भौतिक अनुसंधान प्रयोगशाला, अहमदाबाद
Physical Research Laboratory, Ahmedabad

ASSIGNMENT NO. 6

GROUP - 2
CHANDANA
JINIA

PROBLEM 1:

If $\sin(A+iB) = x+iy$

Prove that $\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$

and $\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$

SOLUTION :

We have $\sin(A+iB) = x+iy$

$$x+iy = \sin A \cos iB + \cos A \sin iB$$

$$x+iy = \sin A \cosh B + i\cos A \sinh B$$

Now equating real & imaginary parts, we get

$x = \sin A \cosh B$ $\quad$ $y = \cos A \sinh B$

$\frac{x}{\sin A} = \cosh B$ —— (1) $\quad$ $\frac{y}{\cos A} = \sinh B$ —— (2)

$\frac{x}{\cosh B} = \sin A$ —— (3) $\quad$ $\frac{y}{\sinh B} = \cos A$ —— (4)

Squaring & subtracting eqn ② from eqn ①, we get

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B$$

$$\boxed{\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1}$$ $\left[\because \cosh^2 B - \sinh^2 B = 1\right]$

Now squaring and adding eqn ③ & ④, we get

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \sin^2 A + \cos^2 A$$

$$\boxed{\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1}$$ $\left[\because \sin^2 A + \cos^2 A = 1\right]$

Assignment 6

Group 5: Apurv & Sanjay

Question

2 i. If $\sin(\theta + i\phi) = \rho(\cos\alpha + i\sin\alpha)$, prove that $\rho^2 = 1/2[\cosh 2\phi - \cos 2\theta]$

Solution

$$\rho(\cos\alpha + i\sin\alpha) = \sin(\theta + i\phi) \tag{0.0.1}$$
$$= \sin\theta\cos i\phi + \cos\theta\sin i\phi \tag{0.0.2}$$
$$= \sin\theta\cosh\phi + i\cos\theta\sinh\phi \tag{0.0.3}$$

For two complex numbers to be equal one of the necessary condition is that their magnitude must be equal. Then,

$$\rho^2(\cos^2\alpha + \sin^2\alpha) = \sin^2\theta\cosh^2\phi + \cos^2\theta\sinh^2\phi \tag{0.0.4}$$
$$\rho^2 = (1 - \cos^2\theta)\cosh^2\phi + \cos^2\theta(1 - \cosh^2\phi) \tag{0.0.5}$$
$$= \cosh^2\phi - \cos^2\theta \tag{0.0.6}$$
$$= \frac{1 + \cosh 2\phi}{2} - \frac{1 + \cos 2\theta}{2} \tag{0.0.7}$$
$$= \frac{1}{2}[\cosh 2\phi - \cos 2\theta] \qquad Q.E.D \tag{0.0.8}$$

Group - 4 (Assignment - 6)

Q. If $\sin(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$ prove that

$\tan\alpha = \tanh\phi \cdot \cot\theta$.

Proof: Given that $\sin(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$

$\Rightarrow \sin\theta \cdot \cos i\phi + \cos\theta \cdot \sin i\phi = r(\cos\alpha + i\sin\alpha)$

$\Rightarrow \sin\theta \cosh\phi + \cos\theta \cdot (i\sinh\phi) = r(\cos\alpha + i\sin\alpha)$

$\Rightarrow \sin\theta \cdot \cosh\phi + i\cos\theta \cdot \sinh\phi = r\cos\alpha + i\, r\sin\alpha$

Comparing the real & imaginary parts we get

$r\cos\alpha = \cosh\phi \cdot \sin\theta$ ——— (1)

$r\sin\alpha = \sinh\phi \cdot \cos\theta$ ——— (2)

Dividing eqn (2) by eqn (1) we get

$\tan\alpha = \tanh\phi \cdot \cot\theta$ Proved

Assignment – 6 Group–3

Prob–3(i)

If $\cos(\theta + i\phi) = \cos\alpha + i\sin\alpha$, prove that

(i) $\sin^2\theta = \pm\sin\alpha$

Solⁿ :–

$$\cos(\theta + i\phi) = \cos\alpha + i\sin\alpha$$

$$\cos\theta\cos(i\phi) - \sin\theta\sin(i\phi) = \cos\alpha + i\sin\alpha \quad \{\cos(A+B) = \cos A\cos B - \sin A\sin B\}$$

$$\cos\theta\cosh\phi - i\sin\theta\sinh\phi = \cos\alpha + i\sin\alpha \quad \{\cos(i\theta) = \cosh\theta,\ \sin(i\theta) = i\sinh\theta\}$$

