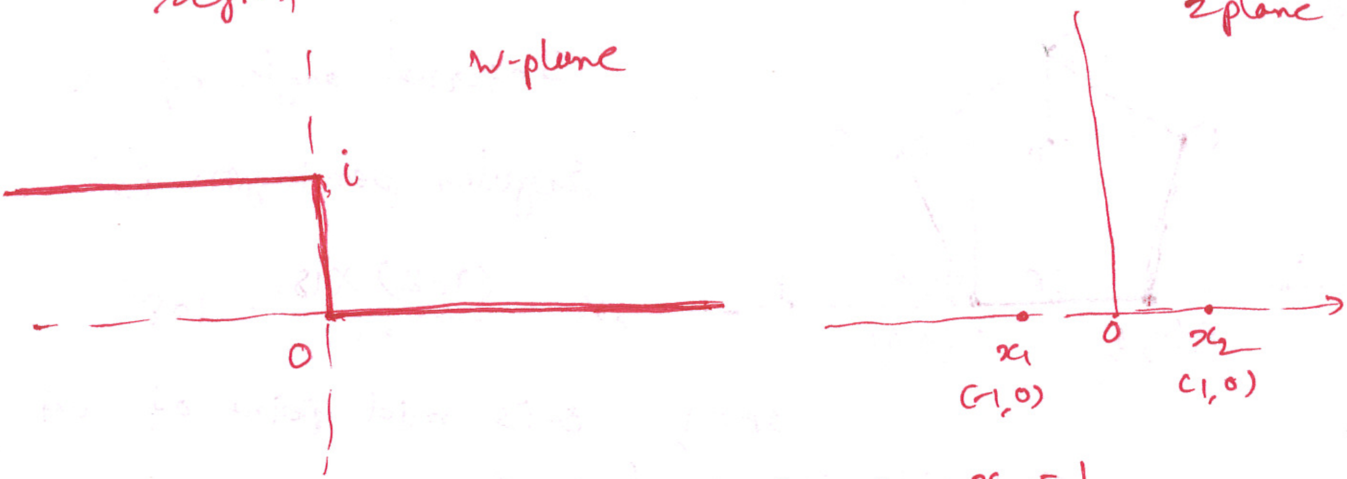
$$\therefore i\sqrt{3}(z_3 - z_2) = 2z_1 - z_2 - z_3$$

Squaring $-3(z_3 - z_2)^2 = (2z_1 - z_2 - z_3)^2$

$$\therefore 4[z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1] = 0$$

$$\therefore z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

The ~~region~~ boundary of region to be mapped is shown in the figure. Ans. 2



Let $w_1 = i$, $w_2 = 0$, $x_1 = -1$, $x_2 = 1$

$\alpha_1 = \frac{3\pi}{2}$, $\alpha_2 = \frac{\pi}{2}$

From S-C transformation

$$\frac{dw}{dz} = A\left[z - x_1\right]^{\frac{\alpha_1}{\pi} - 1}\left[z - x_2\right]^{\frac{\alpha_2}{\pi} - 1}$$

$$= A(z+1)^{1/2}(z-1)^{-1/2} = A\sqrt{\frac{z+1}{z-1}}$$

$$w = A\int\sqrt{\frac{z+1}{z-1}}\,dz + B = A\int\frac{z+1}{\sqrt{z^2-1}}\,dz + B$$

$$w = A\int\left[\frac{1}{2}\cdot\frac{2z}{\sqrt{z^2-1}} + \frac{1}{\sqrt{z^2-1}}\right]dz + B$$

$$\boxed{w = A\left[\sqrt{z^2-1} + \cosh^{-1} z\right] + B}$$

When $w_2 = 0$, $z = 1$ $\therefore 0 = A[0] + B$ $\therefore \boxed{B = 0}$

When $w_1 = i$, $z = -1$, $\therefore i = A[\cosh^{-1}(-1)]$

$\therefore \cosh\left(\frac{i}{A}\right) = -1$

$\therefore \cos\left(\frac{1}{A}\right) = \cos\pi$ $\therefore \boxed{A = \frac{1}{\pi}}$

$$\therefore \boxed{w = \frac{1}{\pi}\left[\sqrt{z^2-1} + \cosh^{-1} z\right]}$$ is the desired transformation.

