Assignment - 9

Newton Nath & Vikas Chand

(2) If λ is an e.v of a non-singular matrix A. Find out e.v of matrix adj A.

Soln/ Since λ is e.v of matrix A.

Let, 'x' be a eigen vector of A

then, $AX = \lambda X$ —(1)

Multiplying both side by A^{-1} we get,

$$A^{-1}AX = A^{-1}\lambda X$$

$$X = \lambda A^{-1}X \qquad [\because A^{-1}A = I]$$

$$\Rightarrow A^{-1}X = \frac{1}{\lambda}X \quad \text{—(2)}$$

$\therefore \frac{1}{\lambda}$ is the eigen value of the matrix A^{-1}.

Since we know,

$$A^{-1} = \frac{adj(A)}{|A|} \qquad [\text{from def}^n]$$

$$\Rightarrow adj(A) = A^{-1}|A|$$

$$= \cancel{\frac{1}{\lambda}|A|}$$

So the eigen value of adj (A) is eigen value of ~~A^{-1} multi~~ $A^{-1}|A|$ which is $\frac{1}{\lambda}|A|$

using (2)

//

Assignment 9

Apurv Chaitanya N & Sanjay Saini

September 2, 2012

Question: 9. For a symmetical square matrix, show that the eigen vectors corresponding to two unequal eigenvalues are orthogonal.

Answer:
Convention: Let $|a\rangle$ represent a column vector and $\langle a|$ represent a row vector.

Given:

$$A = A^T \tag{1}$$

Claim: $|\lambda_i\rangle$ and $|\lambda_i\rangle$ are eigenvectors of A with distinct eigenvalues λ_i and λ_j respectively then $\langle\lambda_i|\lambda_j\rangle = 0$

Proof: As $|\lambda_i\rangle$ is an eigen vector of A

$$\mathrm{A}|\lambda_\mathrm{i}\rangle = \lambda_\mathrm{i}|\lambda_\mathrm{i}\rangle \tag{2}$$

We take inner product of eq.2 with $\langle\lambda_j|$,

$$\langle\lambda_j|\mathrm{A}|\lambda_i\rangle = \lambda_i\langle\lambda_j|\lambda_i\rangle \tag{3}$$

Now taking transpose on both sides of eq.3 , we get

$$\langle\lambda_i|\mathrm{A}|\lambda_j\rangle = \lambda_i\langle\lambda_i|\lambda_j\rangle \tag{4}$$

where we have used the facts that $\mathrm{A^T} = \mathrm{A}$, $|a\rangle^T = \langle a|$ and $\langle a|^T = |a\rangle$. We have also used the fact that $\lambda^T = \lambda$ for any symmetric matrix as $\lambda \in \mathbb{R}$. (Though transposing dosent have any effect on the eigenvalue of symmetric matrix we emphasis this point so that we can easily generalise this proof to Hermitian matrices as well). As λ_j is also an eigen vector, we get from eq. 4

$$\lambda_j\langle\lambda_i|\lambda_j\rangle = \lambda_i\langle\lambda_i|\lambda_j\rangle \tag{5}$$

Taking the terms of eq.5 to one side,

$$(\lambda_j - \lambda_i)\langle\lambda_i|\lambda_j\rangle = 0 \tag{6}$$

as $\lambda_j \neq \lambda_i$, the first term in eq.6 can not be zero i. e. $(\lambda_j - \lambda_i) \neq 0$. So the second term in eq.6 must be zero. So

$$\langle \lambda_i | \lambda_j \rangle = 0 \qquad Q.E.D$$

Assignment - 9 (Group-4)