Comparing real & Imaginary parts.

$$\cos\theta\cosh\phi = \cos\alpha \quad \& \quad -\sin\theta\sinh\phi = \sin\alpha$$

To eliminate ϕ, Squaring & ~~Subst.~~

$$\cos^2\theta\cosh^2\phi = \cos^2\alpha \qquad \sin^2\theta\sinh^2\phi = \sin^2\alpha$$

$$\cosh^2\phi = \frac{\cos^2\alpha}{\cos^2\theta} \qquad \sinh^2\phi = \frac{\sin^2\alpha}{\sin^2\theta}$$

As $\cosh^2\phi - \sinh^2\phi = 1$

$$\therefore \quad \frac{\cos^2\alpha}{\cos^2\theta} - \frac{\sin^2\alpha}{\sin^2\theta} = 1$$

$$\frac{1-\sin^2\alpha}{1-\sin^2\theta} - \frac{\sin^2\alpha}{\sin^2\theta} = 1$$

$$\frac{1-\sin^2\alpha}{1-\sin^2\theta} - 1 = \frac{\sin^2\alpha}{\sin^2\theta}$$

$$\frac{\cancel{1} - \sin^2\alpha - \cancel{1} + \sin^2\theta}{1-\sin^2\theta} = \frac{\sin^2\alpha}{\sin^2\theta}$$

$$\sin^2\theta(\sin^2\theta - \sin^2\alpha) = \sin^2\alpha(1-\sin^2\theta)$$

$$\sin^4\theta - \cancel{\sin^2\theta\sin^2\alpha} = \sin^2\alpha - \cancel{\sin^2\alpha\sin^2\theta}$$

$$\sin^4\theta = \sin^2\alpha$$

$\boxed{\sin^2\theta = \pm\sin\alpha}$ Hence proved

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MNM (Assignment # 6) Group-6

Problem. If $\cos(\theta+i\phi) = \cos\alpha + i\sin\alpha$

Prove that

$$\cos 2\theta + \cosh 2\phi = 2$$

Sol^n We have

$$\cos(\theta+i\phi) = \cos\alpha + i\sin\alpha$$

$$\Rightarrow \cos\theta\cos(i\phi) - \sin\theta\sin(i\phi) = \cos\alpha + i\sin\alpha$$

$$\Rightarrow \cos\theta\cosh\phi - i\sin\theta\sinh\phi = \cos\alpha + i\sin\alpha$$

$$\Rightarrow \cos\alpha = \cos\theta\cosh\phi, \quad \sin\alpha = -\sin\theta\sinh\phi$$

$$\because \cos^2\alpha + \sin^2\alpha = 1$$

$$\cos^2\theta\cosh^2\phi + \sin^2\theta\sinh^2\phi = 1$$

$$\Rightarrow \cos^2\theta\cosh^2\phi + (1-\cos^2\theta)\sinh^2\phi = 1$$

$$\Rightarrow (\cos^2\theta)(\cosh^2\phi - \sinh^2\phi) + \sinh^2\phi = 1$$

$$\Rightarrow \cos^2\theta + \sinh^2\phi = 1 \quad (\because \cosh^2\phi - \sinh^2\phi = 1)$$

$$\Rightarrow 2\cos^2\theta + 2\sinh^2\phi = 2$$

$$\Rightarrow (2\cos^2\theta - 1) + (1 + 2\sinh^2\phi) = 2$$

$$\Rightarrow \boxed{\cos 2\theta + \cosh 2\phi = 2}$$

$$\left(\text{Since } \begin{array}{l} \cos 2\theta = 2\cos^2\theta - 1 \\ \cosh 2\phi = 1 + 2\sinh^2\phi \end{array}\right)$$

Proved

ASSIGNMENT-06 GROUP-07

Given that $\tan(A+iB) = x+iy$. P.T. $x^2+y^2+2x\cot 2A = 1$

$$\frac{\tan A + i\tanh B}{1 - i\tan A\tanh B} = x+iy, \quad \frac{\tan A + i\tanh B}{1-i\tan A\tanh B} \times \frac{1+i\tan A\tanh B}{1+i\tan A\tanh B} = x+iy \quad [\because \text{Rationalization}]$$

$$\Rightarrow \quad x = \frac{\tan(A) - \tan A\tanh^2(B)}{1+\tan^2(A)\tanh^2(B)}, \quad y = \frac{\tanh(B) + \tan^2(A)\tanh(B)}{1+\tan^2(A)\tanh^2(B)}$$

$$x^2+y^2+2x\cot 2A = \frac{\tan^2A + \tan^2A\tanh^4B - 2\tan^2A\cancel{\tanh^2B} + \tanh^2B + \tan^4A\tanh^2B + 2\,\cancel{\tan^2A\tanh^2B} + 2[\cancel{\tan A} - \cancel{\tan A}\tanh^2B][1+\tan^2A\tanh^2B]\left[\frac{1-\tan^2A}{\cancel{2\tan A}}\right]}{[1+\tan^2(A)\tanh^2(B)]^2}$$