Ans. 3(a) We need power series of $\sin^{-1} z$ about $z_0 = 0$.

Function is analytic in a disk of non-zero radius around the point of interest.

Let $w = \sin^{-1} z \quad \therefore z = \sin w$

$$\frac{dz}{dw} = \cos w = \sqrt{1-\sin^2 w} = \sqrt{1-z^2}$$

$$\therefore \frac{dw}{dz} = \frac{1}{\sqrt{1-z^2}} = \left[1+(-z^2)\right]^{-1/2} \quad \text{——— (1)}$$

Expand Eq. (1) and integrate term by term,

$$\boxed{w = z + \frac{1}{6} z^3 + \frac{3}{40} z^5 + \frac{5}{112} z^7 + \cdots}$$

Because series converges uniformly for $|-z^2|<1$, a condition which is satisfied for z in the neighborhood of zero. Also, when $z=0$, $w=0$ so Constant of integration must be zero.

Ans. 3(b) $f(z) = z^7 + 5z^3 - z - 2$

Compare the function $f(z)$ to the holomorphic function $g(z) = 5z^3$ on the unit circle. For $|z|=1$,

$$|f(z)-g(z)| = |z^7 - z - 2| \leq 4 < |g(\zeta)| \leq |f(\zeta)| + |g(\zeta)|$$

By Rouche's theorem, f and g have the same no. of zeros, counting multiplicity, in the unit disc. Since g has three zeros, so does f.

Ans. 4

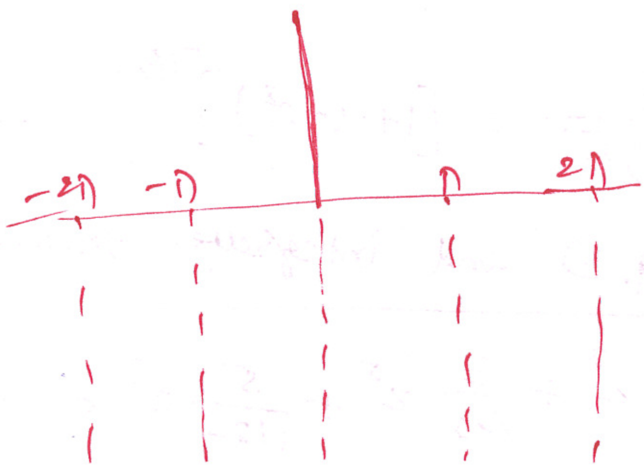
$\int_0^{\pi} \log \sin x \, dx$

Consider the function

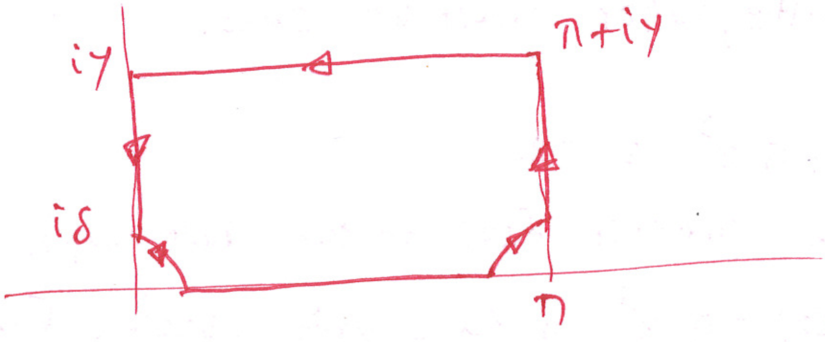
$$1 - e^{2iz} = e^{iz}\left[e^{-iz} - e^{iz}\right] = -2ie^{iz} \sin z \qquad (1)$$

If $z = x + iy$, $\quad 1 - e^{2iz} = 1 - e^{-2y}\left[\cos 2x + i \sin 2x\right]$

So function is real and non-positive if and only if $x = n\pi$ and $y \le 0$. Consider domain by deleting all half lines of the form $\{z = n\pi + iy;\ y \le 0\}$



In the domain, principal branch of $\log\left[1 - e^{2iz}\right]$ with imaginary part between $-\pi$ and π, is single valued and analytic.



