Assignment - 5 — Group-3

Prob-1

Prove that $\cosh(\alpha+\beta) - \cosh(\alpha-\beta) = 2\sinh\alpha \sinh\beta$

Solⁿ:- L.H.S

$\cosh(\alpha+\beta) - \cosh(\alpha-\beta)$

$$= \frac{e^{(\alpha+\beta)} + e^{-(\alpha+\beta)}}{2} - \frac{e^{(\alpha-\beta)} + e^{-(\alpha-\beta)}}{2}$$

$$= \frac{e^{\alpha}e^{\beta} + e^{-\alpha}e^{-\beta}}{2} - \frac{e^{\alpha}e^{-\beta} + e^{-\alpha}e^{+\beta}}{2}$$

$$= \frac{e^{\alpha}e^{\beta} + e^{-\alpha}e^{-\beta} - e^{\alpha}e^{-\beta} - e^{-\alpha}e^{+\beta}}{2}$$

$$= \frac{e^{\alpha}(e^{\beta} - e^{-\beta}) + e^{-\alpha}(e^{-\beta} - e^{\beta})}{2}$$

$$= \frac{(e^{\beta} - e^{-\beta})(e^{\alpha} - e^{-\alpha})}{2} = 2\left(\frac{e^{\beta} - e^{-\beta}}{2}\right)\left(\frac{e^{\alpha} - e^{-\alpha}}{2}\right)$$

$$= 2\sinh\beta \sinh\alpha$$

$$= R.H.S$$

Hence proved

Assignment - 5

Newton Nath & Vikas Chand

(2) S.t.

$$\left(\frac{1+\tanh\theta}{1-\tanh\theta}\right)^3 = \cosh 6\theta + \sinh 6\theta$$

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L.H.S

$$\left(\frac{1+\tanh\theta}{1-\tanh\theta}\right)^3$$

$$= \left(\frac{1+\frac{\sinh\theta}{\cosh\theta}}{1-\frac{\sinh\theta}{\cosh\theta}}\right)^3$$

$$= \left(\frac{\cosh\theta+\sinh\theta}{\cosh\theta-\sinh\theta}\right)^3 .$$

$$= \left(\frac{\frac{e^{\theta}+e^{-\theta}}{2}+\frac{e^{\theta}-e^{-\theta}}{2}}{\frac{e^{\theta}+e^{-\theta}}{2}-\frac{e^{\theta}-e^{-\theta}}{2}}\right)^3$$

$$= \left(\frac{e^{\theta}+\cancel{e^{-\theta}}+e^{\theta}-\cancel{e^{-\theta}}}{\cancel{e^{\theta}}+e^{-\theta}-\cancel{e^{\theta}}+e^{-\theta}}\right)^3$$

$\because \cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$

$\& \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$

$$= \left(\frac{2e^{\theta}}{2e^{-\theta}}\right)^3$$

$$= (e^{2\theta})^3 = e^{6\theta}$$

$= e^{6\theta}$

$= e^{6\theta} + e^{-6\theta} - e^{6\theta} - e^{-6\theta}$

$= \frac{1}{2}\left(2e^{6\theta}\right)$

$= \frac{1}{2}\left[e^{6\theta} + e^{-6\theta} + e^{6\theta} - e^{-6\theta}\right]$ [adding & subtracting $e^{-6\theta}$]

$= \frac{e^{6\theta} + e^{-6\theta}}{2} + \frac{e^{6\theta} - e^{-6\theta}}{2}$

$= \cosh 6\theta + \sinh 6\theta$

R.H.S.

Proved

Group-4 (Assignment-5)

Q→ Express $\cosh^7\theta$ in terms of hyperbolic cosines of multiples of θ.

Solution→ We know $\cosh\theta = \dfrac{e^{\theta}+e^{-\theta}}{2}$

$$\Rightarrow \cosh^7\theta = \frac{1}{2^7}\left(e^{\theta}+e^{-\theta}\right)^7$$

$$= \frac{1}{2^7}\Big\{e^{7\theta} + {}^7C_1\, e^{6\theta} e^{-\theta} + {}^7C_2\, e^{5\theta}\cdot e^{-2\theta}$$
$$+ {}^7C_3\, e^{4\theta} e^{-3\theta} + {}^7C_4\, e^{3\theta}\cdot e^{-4\theta} + {}^7C_5\, e^{2\theta}\cdot e^{-5\theta}$$
$$+ {}^7C_6\, e^{\theta}\cdot e^{-6\theta} + {}^7C_7\, e^{-7\theta}\Big\}$$