Q.4. Using Cayley - Hamilton theorem, find the inverse of

$$\begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

Ans→ Let us take $A = \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$

Now we can write

$$A - \lambda I = \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-\lambda & -1 & 3 \\ 6 & 1-\lambda & 4 \\ 2 & 4 & 8-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 7-\lambda & -1 & 3 \\ 6 & 1-\lambda & 4 \\ 2 & 4 & 8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (7-\lambda)\{(1-\lambda)(8-\lambda)-16\} - (-1)\{6(8-\lambda)-8\} + 3\{24-2(1-\lambda)\} = 0$$

$$\Rightarrow (7-\lambda)(8-\lambda-8\lambda+\lambda^2-16) + (48-6\lambda-8) + 3(24-2+2\lambda) = 0$$

$$\Rightarrow (7-\lambda)(\lambda^2-9\lambda-8) + 40 - \cancel{6\lambda} + 72 - 6 + \cancel{6\lambda} = 0$$

$$\Rightarrow 7\lambda^2 - 63\lambda - 56 - \lambda^3 + 9\lambda^2 + 8\lambda + 106 = 0$$

$$\Rightarrow -\lambda^3 + 16\lambda^2 - 55\lambda + 50 = 0$$

$$\Rightarrow \lambda^3 - 16\lambda^2 + 55\lambda - 50 = 0 \quad \text{——— (1)}$$

∴ Hence the characteristic eqn is $\lambda^3 - 16\lambda^2 + 55\lambda - 50 = 0$

According to Cayley - Hamilton theorem the matrix A also satisfies the above charactoristic eqn. So we can write

$$A^3 - 16A^2 + 55A - 50I = 0$$

$$\Rightarrow 50I = A^3 - 16A^2 + 55A$$

$$\Rightarrow I = \frac{1}{50}(A^3 - 16A^2 + 55A)$$

Premultiplying A^{-1} on both sides of the above eq^n we've

$$A^{-1}I = \frac{A^{-1}}{50}\left(A^3 - 16A^2 + 55A\right)$$

$$\Rightarrow A^{-1} = \frac{1}{50}\left(A^2 - 16A + 55I\right) \quad \text{------} (2)$$

$A^2 = A \cdot A$

$$= \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}\begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 49-6+6 & -7-1+12 & 21-4+24 \\ 42+6+8 & -6+1+16 & 18+4+32 \\ 14+24+16 & -2+4+\cancel{86}32 & 6+16+64 \end{bmatrix}$$

$$= \begin{bmatrix} 49 & 4 & 41 \\ 56 & 11 & 54 \\ 54 & 34 & 86 \end{bmatrix}$$

Now eq^n (2) becomes

$$A^{-1} = \frac{1}{50}\left\{\begin{bmatrix} 49 & 4 & 41 \\ 56 & 11 & 54 \\ 54 & 34 & 86 \end{bmatrix} - 16\begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix} + 55\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{50}\left\{\begin{bmatrix} 49 & 4 & 41 \\ 56 & 11 & 54 \\ 54 & 34 & 86 \end{bmatrix} - \begin{bmatrix} 112 & -16 & 48 \\ 96 & 16 & 64 \\ 32 & 64 & 128 \end{bmatrix} + \begin{bmatrix} 55 & 0 & 0 \\ 0 & 55 & 0 \\ 0 & 0 & 55 \end{bmatrix}\right\}$$

$$= \frac{1}{50}\begin{bmatrix} 104-112 & 20 & -7 \\ -40 & 66-16 & -10 \\ 22 & -30 & 141-128 \end{bmatrix}$$

$$= \frac{1}{50}\begin{bmatrix} -8 & 20 & -7 \\ -40 & 50 & -10 \\ 22 & -30 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{8}{50} & \frac{2}{5} & \frac{-7}{50} \\ \frac{-4}{5} & 1 & \frac{-1}{5} \\ \frac{11}{25} & \frac{-3}{5} & \frac{13}{50} \end{bmatrix}$$

Ans

Extra

To check:

$\therefore AA^{-1}$

$$= \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} -\frac{8}{50} & \frac{2}{5} & -\frac{7}{50} \\ -\frac{4}{5} & 1 & -\frac{1}{5} \\ \frac{11}{25} & -\frac{3}{5} & \frac{13}{50} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{56}{50} + \frac{4}{5} + \frac{33}{25} & \frac{14}{5} - 1 = \frac{9}{5} & -\frac{49}{50} + \frac{1}{5} + \frac{39}{50} \\ -\frac{48}{50} - \frac{4}{5} + \frac{44}{25} & \frac{12}{5} + 1 = \frac{12}{5} & -\frac{42}{50} - \frac{1}{5} + \frac{52}{50} \\ -\frac{16}{50} - \frac{16}{5} + \frac{88}{25} & \frac{4}{5} + 4 = \frac{24}{5} & -\frac{14}{50} - \frac{4}{5} + \frac{104}{50} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-56+40+66}{50} & \frac{14-5=9}{5} & \frac{-49+10+39}{50} \\ \frac{-48-40+88}{50} & \frac{12+5=12}{5} & \frac{-42-10+52}{50} \\ \frac{-16-160+176}{50} & \frac{4+20=24}{5} & \frac{-14-40+104}{50} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Proved

Assignment No-9 Group-3

भौतिक अनुसंधान प्रयोगशाला, अहमदाबाद
Physical Research Laboratory, Ahmedabad

Problem-5

The given matrix is $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Let λ be the eigen value of of this matrix.

Then $\begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$

$$(-1-\lambda)\{(2-\lambda).(-\lambda)+1\} + 2\{-1+\lambda\} - 2\{-1+(2-\lambda)\} = 0$$

$$(-1-\lambda)(-2\lambda+\lambda^2+1) + 2(-1+\lambda) - 2(-1+2-\lambda) = 0$$

$$(2\lambda-\lambda^2-1+2\lambda^2-\lambda^3-\lambda) + 2(-1+\lambda) + 2 - 4 + 2\lambda = 0$$

$$-\lambda^3+\lambda^2+\lambda-1+2\lambda-2+2\lambda-2=0$$

$$-\lambda^3+\lambda^2+5\lambda-5=0$$

$$-\lambda^2(\lambda-1)+5(\lambda-1)=0$$

$$(\lambda-1)(5-\lambda^2)=0$$

$$\lambda=1$$

$$\lambda=\pm\sqrt{5}$$

From the eigen value we can directly ~~writh~~ write the diagonal matrix of A. Because we know that the eigenvalue of a matrix, appears in the diagonal of a diagonal matrix.

So the diagonal form of a matrix A is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & +\sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

Assignment No - 9 Group - 3

भौतिक अनुसंधान प्रयोगशाला, अहमदाबाद
Physical Research Laboratory, Ahmedabad

Problem No - 5 The given matrix is $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

~~Soln~~ To find the eigen value we do

$|A - \lambda I| = 0$, λ is the eigen values.

$$\begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)\{(2-\lambda)\cdot(-\lambda)+1\} + 2\{-1+\lambda\} - 2\{-1+2-\lambda\} = 0$$

$$\Rightarrow (-1-\lambda)(-2\lambda+\lambda^2+1) - 2+2\lambda+2-4+2\lambda = 0$$

$$-\lambda^3+2\lambda^2-\lambda+2\lambda+\lambda^2-1+4\lambda-4 = 0$$

$$-\lambda^3+\lambda^2+5\lambda-5 = 0$$

$$-\lambda^2(\lambda-1)+5(\lambda-1) = 0$$

$$(\lambda-1)(\lambda^2-5) = 0$$

$$\lambda = 1, \lambda = \pm\sqrt{2}$$

The eigen values are $\lambda_1 = 1$, $\lambda_2 = -\sqrt{5}$, $\lambda_3 = \sqrt{5}$

Eigen vector corresponding to $\lambda_1 = 1$ is given by

$$\begin{bmatrix} -1-1 & 2 & -2 \\ 1 & 2-1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-2x_1 + 2x_2 - 2x_3 = 0 \quad \text{---} (1)$$

$$x_1 + x_2 + x_3 = 0 \quad \text{---} (2)$$

From (1) $\Rightarrow$ $x_1 - x_2 + x_3 = 0$ — (3)