For the contour,

$$C = [\delta, \pi - \delta] \cup T_1(\delta) \cup [\pi + i\delta, \pi + iY]$$
$$\cup\ [\pi + iY, iY] \cup [iY, i\delta] \cup T_2(\delta)$$

where $\delta > 0$ is small and $Y > 0$ is large.

$-T_1(\delta)$ denotes circular arc $z = \pi + \delta e^{it}$ where $t \in [\pi/2, \pi]$ and $-T_2(\delta)$ denotes the circular arc $z = \delta e^{it}$, where $t \in [0, \pi/2]$

By Cauchy's theorem,

$$\int_{[\delta, \pi-\delta]} \log(1-e^{2iz})\,dz + \int_{T_1(\delta)} \log(1-e^{2iz})\,dz$$

$$+ \int_{[\pi+i\delta, \pi+iY]} \log(1-e^{2iz})\,dz + \int_{[\pi+iY, iY]} \log(1-e^{2iz})\,dz$$

$$+ \int_{[iY, i\delta]} \log(1-e^{2iz})\,dz + \int_{T_2(\delta)} \log(1-e^{2iz})\,dz = 0$$

Addition of 3rd and fifth terms is zero.
second term $\to 0$ and sixth term $\to 0$, as $\delta \to 0$.
Fourth term $\to 0$ as $Y \to \infty$

So, $\displaystyle\int_0^{\pi} \log(1-e^{2ix})\,dx = 0$ ——— (2)

Consider the function $\log(-2i\,e^{ix}\sin x)$
If $\log e^{ix} = ix$, then imaginary part lies between 0 and π. To obtain principal branch of log with imaginary part between $-\pi$ and π, choose

$\log(-i) = -i\,\pi/2$

$\log[-2i\,e^{ix}\sin x] = \log 2 - i\frac{\pi}{2} + ix + \log\sin x$

$$\int_0^{\pi} \log\left[-2ie^{ix}\sin x\right]dx = \pi\log 2 - i\frac{\pi^2}{2} + i\frac{\pi^2}{2} + \int_0^{\pi}\log\sin x\,dx \quad \text{--- (3)}$$

From (1) and (3),

$$\boxed{\int_0^{\pi} \log\sin x\,dx = -\pi\log 2}$$

Ans. 5

t	0	1	2
y	1	3	5

model function is
$y = a_1 t + a_2 (t-1)^2$

The general system is $Ax = b$ and for the given experiment we have

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

$|AA^T - \lambda I| = 0$ $\quad\therefore \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 2 \\ 1 & 2 & 5-\lambda \end{vmatrix} = 0 \quad \text{--- (1)}$

$\therefore \lambda^3 - 7\lambda^2 + 6\lambda = 0$ $\quad\therefore \boxed{\lambda = 6} \quad \boxed{\lambda = 1} \quad \boxed{\lambda = 0}$

$\boxed{\lambda = 6}$ From Eq. (1) $\begin{bmatrix} -5 & 0 & 1 \\ 0 & -5 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$\therefore \quad -5x + z = 0$
$-5y + 2z = 0$
$x + 2y - z = 0$

$\therefore \quad z = 5x$
$y = 2x$

If $x = 1, \quad y = 2, \quad z = 5$

$$\therefore \quad u_1 = \begin{bmatrix} \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ \frac{5}{\sqrt{30}} \end{bmatrix} = \begin{bmatrix} 0.1825 \\ 0.3651 \\ 0.9128 \end{bmatrix} \quad \text{--- (2)}$$

$\boxed{\lambda = 1}$ From (1)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\therefore \quad \boxed{z = 0} \qquad 2z = 0, \qquad x + 2y + 4z = 0$