Numerator only

$$\underline{\tan^2A(1+\tan^2A\tanh^2B)} + \tanh^2B(1+\tan^2A\tanh^2B) + \underline{[1-\tanh^2B][1+\tan^2A\tanh^2B][1-\tan^2A]}$$

$$[1+\tan^2(A)\tanh^2B][\cancel{\tan^2A} + 1 - \cancel{\tan^2A} - \tanh^2B + \tanh^2B\tan^2A] + \tanh^2(B)(1+\tan^2(A)\tanh^2(B))$$

$$= \frac{[1+\tan^2(A)\tanh^2(B)][1+\tan^2(A)\tanh^2(B)]}{[1+\tan^2(A)\tanh^2(B)]^2}$$

$$= 1. \quad \text{Proved}$$

G-11 Assignment - 6

Venkatesh chinni
Durgaprasad

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4(iii) If $\tan(A+iB) = x+iy$, P.T $x^2+y^2-2y\coth 2B+1=0$.

Sol: Given $\tan(A+iB) = x+iy \longrightarrow (1)$

Consider,

$$\tan[(A+iB)-(A-iB)] = \tan i2B$$

$$\left[\because \tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B}\right]$$

$$\Rightarrow \frac{\tan(A+iB)-\tan(A-iB)}{1+\tan(A+iB)\tan(A-iB)} = \tan i2B$$

$$\Rightarrow \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)} = i\tanh(2B) \quad [\because \text{By } (1)]$$

$$\Rightarrow \frac{\cancel{x}+iy-\cancel{x}+iy}{1+x^2+y^2} = i\tanh(2B)$$

$$\Rightarrow \frac{2y}{1+x^2+y^2} = \tanh(2B)$$

$$\Rightarrow 2y = (1+x^2+y^2)[\tanh(2B)] \longrightarrow (2)$$

Take L.H.S:

$$= x^2+y^2-2y\coth 2B+1$$

$$= x^2+y^2-(1+x^2+y^2)\cancel{\tanh(2B)}\,\cancel{\coth(2B)}+1 \quad [\because \text{By } (2)]$$

$$= x^2+y^2-1-x^2-y^2+1$$

$= 0$ R.H.S Hence proved.

Group-8
assignment-6

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If $\tan(x+iy) = \sin(u+iv)$ then prove that

$$\frac{\sin 2x}{\sinh 2y} = \frac{\tan u}{\tanh v}.$$

Now, $\tan(x+iy) = \sin(u+iv)$

Take complex conjugate of both side

$$\tan(x-iy) = \sin(u-iv)$$

So, $$\frac{\tan(x+iy)}{\tan(x-iy)} = \frac{\sin(u+iv)}{\sin(u-iv)}$$

or, $$\frac{\tan(x+iy)+\tan(x-iy)}{\tan(x+iy)-\tan(x-iy)} = \frac{\sin(u+iv)+\sin(u-iv)}{\sin(u+iv)-\sin(u-iv)}$$

or, $$\frac{\frac{\sin(x+iy)}{\cos(x+iy)}+\frac{\sin(x-iy)}{\cos(x-iy)}}{\frac{\sin(x+iy)}{\cos(x+iy)}-\frac{\sin(x-iy)}{\cos(x-iy)}} = \frac{2\sin u\cos(iv)}{2\cos u\sin(iv)}$$

or, $$\frac{\sin(x+iy)\cos(x-iy)+\cos(x+iy)\sin(x-iy)}{\sin(x+iy)\cos(x-iy)-\cos(x+iy)\sin(x-iy)} = \frac{\tan u\cosh v}{i\sinh v}$$

or, $$\frac{\sin[x+iy+x-iy]}{\sin[(x+iy)-(x-iy)]} = \frac{\tan u}{i\tanh v}$$

or, $$\frac{\sin 2x}{\sin(i2y)} = \frac{\tan u}{i\tanh v} \Rightarrow \frac{\sin 2x}{\sinh(2y)} = \frac{\tan u}{\tanh v}$$ (proved)

Date: 22.08.2012 Assignment-6 Group - 9

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(6) If $x = 2\cos\alpha\cosh\beta$

$y = 2\sin\alpha\sinh\beta$, prove that

$$\sec(\alpha+i\beta) + \sec(\alpha-i\beta) = \frac{4x}{x^2+y^2}.$$

$\Rightarrow$ $x = 2\cos\alpha\cosh\beta$.