$$= \frac{1}{2^7}\Big(e^{7\theta} + 7e^{5\theta} + 21e^{3\theta} + 35e^{\theta} + 35e^{-\theta}$$
$$+ 21e^{-3\theta} + 7e^{-5\theta} + e^{-7\theta}\Big)$$

$$= \frac{1}{2^7}\Big\{\left(e^{7\theta}+e^{-7\theta}\right) + 7\left(e^{5\theta}+e^{-5\theta}\right) + 35\left(e^{\theta}+e^{-\theta}\right)$$
$$+ 21\left(e^{3\theta}+e^{-3\theta}\right)\Big\}$$

$$= \frac{1}{2^6}\Big\{\left(\frac{e^{7\theta}+e^{-7\theta}}{2}\right) + 7\left(\frac{e^{5\theta}+e^{-5\theta}}{2}\right) + 21\left(\frac{e^{3\theta}+e^{-3\theta}}{2}\right)$$
$$+ 35\left(\frac{e^{\theta}+e^{-\theta}}{2}\right)\Big\}$$

$$= \frac{1}{2^6}\left(\cosh 7\theta + 7\cosh 5\theta + 21\cosh 3\theta + 35\cosh\theta\right)$$

$$= \frac{1}{2^6}\left(\cosh 7\theta + 7\cosh 5\theta + 21\cosh 3\theta + 35\cosh\theta\right)$$

MNM (Assignment ~~#~~5) Group-6

Problem- If $\sin\theta = \tanh x$,
Prove that
$$\tan\theta = \sinh x$$

Solution- We have

$$\sin\theta = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \text{--- (i)}$$

& $$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2}$$

$$= \sqrt{\frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}}$$

$$= \sqrt{\frac{4}{(e^x + e^{-x})^2}}$$

$$\cos\theta = \frac{2}{(e^x + e^{-x})} \quad \text{--- (ii)}$$

Now

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

using eqn (i) & (ii)

$$\tan\theta = \frac{e^x - e^{-x}}{2}$$

$$\boxed{\tan\theta = \sinh x} \qquad \left(\because \sinh x = \frac{e^x - e^{-x}}{2}\right)$$

Proved.

Group-8
Assignment-5

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If $\tan x/2 = \tanh(u/2)$ prove that i) $\tan x = \sinh u$ and $\cos x \cosh u = 1$

$$L.H.S = \tan x = \frac{2\tan x/2}{1-\tan^2 x/2}$$

$$= \frac{2\tanh(u/2)}{1-\tanh^2 u/2} = \frac{2\sinh(u/2)\cosh(u/2)}{\cosh^2(u/2)-\sinh^2(u/2)}$$

$$= \frac{2\left(\frac{e^{u/2}-e^{-u/2}}{2}\right)\left(\frac{e^{u/2}+e^{-u/2}}{2}\right)}{\left(\frac{e^{u/2}+e^{-u/2}}{2}\right)^2-\left(\frac{e^{u/2}-e^{-u/2}}{2}\right)^2}$$

$$= \frac{2\,(e^{u}-e^{-u})}{(e^{u/2}+e^{-u/2})^2-(e^{u/2}-e^{-u/2})^2}, \quad \left[(a+b)(a-b) = a^2-b^2\right]$$

$$= \frac{2\,(e^{u}-e^{-u})}{\left(e^{u/2}+e^{-u/2}+e^{u/2}-e^{-u/2}\right)\left(e^{u/2}+e^{-u/2}-e^{u/2}+e^{-u/2}\right)}$$

$$= \frac{2\,(e^{u}-e^{-u})}{2e^{u/2}\cdot 2e^{-u/2}} \quad \left[a^2-b^2 = (a+b)(a-b)\right]$$

$$= \frac{e^{u}-e^{-u}}{2} = \sinh(u) = R.H.S \text{ proved}$$

Now, $\cosh^2 u - \sinh^2 u = 1$

or, $\cosh^2 u = 1+\sinh^2 u = 1+\tan^2 x$ [from the above eqn]

$= \sec^2 x$

$\Rightarrow \cosh u = \sec x$

or, $\cos x \cosh u = 1$ (proved)