$\therefore \quad z = 0$
$y = -\frac{1}{2}x$

For $x = -1, \quad y = \frac{1}{2}, \quad z = 0$

$$\therefore \quad u_2 = \begin{bmatrix} \frac{-1}{\sqrt{5}/2} \\ \frac{1/2}{\sqrt{5}/2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} = \begin{bmatrix} -0.8944 \\ 0.4472 \\ 0 \end{bmatrix} \quad \text{--- (3)}$$

$\boxed{\lambda = 0}$ From (1),

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\therefore \quad x + z = 0$
$y + 2z = 0$
$x + 2y + 5z = 0$

$\therefore \quad \boxed{z = -x}$
$\boxed{y = 2x}$

For $\boxed{x = 1}$
$\boxed{y = 2}$
$\boxed{z = -1}$

$$\therefore\ u_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 0.4082 \\ 0.8165 \\ -0.4082 \end{bmatrix} \quad \text{—— } (4)$$

$$A^T A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$$

$$|A^T A - \lambda I| = 0 \qquad \therefore \begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0 \quad \text{—— } (5)$$

$$\therefore\ \boxed{\lambda = 6}\ ,\ \boxed{\lambda = 1}$$

$\boxed{\lambda = 6}$ From (5),

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$\therefore\ -x + 2y = 0$, $2x - 4y = 0$ | $\therefore\ y = \frac{1}{2}x$ | For $x = 1$, $y = \frac{1}{2}$

$$\therefore\ v_1 = \begin{bmatrix} \frac{1}{\sqrt{5/2}} \\ \frac{1/2}{\sqrt{5/2}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 0.8944 \\ 0.4472 \end{bmatrix} \quad \text{—— } (6)$$

$\boxed{\lambda = 1}$ From (5),

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$\therefore\ 4x + 2y = 0$, $2x + y = 0$ | $\therefore\ y = -2x$ | For $x = 1$, $y = -2$

$$\therefore\ v_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 0.4472 \\ -0.8944 \end{bmatrix} \qquad \text{———} \ (7)$$

$$S = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \Bigg\| \quad \therefore\ S_p^{-1} = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.4082 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\text{———} \ (8)$$

$$A_p^{-1} = V S_p^{-1} U^T$$

$$\therefore\ A_p^{-1} = \begin{bmatrix} 0.8944 & 0.4472 \\ 0.4472 & -0.8944 \end{bmatrix} \begin{bmatrix} 0.4082 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.1825 & 0.3651 & 0.9128 \\ -0.8944 & 0.4472 & 0 \\ 0.4082 & 0.8165 & -0.4082 \end{bmatrix}.$$

$$\therefore\ A_p^{-1} = \begin{bmatrix} -0.3333 & 0.3333 & 0.3333 \\ 0.8333 & -0.3333 & 0.1666 \end{bmatrix} \qquad \text{———} \ (9)$$

$$X = A_p^{-1} b$$

$$\therefore\ X = \begin{bmatrix} -0.3333 & 0.3333 & 0.3333 \\ 0.8333 & -0.3333 & 0.1666 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$\therefore \quad X = \begin{bmatrix} 2.3330 \\ 0.6665 \end{bmatrix} \quad \therefore \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2.3330 \\ 0.6665 \end{bmatrix}$$

$\therefore$ $\boxed{y = 2.3330t + 0.6665\,(t-1)^2}$ is the desired model function.

When $y = 2$,

$$0.6665\,(t-1)^2 + 2.3330t - 2 = 0$$

$$\therefore \quad t^2 - 2t + 1 + 3.5t - 3 = 0$$

$$\therefore \quad t^2 + 1.5t - 2 = 0$$

$$\therefore \quad t = \frac{-1.5 \pm \sqrt{(1.5)^2 + 8}}{2}$$

$$\therefore \quad t = \frac{-1.5 \pm 3.2}{2}$$

$$\therefore \quad \boxed{t = 0.85}$$