Now, $\cosh\theta = \dfrac{e^{\theta}+e^{-\theta}}{2} = \dfrac{e^{-i(i\theta)}+e^{i(i\theta)}}{2} = \dfrac{e^{i(i\theta)}+e^{-i(i\theta)}}{2}$

$= \cos(i\theta)$

and $\sinh\theta = \dfrac{e^{\theta}-e^{-\theta}}{2} = \dfrac{e^{-i(i\theta)}-e^{i(i\theta)}}{2} = (-i)\dfrac{e^{i(i\theta)}-e^{-i(i\theta)}}{2i}$

$= (-i)\sin(i\theta)$.

So, $x = 2\cos\alpha\cos(i\beta) = \cos(\alpha+i\beta) + \cos(\alpha-i\beta)$

$y = -2i\sin\alpha\sin(i\beta) = i\,[\cos(\alpha+i\beta) - \cos(\alpha-i\beta)]$

~~So $x+y=$~~

So, $\cos(\alpha+i\beta) = \frac{1}{2}(x-iy)$.

$\cos(\alpha-i\beta) = \frac{1}{2}(x+iy)$.

So,

L.H.S. $= \sec(\alpha+i\beta) + \sec(\alpha-i\beta)$

$= \dfrac{1}{\cos(\alpha+i\beta)} + \dfrac{1}{\cos(\alpha-i\beta)} = \dfrac{2}{x-iy} + \dfrac{2}{x+iy}$

$= 2\cdot\dfrac{x+iy+x-iy}{(x+iy)(x-iy)} = \dfrac{4x}{x^2+y^2}$ (proved)

ASSIGNMENT-6

Group-10(Bivin Geo George and Ritwik Mondal)

Question 7:

If $(a + ib) = \tanh\left(v + \frac{i\pi}{4}\right)$, then prove that $a^2 + b^2 = 1$.

Answer:

$$a + ib = \tanh\left(v + \frac{i\pi}{4}\right)$$

$$= \frac{\sinh\left(v + \frac{i\pi}{4}\right)}{\cosh\left(v + \frac{i\pi}{4}\right)}$$

Now, $\sinh(x + iy) = \sinh x \ \cos y + i \cosh x \sin y$

Using the equation,

$$\sinh(v + \frac{i\pi}{4}) = \frac{1}{\sqrt{2}}(\sinh v + i \cosh v)$$

And similarly using $\cosh(x + iy) = \cosh x \, cosy + i \sinh x \sin y$

$$\cosh\left(v + \frac{i\pi}{4}\right) = \frac{1}{\sqrt{2}}(\cosh v + i \sinh v)$$

Therefore from previous,

$$a + ib = \frac{(\sinh v + i \cosh v)}{(\cosh v + i \sinh v)}$$

$$= \frac{(\sinh v + i \cosh v)(\cosh v - i \sinh v)}{\cosh^2 v + \sinh^2 v}$$

$$= \frac{(\sinh v \cosh v + \cosh v \sinh v) + i(\cosh^2 v - \sinh^2 v)}{\cosh^2 v + \sinh^2 v}$$

$$= \frac{2 \sinh v \cosh v + i}{\cosh^2 v + \sinh^2 v}$$

Then take the conjugate of that,

$$a - ib = \frac{2 \sinh v \cosh v - i}{\cosh^2 v + \sinh^2 v}$$

Multiplying the two last equations we got that

$$(a+ib)(a-ib) = \left[\frac{2\sinh v\cosh v + i}{\cosh^2 v + \sinh^2 v}\right]\left[\frac{2\sinh v\cosh v - i}{\cosh^2 v + \sinh^2 v}\right]$$

$$\Rightarrow a^2 + b^2 = \frac{4\sinh^2 v\cosh^2 v + 1}{(\cosh^2 v + \sinh^2 v)^2}$$

$$= \frac{4\sinh^2 v\,(\sinh^2 v + 1) + 1}{(\cosh^2 v + \sinh^2 v)^2}$$

$$= \frac{4\sinh^4 v + 4\sinh^2 v + 1}{(\cosh^2 v + \sinh^2 v)^2}$$

$$= \frac{(2\sinh^2 v + 1)^2}{(\cosh^2 v + \sinh^2 v)^2}$$

$$= \frac{(\cosh^2 v + \sinh^2 v)^2}{(\cosh^2 v + \sinh^2 v)^2}$$

$$\boxed{\Rightarrow a^2 + b^2 = 1}$$

That’s what it was in the question.