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Assignment – 5

(Bivin Geo George and Ritwik) Group 10

Q.
5. (ii) if $\tan\frac{x}{2} = \tanh\frac{u}{2}$ prove that,

$$u = \log_e \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

Ans :-

$$\tan\frac{x}{2} = \tanh\frac{u}{2}$$

$$\Rightarrow \frac{u}{2} = \tanh^{-1}\left(\tan\frac{x}{2}\right)$$

Let, $\tanh^{-1}(a) = b$

$$a = \tanh b = \frac{e^b - e^{-b}}{e^b + e^{-b}}$$

$$\Rightarrow a(e^b + e^{-b}) = e^b - e^{-b}$$

$$\Rightarrow a e^{2b} + a = e^{2b} - 1$$

$$\Rightarrow (a-1)e^{2b} = -(a+1)$$

$$\Rightarrow e^{2b} = \frac{1+a}{1-a}$$

$$\Rightarrow 2b = \ln\left(\frac{1+a}{1-a}\right)$$

$$\Rightarrow b = \tanh^{-1}(a) = \frac{1}{2}\ln\left(\frac{1+a}{1-a}\right).$$

This is the formula we need actually

so, using that,

$$\tanh^{-1}\left(\tan\frac{x}{2}\right)$$

$$= \frac{1}{2}\ln\left[\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right]$$

$$\Rightarrow \frac{u}{2} = \frac{1}{2}\ln\left[\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right]$$

$$\Rightarrow u = \ln_e\left[\frac{\tan\frac{x}{2}+\tan\frac{\pi}{4}}{1-\tan\frac{x}{2}\tan\frac{\pi}{4}}\right]$$

$$= \ln_e\left[\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right]$$

(Proved)

ASSIGNMENT-05
GROUP - 07

If $\cosh x = \sec\theta$, prove that

$$\tanh^2\left(\frac{x}{2}\right) = \tan^2\left(\frac{\theta}{2}\right)$$

Given that

$$\cosh x = \sec\theta$$

$$\Rightarrow \cosh^2\left(\frac{x}{2}\right) + \sinh^2\left(\frac{x}{2}\right) = \frac{1}{\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \frac{1 + \tanh^2\left(\frac{x}{2}\right)}{\operatorname{sech}^2\left(\frac{x}{2}\right)} = \frac{\sec^2\left(\frac{\theta}{2}\right)}{1 - \tan^2\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \frac{1 + \tanh^2\left(\frac{x}{2}\right)}{1 - \tanh^2\left(\frac{x}{2}\right)} = \frac{1 + \tan^2\left(\frac{\theta}{2}\right)}{1 - \tan^2\left(\frac{\theta}{2}\right)}$$

Now applying componendo and dividendo

$$\boxed{\tanh^2\left(\frac{x}{2}\right) = \tan^2\left(\frac{\theta}{2}\right)}$$

Proved.

G-11 Assignment: 5 { Venkatesh chinni / Durgaprasad

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Q(ii): If $\cosh x = \sec\theta$, prove that $x = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

Sol Given $\cosh x = \sec\theta$

$$\Rightarrow \frac{e^x+e^{-x}}{2} = \frac{1}{\cos 2\frac{\theta}{2}} \qquad \left[\because \cosh x = \frac{e^x+e^{-x}}{2}\right]$$

$$= \frac{1}{\cos^2\theta/2 - \sin^2\theta/2}$$

$$= \frac{\sec^2\theta/2}{1-\tan^2\theta/2}$$

$$= \frac{2\,(1+\tan^2\theta/2)}{1-\tan^2\theta/2}$$

$$= \frac{1}{2}\left[\frac{(1-\tan\theta/2)^2+(1+\tan\theta/2)^2}{1-\tan^2\theta/2}\right]$$

$$= \frac{1}{2}\left[\frac{1-\tan\theta/2}{1+\tan\theta/2} + \frac{1+\tan\theta/2}{1-\tan\theta/2}\right]$$

$$\Rightarrow \frac{e^x+e^{-x}}{2} = \frac{1}{2}\left[\frac{1+\tan\theta/2}{1-\tan\theta/2} + \left(\frac{1+\tan\theta/2}{1-\tan\theta/2}\right)^{-1}\right]$$

$$\Rightarrow e^x+e^{-x} = \frac{1+\tan\theta/2}{1-\tan\theta/2} + \left(\frac{1+\tan\theta/2}{1-\tan\theta/2}\right)^{-1}$$

By comparing the terms we get

$$e^x = \frac{1+\tan\theta/2}{1-\tan\theta/2}$$

$$\Rightarrow \quad e^x = \frac{1+\tan\theta/2}{1-\tan\theta/2}$$

$$\Rightarrow \quad e^x = \frac{\tan\left(\frac{\pi}{4}\right)+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \quad e^x = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) \qquad \left[\because \tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}\right]$$

$\Rightarrow$ Take log on both sides

$$\Rightarrow \quad \log(e^x) = \log\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right)$$

$$\Rightarrow \quad x\log_e e = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$\Rightarrow \quad \boxed{x = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}$$

Hence, proved.