From (2) and (3)

$$x_1 - x_2 - x_1 - x_2 = 0$$

$$-2x_2 = 0$$

$$x_2 = 0$$

$$\Rightarrow \quad x_1 = -x_3$$

Taking $x_3 = 1$

Eigen vector corresponding to $\lambda = 1$ is $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Now eigenvector corresponding to $\lambda_2 = +\sqrt{5}$ is given by

$$\begin{bmatrix} -1-\sqrt{5} & 2 & -2 \\ 1 & 2-\sqrt{5} & 1 \\ -1 & -1 & -\sqrt{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$(-1-\sqrt{5})x_1 + 2x_2 - 2x_3 = 0 \quad \text{---} (4)$$

$$x_1 + (2-\sqrt{5})x_2 + x_3 = 0 \quad \text{---} (5)$$

$$-x_1 - x_2 - \sqrt{5}x_3 = 0 \quad \text{---} (6)$$

From (6) $x_1 = -x_2 - \sqrt{5}\,x_3$

Putting in (4) and (5)

$$(-1-\sqrt{5})(-x_2-\sqrt{5}\,x_3)+2x_2-2x_3=0 \quad \text{— (7)}$$

$$-x_2-\sqrt{5}\,x_3+(2-\sqrt{5})x_2+x_3=0 \quad \text{— (8)}$$

From (8) (7)

$$x_2(1+\sqrt{5}+2)+x_3(\sqrt{5}+5-2)\cancel{x_3}=0$$

$$x_2(\sqrt{5}+3)+x_3(\sqrt{5}+3)=0$$

$$\Rightarrow \quad x_2=-x_3$$

Now $\quad x_1 = x_3(1-\sqrt{5})$

Therefore the eigenvector is given by, taking $x_3=1$

$$X_2=\begin{bmatrix}1-\sqrt{5}\\ -1\\ 1\end{bmatrix}$$

The eigenvector corresponding to $\lambda_3=-\sqrt{5}$ is given by

$$\begin{bmatrix}-1+\sqrt{5} & 2 & -2\\ 1 & 2+\sqrt{5} & 1\\ -1 & -1 & \sqrt{5}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=0$$

$$x_1(-1+\sqrt{5})+2x_2-2x_3=0 \quad \text{— (9)}$$

$$x_1+(2+\sqrt{5})x_2+x_3=0 \quad \text{— (10)}$$

$$-x_1-x_2+\sqrt{5}\,x_3=0 \quad \text{— (11)}$$

$$x_1=-x_2+\sqrt{5}\,x_3$$

Putting this (9) o

$$-x_2(-1+\sqrt{5})+x_3\sqrt{5}(-1+\sqrt{5})+2x_2-2x_3=0$$

$$x_2(1-\sqrt{5}+2)+x_3(-\sqrt{5}+5-2)=0$$

$$x_2(3-\sqrt{5})+x_3(3-\sqrt{5})=0$$

$$\Rightarrow \quad x_2=-x_3$$

$$x_1=x_3(1+\sqrt{5})$$

Taking $x_3=1$ we get eigenvector corresponding to $-\sqrt{5}$

i.e. $X_3=\begin{bmatrix}1+\sqrt{5}\\-1\\1\end{bmatrix}$

Now construct the matrix from the eigenvectors

$$\Rightarrow \quad P=\begin{bmatrix}-1 & 1-\sqrt{5} & 1+\sqrt{5}\\0 & -1 & -1\\1 & 1 & 1\end{bmatrix}$$

Now take the inverse of matrix P, P^{-1} comes as

$$P^{-1}=\begin{bmatrix}0 & 1 & 1\\-1/2\sqrt{5} & \frac{-2-5}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}}\\1/2\sqrt{5} & \frac{2-\sqrt{5}}{2\sqrt{5}} & \frac{1}{2\sqrt{5}}\end{bmatrix}=\frac{1}{2\sqrt{5}}\begin{bmatrix}0 & 2\sqrt{5} & 2\sqrt{5}\\-1 & -2-\sqrt{5} & -1\\1 & 2-\sqrt{5} & 1\end{bmatrix}$$