Q-1 Assignment-6

Newton

Vikas

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Problem

Write $\cos^{-1}(\cos\theta + i\sin\theta)$ in Real & imaginary parts. where θ is an acute angle.

Solution

$$\cos^{-1}(\cos\theta + i\sin\theta) = \alpha + i\beta \quad \text{—(I)}$$

taking complex conjugate

$$\Rightarrow \cos^{-1}(\cos\theta - i\sin\theta) = \alpha - i\beta \quad \text{—(II)}$$

Adding (I) + (II)

$$\Rightarrow 2\alpha = \cos^{-1}(\cos\theta + i\sin\theta) + \cos^{-1}(\cos\theta - i\sin\theta)$$

$$\alpha = \frac{1}{2}\left[\cos^{-1}\left((\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) - \sqrt{1-(\cos\theta+i\sin\theta)^2}\sqrt{1-(\cos\theta - i\sin\theta)^2}\right)\right]$$

$$= \frac{1}{2}\left[\cos^{-1}\left(\cos^2\theta + \sin^2\theta - \sqrt{\cancel{1-\cos^2\theta+\sin^2\theta - 2i\cos\theta\sin\theta}}\sqrt{\cancel{1-\cos^2\theta+\sin^2\theta}}\right)\right]$$

$$= \frac{1}{2}\cos^{-1}\left[\cos^2\theta + \sin^2\theta - \sqrt{1-(\cos\theta+i\sin\theta)^2 - (\cos\theta - i\sin\theta)^2 + (\cos^2\theta+\sin^2\theta)^2}\right]$$

$$= \frac{1}{2}\cos^{-1}\left[1 - \sqrt{\underbrace{1-\cos^2\theta}_{\sin^2\theta} + \sin^2\theta - 2i\cancel{\cos\theta}\sin\theta - \cos^2\theta + \underbrace{\sin^2\theta}_{2\sin^2\theta} + 2i\cancel{\cos\theta}\sin\theta + 1}\right]$$

$$= \frac{1}{2}\cos^{-1}\left[1 - \sqrt{4\sin^2\theta}\right] = \frac{1}{2}\cos^{-1}(1-2\sin\theta)$$

$\cos(x+y)$

$= \cos x\cos y - \sin x\sin y$

$= \cos x\cos y - \sqrt{1-\cos^2 x}\sqrt{1-\cos^2 y}$

put $x = \cos^{-1}a$, $y = \cos^{-1}b$

$\Rightarrow \cos^{-1}a + \cos^{-1}b = \cos^{-1}(ab - \sqrt{1-a^2}\sqrt{1-b^2})$

similarly

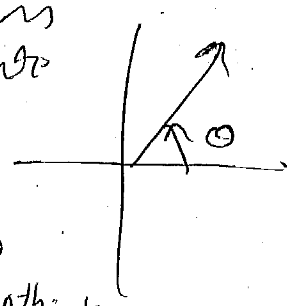
$\cos^{-1}a - \cos^{-1}b = \cos^{-1}(ab + \sqrt{1-a^2}\sqrt{1-b^2})$

$\alpha = \frac{1}{2}\cos^{-1}[1-2\sin\theta]$

Similarly subtracting ① & ②

① − ② ⇒

θ is acute so $\cos\theta$ & $\sin\theta$ are both +ve



$$2i\beta = \cos^{-1}(\cos\theta + i\sin\theta) - \cos^{-1}(\cos\theta - i\sin\theta)$$

$$2i\beta = \cos^{-1}(1+2\sin\theta)$$

$\cos ix = \cosh x$

$$\Rightarrow \cos(2i\beta) = (1+2\sin\theta)$$

$$\Rightarrow \cosh(2\beta) = (1+2\sin\theta)$$

$$\Rightarrow 2\beta = \cosh^{-1}(1+2\sin\theta) \Rightarrow \boxed{\beta = \frac{1}{2}\cosh^{-1}(1+2\sin\theta)}$$

$$\therefore \boxed{\cos^{-1}(\cos\theta + i\sin\theta) = \frac{1}{2}\left[\cos^{-1}(1-2\sin\theta) + i\cosh^{-1}(1+2\sin\theta)\right]}$$

Ans