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8. Prove that $\sinh^{-1} x = \tanh^{-1}\left|\frac{x}{\sqrt{1+x^2}}\right|$.

$\Rightarrow$ Let $\sinh^{-1} x = u$.

$\therefore x = \sinh u = \frac{e^u - e^{-u}}{2}$

or, $e^u - \frac{1}{e^u} = 2x$

or, $(e^u)^2 - 2x e^u - 1 = 0$. the solⁿ to this quadratic eqⁿ is:

So, $e^u = \frac{2x \pm \sqrt{(2x)^2 - 4.1.(-1)}}{2} = x \pm \sqrt{1+x^2}$

So, $e^{-u} = \frac{1}{x \pm \sqrt{1+x^2}} = \frac{x \mp \sqrt{1+x^2}}{(x \pm \sqrt{1+x^2})(x \mp \sqrt{1+x^2})}$

$= \frac{x \mp \sqrt{1+x^2}}{x^2 - (\sqrt{1+x^2})^2} = -x \pm \sqrt{1+x^2}$

So, $\tanh u = \frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{x \pm \sqrt{1+x^2} + x \mp \sqrt{1+x^2}}{x \pm \sqrt{1+x^2} - x \pm \sqrt{1+x^2}}$

$= \pm \frac{x}{\sqrt{1+x^2}}$

So, $u = \tanh^{-1}\left|\frac{x}{\sqrt{1+x^2}}\right|$.

$\therefore$ L.H.S. $= \sinh^{-1} x = u = \tanh^{-1}\left|\frac{x}{\sqrt{1+x^2}}\right|$ = R.H.S. (proved)

[Note: The problem given in the problem sheet was:
RHS $= \tanh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$. I request you to please see to it.]

Assignment 5

Group 5: Apurv & Sanjay

Question

5. Show that $\operatorname{sech}^{-1}\sin\theta = \log\cot\theta/2$

Solution

Digression :

Let

$$\operatorname{sech}^{-1} x = y \tag{0.0.1}$$

Then

$$x = \operatorname{sech} y \tag{0.0.2}$$

$$x = \frac{2e^y}{e^{2y}+1} \tag{0.0.3}$$

Rearranging terms

$$e^{2y}x - 2e^y + x = 0 \tag{0.0.4}$$

Solving for e^y from the above equation(It is a quadratic equation in e^y).

$$e^y = \frac{1 \pm \sqrt{1+x^2}}{x} \tag{0.0.5}$$

We know that Range(e^y) $\in \mathbb{R}^+$, so that gives a restriction on the values x can take, i. e. $0 < x \leq 1$, also the $(1-\sqrt{1+x^2})/x$ solution has to be dropped as it is negative in the interval. Then

$$e^y = \frac{1+\sqrt{1+x^2}}{x} \tag{0.0.6}$$

From Eq.0.0.1 and Eq.0.0.6,

$$\operatorname{sech}^{-1} x = \log\left[\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right] \qquad \forall\, x \in (0,1] \tag{0.0.7}$$

In the question we have to show $\operatorname{sech}^{-1}\sin\theta = \log\cot\theta/2$. From Eq0.0.7

$$\begin{aligned}
\operatorname{sech}^{-1}\sin\theta &= \log\left[\frac{1}{\sin\theta} + \frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right] && (0.0.8)\\
&= \log\left[\frac{1+\cos\theta}{\sin\theta}\right] && (0.0.9)\\
&= \log\frac{2\cos^2\frac{\theta}{2}}{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}} && (0.0.10)\\
&= \log\frac{\cos\theta/2}{\sin\theta/2} && (0.0.11)\\
&= \log\cot\frac{\theta}{2} \qquad Q.E.D && (0.0.12)
\end{aligned}$$