The diagonal matrix of A is given by

$$P^{-1}AP=\frac{1}{2\sqrt{5}}\begin{bmatrix}0 & 2\sqrt{5} & 2\sqrt{5}\\-1 & -2-\sqrt{5} & -1\\1 & 2-\sqrt{5} & 1\end{bmatrix}\begin{bmatrix}-1 & 2 & -2\\1 & 2 & 1\\-1 & -1 & 0\end{bmatrix}\begin{bmatrix}-1 & 1-\sqrt{5} & 1+\sqrt{5}\\0 & -1 & -1\\1 & 1 & 1\end{bmatrix}$$

$$=\frac{1}{2\sqrt{5}}\begin{bmatrix}0 & 2\sqrt{5} & 2\sqrt{5}\\-1 & -2-\sqrt{5} & -1\\1 & 2-\sqrt{5} & 1\end{bmatrix}\begin{bmatrix}-1 & -5+\sqrt{5} & -5-\sqrt{5}\\0 & -\sqrt{5} & \sqrt{5}\\1 & \sqrt{5} & -\sqrt{5}\end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}} \begin{bmatrix} 2\sqrt{5} & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

This is the diagonal matrix of the given matrix.

Assignment - 9 Date: 28.08.2012

Group 9

6. Show that an $n \times n$ matrix A is singular if and only if zero is one of its eigenvalues. Give an example of a singular 3×3 matrix and find out its eigenvalues and eigenvectors.

$\Rightarrow$ Let A be the $n\times n$ matrix: $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

Now, A is a singular matrix, $\det(A) = 0$.

Method 1: We know that, if A is a singular matrix then only

$AX = 0$ for any nonzero arbitrary column matrix X

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Now, in general, if λ is the eigenvalue of a matrix then the characteristic equation can be written as:

$AX = \lambda X$ for any nonzero X.

So, here, $\lambda X = 0$.

or, $\lambda I X = 0$

or, $\lambda = 0$. $[\because I, X \neq 0]$

Hence, one of the eigenvalues of A must be zero for A to become a singular matrix.

Method 2: If λ is the eigenvalue of A then the characteristic eq^n is:

$|A - \lambda I| = 0$

Now if we put $\lambda = 0$, it gives $|A| = 0$ which implies A is a singular matrix.

if we put $|A| = 0$ then putting $\lambda = 0$ satisfies the above eq^n.

Hence $\lambda = 0$ is essentially the "necessary & sufficient condition" for A to become a singular matrix.

xample:

$A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ is a singular matrix

The characteristic eqn is:

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$

or, $(8-\lambda)[(7-\lambda)(3-\lambda) - 16] + 6[-6(3-\lambda) + 8] + 2[24 - 2(7-\lambda)] = 0$

or, $(8-\lambda)[21 + \lambda^2 - 10\lambda - 16] + 6[6\lambda - 10] + 2[2\lambda + 10] = 0$

or, $(8-\lambda)[\lambda^2 - 10\lambda + 5] + 36\lambda - 60 + 4\lambda + 20 = 0$

or, $8\lambda^2 - 80\lambda + \cancel{40} - \lambda^3 + 10\lambda^2 - 5\lambda + 40\lambda - \cancel{40} = 0$

~~or, $-\lambda^3 + 18\lambda^2 - 70\lambda^2 + 35\lambda = 0$~~

or, $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

or, $\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\therefore \lambda = 0$ or, $\lambda^2 - 3\lambda - 15\lambda + 45 = 0$

or, $(\lambda - 3)(\lambda - 15) = 0$

$\therefore \lambda = 3, 15$.

So $\lambda = 0, 3, 15$. $\Rightarrow$ These are the eigenvalues of the matrix A.

Now, let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigenvectors of the given matrix A

So, $(A - \lambda I)X = 0$

(i) for $\lambda = 0$,

$AX = 0$

or, $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \left.\begin{aligned} 8x_1 - 6x_2 + 2x_3 &= 0 \\ -6x_1 + 7x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 + 3x_3 &= 0 \end{aligned}\right\} \rightarrow (1)$

$\therefore$ $-5x_2 + 5x_3 = 0$

$\therefore x_2 = x_3$

$2x_1 = x_2$

$\therefore x_1$ is arbitrary and the eigenvector is! $\vec{X}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. (say $x_1 = 1$).

(ii) for $\lambda = 3$,

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

or, $\left.\begin{aligned} 5x_1 - 6x_2 + 2x_3 &= 0 \\ -6x_1 + 4x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 \quad &= 0 \end{aligned}\right\}$ (2).

$\therefore$ $x_1 = 2x_2$. & x_2 is arbitrary

$x_3 = -2x_2$.

$\therefore$ The eigenvector is! $\vec{X}_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ (say $x_2 = 1$).

(iii) for, $\lambda = 15$

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \left.\begin{aligned} -7x_1 - 6x_2 + 2x_3 &= 0 \\ -6x_1 - 8x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 - 12x_3 &= 0 \end{aligned}\right\} \Rightarrow (3).$$

$\therefore$ $-20x_2 - 40x_3 = 0$.

$x_2 = -2x_3$.

& $x_1 = 2x_2$.

and x_3 is arbitrary. (say $x_3 = 1$).

$\therefore$ The eigenvector is! $\vec{X}_3 = \begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix}$

$\therefore$ The eigenvalues are $\lambda = 0, 3$ & 15 and the corresponding set of eigenvectors are!

$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix}$ respectively. Ans

Assignment # 9

Group - Six

Question # 9 :- Given $A = \begin{bmatrix} 0 & 0 & i \\ -i & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

Show that $A^n = 1$ (with proper choice of n ; $n \neq 0$)

Solution.. Charecteristic equation of given matrix,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 0 & i \\ -i & -\lambda & 0 \\ 0 & -1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(\lambda^2 - 0) + i(i) = 0$$

$$-\lambda^3 + i^2 = 0$$

$$-\lambda^3 - 1 = 0$$

$$\lambda^3 + 1 = 0 \quad \text{--- (1)}$$

By Cayley-Hamilton's theorem,

"Every matrix satisfies its own charecteristic equation."

So

$$A^3 + 1 = 0$$

$$\Rightarrow A^3 = -1$$

So $A^3 \cdot A^3 = (-1)(-1)$

$$\Rightarrow \boxed{A^6 = 1}$$

So for $n = 6$, $A^n = 1$.

ie $A^6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

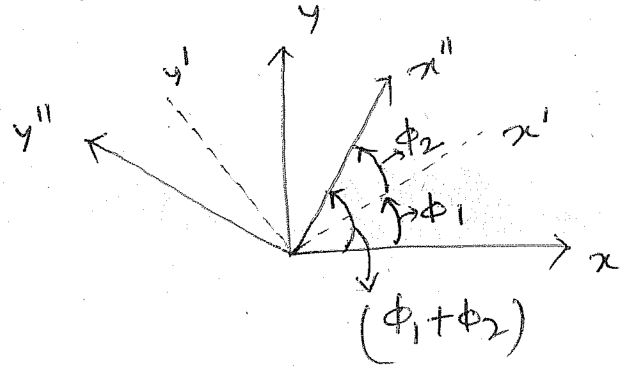
भौतिक अनुसंधान प्रयोगशाला, अहमदाबाद
Physical Research Laboratory, Ahmedabad

Group - 8   assignment - 9

Prob:- (10) A rotation $\phi_1+\phi_2$ about the z-axis is carried out as two successive rotations ϕ_1 and ϕ_2, each about the z-axis. Use the matrix representation of the rotations to derive the trigonometric identities:

$$\cos(\phi_1+\phi_2) = \cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2 \quad \text{and}$$
$$\sin(\phi_1+\phi_2) = \sin\phi_1\cos\phi_2 + \cos\phi_1\sin\phi_2 .$$

Solⁿ



If we rotate the x-y plane about z axis with an angle $\phi_1+\phi_2$, then the new co-ordinate becomes x''- y''. The relation betⁿ ~~the~~ (x'', y'') and (x, y) given by

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{bmatrix} \cos(\phi_1+\phi_2) & \sin(\phi_1+\phi_2) \\ -\sin(\phi_1+\phi_2) & \cos(\phi_1+\phi_2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= T(\phi_1+\phi_2)\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{—(1)}$$

Now, if we rotate x-y plane w.r.t z axis about an angle ϕ_1, then

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = T(\phi_1) \begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } T(\phi_1) = \begin{pmatrix} \cos\phi_1 & \sin\phi_1 \\ -\sin\phi_1 & \cos\phi_1 \end{pmatrix} \quad \text{— } (2)$$

Now, again we rotate x'-y' plane w.r.t z axis, say an angle ϕ_2, then

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = T(\phi_2) \begin{pmatrix} x' \\ y' \end{pmatrix}, \text{ where } T(\phi_2) = \begin{pmatrix} \cos\phi_2 & \sin\phi_2 \\ -\sin\phi_2 & \cos\phi_2 \end{pmatrix}$$

$$= T(\phi_2)\, T(\phi_1) \begin{pmatrix} x \\ y \end{pmatrix} \quad \left[\text{From eq}^n \text{ (2)}\right] \quad \text{— } (3)$$

Comparing (1) and (3), we get

$$T(\phi_1+\phi_2) = T(\phi_2)\, T(\phi_1)$$

$$\Rightarrow \begin{pmatrix} \cos(\phi_1+\phi_2) & \sin(\phi_1+\phi_2) \\ -\sin(\phi_1+\phi_2) & \cos(\phi_1+\phi_2) \end{pmatrix} = \begin{pmatrix} \cos\phi_2 & \sin\phi_2 \\ -\sin\phi_2 & \cos\phi_2 \end{pmatrix} \begin{pmatrix} \cos\phi_1 & \sin\phi_1 \\ -\sin\phi_1 & \cos\phi_1 \end{pmatrix}$$

$$= \begin{pmatrix} (\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2) & (\sin\phi_1\cos\phi_2 + \cos\phi_1\sin\phi_2) \\ -(\cos\phi_1\sin\phi_2 + \sin\phi_1\cos\phi_2) & (-\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2) \end{pmatrix}$$

$$\Rightarrow \cos(\phi_1+\phi_2) = \cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2$$

$$\sin(\phi_1+\phi_2) = \sin\phi_1\cos\phi_2 + \cos\phi_1\sin\phi_2$$

Assignment-9

Group-10 (George and Ritwik)

Question:

Prove that if $A^k = 0$ for some positive integer k (such a matrix is said to be nilpotent), then all the eigen-values of A are zero.

Answer:

Let A be any $n \times n$ nilpotent matrix with the degree k and λ be the eigen values of the matrix corresponding to the Eigen-vector V. Then the eigen value equation would be

$$AV = \lambda V$$

It is given that,

$$A^k = 0$$

Take the term

$$A^k V = A^{k-1} AV = A^{k-1} \lambda V = A^{k-2} \lambda^2 V = \ \dots \dots \dots \dots \dots \dots = \lambda^k V$$

Since A is nilpotent matrix and it is defined that $A^k = 0$,

$$\Rightarrow A^k V = \lambda^k V = 0$$

Now take that V's are not a zero matrices, then we can write

$$\lambda^k = 0$$

This tells that $\lambda = 0$

Here we have chosen a arbitrary eigen-value so it is in general all the eigen values of a nilpotent matrix should be zero. (Q.E.D.)